

# Fanshawe Pre-Health Sciences Mathematics 2



# FANSHAWE PRE-HEALTH SCIENCES MATHEMATICS 2

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Fanshawe College Pressbooks  
London Ontario



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# WELCOME TO PRE-HEALTH SCIENCES MATHEMATICS 2!

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Welcome to Pre-Health Sciences (PHS) term 2, and in particular, to MATH-1028!

In this course, we will cover the following topics.

- Unit 1: Factoring
- Unit 2: Rational Expressions
- Unit 3: Roots and Radical Expressions
- Unit 4: Quadratic Equations and Functions
- Unit 5: Exponential and Logarithmic Equations
- Unit 6: Introduction to Statistics

Through our coverage of these topics, we will continue towards the main goal of developing resilience, critical thinking skills, and self-directed learning skills. In addition to these transferable skills, we will aim to develop your attention to detail, determination, time management skills, as well as your mental mathematics skills.

In health care, making a small calculation error can be life threatening, and therefore, it is essential that we develop our intuition with numbers and mathematics throughout the PHS math program. To do this, we will often work without a calculator to improve on our mathematical abilities.

Mathematics can be challenging, and it is critical that students keep an open-mind and remain positive. Remember that it takes a lot of practice to develop these foundational skills, so if you do not succeed right away, stay patient, ask for help, and keep working hard!

# ACKNOWLEDGEMENTS

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## Attribution

This book is primarily an adaptation of the following resource:

- Intermediate Algebra 2e by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

Unit 6 was adapted from:

- Introduction to Statistics in the Psychological Sciences by Linda R. Cote Ph.D.; Rupa G. Gordon Ph.D.; Chrislyn E. Randell Ph.D.; Judy Schmitt; and Helena Marvin is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License, except where otherwise noted.
- Prealgebra 2e. by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.
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A list of changes from the previous sources can be found at Changes from Adapted Text in the About This Book section.

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## Collaborators

This project was a collaboration between the author and the team in the OER Design Studio at Fanshawe. The following staff and students were involved in the creation of this project:

- David Giesbrecht, *Instructional Design Student*
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- Melanie Mitchell Sparkes – *Instructional Design Student*
- Catherine Steeves – *Instructional Designer*
- Robert Armstrong – *Instructional & Graphic Design*
- Salma Habib- *Graphic Design Student*
- Shauna Roch – *Project Lead*

# ABOUT THIS BOOK

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## Changes From Adapted Resource

Overall significant work was done on this version of the text to improve the accessibility and format from the import from Open Stax and other resources.

All mathematical notation was written in LaTeX to allow for MathJax functionality. MathJax provides the following accessibility features:

- Automatic magnification on hover
- Text-to-speech options
- Collapsible math
- Highlight, foreground, and background customization
- The ability to translate math into other syntaxes
- MathJax options in over 20 languages

See the full extent of MathJax's accessibility features in their documentation.

In addition the following formatting changes were made:

- The majority of images that included math were replaced with LaTeX
- Images were recolored for contrast
- Examples were reformatted into steps for ease of understanding

Specific content changes from the adapted resources can be found in the table below.

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<b>Unit 1</b>	All sections from Intermediate Algebra 2e Section 6.
<b>Unit 2</b>	All sections from Intermediate Algebra 2e Section 7. Removed section on ‘solve similar figure applications’ from section 7.5.
<b>Unit 3</b>	All sections from Intermediate Algebra 2e. Section 8. Removed section on ‘use radicals in functions’ from section 8.7.
<b>Unit 4</b>	All sections from Intermediate Algebra 2e. Section 9. Removed section on ‘solve quadratic inequalities’ from section 9.8.
<b>Unit 5</b>	All sections from Intermediate Algebra 2e. Section 10. Removed section on ‘finding composite and inverse functions’ from section 10.1
	Unit is a compilation of Introduction to Statistics in the Psychological Sciences and sections from Prealgebra 2e and Introductory Statistics Re-arranged content for consistency.
<b>Unit 6</b>	All content reworded as it was from the perspective of psychology students. Split the introduction chapter into an introduction and new section 6.1 Added an example on creating a histogram, content on outliers, Added a z-score table. Changed the section on probability to align with teaching methods. Added the examples from “the standard normal distribution” and “the standardizing process” and the “un-standardizing process” section from 5.2” into 6.5 Probability and the Normal Distribution

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## Accessibility Statement

We are actively committed to increasing the accessibility and usability of the textbooks we produce. Every attempt has been made to make this OER accessible to all learners and is compatible with assistive and adaptive technologies. We have attempted to provide closed captions, alternative text, or multiple formats for on-screen and off-line access.

The web version of this resource has been designed to meet Web Content Accessibility Guidelines 2.0, level AA. In addition, it follows all guidelines in Appendix A: Checklist for Accessibility of the *Accessibility Toolkit – 2nd Edition*.

In addition to the web version, additional files are available in a number of file formats including PDF, EPUB (for eReaders), and MOBI (for Kindles).

If you are having problems accessing this resource, please contact us at [oyer@fanshawec.ca](mailto:oyer@fanshawec.ca).

Please include the following information:

- The location of the problem by providing a web address or page description
- A description of the problem

- The computer, software, browser, and any assistive technology you are using that can help us diagnose and solve your issue (e.g., Windows 10, Google Chrome (Version 65.0.3325.181), NVDA screen reader)

## Feedback & Adoption

To provide feedback on this text please contact **oer@fanshawec.ca**. If you adopt this text, please let us know.

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Pedrosa, L. (2019, July 10). MathJax: now available on PressbooksEDU. Pressbooks. <https://pressbooks.com/new-features/mathjax-now-available-on-pressbooksedu/>

# UNIT 1: FACTORING

## Chapter Outline

- 1.0 Introduction to Factoring
- 1.1 Greatest Common Factor and Factor by Grouping
- 1.2 Factor Trinomials
- 1.3 Factor Special Products
- 1.4 General Strategy for Factoring Polynomials
- 1.5 Polynomial Equations





# 1.0 INTRODUCTION TO FACTORING

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“Pathologists” by Linda Bartlett CCO

Scientists use factoring to calculate growth rates of infectious diseases such as viruses.

An epidemic of a disease has broken out. Where did it start? How is it spreading? What can be done to control it? Answers to these and other questions can be found by scientists known as epidemiologists. They collect data and analyze it to study disease and consider possible control measures. Because diseases can spread at alarming rates, these scientists must use their knowledge of mathematics involving factoring. In this chapter, you will learn how to factor and apply factoring to real-life situations.

# 1.1 GREATEST COMMON FACTOR AND FACTOR BY GROUPING

---

## Learning Objectives

By the end of this section, you will be able to:

- Find the greatest common factor of two or more expressions
- Factor the greatest common factor from a polynomial
- Factor by grouping

## Try It

Before you get started, take this readiness quiz:

- 1) Factor **56** into primes.
- 2) Find the least common multiple (LCM) of **18** and **24**.
- 3) Multiply:  $(-3a)(7a + 8b)$ .

## Find the Greatest Common Factor of Two or More Expressions

Earlier we multiplied factors together to get a product. Now, we will reverse this process; we will start with a product and then break it down into its factors. Splitting a product into factors is called **factoring**.

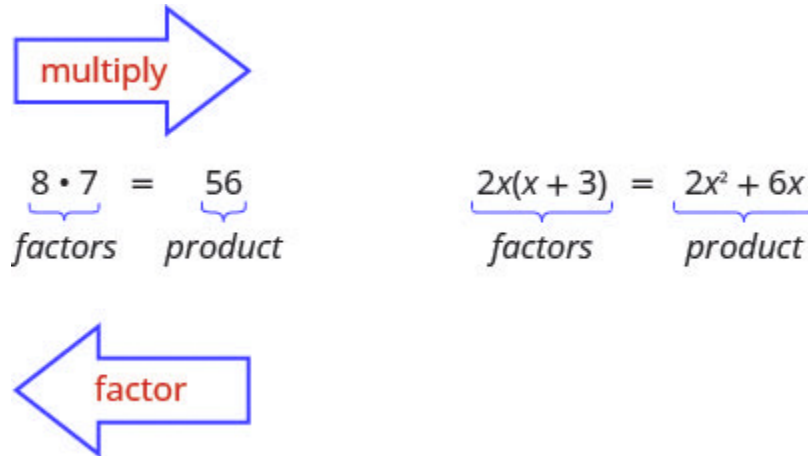


Figure 1.1.1

We have learned how to factor numbers to find the least common multiple (LCM) of two or more numbers. Now we will factor expressions and find the **greatest common factor** of two or more expressions. The method we use is similar to what we used to find the LCM.

### Greatest Common Factor

The **greatest common factor** (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

We summarize the steps we use to find the greatest common factor.

## HOW TO

### Find the greatest common factor (GCF) of two expressions.

1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
2. List all factors—matching common factors in a column. In each column, circle the common factors.
3. Bring down the common factors that all expressions share.
4. Multiply the factors.

The next example will show us the steps to find the greatest common factor of three expressions.

### Example 1.1.1

Find the greatest common factor of  $21x^3$ ,  $9x^2$ ,  $15x$ .

#### Solution

**Step 1: Factor each coefficient into primes and write the variables with exponents in expanded form.**

Circle the common factors in each column.

Bring down the common factors.

$$\begin{array}{r}
 21x^3 = 3 \cdot 7 \cdot x \cdot x \cdot x \\
 9x^2 = 3 \cdot 3 \cdot x \cdot x \\
 15x = 3 \cdot 5 \cdot x \\
 \hline
 \text{GCF} = 3 \cdot x
 \end{array}$$

Figure 1.1.2

**Step 2: Multiply the factors.**

$$GCF = 3x$$

**Step 3: Worded statement.**

The GCF of  $21x^3$ ,  $9x^2$  and  $15x$  is  $3x$ .

### Try It

4) Find the greatest common factor:  $25m^4$ ,  $35m^3$ ,  $20m^2$ .

**Solution**

$$5m^2$$

### Try It

5) Find the greatest common factor:  $14x^3$ ,  $70x^2$ ,  $105x$ .

**Solution**

$$7x$$

## Factor the Greatest Common Factor from a Polynomial

It is sometimes useful to represent a number as a product of factors, for example, 12 as  $2 \cdot 6$  or  $3 \cdot 4$ . In algebra, it can also be useful to represent a polynomial in factored form. We will start with a product, such

as  $3x^2 + 15x$ , and end with its factors,  $3x(x + 5)$ . To do this we apply the Distributive Property “in reverse.”

We state the Distributive Property here just as you saw it in earlier chapters and “in reverse.”

## Distributive Property

If  $a$ ,  $b$ , and  $c$  are real numbers, then

$$a(b + c) = ab + ac \quad \text{and} \quad ab + ac = a(b + c)$$

The form on the left is used to multiply. The form on the right is used to factor.

So how do you use the Distributive Property to factor a polynomial? You just find the GCF of all the terms and write the polynomial as a product!

### Example 1.1.2

How to Use the Distributive Property to factor a polynomials. Factor:

$$8m^3 - 12m^2n + 20mn^2.$$

#### Solution

**Step 1.** Find the GCF of all the terms of the polynomial.

Find the GCF of  
 $8m^3, 12m^2n, 20mn^2$

$$\begin{array}{l} 8m^3 = 2 \cdot 2 \cdot 2 \cdot m \cdot m \cdot m \\ 12m^2n = 2 \cdot 2 \cdot 3 \cdot m \cdot m \cdot n \\ 20mn^2 = 2 \cdot 2 \cdot 5 \cdot m \cdot n \cdot n \\ \hline \text{GCF} = 2 \cdot 2 \cdot m \\ \text{GCF} = 4m \end{array}$$

Figure 1.1.3

**Step 2.** Rewrite each term as a product using the GCF.

Rewrite  $8m^3$ ,  $12m^2n$ ,  $20mn^2$  as products of their GCF,  $4m$ .

$$\begin{aligned}8m^3 &= 4m \cdot 2m^2 \\12m^2n &= -4m \cdot 3m n \\20mn^2 &= 4m \cdot 5n^2\end{aligned}$$

$$\begin{aligned}8m^3 - 12m^2n + 20mn^2 \\4m \cdot 2m^2 - 4m \cdot 3m n + 4m \cdot 5n^2\end{aligned}$$

Figure 1.1.4

**Step 3.** Use the “reverse” Distributive Property to factor the expression.

$$4m(2m^2 - 3m n + 5n^2)$$

Figure 1.1.5

**Step 4.** Check by multiplying the factors.

$$\begin{aligned}4m(2m^2 - 3m n + 5n^2) \\4m \cdot 2m^2 - 4m \cdot 3m n + 4m \cdot 5n^2 \\8m^3 - 12m^2n + 20mn^2 \checkmark\end{aligned}$$

Figure 1.1.6

## Try It

6) Factor:  $9xy^2 + 6x^2y^2 + 21y^3$ .

**Solution**

$$3y^2 (3x + 2x^2 + 7y)$$

## Try It

7) Factor:  $3p^3 - 6p^2q + 9pq^3$ .

### Solution

$$3p(p^2 - 2pq + 3q^2)$$

## Factor the greatest common factor from a polynomial.

1. Find the GCF of all the terms of the polynomial.
2. Rewrite each term as a product using the GCF.
3. Use the “reverse” Distributive Property to factor the expression.
4. Check by multiplying the factors.

## Factor as a Noun and a Verb

We use “factor” as both a noun and a verb:

Noun:

7 is a factor of 14

Verb:

factor 3 from  $3a + 3$



### Example 1.1.3

Factor:  $5x^3 - 25x^2$ .

#### Solution

**Step 1:** Find the GCF of  $5x^3$  and  $25x^2$ .

$$\begin{array}{r} 5x^3 = 5 \cdot x \cdot x \cdot x \\ 25x^2 = 5 \cdot 5 \cdot x \cdot x \\ \hline \text{GCF} = 5 \cdot x \cdot x \end{array}$$

Figure 1.1.7

$$\begin{array}{l} \text{GCF} \\ \text{Rewrite} \\ \text{each term.} \\ \text{Factor the GCF.} \end{array} \quad \begin{array}{l} 5x^2 \\ 5x^3 \\ 25x^2 \\ 5x^2(x-5) \end{array}$$

**Step 2: Check.**

$$\begin{aligned} &= 5x^2(x - 5) \\ &= 5x^2 \times x - 5x^2 \times 5 \\ &= 5x^3 - 25x^2 \checkmark \end{aligned}$$

### Try It

8) Factor:  $2x^3 + 12x^2$ .

#### Solution

$$2x^2(x + 6)$$

## Try It

9) Factor:  $6y^3 - 15y^2$ .

**Solution**

$$3y^2(2y - 5)$$

## Example 1.1.4

Factor:  $8x^3y$ ,  $-10x^2y^2$  and  $12xy^3$ .

**Solution**

**Step 1: Find the GCF.**

$8x^3y = 2 \cdot 2 \cdot 2$	$(x \cdot x \cdot x \cdot y)$
$10x^2y^2 = 2 \cdot 5$	$(x \cdot x \cdot y \cdot y)$
$12xy^3 = 2 \cdot 2 \cdot 3$	$(x \cdot y \cdot y \cdot y)$
<b>GCF = <math>2 \cdot</math></b>	<b><math>x \cdot y</math></b>

Figure 1.1.8

$$\text{GCF} = 2xy \cdot (4x^2 - 5xy + 6y^2)$$
 \text{Rewrite using GCF.}

**Step 2: Check.**

$$\begin{aligned}
 &= 2xy(4x^2 - 5xy + 6y^2) \\
 &= 2xy \times 4x^2 - 2xy \times 5xy + 2xy \times 6y^2 \\
 &= 8x^3y - 10x^2y^2 + 12xy^3 \checkmark
 \end{aligned}$$

## Try It

10) Factor:  $15x^3y - 3x^2y^2 + 6xy^3$ .

**Solution**

$$3xy(5x^2 - xy + 2y^2)$$

## Try It

11) Factor:  $8a^3b + 2a^2b^2 - 6ab^3$ .

**Solution**

$$2ab(4a^2 + ab - 3b^2)$$

When the leading coefficient is negative, we factor the negative out as part of the GCF.

### Example 1.1.5

Factor:  $-4a^3 + 36a^2 - 8a$ .

#### Solution

**Step 1: Rewrite each term using the GCF,  $-4a$ .**

The leading coefficient is negative, so the GCF will be negative.

$$\begin{aligned} & \begin{aligned} & \text{Factor the} \\ & \text{GCF.} \end{aligned} \\ & \begin{aligned} & = -4a(a^2 - 9a + 2) \end{aligned} \end{aligned}$$

**Step 2: Check.**

$$\begin{aligned} & = -4a(a^2 - 9a + 2) \\ & = -4a \times a^2 - (-4a) \times 9a + (-4a) \times 2 \\ & = -4a^3 + 36a^2 - 8a \checkmark \end{aligned}$$

### Try It

12) Factor:  $-4b^3 + 16b^2 - 8b$ .

#### Solution

$$-4b(b^2 - 4b + 2)$$

## Try It

13) Factor:  $-7a^3 + 21a^2 - 14a$ .

### Solution

$$-7a(a^2 - 3a + 2)$$

So far our greatest common factors have been monomials. In the next example, the greatest common factor is a binomial.

## Example 1.1.6

Factor:  $3y(y + 7) - 4(y + 7)$ .

### Solution

#### Step 1: Factor the GCF.

The GCF is the binomial  $y + 7$ .

$= 3yrgb[1.0, 0.0, 0.0](rgb[1.0, 0.0, 0.0]yrgb[1.0, 0.0, 0.0]+rgb[1.0, 0.0, 0.0]7rgb[1.0, 0.0, 0.0]) - 4rgb[1.0, 0.0, 0.0](rgb[1.0, 0.0, 0.0]yrgb[1.0, 0.0, 0.0]+rgb[1.0, 0.0, 0.0]7rgb[1.0, 0.0, 0.0])$   
 $rgb[0.1, 0.1, 0.1]$ Factor the GCF. $rgb[0.1, 0.1, 0.1] = rgb[0.1, 0.1, 0.1](rgb[0.1, 0.1, 0.1]yrgb[0.1, 0.1, 0.1]+rgb[0.1, 0.1, 0.1]7rgb[0.1, 0.1, 0.1]) - 4rgb[0.1, 0.1, 0.1](rgb[0.1, 0.1, 0.1]yrgb[0.1, 0.1, 0.1]+rgb[0.1, 0.1, 0.1]7rgb[0.1, 0.1, 0.1])$

#### Step 2: Check on your own by multiplying.

## Try It

14) Factor:  $4m(m + 3) - 7(m + 3)$ .

**Solution**

$$(m + 3)(4m - 7)$$

## Try It

15) Factor:  $8n(n - 4) + 5(n - 4)$ .

**Solution**

$$(n - 4)(8n + 5)$$

## Factor by Grouping

Sometimes there is no common factor of all the terms of a polynomial. When there are four terms we separate the polynomial into two parts with two terms in each part. Then look for the GCF in each part. If the polynomial can be factored, you will find a common factor emerges from both parts. Not all polynomials can be factored. Just like some numbers are prime, some polynomials are prime.

## Example 1.1.7

How to Factor a Polynomial by Grouping. Factor by grouping:  $xy + 3y + 2x + 6$ .

### Solution

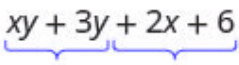
<b>Step 1.</b> Group terms with common factors.	Is there a greatest common factor of all four terms?	$xy + 3y + 2x + 6$
	No, so let's separate the first two terms from the second two.	$xy + 3y + 2x + 6$ 

Figure 1.1.9

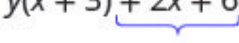
<b>Step 2.</b> Factor out the common factor in each group.	Factor the GCF from the first two terms.	$y(x + 3) + 2x + 6$ 
	Factor the GCF from the second two terms.	$y(x + 3) + 2(x + 3)$

Figure 1.1.10

<b>Step 3.</b> Factor the common factor from the expression.	Notice that each term has a common factor of $(x + 3)$ .	$y(x + 3) + 2(x + 3)$
	Factor out the common factor.	$(x + 3)(y + 2)$

Figure 1.1.11

**Step 4. Check.**

Multiply  $(x + 3)(y + 2)$ . Is the product the original expression?

$$\begin{aligned}(x + 3)(y + 2) \\ xy + 2x + 3y + 6 \\ xy + 3y + 2x + 6 \checkmark\end{aligned}$$

Figure 1.1.12

**Try It**

16) Factor by grouping:  $xy + 8y + 3x + 24$ .

**Solution**

$$(x + 8)(y + 3)$$

**Try It**

17) Factor by grouping:  $ab + 7b + 8a + 56$ .

**Solution**

$$(a + 7)(b + 8)$$



## Factor by grouping.

1. Group terms with common factors.
2. Factor out the common factor in each group.
3. Factor the common factor from the expression.
4. Check by multiplying the factors.

### Example 1.1.8

Factor by Grouping:

a.  $x^2 + 3x - 2x - 6$

b.  $6x^2 - 3x - 4x + 2$

#### Solution

a.

#### Step 1: Find the GCF.

There is no GCF for all four terms, separate into two parts.

#### Step 2: Factor the GCF from both parts. Be careful with the signs when factoring the GCF from the last two terms.

$$= rgb]0.0, 0.0, 1.0x(x + 3)rgb]0.0, 0.0, 1.0-rgb]0.0, 0.0, 1.02(x + 3)$$

Factor out common factor.  $= (x + 3)(rgb]0.0, 0.0, 1.0xrgb]0.0, 0.0, 1.0-rgb]0.0, 0.0, 1.02)$

#### Step 3: Check on your own by multiplying.

b.

#### Step 1: Find the GCF.

There is no GCF for all four terms, separate into two parts.

**Step 2: Factor the GCF from both parts.**

$$= rgb]0.0, 0.0, 1.03rgb]0.0, 0.0, 1.0x(2x - 1)rgb]0.0, 0.0, 1.0 - rgb]0.0, 0.0, 1.02(2x - 1)$$

Factor out common factor. =  $(2x - 1)(rgb]0.0, 0.0, 1.03rgb]0.0, 0.0, 1.0xrgb]0.0, 0.0, 1.0 - rgb]0.0, 0.0, 1.02)$

**Step 3: Check on your own by multiplying.**

## Try It

18) Factor by grouping:

a.  $x^2 + 2x - 5x - 10$

b.  $20x^2 - 16x - 15x + 12.$

**Solution**

a.  $(x - 5)(x + 2)$

b.  $(5x - 4)(4x - 3)$

## Try It

19) Factor by grouping:

a.  $y^2 + 4y - 7y - 28$

b.  $42m^2 - 18m - 35m + 15.$

**Solution**

- a.  $(y + 4)(y - 7)$   
 b.  $(7m - 3)(6m - 5)$

## Key Concepts

- **How to find the greatest common factor (GCF) of two expressions.**

1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
2. List all factors—matching common factors in a column. In each column, circle the common factors.
3. Bring down the common factors that all expressions share.
4. Multiply the factors.

- **Distributive Property:** If  $a$ ,  $b$ , and  $c$  are real numbers, then

$$a(b + c) = ab + ac \quad \text{and} \quad ab + ac = a(b + c)$$

The form on the left is used to multiply. The form on the right is used to factor.

- **How to factor the greatest common factor from a polynomial.**

1. Find the GCF of all the terms of the polynomial.
2. Rewrite each term as a product using the GCF.
3. Use the “reverse” Distributive Property to factor the expression.
4. Check by multiplying the factors.

- **Factor as a Noun and a Verb:** We use “factor” as both a noun and a verb.

Noun:           7 is a factor of 14

Verb:           factor 3 from  $3a + 3$

- **How to factor by grouping.**

1. Group terms with common factors.

2. Factor out the common factor in each group.
3. Factor the common factor from the expression.
4. Check by multiplying the factors.

## Self Check

After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

*<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=56#h5p-10>*

## Glossary

### **factoring**

Splitting a product into factors is called factoring.

### **greatest common factor**

The greatest common factor (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

# 1.2 FACTOR TRINOMIALS

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## Learning Objectives

By the end of this section, you will be able to:

- Factor trinomials of the form  $x^2 + bx + c$
- Factor trinomials of the form  $ax^2 + bx + c$  using trial and error
- Factor trinomials of the form  $ax^2 + bx + c$  using the 'ac' method
- Factor using substitution

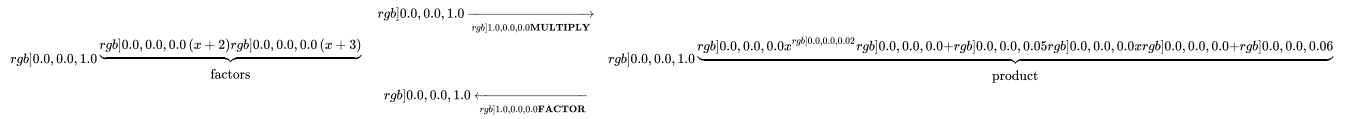
## Try It

Before you get started, take this readiness quiz:

- 1) Find all the factors of **72**.
- 2) Find the product:  $(3y + 4)(2y + 5)$ .
- 3) Simplify:  $-9(6)$  and  $-9(-6)$ .

## Factor Trinomials of the Form $x^2 + bx + c$

You have already learned how to multiply binomials using FOIL. Now you'll need to "undo" this multiplication. To factor the trinomial means to start with the product, and end with the factors.



## HOW TO

### Factor a Trinomial of the Form $x^2 + bx + c$

To figure out how we would factor a trinomial of the form  $x^2 + bx + c$ , such as  $x^2 + 5x + 6$  and factor it to  $(x + 2)(x + 3)$ , let's start with two general binomials of the form  $(x + m)$  and  $(x + n)$ .

	$(x + m)(x + n)$
Foil to find the product.	$x^2 + mx + nx + mn$
Factor the GCF from the middle terms.	$x^2 + (m + n)x + mn$
Our trinomial is of the form $x^2 + bx + c$ .	$\overbrace{x^2 + (m + n)x + mn}^{x^2 + bx + c}$

This tells us that to factor a trinomial of the form  $x^2 + bx + c$ , we need two factors  $(x + m)$  and  $(x + n)$  where the two numbers  $m$  and  $n$  multiply to  $c$  and add to  $b$ .

### Example 1.2.1

How to Factor a Trinomial of the form  $x^2 + bx + c$

Factor:  $x^2 + 11x + 24$ .

**Solution****Step 1: Write the factors as two binomials with first terms  $x$ .**

Write two sets of parentheses and put  $x$  as the first term.

$$(x^2 + 11x + 24)$$

$$(x \quad)(x \quad)$$

**Step 2: Find two numbers  $m$  and  $n$  that multiply to  $c$ ,  $m \cdot n = c$  and add to  $b$ ,  $m + n = b$ .**

Find two numbers that multiply to **24** and add to **11**.

Factors of 24	Sum of Factors
1, 24	$1 + 24 = 25$
2, 12	$2 + 12 = 14$
3, 8	$3 + 8 = 11^*$
4, 6	$4 + 6 = 10$

**Step 3: Use  $m$  and  $n$  as the last terms of the factors.**

Use **3** and **8** as the last terms of the binomials.

$$(x + 3)(x + 8)$$

**Step 4: Check by multiplying the factors.**

$$\begin{array}{rcl} & & (x+3)(x+8) \\ \hline & & = x^2 + 8x + 3x + 24 \\ & & = x^2 + 11x + 24 \end{array} \quad \checkmark$$

**Try It**

4) Factor:  $q^2 + 10q + 24$ .

**Solution**

$$(q + 4)(q + 6)$$

## Try It

5) Factor:  $t^2 + 14t + 24$ .

### Solution

$$(t + 2)(t + 12)$$

Let's summarize the steps we used to find the factors.

## HOW TO

Factor trinomials of the form  $x^2 + bx + c$ .

1. Write the factors as two binomials with first terms  $x$ .

$$\begin{array}{l} x^2 + bx + c \\ (x \quad)(x \quad) \end{array}$$

2. Find two numbers  $m$  and  $n$  that

- multiply to  $c$ ,  $m \cdot n = c$
- add to  $b$ ,  $m + n = b$

3. Use  $m$  and  $n$  as the last terms of the factors.

$$(x + m)(x + n)$$

4. Check by multiplying the factors.

In the first example, all terms in the trinomial were positive. What happens when there are negative terms? Well, it depends which term is negative. Let's look first at trinomials with only the middle term negative.



How do you get a *positive product* and a *negative sum*? We use two negative numbers.

### Example 1.2.2

Again, with the positive last term, 28, and the negative middle term,  $-11y$ , we need two negative factors. Find two numbers that multiply 28 and add to  $-11$ .

#### Solution

**Step 1: Write the factors as two binomials with first terms  $y$ .**

$$\begin{array}{c} y^2 - 11y + 28 \\ (y \quad)(y \quad) \end{array}$$

**Step 2: Find two numbers that multiply to 28 and add to  $-11$ .**

Factors of 28	Sum of factors
$-1, -28$	$-1 + (-28) = -29$
$-2, -14$	$-2 + (-14) = -16$
$-4, -7$	$-4 + (-7) = -11^*$

**Step 3: Use  $-4, -7$  as the last terms of the binomials.**

$$(y - 4)(y - 7)$$

**Step 4: Check.**

$$\begin{array}{l} \text{rc} \\ \&=(y-4)(y-7) \\ \&=y^2-7y-4y+28 \\ \&=y^2-11y+28 \\ \checkmark \end{array}$$

### Try It

6) Factor:  $u^2 - 9u + 18$ .

**Solution**

$$(u - 3)(u - 6)$$

**Try It**

7) Factor:  $y^2 - 16y + 63$ .

**Solution**

$$(y - 7)(y - 9)$$

Now, what if the last term in the trinomial is negative? Think about FOIL. The last term is the product of the last terms in the two binomials. A negative product results from multiplying two numbers with opposite signs. You have to be very careful to choose factors to make sure you get the correct sign for the middle term, too.

How do you get a *negative product* and a *positive sum*? We use one positive and one negative number.

When we factor trinomials, we must have the terms written in descending order—in order from highest degree to lowest degree.

**Example 1.2.3**

Factor:  $2x + x^2 - 48$ .

**Solution**

**Step 1:** First we put the terms in decreasing order.

$$x^2 + 2x - 48$$

**Step 2: Factors will be two binomials with first terms  $x$ .**

$$(x \quad)(x \quad)$$

Factors of $-48$	Sum of factors
$-1, 48$	$-1 + 48 = 47$
$-2, 24$	$-2 + 24 = 22$
$-3, 16$	$-3 + 16 = 13$
$-4, 12$	$-4 + 12 = 8$
$-6, 8$	$-6 + 8 = 2^*$

**Step 3: Use  $-6, -8$  as the last terms of the binomials.**

$$(x - 6)(x + 8)$$

**Step 4: Check.**

$$\begin{array}{l} \&= \&(x-6)(x+8) \\ \&= \&x^2-6x+8x-48 \\ \&= \&x^2+2x-48 \end{array}; \checkmark$$

## Try It

8) Factor:  $9m + m^2 + 18$ .

**Solution**

$$(m + 3)(m + 6)$$

## Try It

9) Factor:  $-7n + 12 + n^2$ .

### Solution

$$(n - 3)(n - 4)$$

Sometimes you'll need to factor trinomials of the form  $x^2 + bxy + cy^2$  with two variables, such as  $x^2 + 12xy + 36y^2$ . The first term,  $x^2$ , is the product of the first terms of the binomial factors,  $x \cdot x$ . The  $y^2$  in the last term means that the second terms of the binomial factors must each contain  $y$ . To get the coefficients  $b$  and  $c$ , you use the same process summarized in How To Factor trinomials.

## Example 1.2.4

We need  $r$  in the first term of each binomial and  $s$  in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

Factor:  $r^2 - 8rs - 9s^2$ .

### Solution

**Step 1:** Note that the first terms are  $r$ , and last terms contain  $s$ .

$$(r \quad s)(r \quad s)$$

**Step 2:** Find the numbers that multiply to  $-9$  and add to  $-8$ .

Factors of $-9$	Sum of factors
1, $-9$	$-1 + 9 = 8$
$-1, 9$	$1 + (-9) = -8^*$
3, $-3$	$3 + (-3) = 0$

**Step 3:** Use 1,  $-9$  as coefficients of the last terms.

$$(r + s)(r - 9s)$$

**Step 4:** Check.

$$\begin{array}{l} \text{\textbackslash begin\{array\}\{rcl\}\&=&\&(r-9s)(r+s)\&\&=&\&r^2+rs-9rs-9s^2\&\&=&r^2-8rs-9s^2;\&\checkmark\end{array}$$

## Try It

10) Factor:  $a^2 - 11ab + 10b^2$ .

**Solution**

$$(a - b)(a - 10b)$$

## Try It

11) Factor:  $m^2 - 13mn + 12n^2$ .

**Solution**

$$(m - n)(m - 12n)$$

Some trinomials are prime. The only way to be certain a trinomial is prime is to list all the possibilities and show that none of them work.

### Example 1.2.5

We need  $u$  in the first term of each binomial and  $v$  in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

Factor:  $u^2 - 9uv - 12v^2$ .

#### Solution

**Step 1:** Note that the first terms are  $u$ , and last terms contain  $v$ .

$$(u \quad v)(u \quad v)$$

**Step 2:** Find the numbers that multiply to  $-12$  and add to  $-9$ .

Factors of $-12$	Sum of factors
1, $-12$	$1 + (-12) = -11$
$-1$ , 12	$-1 + 12 = 11$
2, $-6$	$2 + (-6) = -4$
$-2$ , 6	$-2 + 6 = 4$
3, $-4$	$3 + (-4) = -1$
$-3$ , 4	$-3 + 4 = 1$

**Step 3:** Note there are no factor pairs that give us  $-9$  as a sum.

The trinomial is prime.

**Try It**

12) Factor:  $x^2 - 7xy - 10y^2$ .

**Solution**

Prime

**Try It**

13) Factor:  $p^2 + 15pq + 20q^2$ .

**Solution**

Prime

Let's summarize the method we just developed to factor trinomials of the form  $x^2 + bx + c$ .

**Strategy for Factoring Trinomials of the Form  $x^2 + bx + c$** 

When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

$$\begin{array}{c} x^2 + bx + c \\ (x + m)(x + n) \end{array}$$

**When  $c$  is positive,  $m$  and  $n$  have the same sign.**

$b$ positive	$b$ negative
$m, n$ positive	$m, n$ negative
$x^2 + 5x + 6$	$x^2 - 6x + 8$
$(x + 2)(x + 3)$	$(x - 4)(x - 2)$
same signs	same signs

**When  $c$  is negative,  $m$  and  $n$  have the same sign.**

$x^2 + x - 12$	$x^2 - 2x - 15$
$(x + 4)(x - 3)$	$(x - 5)(x + 3)$
opposite signs	opposite signs

Notice that, in the case when  $m$  and  $n$  have opposite signs, the sign of the one with the larger absolute value matches the sign of  $b$ .

## Factor Trinomials of the form $ax^2 + bx + c$ using Trial and Error

Our next step is to factor trinomials whose leading coefficient is not 1, trinomials of the form  $ax^2 + bx + c$ .

Remember to always check for a GCF first! Sometimes, after you factor the GCF, the leading coefficient of the trinomial becomes 1 and you can factor it by the methods we've used so far. Let's do an example to see how this works.



## Example 1.2.6

Factor:  $4x^3 + 16x^2 - 20x$ .

### Solution

#### Step 1: Is there a greatest common factor?

Yes,  $\text{GCF} = 4x$ . Factor it.

$$4x(x^2 + 4x - 5)$$

#### Step 2: Is it a binomial, trinomial, or more than three terms?

It is a trinomial. So “undo FOIL.”

$$4x(x \quad)(x \quad)$$

#### Step 3: Use a table like the one shown to find two numbers that multiply to $-5$ and add to $4$ .

Factors of $-5$	Sum of factors
$-1, 5$	$-1 + 5 = 4^*$
$1, -5$	$1 + (-5) = -4$

#### Step 4: Check.

$$\begin{aligned} &= 4x(x - 1)(x + 5) \\ &= 4x(x^2 + 5x - x - 5) \\ &= 4x(x^2 + 4x - 5) \\ &= 4x^3 + 16x^2 - 20x \checkmark \end{aligned}$$

**Try It**

14) Factor completely:  $5x^3 + 15x^2 - 20x$ .

**Solution**

$$5x(x - 1)(x + 4)$$

**Try It**

15) Factor completely:  $6y^3 + 18y^2 - 60y$ .

**Solution**

$$6y(y - 2)(y + 5)$$

What happens when the leading coefficient is not 1 and there is no GCF? There are several methods that can be used to factor these trinomials. First we will use the Trial and Error method.

Let's factor the trinomial  $3x^2 + 5x + 2$ .

From our earlier work, we expect this will factor into two binomials.

$$\begin{array}{c} 3x^2 + 5x + 2 \\ () () \end{array}$$

We know the first terms of the binomial factors will multiply to give us  $3x^2$ . The only factors of  $3x^2$  are  $1x$ ,  $3x$ . We can place them in the binomials.

$$\begin{array}{c}
 3x^2 + 5x + 2 \\
 \text{1x, 3x} \\
 \text{---} \\
 (x \quad ) (3x \quad )
 \end{array}$$

Figure 1.2.1

Check: Does  $1x \cdot 3x = 3x^2$ ?

We know the last terms of the binomials will multiply to 2. Since this trinomial has all positive terms, we only need to consider positive factors. The only factors of 2 are 1, 2. But we now have two cases to consider as it will make a difference if we write 1, 2 or 2, 1.

$$\begin{array}{c}
 3x^2 + 5x + 2 \\
 \text{1x, 3x} \quad \quad \text{1, 2} \\
 \text{---} \\
 (x + 1) (3x + 2)
 \end{array}
 \quad \text{or} \quad
 \begin{array}{c}
 3x^2 + 5x + 2 \\
 \text{1x, 3x} \quad \quad \text{1, 2} \\
 \text{---} \\
 (x + 2) (3x + 1)
 \end{array}$$

Figure 1.2.2

Which factors are correct? To decide that, we multiply the inner and outer terms.

$$\begin{array}{c}
 3x^2 + 5x + 2 \\
 \text{1x, 3x} \quad \quad \text{1, 2} \\
 \text{---} \\
 (x + 1) (3x + 2) \\
 \text{---} \\
 \text{3x} \\
 \text{2x} \\
 \text{---} \\
 \text{5x}
 \end{array}
 \quad \text{or} \quad
 \begin{array}{c}
 3x^2 + 5x + 2 \\
 \text{1x, 3x} \quad \quad \text{1, 2} \\
 \text{---} \\
 (x + 2) (3x + 1) \\
 \text{---} \\
 \text{6x} \\
 \text{1x} \\
 \text{---} \\
 \text{7x}
 \end{array}$$

Figure 1.2.3

Since the middle term of the trinomial is  $5x$ , the factors in the first case will work. Let's use FOIL to check.

$$\begin{aligned}
 &= (x + 1)(3x + 2) \\
 &= 3x^2 + 2x + 3x + 2 \\
 &= 3x^2 + 5x + 2 \quad \checkmark
 \end{aligned}$$

Our result of the factoring is:

$$\begin{array}{c}
 3x^2 + 5x + 2 \\
 (x + 1)(3x + 2)
 \end{array}$$

## Example 1.2.7

How to Factor a Trinomial Using Trial and Error

Factor completely using trial and error:  $3y^2 + 22y + 7$ .

### Solution

#### Step 1: Write the trinomial in descending order.

The trinomial is already in descending order.

$$3y^2 + 22y + 7$$

#### Step 2: Factor any GCF.

There is no GCF.

#### Step 3: Find all the factor pairs of the first term.

The only factors of  $3y^2$  are  $1y$ ,  $3y$ .

$$\begin{array}{c} 3y^2 + 22y + 7 \\ \color{red}{1y, 3y} \end{array}$$

Since there is only one pair, we can put them in the parentheses.

$$(y \quad)(3y \quad)$$

#### Step 4: Find all the factor pairs of the third term.

The only factors of 7 are 1, 7.

$$\begin{array}{c} 3y^2 + 22y + 7 \\ \color{red}{1y, 3y} \quad \color{red}{1, 7} \end{array}$$

$$(y \quad)(3y \quad)$$

#### Step 5: Test all the possible combinations of the factors until the correct product is found.

$$\begin{array}{c}
 3y^2 + 22y + 7 \\
 \text{1y, 3y} \qquad \qquad \text{1, 7} \\
 \\
 (y + 1)(3y + 7) \\
 \begin{array}{c}
 \text{3y} \\
 \text{7y} \\
 \hline
 10y
 \end{array} \\
 \text{10y No! We need 22y} \\
 \\
 3y^2 + 22y + 7 \\
 \text{1y, 3y} \qquad \qquad \text{1, 7} \\
 \\
 (y + 7)(3y + 1) \\
 \begin{array}{c}
 \text{21y} \\
 \text{+y} \\
 \hline
 22y
 \end{array}
 \end{array}$$

Figure 1.2.4

---

$3y^2 + 22y + 7$	
Possible Factors	Product
$(y + 1)(3y + 7)$	$3y^2 + 10y + 7$
$(y + 7)(3y + 1)$	$3y^2 + 22y + 7$

---

**Step 6: Check by multiplying.**

$$\begin{aligned}
 &= (y + 7)(3y + 1) \\
 &= 3y^2 + 22y + 7 \checkmark
 \end{aligned}$$

## Try It

16) Factor completely using trial and error:  $2a^2 + 5a + 3$ .

**Solution**

$$(a + 1)(2a + 3)$$

## Try It

17) Factor completely using trial and error:  $4b^2 + 5b + 1$ .

**Solution**

$$(b + 1)(4b + 1)$$

## HOW TO

**Factor trinomials of the form  $ax^2 + bx + c$  using trial and error.**

1. Write the trinomial in descending order of degrees as needed.
2. Factor any GCF.
3. Find all the factor pairs of the first term.
4. Find all the factor pairs of the third term.
5. Test all the possible combinations of the factors until the correct product is found.

6. Check by multiplying.

Remember, when the middle term is negative and the last term is positive, the signs in the binomials must both be negative.

### Example 1.2.8

Factor:  $6b^2 - 13b + 5$ .

#### Solution

**Step 1: The trinomial is already in descending order.**

$$6b^2 - 13b + 5$$

**Step 2: Find the factors of the first term.**

$$6b^2 - 13b + 5$$

**Step 3: Find the factors of the last term.**

Consider the signs. Since the last term, 5, is positive its factors must both be positive or both be negative. The coefficient of the middle term is negative, so we use the negative factors.

$$6b^2 - 13b + 5$$

**Step 4: Consider all the combinations of factors.**

$6b^2 - 13b + 5$	
Possible factors	Product
$(b - 1)(6b - 5)$	$6b^2 - 11b + 5$
$(b - 5)(6b - 1)$	$6b^2 - 31b + 5$
$(2b - 1)(3b - 5)$	$6b^2 - 13b + 5^*$
$(2b - 5)(3b - 1)$	$6b^2 - 17b + 5$

The correct factors are those whose product is the original trinomial.

$$(2b - 1)(3b - 5)$$

**Step 6: Check by multiplying.**

$$\begin{aligned}
 &= (2b - 1)(3b - 5) \\
 &= 6b^2 - 10b - 3b + 5 \\
 &= 6b^2 - 13b + 5 \checkmark
 \end{aligned}$$

## Try It

18) Factor completely using trial and error:  $8x^2 - 13x + 3$ .

**Solution**

$$(2x - 3)(4x - 1)$$



## Try It

19) Factor completely using trial and error:  $10y^2 - 37y + 7$ .

### Solution

$$(2y - 7)(5y - 1)$$

When we factor an expression, we always look for a greatest common factor first. If the expression does not have a greatest common factor, there cannot be one in its factors either. This may help us eliminate some of the possible factor combinations.

## Example 1.2.9

Factor:  $18x^2 - 37xy + 15y^2$ .

### Solution

**Step 1: The trinomial is already in descending order.**

$$18x^2 - 37xy + 15y^2$$

**Step 2: Find the factors of the first term.**

Formula does not parse

**Step 3: Find the factors of the last term.**

Consider the signs. Since 15 is positive and the coefficient of the middle term is negative, we use the negative factors.

Formula does not parse

**Step 4: Consider all the combinations of factors.**

$18x^2 - 37xy + 15y^2$	
Possible factors	Product
$(x - 1y)(18x - 15y)$	Not an option
$(x - 15y)(18x - 1y)$	$18x^2 - 271xy + 15y^2$
$(x - 3y)(18x - 5y)$	$18x^2 - 59xy + 15y^2$
$(x - 5y)(18x - 3y)$	Not an option
$(2x - 1y)(9x - 15y)$	Not an option
$(2x - 15y)(9x - 1y)$	$18x^2 - 137xy + 15y^2$
$(2x - 3y)(9x - 5y)$	$18x^2 - 37xy + 15y^2^*$
$(2x - 5y)(9x - 3y)$	Not an option
$(3x - 1y)(6x - 15y)$	Not an option
$(3x - 15y)(6x - 1y)$	Not an option
$(3x - 3y)(6x - 5y)$	Not an option

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

Figure 1.2.5

The correct factors are those whose product is the original trinomial.

$$(2x - 3y)(9x - 5y)$$

**Step 6: Check by multiplying.**

$$\begin{aligned}
 &= (2x - 3y)(9x - 5y) \\
 &= 18x^2 - 10xy - 27xy + 15y^2 \\
 &= 18x^2 - 37xy + 15y^2 \quad \checkmark
 \end{aligned}$$

**Try It**

20) Factor completely using trial and error  $18x^2 - 3xy - 10y^2$ .

**Solution**

$$(3x + 2y)(6x - 5y)$$

**Try It**

21) Factor completely using trial and error:  $30x^2 - 53xy - 21y^2$ .

**Solution**

$$(3x + y)(10x - 21y)$$

Don't forget to look for a GCF first and remember if the leading coefficient is negative, so is the GCF.

**Example 1.2.10**

Factor:  $-10y^4 - 55y^3 - 60y^2$ .

**Solution**

**Step 1: Notice the greatest common factor, so factor it first.**

$$-5y^2 (2y^2 + 11y + 12)$$

**Step 2: Factor the trinomial.**

Formula does not parse

**Step 3: Consider all the combinations.**

$2y^2 + 11y + 12$	
Possible factors	Product
$(y + 1)(2y + 12)$	Not an option
$(y + 12)(2y + 1)$	$2y^2 + 25y + 12$
$(y + 2)(2y + 6)$	Not an option
$(y + 6)(2y + 2)$	Not an option
$(y + 3)(2y + 4)$	Not an option
$(y + 4)(2y + 3)$	$2y^2 + 11y + 12^*$

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

Figure 1.2.6

The correct factors are those whose product is the original trinomial. Remember to include the factor  $-5y^2$ .

$$-5y^2 (y + 4)(2y + 3)$$

**Step 4: Check by multiplying.**

$$\begin{aligned} &= -5y^2 (y + 4)(2y + 3) \\ &= -5y^2 (2y^2 + 8y + 3y + 12) \\ &= -10y^4 - 55y^3 - 60y^2 \quad \checkmark \end{aligned}$$

**Try It**

22) Factor completely using trial and error:  $15n^3 - 85n^2 + 100n$ .

**Solution**

$$5n(n - 4)(3n - 5)$$

**Try It**

23) Factor completely using trial and error:  $56q^3 + 320q^2 - 96q$ .

**Solution**

$$8q(q + 6)(7q - 2)$$

## Factor Trinomials of the Form $ax^2 + bx + c$ using the “ac” Method

Another way to factor trinomials of the form  $ax^2 + bx + c$  is the “ac” method. (The “ac” method is sometimes called the grouping method.) The “ac” method is actually an extension of the methods you used in the last section to factor trinomials with leading coefficient one. This method is very structured (that is step-by-step), and it always works!

### Example 1.2.11

How to Factor Trinomials using the “ $ac$ ” Method

Factor using the ‘ $ac$ ’ method:  $6x^2 + 7x + 2$ .

#### Solution

##### Step 1: Factor any GCF.

Is there a greatest common factor? No!

$$6x^2 + 7x + 2$$

##### Step 2: Find the product $ac$ .

$$a \cdot c$$

$$6 \cdot 2$$

$$12$$

$$\begin{array}{c} ax^2 + bx + c \\ 6x^2 + 7x + 2 \end{array}$$

##### Step 3: Find two numbers $m$ and $n$ that:

**Multiply to  $ac$ .**       $m \cdot n = a \cdot c$

**Add to  $b$ .**           $m + n = b$

Find two numbers that multiply to 12 and add to 7. Both factors must be positive.

$$\underline{\underline{3 \cdot 4 = 12 \quad 3 + 4 = 7}}$$

##### Step 4: Split the middle term using $m$ and $n$ .

$$\begin{array}{c} ax^2 + bx + c \\ ax^2 + \overbrace{mx + nx}^{bx} + c \end{array}$$

Rewrite  $7x$  as  $3x + 4x$ . It would also give the same result if we used  $4x + 3x$ .

$$6x^2 + 7x + 2$$

$$6x^2 + 3x + 4x + 2$$

Figure 1.2.7

Notice that  $6x^2 + 3x + 4x + 2$  is equal to  $6x^2 + 7x + 2$ . We just split the middle term to get a more useful form.

**Step 5: Factor by grouping.**

$$3x(2x + 1) + 2(2x + 1)$$

$$(2x + 1)(3x + 2)$$

**Step 6: Check by multiplying the factors.**

$$= (2x + 1)(3x + 2)$$

$$= 6x^2 + 4x + 3x + 2$$

$$= 6x^2 + 7x + 2 \checkmark$$

## Try It

24) Factor using the 'ac' method:  $6x^2 + 13x + 2$ .

**Solution**

$$(x + 2)(6x + 1)$$

## Try It

25) Factor using the 'ac' method:  $4y^2 + 8y + 3$ .

### Solution

$$(2y + 1)(2y + 3)$$

## HOW TO

**Factor trinomials of the form  $ax^2 + bx + c$  using the "ac" method.**

The "ac" method is summarized here.

1. Factor any GCF.
2. Find the product  $ac$ .
3. Find two numbers  $m$  and  $n$  that:

Multiply to  $ac$

Add to  $b$

$$m \cdot n = a \cdot c$$

$$m + n = b$$

$$ax^2 + bx + c$$

4. Split the middle term using  $m$  and  $n$ .  $ax^2 + mx + nx + c$
5. Factor by grouping.
6. Check by multiplying the factors.

Don't forget to look for a common factor!



### Example 1.2.12

Factor:  $10y^2 - 55y + 70$ .

#### Solution

##### Step 1: Is there a greatest common factor?

Yes. The GCF is 5.

##### Step 2: Factor it.

$$5(2y^2 - 11y + 14)$$

The trinomial inside the parentheses has a leading coefficient that is not 1.

$$5(\overset{ax^2 + bx + c}{2y^2 - 11y + 14})$$

##### Step 3: Find the product $ac$ .

$$ac = 28$$

##### Step 4: Find two numbers that multiply to $ac$ and add to $b$ .

---


$$(-4)(-7) = 28 \quad -4 + (-7) = -11$$


---

##### Step 5: Split the middle term.

$$5(2y^2 - 11y + 14)$$

$$5(\underbrace{2y^2 - 7y}_{-7y} - \underbrace{4y + 14}_{-4y} + 14)$$

##### Step 6: Factor the trinomial by grouping.

$$5(y(2y - 7) - 2(y - 7))$$

##### Step 7: Check by multiplying all three factors.

$$\begin{aligned}
 &= 5(y - 2)(2y - 7) \\
 &= 5(2y^2 - 7y - 4y + 14) \\
 &= 5(2y^2 - 11y + 14) \\
 &= 10y^2 - 55y + 70 \checkmark
 \end{aligned}$$

### Try It

26) Factor using the 'ac' method:  $16x^2 - 32x + 12$ .

**Solution**

$$4(2x - 3)(2x - 1)$$

### Try It

27) Factor using the 'ac' method:  $18w^2 - 39w + 18$ .

**Solution**

$$3(3w - 2)(2w - 3)$$

## Factor Using Substitution

Sometimes a trinomial does not appear to be in the  $ax^2 + bx + c$  form. However, we can often make a

thoughtful substitution that will allow us to make it fit the  $ax^2 + bx + c$  form. This is called factoring by substitution. It is standard to use  $u$  for the substitution.

In the  $ax^2 + bx + c$ , the middle term has a variable,  $x$ , and its square,  $x^2$ , is the variable part of the first term. Look for this relationship as you try to find a substitution.

### Example 1.2.13

Factor:  $x^4 - 4x^2 - 5$ .

#### Solution

The variable part of the middle term is  $x^2$  and its square,  $x^4$ , is the variable part of the first term. (We know  $(x^2)^2 = x^4$ ). If we let  $u = x^2$ , we can put our trinomial in the  $ax^2 + bx + c$  form we need to factor it.

**Step 1: Rewrite the trinomial to prepare for the substitution.**

$$(x^2)^2 - 4(x^2) - 5$$

**Step 2: Let  $u = x^2$  and substitute.**

$$u^2 - 4u - 5$$

**Step 3: Factor the trinomial.**

$$(u + 1)(u - 5)$$

**Step 4: Replace  $u$  with  $x^2$ .**

$$(x^2 + 1)(x^2 - 5)$$

**Step 5: Check.**

$$\begin{aligned} &= (x^2 + 1)(x^2 - 5) \\ &= x^4 - 5x^2 + x^2 - 5 \\ &= x^4 - 4x^2 - 5 \quad \checkmark \end{aligned}$$

## Try It

28) Factor by substitution:  $h^4 + 4h^2 - 12$ .

**Solution**

$$(h^2 - 2)(h^2 + 6)$$

## Try It

29) Factor by substitution:  $y^4 - y^2 - 20$ .

**Solution**

$$(y^2 + 4)(y^2 - 5)$$

Sometimes the expression to be substituted is not a monomial.

## Example 1.2.14

Factor:  $(x - 2)^2 + 7(x - 2) + 12$ .

**Solution**

The binomial in the middle term,  $(x - 2)$  is squared in the first term. If we let  $u = x - 2$  and substitute, our trinomial will be in  $ax^2 + bx + c$  form.

**Step 1: Rewrite the trinomial to prepare for the substitution.**

$$(x - 2)^2 + 7(x - 2) + 12$$

**Step 2: Let  $u = x - 2$  and substitute.**

$$u^2 + 7u + 12$$

**Step 3: Factor the trinomial.**

$$(u + 3)(u + 4)$$

**Step 4: Replace  $u$  with  $x - 2$ .**

$$((x - 2) + 3)((x - 2) + 4)$$

**Step 5: Simplify inside the parentheses.**

$$(x + 1)(x + 2)$$

This could also be factored by first multiplying out the  $(x - 2)^2$  and the  $7(x - 2)$  and then combining like terms and then factoring. Most students prefer the substitution method.

## Try It

30) Factor by substitution:  $(x - 5)^2 + 6(x - 5) + 8$ .

**Solution**

$$(x - 3)(x - 1)$$

## Try It

31) Factor by substitution:  $(y - 4)^2 + 8(y - 4) + 15$ .

### Solution

$$(y - 1)(y + 1)$$

Access this online resource for additional instruction and practice with factoring.

- Factor a trinomial using the AC method

## Key Concepts

- **How to factor trinomials of the form  $x^2 + bx + c$ .**

1. Write the factors as two binomials with first terms  $x$ .

$$\begin{array}{l} x^2 + bx + c \\ (x \quad)(x \quad) \end{array}$$

2. Find two numbers  $m$  and  $n$  that

- multiply to  $c$ ,  $m \cdot n = c$
- add to  $b$ ,  $m + n = b$

3. Use  $m$  and  $n$  as the last terms of the factors.

$$(x + m)(x + n)$$

4. Check by multiplying the factors.

• **Strategy for Factoring Trinomials of the Form  $x^2 + bx + c$ .**

When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

$$\begin{array}{c} x^2 + bx + c \\ (x + m)(x + n) \end{array}$$

**When  $c$  is positive,  $m$  and  $n$  have the same sign.**

$b$ positive	$b$ negative
$m, n$ positive	$m, n$ negative
$x^2 + 5x + 6$	$x^2 - 6x + 8$
$(x + 2)(x + 3)$	$(x - 4)(x - 2)$
same signs	same signs

**When  $c$  is negative,  $m$  and  $n$  have opposite signs.**

$x^2 + x - 12$	$x^2 - 2x - 15$
$(x + 4)(x - 3)$	$(x - 5)(x + 3)$
opposite signs	opposite signs

Notice that, in the case when  $m$  and  $n$  have opposite signs, the sign of the one with the larger absolute value matches the sign of  $b$ .

• **How to factor trinomials of the form  $ax^2 + bx + c$  using trial and error.**

1. Write the trinomial in descending order of degrees as needed.
2. Factor any GCF.
3. Find all the factor pairs of the first term.
4. Find all the factor pairs of the third term.
5. Test all the possible combinations of the factors until the correct product is found.
6. Check by multiplying.

• **How to factor trinomials of the form  $ax^2 + bx + c$  using the “ $ac$ ” method.**

1. Factor any GCF.
2. Find the product  $ac$ .
3. Find two numbers  $m$  and  $n$  that:

**Multiply to  $ac$ .**

$$m \cdot n = a \cdot c$$

**Add to  $b$ .**

$$m + n = b$$

$$ax^2 + bx + c$$

4. Split the middle term using  $m$  and  $n$ .  $ax^2 + mx + nx + c$
5. Factor by grouping.
6. Check by multiplying the factors.

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

*<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=111#h5p-11>*

b) After reviewing this checklist, what will you do to become confident for all objectives?



# 1.3 FACTOR SPECIAL PRODUCTS

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## Learning Objectives

By the end of this section, you will be able to:

- Factor perfect square trinomials
- Factor differences of squares
- Factor sums and differences of cubes

## Try It

Before you get started, take this readiness quiz:

- 1) Simplify:  $(3x^2)^3$ .
- 2) Multiply:  $(m + 4)^2$ .
- 3) Multiply:  $(x - 3)(x + 3)$ .

We have seen that some binomials and trinomials result from special products—squaring binomials and multiplying conjugates. If you learn to recognize these kinds of polynomials, you can use the special products patterns to factor them much more quickly.

## Factor Perfect Square Trinomials

Some trinomials are perfect squares. They result from multiplying a binomial times itself. We squared a binomial using the Binomial Squares pattern in a previous chapter.

$$\begin{aligned} & (a+b)^2 \\ & (3x + 4)^2 \\ & a^2 + 2 \cdot a \cdot b + b^2 \\ & (3x)^2 + 2(3x \cdot 4) + 4^2 \\ & 9x^2 + 24x + 16 \end{aligned}$$

The trinomial  $9x^2 + 24x + 16$  is called a *perfect square trinomial*. It is the square of the binomial  $3x + 4$ .

In this chapter, you will start with a perfect square trinomial and factor it into its prime factors.

You could factor this trinomial using the methods described in the last section, since it is of the form  $ax^2 + bx + c$ . But if you recognize that the first and last terms are squares and the trinomial fits the perfect square trinomials pattern, you will save yourself a lot of work.

Here is the pattern—the reverse of the binomial squares pattern.

### Perfect Square Trinomials Pattern

If  $a$  and  $b$  are real numbers

$$\begin{aligned} a^2 + 2ab + b^2 &= (a + b)^2 \\ a^2 - 2ab + b^2 &= (a - b)^2 \end{aligned}$$

To make use of this pattern, you have to recognize that a given trinomial fits it. Check first to see if the leading coefficient is a perfect square,  $a^2$ . Next check that the last term is a perfect square,  $b^2$ . Then check the middle term—is it the product,  $2ab$ ? If everything checks, you can easily write the factors.

### Example 1.3.1

How to Factor Perfect Square Trinomials

Factor:  $9x^2 + 12x + 4$ .

#### Solution

**Step 1: Does the trinomial fit the perfect square trinomial pattern,  $a^2 + 2ab + b^2$ ?**

Is the first term a perfect square? Write it as a square,  $a^2$ .

Is the last term a perfect square? Write it as a square,  $b^2$ .

Check the middle term. Is it  $2ab$ ?

<p>Is <math>9x^2</math> a perfect square? Yes—write it as <math>(3x)^2</math>.</p>	$9x^2 + 12x + 4$ $(3x)^2$
<p>Is 4 a perfect square? Yes—write it as <math>(2)^2</math>.</p>	$(3x)^2$ $(2)^2$
<p>Is <math>12x</math> twice the product of <math>3x</math> and <math>2</math>? Does it match? Yes, so we have a perfect square trinomial!</p>	$(3x)^2$ $(2)^2$ $2(3x)(2)$ $12x$

Figure 1.3.1

**Step 2: Write the square of the binomial.**

$$= 9x^2 + 12x + 4$$

$$= (rgb|1.0, 0.0, 0.03rgb|1.0, 0.0, 0.0x)^2 + 2 \times rgb|1.0, 0.0, 0.03rgb|1.0, 0.0, 0.0x \times rgb|0.1, 0.1, 0.1 \times rgb|1.0, 0.0, 0.0 \times rgb|0.0, 0.0, 1.02rgb|0.0, 0.0, 1.0 + rgb|0.0, 0.0, 1.02^2$$

$$= (rgb|1.0, 0.0, 0.03rgb|1.0, 0.0, 0.0x + rgb|0.0, 0.0, 1.02)^2$$

**Step 3: Check.**

$$\begin{aligned}
 &= (3x + 2)^2 \\
 &= (3x)^2 + 2 \times 3x \times 2 + 2^2 \\
 &= 9x^2 + 12x + 4 \checkmark
 \end{aligned}$$

## Try It

4) Factor:  $4x^2 + 12x + 9$ .

**Solution**

$$(2x + 3)^2$$

## Try It

5) Factor:  $9y^2 + 24y + 16$ .

**Solution**

$$(3y + 4)^2$$

The sign of the middle term determines which pattern we will use. When the middle term is negative, we use the pattern  $a^2 - 2ab + b^2$ , which factors to  $(a - b)^2$ .

The steps are summarized here.

## HOW TO

### Factor Perfect Square Trinomials

**Step 1:** Does the trinomial fit the pattern?

Is the first term a perfect square?

Write it as a square.

Is the last term a perfect square?

Write it as a square.

Check the middle term. Is it  $2ab$ ?

**Step 2:** Write the square of the binomial.

**Step 3:** Check by multiplying.

$$a^2 + 2ab + b^2$$

$$(a)^2$$

$$(a)^2(b)^2$$

$$(a)^2 \searrow_{2 \cdot a \cdot b} \swarrow (b)^2$$

$$(a + b)^2$$

$$a^2 - 2ab + b^2$$

$$(a)^2$$

$$(a)^2(b)^2$$

$$(a)^2 \searrow_{2 \cdot a \cdot b} \swarrow (b)^2$$

$$(a - b)^2$$

We'll work one now where the middle term is negative.

### Example 1.3.2

Factor:  $81y^2 - 72y + 16$

#### Solution

The first and last terms are squares. See if the middle term fits the pattern of a perfect square trinomial. The middle term is negative, so the binomial square would be  $(a - b)^2$ .

#### Step 1: Are the first and last terms perfect squares?

Check the middle term.

$$(9y^2)$$

$$(4^2)$$

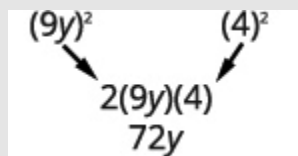


Figure 1.3.2

**Step 2: Does it match  $(a - b)^2$ ?**

Yes.

$$\begin{array}{c} a^2 + 2 \cdot a \cdot b + b^2 \\ (9y)^2 - 2 \cdot 9y \cdot 4 + 4^2 \end{array}$$

**Step 3: Write as the square of a binomial.**

$$(9y - 4)^2$$

**Step 4: Check by multiplying.**

$$\begin{aligned} &= (9y - 4)^2 \\ &= (9y)^2 - 2 \times 9y \times 4 + 4^2 \\ &= 81y^2 - 72y + 16 \checkmark \end{aligned}$$

## Try It

6) Factor:  $64y^2 - 80y + 25$ .

**Solution**

$$(8y - 5)^2$$

## Try It

7) Factor:  $16z^2 - 72z + 81$ .

**Solution**

$$(4z - 9)^2$$

The next example will be a perfect square trinomial with two variables.

## Example 1.3.3

Factor:  $36x^2 + 84xy + 49y^2$ .

**Solution**

**Step 1: Test each term to verify the pattern.**

$$\begin{array}{c} a^2 + 2 \cdot a \cdot b + b^2 \\ (6x)^2 + 2 \cdot 6x \cdot 7y + (7y)^2 \end{array}$$

**Step 2: Factor.**

$$(6x + 7y)^2$$

**Step 3: Check by multiplying.**

$$\begin{aligned} &= (6x + 7y)^2 \\ &= (6x)^2 + 2 \times 6x \times 7y + (7y)^2 \\ &= 36x^2 + 84xy + 49y^2 \quad \checkmark \end{aligned}$$

### Try It

8) Factor:  $49x^2 + 84xy + 36y^2$ .

**Solution**

$$(7x + 6y)^2$$

### Try It

9) Factor:  $64m^2 + 112mn + 49n^2$ .

**Solution**

$$(8m + 7n)^2$$

Remember the first step in factoring is to look for a greatest common factor. Perfect square trinomials may have a GCF in all three terms and it should be factored out first. And, sometimes, once the GCF has been factored, you will recognize a perfect square trinomial.

### Example 1.3.4

Factor:  $100x^2y - 80xy + 16y$ .



**Solution****Step 1: Is there a GCF?**

Yes,  $4y$ , so factor it out.

$$4y(25x^2 - 20x + 4)$$

**Step 2: Is this a perfect square trinomial?**

Verify the pattern.

$$4y \left[ \overset{a^2}{(5x)^2} - \overset{2 \cdot a \cdot b}{2 \cdot 5x \cdot 2} + \overset{b^2}{2^2} \right]$$

**Step 3: Factor.**

$$4y(5x - 2)^2$$

Remember: Keep the factor  $4y$  in the final product.

**Step 5: Check.**

$$\begin{aligned} &= 4y(5x - 2)^2 \\ &= 4y \left[ (5x)^2 - 2 \cdot 5x \cdot 2 + 2^2 \right] \\ &= 4y(25x^2 - 20x + 4) \\ &= 100x^2y - 80xy + 16y \checkmark \end{aligned}$$

**Try It**

10) Factor:  $8x^2y - 24xy + 18y$ .

**Solution**

$$2y(2x - 3)^2$$

## Try It

11) Factor:  $27p^2q + 90pq + 75q$ .

**Solution**

$$3q(3p + 5)^2$$

## Factor Differences of Squares

The other special product you saw in the previous chapter was the Product of Conjugates pattern. You used this to multiply two binomials that were conjugates. Here's an example:

$$\begin{aligned} & \text{rgb]1.0,0.0,0.0}(a - b)(a + b) \\ = & (3x - 4)(3x + 4) \\ & \text{rgb]1.0,0.0,0.0}(a)^2 - (b)^2 \\ = & (3x)^2 - (4)^2 \\ = & 9x^2 - 16 \end{aligned}$$

A difference of squares factors to a product of conjugates.

### Difference of Squares Pattern

If  $a$  and  $b$  are real numbers,

$$a^2 - b^2 = (a - b)(a + b)$$

Figure 1.3.3

Remember, “difference” refers to subtraction. So, to use this pattern you must make sure you have a binomial in which two squares are being subtracted.

### Example 1.3.5

How to Factor a Trinomial Using the Difference of Squares

Factor:  $64y^2 - 1$ .

#### Solution

##### Step 1: Does the binomial fit the pattern?

Is this a difference?

Yes.

$$64y^2 - 1$$

Are the first and last terms perfect squares?

Yes.

##### Step 2: Write them as squares.

Write them as  $x^2$  and  $y^2$ .

$$\begin{array}{c} a^2 - b^2 \\ (8y)^2 - 1^2 \end{array}$$

##### Step 3: Write the product of conjugates.

$$\begin{array}{c} (a - b) (a + b) \\ (8y - 1)(8y + 1) \end{array}$$

##### Step 4: Check.

$$\begin{aligned} &= (8y - 1)(8y + 1) \\ &= 64y^2 - 1 \quad \checkmark \end{aligned}$$

**Try It**

12) Factor:  $121m^2 - 1$ .

**Solution**

$$(11m - 1)(11m + 1)$$

**Try It**

13) Factor:  $81y^2 - 1$ .

**Solution**

$$(9y - 1)(9y + 1)$$

## HOW TO

### Factor Differences of Squares

<b>Step 1:</b> Does the binomial fit the pattern? Is this a difference? Are the first and last terms perfect squares?	$a^2 - b^2$ ____ - ____
<b>Step 2:</b> Write them as squares.	$(a)^2 - (b)^2$
<b>Step 3:</b> Write the product of conjugates.	$(a - b)(a + b)$
<b>Step 4:</b> Check by multiplying.	

It is important to remember that *sums of squares do not factor into a product of binomials*. There are no binomial factors that multiply together to get a sum of squares. After removing any GCF, the expression  $a^2 + b^2$  is prime!

The next example shows variables in both terms.

### Example 1.3.6

Factor:  $144x^2 - 49y^2$ .

#### Solution

**Step 1: Is this a difference of squares?**

Yes.

$$(12x)^2 - (7y)^2$$

**Step 2: Factor as the product of conjugates.**

$$(12x - 7y)(12x + 7y)$$

**Step 3: Check by multiplying.**

$$\begin{aligned} &= (12x - 7y)(12x + 7y) \\ &= 144x^2 - 49y^2 \checkmark \end{aligned}$$

### Try It

14) Factor:  $196m^2 - 25n^2$ .

**Solution**

$$(16m - 5n)(16m + 5n)$$

### Try It

15) Factor:  $121p^2 - 9q^2$ .

**Solution**

$$(11p - 3q)(11p + 3q)$$

As always, you should look for a common factor first whenever you have an expression to factor. Sometimes a common factor may “disguise” the difference of squares and you won’t recognize the perfect squares until you factor the GCF.

Also, to completely factor the binomial in the next example, we’ll factor a difference of squares twice!

### Example 1.3.7

Factor:  $48x^4y^2 - 243y^2$ .

#### Solution

##### Step 1: Is there a GCF?

Yes,  $3y^2$ —factor it out!

$$3y^2(16x^4 - 81)$$

##### Step 2: Is the binomial a difference of squares?

Yes.

$$3y^2((4x^2)^2 - (9)^2)$$

##### Step 3: Factor as a product of conjugates.

$$3y^2(4x^2 - 9)(4x^2 + 9)$$

##### Step 4: Notice the first binomial is also a difference of squares!

$$3y^2((2x)^2 - (3)^2)(4x^2 + 9)$$

##### Step 5: Factor it as the product of conjugates.

$$3y^2(2x - 3)(2x + 3)(4x^2 + 9)$$

The last factor, the sum of squares, cannot be factored.

##### Step 6: Check by multiplying.

$$\begin{aligned} &= 3y^2(2x - 3)(2x + 3)(4x^2 + 9) \\ &= 3y^2(4x^2 - 9)(4x^2 + 9) \\ &= 3y^2(16x^4 - 81) \\ &= 48x^4y^2 - 243y^2 \quad \checkmark \end{aligned}$$

## Try It

16) Factor:  $2x^4y^2 - 32y^2$ .

**Solution**

$$2y^2(x - 2)(x + 2)(x^2 + 4)$$

## Try It

17) Factor:  $7a^4c^2 - 7b^4c^2$ .

**Solution**

$$7c^2(a - b)(a + b)(a^2 + b^2)$$

The next example has a polynomial with 4 terms. So far, when this occurred we grouped the terms in twos and factored from there. Here we will notice that the first three terms form a perfect square trinomial.

## Example 1.3.8

Factor:  $x^2 - 6x + 9 - y^2$ .

**Solution**



**Step 1:** Notice that the first three terms form a perfect square trinomial. Factor by grouping the first three terms.

$$\underbrace{x^2 - 6x + 9}_{(x-3)^2} - y^2$$

**Step 2:** Use the perfect square trinomial pattern.

$$(x - 3)^2 - y^2$$

**Step 3:** Is this a difference of squares?

Yes—write them as squares.

$$(x - 3)^2 - y^2$$

**Step 4:** Factor as the product of conjugates.

$$((x - 3) - y) \overset{(a - b)}{(x - 3)} \overset{(x + b)}{+ y}$$

**Step 5:** Factor as the product of conjugates.

$$(x - 3 - y)(x - 3 + y)$$

You may want to rewrite the solution as  $(x - y - 3)(x + y - 3)$ .

## Try It

18) Factor:  $x^2 - 10x + 25 - y^2$ .

**Solution**

$$(x - 5 - y)(x - 5 + y)$$

## Try It

19) Factor:  $x^2 + 6x + 9 - 4y^2$ .

### Solution

$$(x + 3 - 2y)(x + 3 + 2y)$$

## Factor Sums and Differences of Cubes

There is another special pattern for factoring, one that we did not use when we multiplied polynomials. This is the pattern for the sum and difference of cubes. We will write these formulas first and then check them by multiplication.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

We'll check the first pattern and leave the second to you.

---

	$(a + b)(a^2 - ab + b^2)$
Distribute.	$a(a^2 - ab + b^2) + b(a^2 - ab + b^2)$
Multiply.	$a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3$
Combine like terms.	$a^3 + b^3$

---

### Sum and Difference of Cubes Pattern

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

The two patterns look very similar, don't they? But notice the signs in the factors. The sign of the binomial factor matches the sign in the original binomial. And the sign of the middle term of the trinomial factor is the opposite of the sign in the original binomial. If you recognize the pattern of the signs, it may help you memorize the patterns.

Formula does not parse

Formula does not parse

The trinomial factor in the sum and difference of cubes pattern cannot be factored.

It be very helpful if you learn to recognize the cubes of the integers from 1 to 10, just like you have learned to recognize squares. We have listed the cubes of the integers from 1 to 10.

$n$	1	2	3	4	5	6	7	8	9	10
$n^3$	1	8	27	64	125	216	343	512	729	1000

### Example 1.3.9

How to Factor the Sum or Difference of Cubes

Factor:  $x^3 + 64$ .

#### Solution

**Step 1: Does the binomial fit the sum or difference of cubes pattern?**

Is it a sum or a difference?

$$x^3 + 64$$

This is a sum.

Are the first and last terms perfect cubes?

Yes.

**Step 2: Write the terms as cubes.**

Write them as  $x^3$  and  $4^3$ .

$$\begin{array}{c} a^3 + b^3 \\ x^3 + 4^3 \end{array}$$

**Step 3: Use either the sum or difference of cubes pattern.**

This is a sum of cubes.

$$\begin{array}{c} (a + b) \quad (a^2 - ab + b^2) \\ (x + 4)(x^2 - 4x + 4^2) \end{array}$$

**Step 4: Simplify inside the parentheses.**

It is already simplified.

$$(x + 4)(x^2 - 4x + 4^2)$$

**Step 5: Check by multiplying the factors.**

$$\begin{array}{r} x^2 - 4x \quad + \quad 16 \\ \quad \quad \quad x \quad + \quad 4 \\ \hline 4x^2 - 16x \quad + \quad 64\checkmark \\ x^3 - 4x^2 + 16x \\ \hline x^3 \quad \quad \quad + \quad 64 \end{array}$$

## Try It

20) Factor:  $x^3 + 27$ .

**Solution**

$$(x + 3)(x^2 - 3x + 9)$$

**Try It**

21) Factor:  $y^3 + 8$ .

**Solution**

$$(y + 2)(y^2 - 2y + 4)$$

**HOW TO****Factor the Sum or Difference of Cubes**

1. Does the binomial fit the sum or difference of cubes pattern?  
Is it a sum or difference?  
Are the first and last terms perfect cubes?
2. Write them as cubes.
3. Use either the sum or difference of cubes pattern.
4. Simplify inside the parentheses.
5. Check by multiplying the factors.

### Example 1.3.10

Factor:  $27u^3 - 125v^3$ .

#### Solution

##### Step 1: Does the binomial fit the sum or difference of cubes pattern?

This binomial is a difference. The first and last terms are perfect cubes.

##### Step 2: Write the terms as cubes.

$$\begin{array}{c} a^3 - b^3 \\ (3u)^3 - (5v)^3 \end{array}$$

##### Step 3: Use the difference of cubes pattern.

$$\begin{array}{c} (a - b) \qquad (a^2 + ab + b^2) \\ (3u - 5v) \left( (3u)^2 + 3u \cdot 5v + (5v)^2 \right) \end{array}$$

##### Step 4: Simplify.

$$\begin{array}{c} (a - b) \qquad (a^2 + ab + b^2) \\ (3u - 5v)(9u^2 + 15uv + 25v^2) \end{array}$$

##### Step 5: Check by multiplying.

We'll leave the check to you.

### Try It

22) Factor:  $8x^3 - 27y^3$ .

#### Solution

$$(2x - 3y)(4x^2 - 6xy + 9y^2)$$

## Try It

23) Factor:  $1000m^3 - 125n^3$ .

**Solution**

$$(10m - 5n)(100m^2 - 50mn + 25n^2)$$

In the next example, we first factor out the GCF. Then we can recognize the sum of cubes.

## Example 1.3.11

Factor:  $6x^3y + 48y^4$ .

**Solution**

**Step 1: Factor the common factor.**

$$6y(x^3 + 8y^3)$$

**Step 2: Does the binomial fit the sum or difference of cubes pattern?**

This binomial is a sum. The first and last terms are perfect cubes.

**Step 3: Write the terms as cubes.**

$$6y(x^3 + (2y)^3)$$

**Step 4: Use the sum of cubes pattern.**

$$6y(x + 2y) \overset{(a + b)}{\left(x^2 - x \cdot 2y + (2y)^2\right)} \overset{(a^2 - ab + b^2)}{}$$

**Step 5: Simplify.**

$$6y(x + 2y)(x^2 - 2xy + 4y^2)$$

**Step 6: Check.**

To check, you may find it easier to multiply the sum of cubes factors first, then multiply that product by  $6y$ . We'll leave the multiplication for you.

## Try It

24) Factor:  $500p^3 + 4q^3$ .

**Solution**

$$4(5p + q)(25p^2 - 5pq + q^2)$$

## Try It

25) Factor:  $432c^3 + 686d^3$ .

**Solution**

$$2(6c + 7d)(36c^2 - 42cd + 49d^2)$$



The first term in the next example is a binomial cubed.

### Example 1.3.12

Factor:  $(x + 5)^3 - 64x^3$ .

#### Solution

##### **Step 1: Does the binomial fit the sum or difference of cubes pattern?**

This binomial is a difference. The first and last terms are perfect cubes.

##### **Step 2: Write the terms as cubes.**

$$(x + 5)^3 - (4x)^3$$

##### **Step 3: Use the difference of cubes pattern.**

$$((x + 5) - 4x)((x + 5)^2 + (x + 5) \cdot 4x + (4x)^2)$$

##### **Step 4: Simplify.**

$$\begin{aligned} &= (x + 5 - 4x)(x^2 + 10x + 25 + 4x^2 + 20x + 16x^2) \\ &= (-3x + 5)(21x^2 + 30x + 25) \end{aligned}$$

##### **Step 5: Check by multiplying.**

We'll leave the check to you.

## Try It

26) Factor:  $(y + 1)^3 - 27y^3$ .

**Solution**

$$(-2y + 1)(13y^2 + 5y + 1)$$

## Try It

27) Factor:  $(n + 3)^3 - 125n^3$ .

**Solution**

$$(-4n + 3)(31n^2 + 21n + 9)$$

Access this online resource for additional instruction and practice with factoring special products.

- Factoring Binomials-Cubes #2

## Key Concepts

- **Perfect Square Trinomials Pattern:** If  $a$  and  $b$  are real numbers,

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

- **How to factor perfect square trinomials.**

**Step 1:** Does the trinomial fit the pattern?

Is the first term a perfect square?

Write it as a square.

Is the last term a perfect square?

Write it as a square.

Check the middle term. Is it  $2ab$ ?

$$a^2 + 2ab + b^2$$

$$(a)^2$$

$$(a)^2(b)^2$$

$$(a)^2 \searrow_{2 \cdot a \cdot b} \swarrow (b)^2$$

$$(a + b)^2$$

$$a^2 - 2ab + b^2$$

$$(a)^2$$

$$(a)^2(b)^2$$

$$(a)^2 \searrow_{2 \cdot a \cdot b} \swarrow (b)^2$$

$$(a - b)^2$$

**Step 2:** Write the square of the binomial.

**Step 3:** Check by multiplying.

- **Difference of Squares Pattern:** If  $a$ ,  $b$  are real numbers,

$$a^2 - b^2 = (a - b)(a + b)$$

Figure 1.3.4

- **How to factor differences of squares.**

**Step 1:** Does the binomial fit the pattern?

Is this a difference?

Are the first and last terms perfect squares?

$$a^2 - b^2$$

$$\underline{\quad} - \underline{\quad}$$

**Step 2:** Write them as squares.

$$(a)^2 - (b)^2$$

**Step 3:** Write the product of conjugates.

$$(a - b)(a + b)$$

**Step 4:** Check by multiplying.

- **Sum and Difference of Cubes Pattern**

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

- **How to factor the sum or difference of cubes.**

1. Does the binomial fit the sum or difference of cubes pattern?  
Is it a sum or difference? Are the first and last terms perfect cubes?
2. Write them as cubes.
3. Use either the sum or difference of cubes pattern.
4. Simplify inside the parentheses
5. Check by multiplying the factors.

## Self Check

After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

*<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=166#h5p-12>*

# 1.4 GENERAL STRATEGY FOR FACTORING POLYNOMIALS

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## Learning Objectives

By the end of this section, you will be able to:

- Recognize and use the appropriate method to factor a polynomial completely

## Recognize and Use the Appropriate Method to Factor a Polynomial Completely

You have now become acquainted with all the methods of factoring that you will need in this course. The following chart summarizes all the factoring methods we have covered, and outlines a strategy you should use when factoring polynomials.

## General Strategy for Factoring Polynomials

Binomial	Trinomial	More than 3 terms
<b>Difference of Squares</b> $a^2 - b^2 = (a - b)(a + b)$	$x^2 + bx + c$ $(x \quad)(x \quad)$	Grouping
<b>Sum of Squares</b> Sums of squares do not factor	$ax^2 + bx + c$ <p>‘a’ and ‘c’ squares</p> $(a + b)^2 = a^2 + 2ab + b^2$ $(a - b)^2 = a^2 - 2ab + b^2$	
<b>Sum of Cubes</b> $a^3 + b^3 = (a + b)(a^2 + ab + b^2)$	$ax^2 + bx + c$ <p>‘ac’ method</p>	

### HOW TO

Use a general strategy for factoring polynomials.

**Step 1: Is there a greatest common factor?**

Factor it out.

**Step 2: Is the polynomial a binomial, trinomial, or are there more than three terms?**

*If it is a binomial:*

- Is it a sum?
  - Of squares? Sums of squares do not factor.
  - Of cubes? Use the sum of cubes pattern.
- Is it a difference?
  - Of squares? Factor as the product of conjugates.

- Of cubes? Use the difference of cubes pattern.

*If it is a trinomial:*

- Is it of the form  $x^2 + bx + c$ ? Undo FOIL.
- Is it of the form  $ax^2 + bx + c$ ?  
If  $a$  and  $c$  are squares, check if it fits the trinomial square pattern.  
Use the trial and error or “ $ac$ ” method.

*If it has more than three terms:*

- Use the grouping method.

### **Step 3: Check.**

- Is it factored completely?
- Do the factors multiply back to the original polynomial?

Remember, a polynomial is completely factored if, other than monomials, its factors are prime!

## **Example 1.4.1**

Factor completely:  $7x^3 - 21x^2 - 70x$ .

### **Solution**

#### **Step 1: Is there a GCF?**

Yes,  $7x$ .

#### **Step 2: Factor out the GCF.**

$$7x(x^2 - 3x - 10)$$

#### **Step 3: In the parentheses, is it a binomial, trinomial, or are there more terms?**

Trinomial with leading coefficient 1.

**Step 4: “Undo” FOIL.**

$$\begin{aligned} & 7x(x \quad)(x \quad) \\ & 7x(x + 2)(x - 5) \end{aligned}$$

**Step 5: Is the expression factored completely?**

Yes. Neither binomial can be factored completely.

**Step 6: Check your answer. Multiply.**

$$\begin{aligned} &= 7x(x + 2)(x - 5) \\ &= 7x(x^2 - 5x + 2x - 10) \\ &= 7x(x^2 - 3x - 10) \\ &= 7x^3 - 21x^2 - 70x \checkmark \end{aligned}$$

## Try It

1) Factor completely:  $8y^3 + 16y^2 - 24y$ .

**Solution**

$$8y(y - 1)(y + 3)$$

## Try It

2) Factor completely:  $5y^3 - 15y^2 - 270y$ .



**Solution**

$$5y(y - 9)(y + 6)$$

Be careful when you are asked to factor a binomial as there are several options!

**Example 1.4.2**

Factor completely:  $24y^2 - 150$ .

**Solution****Step 1: Is there a GCF?**

Yes, 6.

**Step 2: Factor out the GCF.**

$$6(4y^2 - 25)$$

**Step 3: In the parentheses, is it a binomial, trinomial, or are there more than three terms?**

Binomial.

**Step 4: Is it a sum?**

No.

**Step 5: Is it a difference? Of squares or cubes?**

Yes, squares.

**Step 6: Write as a product of conjugates.**

$$6((2y)^2 - (5)^2)$$

$$6(2y - 5)(2y + 5)$$

**Step 7: Is the expression factored completely?**

Yes. Neither binomial can be factored.

**Step 8: Check by multiplying.**

$$\begin{aligned} &= 6(2y - 5)(2y + 5) \\ &= 6(4y^2 - 25) \\ &= 24y^2 - 150 \checkmark \end{aligned}$$

### Try It

3) Factor completely:  $16x^3 - 36x$ .

**Solution**

$$4x(2x - 3)(2x + 3)$$

### Try It

4) Factor completely:  $27y^2 - 48$ .

**Solution**

$$3(3y - 4)(3y + 4)$$

The next example can be factored using several methods. Recognizing the trinomial squares pattern will make your work easier.

### Example 1.4.3

Factor completely:  $4a^2 - 12ab + 9b^2$ .

#### Solution

##### Step 1: Is there a GCF?

No.

##### Step 2: Is it a binomial, trinomial, or are there more terms?

Trinomial with  $a \neq 1$ , but the first term is a perfect square.

##### Step 3: Is the last term a perfect square?

Yes.

$$(2a)^2 - 12ab + (3b)^2$$

##### Step 4: Does it fit the pattern $a^2 - 2ab + b^2$ ?

Yes.

$$(2a)^2 - \underset{2(2a)(3b)}{12ab} + (3b)^2$$

##### Step 5: Write it as a square.

$$(2a - 3b)^2$$

##### Step 6: Is the expression factored completely?

Yes. The binomial cannot be factored.

##### Step 7: Check your answer. Multiply.

$$\begin{aligned} &= (2a - 3b)^2 \\ &= (2a)^2 - 2 \cdot 2a \cdot 3b + (3b)^2 \\ &= 4a^2 - 12ab + 9b^2 \quad \checkmark \end{aligned}$$

### Try It

5) Factor completely:  $4x^2 + 20xy + 25y^2$ .

**Solution**

$$(2x + 5y)^2$$

### Try It

6) Factor completely:  $9x^2 - 24xy + 16y^2$ .

**Solution**

$$(3x - 4y)^2$$

Remember, sums of squares do not factor, but sums of cubes do!

### Example 1.4.4

Factor completely  $12x^3y^2 + 75xy^2$ .

**Solution**

**Step 1: Is there a GCF?**

Yes,  $3xy^2$ .

**Step 2: Factor out the GCF.**

$$3xy^2(4x^2 + 25)$$

**Step 3: In the parentheses, is it a binomial, trinomial, or are there more than three terms?**

Binomial.

**Step 4: Is it a sum? Of squares?**

Yes. Sums of squares are prime.

**Step 5: Is the expression factored completely?**

Yes.

**Step 6: Check by multiplying.**

$$\begin{aligned} &= 3xy^2(4x^2 + 25) \\ &= 12x^3y^2 + 75xy^2 \quad \checkmark \end{aligned}$$

## Try It

7) Factor completely:  $50x^3y + 72xy$ .

**Solution**

$$2xy(25x^2 + 36)$$

## Try It

8) Factor completely:  $27xy^3 + 48xy$ .

**Solution**

$$3xy(9y^2 + 16)$$

When using the sum or difference of cubes pattern, being careful with the signs.

## Example 1.4.5

Factor completely:  $24x^3 + 81y^3$ .

**Solution**

**Step 1: Is there a GCF?**

Yes, 3.

**Step 2: Factor it out.**

$$3(8x^3 + 27y^3)$$

**Step 3: In the parentheses, is it a binomial, trinomial, or are there more than three terms?**

Binomial.

**Step 4: Is it a sum or a difference?**

Sum.

**Step 5: Of squares or cubes?**

Sum of cubes.

$$3((2x)^3 + (3y)^3)$$

**Step 6: Write it using the sum of cubes pattern.**

$$3(2x + 3y)(2x)^2 - 2x \cdot 3y + (3y)^2$$

**Step 7: Is the expression factored completely?**

Yes.

$$3(2x + 3y)(4x^2 - 6xy + 9y^2)$$

**Step 8: Check by multiplying.**

We leave the check for you!

## Try It

9) Factor completely:  $250m^3 + 432n^3$ .

**Solution**

$$2(5m + 6n)(25m^2 - 30mn + 36n^2)$$

## Try It

10) Factor completely:  $2p^3 + 54q^3$ .

**Solution**

$$2(p + 3q)(p^2 - 3pq + 9q^2)$$

### Example 1.4.6

Factor completely:  $3x^5y - 48xy$ .

#### Solution

**Step 1: Is there a GCF?**

Yes,  $3xy$ .

**Step 2: Factor it out.**

$$3xy(x^4 - 16)$$

**Step 3: Is the binomial a sum or difference? Of squares or cubes?**

Difference of squares.

**Step 4: Write it as a difference of squares.**

$$3xy((x^2)^2 - (4)^2)$$

**Step 5: Factor it as a product of conjugates.**

$$3xy(x^2 - 4)(x^2 + 4)$$

**Step 6: The first binomial is again a difference of squares.**

$$3xy((x)^2 - (2)^2)(x^2 + 4)$$

**Step 7: Factor it as a product of conjugates.**

$$3xy(x - 2)(x + 2)(x^2 + 4)$$

**Step 8: Is the expression factored completely?**

Yes.

**Step 9: Check your answer. Multiply.**



$$\begin{aligned}
 &= 3xy(x-2)(x+2)(x^2+4) \\
 &= 3xy(x^2-4)(x^2+4) \\
 &= 3xy(x^4-16) \\
 &= 3x^5y - 48xy \checkmark
 \end{aligned}$$

### Try It

11) Factor completely:  $4a^5b - 64ab$ .

**Solution**

$$4ab(a^2 + 4)(a - 2)(a + 2)$$

### Try It

12) Factor completely:  $7xy^5 - 7xy$ .

**Solution**

$$7xy(y^2 + 1)(y - 1)(y + 1)$$

### Example 1.4.7

Factor completely:  $4x^2 + 8bx - 4ax - 8ab$ .

#### Solution

**Step 1: Is there a GCF?**

Yes, 4.

**Step 2: Factor it out.**

$$4(x^2 + 2bx - ax - 2ab)$$

**Step 3: There are four terms. Use grouping.**

$$\begin{aligned} &4[x(x + 2b) - a(x + 2b)] \\ &4(x + 2b)(x - a) \end{aligned}$$

**Step 4: Is the expression factored completely?**

Yes.

**Step 5: Check your answer. Multiply.**

$$\begin{aligned} &= 4(x + 2b)(x - a) \\ &= 4(x^2 - ax + 2bx - 2ab) \\ &= 4x^2 + 8bx - 4ax - 8ab \checkmark \end{aligned}$$

### Try It

13) Factor completely:  $6x^2 - 12xc + 6bx - 12bc$ .

#### Solution

$$6(x + b)(x - 2c)$$

## Try It

14) Factor completely:  $16x^2 + 24xy - 4x - 6y$ .

### Solution

$$2(4x - 1)(2x + 3y)$$

Taking out the complete GCF in the first step will always make your work easier.

## Example 1.4.8

Factor completely:  $40x^2y + 44xy - 24y$ .

### Solution

**Step 1: Is there a GCF?**

Yes,  $4y$ .

**Step 2: Factor it out.**

$$4y(10x^2 + 11x - 6)$$

**Step 3: Factor the trinomial with  $a \neq 1$ .**

$$4y(5x - 2)(2x + 3)$$

**Step 4: Is the expression factored completely?**

Yes.

**Step 5: Check your answer. Multiply.**

$$\begin{aligned}
 &= 4y(5x - 2)(2x + 3) \\
 &= 4y(10x^2 + 11x - 6) \\
 &= 40x^2y + 44xy - 24y \checkmark
 \end{aligned}$$

## Try It

15) Factor completely:  $4p^2q - 16pq + 12q$ .

**Solution**

$$4q(p - 3)(p - 1)$$

## Try It

16) Factor completely:  $6pq^2 - 9pq - 6p$ .

**Solution**

$$3p(2q + 1)(q - 2)$$

When we have factored a polynomial with four terms, most often we separated it into two groups of two terms. Remember that we can also separate it into a trinomial and then one term.

## Example 1.4.9

Factor completely:  $9x^2 - 12xy + 4y^2 - 49$ .

### Solution

#### Step 1: Is there a GCF?

No.

#### Step 2: With more than 3 terms, use grouping.

Last 2 terms have no GCF. Try grouping first 3 terms.

$$9x^2 - 12xy + 4y^2 - 49$$

#### Step 3: Factor the trinomial with $a \neq 1$ .

The first term is a perfect square.

$$(3x)^2 - 12xy + (2y)^2 - 49$$

#### Step 4: Is the last term of the trinomial a perfect square?

Yes.

#### Step 5: Does the trinomial fit the pattern $a^2 - 2ab + b^2$ ?

Yes.

$$(3x)^2 - \underset{2(3x)(2y)}{12xy} + (2y)^2 - 49$$

#### Step 6: Write the trinomial as a square.

$$(3x - 2y)^2 - 49$$

#### Step 7: Is this binomial a sum or difference? Of squares or cubes?

Difference of squares.

#### Step 8: Write it as a difference of squares.

$$(3x - 2y)^2 - 7^2$$

#### Step 9: Write it as a product of conjugates.

$$\begin{aligned}
 &= ((3x - 2y) - 7)((3x - 2y) + 7) \\
 &= (3x - 2y - 7)(3x - 2y + 7)
 \end{aligned}$$

**Step 10: Is the expression factored completely?**

Yes.

**Step 11: Check your answer. Multiply.**

$$\begin{aligned}
 &= (3x - 2y - 7)(3x - 2y + 7) \\
 &= 9x^2 - 6xy - 21x - 6xy + 4y^2 + 14y + 21x - 14y - 49 \\
 &= 9x^2 - 12xy + 4y^2 - 49 \checkmark
 \end{aligned}$$

## Try It

17) Factor completely:  $4x^2 - 12xy + 9y^2 - 25$ .

**Solution**

$$(2x - 3y - 5)(2x - 3y + 5)$$

## Try It

18) Factor completely:  $16x^2 - 24xy + 9y^2 - 64$ .

**Solution**

$$(4x - 3y - 8)(4x - 3y + 8)$$

## Key Concepts

- **General Strategy for Factoring Polynomials**

Binomial	Trinomial	More than 3 terms
<b>Difference of Squares</b> $a^2 - b^2 = (a - b)(a + b)$	$x^2 + bx + c$ $(x \quad)(x \quad)$	Grouping
<b>Sum of Squares</b> Sums of squares do not factor	$ax^2 + bx + c$ 'a' and 'c' squares $(a + b)^2 = a^2 + 2ab + b^2$ $(a - b)^2 = a^2 - 2ab + b^2$	
<b>Sum of Cubes</b> $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$	$ax^2 + bx + c$ 'ac' method	

- **How to use a general strategy for factoring polynomials.**

**Step 1: Is there a greatest common factor?**

Factor it out.

**Step 2: Is the polynomial a binomial, trinomial, or are there more than three terms?**

*If it is a binomial:*

- Is it a sum?
  - Of squares? Sums of squares do not factor.
  - Of cubes? Use the sum of cubes pattern.
- Is it a difference?
  - Of squares? Factor as the product of conjugates.
  - Of cubes? Use the difference of cubes pattern.

*If it is a trinomial:*

- Is it of the form  $x^2 + bx + c$ ? Undo FOIL.
- Is it of the form  $ax^2 + bx + c$ ?  
If  $a$  and  $c$  are squares, check if it fits the trinomial square pattern.  
Use the trial and error or “ $ac$ ” method.

*If it has more than three terms:*

- Use the grouping method.

**Step 3: Check.**

- Is it factored completely?
- Do the factors multiply back to the original polynomial?

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=175#h5p-13>

b) On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?



# 1.5 POLYNOMIAL EQUATIONS

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## Learning Objectives

By the end of this section, you will be able to:

- Use the Zero Product Property
- Solve quadratic equations by factoring
- Solve equations with polynomial functions
- Solve applications modelled by polynomial equations

## Try It

Before you get started, take this readiness quiz:

- 1) Solve:  $5y - 3 = 0$ .
- 2) Factor completely:  $n^3 - 9n^2 - 22n$ .
- 3) If  $f(x) = 8x - 16$ , find  $f(3)$  and solve  $f(x) = 0$ .

We have spent considerable time learning how to factor polynomials. We will now look at **polynomial equations** and solve them using factoring, if possible.

A polynomial equation is an equation that contains a polynomial expression. The **degree of the polynomial equation** is the degree of the polynomial.

## Polynomial Equation

A polynomial equation is an equation that contains a polynomial expression.

The degree of the polynomial equation is the degree of the polynomial.

We have already solved polynomial equations of degree one. Polynomial equations of degree one are linear equations of the form  $ax + b = c$ .

We are now going to solve polynomial equations of degree two. A polynomial equation of degree two is called a **quadratic equation**. Listed below are some examples of quadratic equations:

$$x^2 + 5x + 6 = 0 \quad 3y^2 + 4y = 10 \quad 64u^2 - 81 = 0 \quad n(n + 1) = 42$$

The last equation doesn't appear to have the variable squared, but when we simplify the expression on the left we will get  $n^2 + n$ .

The general form of a quadratic equation is  $ax^2 + bx + c = 0$ , with  $a \neq 0$ . (If  $a = 0$ , then  $0 \cdot x^2 = 0$  and we are left with no quadratic term.)

## Quadratic Equation

An equation of the form  $ax^2 + bx + c = 0$  is called a quadratic equation.

$a$ ,  $b$ , and  $c$  are real numbers and  $a \neq 0$ .

To solve quadratic equations we need methods different from the ones we used in solving linear equations. We will look at one method here and then several others in a later chapter.

## Use the Zero Product Property

We will first solve some quadratic equations by using the **Zero Product Property**. The Zero Product Property says that if the product of two quantities is zero, then at least one of the quantities is zero. The only way to get a product equal to zero is to multiply by zero itself.

## Zero Product Property

If  $a \cdot b = 0$ , then either  $a = 0$  or  $b = 0$  or both.

We will now use the Zero Product Property, to solve a quadratic equation.

### Example 1.5.1

How to Solve a Quadratic Equation Using the Zero Product Property

Solve:  $(5n - 2)(6n - 1) = 0$ .

#### Solution

**Step 1: Set each factor equal to zero.**

The product equals zero, so at least one factor must equal zero.

$$(5n - 2)(6n - 1) = 0$$

$$5n - 2 = 0$$

$$\text{or } 6n - 1 = 0$$

**Step 2: Solve the linear equations.**

Solve each equation.

$$n = \frac{2}{5} \quad n = \frac{1}{6}$$

**Step 3: Check.**

Substitute each solution separately into the original equation.

Formula does not parse

Formula does not parse

## Try It

4) Solve:  $(3m - 2)(2m + 1) = 0$ .

### Solution

$$m = \frac{2}{3}, m = -\frac{1}{2}$$

## Try It

5) Solve:  $(4p + 3)(4p - 3) = 0$ .

### Solution

$$p = -\frac{3}{4}, p = \frac{3}{4}$$

## HOW TO

### Use the Zero Product Property

1. Set each factor equal to zero.
2. Solve the linear equations.
3. Check.

## Solve Quadratic Equations by Factoring

The Zero Product Property works very nicely to solve quadratic equations. The quadratic equation must be factored, with zero isolated on one side. So we be sure to start with the quadratic equation in standard form,  $ax^2 + bx + c = 0$ . Then we factor the expression on the left.

### Example 1.5.2

How to Solve a Quadratic Equation by Factoring

Solve:  $2y^2 = 13y + 45$ .

#### Solution

**Step 1: Write the quadratic equation in standard form,  $ax^2 + bx + c = 0$ .**

Write the equation in standard form.

$$\begin{aligned} 2y^2 &= 13y + 45 \\ 2y^2 - 13y - 45 &= 0 \end{aligned}$$

**Step 2: Factor the quadratic expression.**

$$\begin{aligned} 2y^2 - 13y - 45 &= 0 \\ (2y + 5)(y - 9) &= 0 \end{aligned}$$

**Step 3: Use the Zero Product Property.**

Set each factor equal to zero. We have two linear equations.

$$2y + 5 = 0 \quad y - 9 = 0$$

**Step 4: Solve the linear equations.**

$$y = -\frac{5}{2} \quad y = 9$$

**Step 5: Check.**

Substitute each solution separately into the original equation.

$$y = -\frac{5}{2}$$

$$2y^2 = 13y + 45$$

$$2\left(-\frac{5}{2}\right)^2 \stackrel{?}{=} 13\left(-\frac{5}{2}\right) + 45$$

$$2\left(\frac{25}{4}\right) \stackrel{?}{=} \left(-\frac{65}{2}\right) + \frac{90}{2}$$

$$\frac{25}{2} = \frac{25}{2} \checkmark$$

$$y = 9$$

$$2y^2 = 13y + 45$$

$$2(9)^2 \stackrel{?}{=} 13(9) + 45$$

$$2(81) \stackrel{?}{=} 117 + 45$$

$$162 = 162 \checkmark$$

## Try It

6) Solve:  $3c^2 = 10c - 8$ .

**Solution**

$$c = 2, c = \frac{4}{3}$$

## Try It

7) Solve:  $2d^2 - 5d = 3$ .

**Solution**

$$d = 3, d = -\frac{1}{2}$$

## HOW TO

### Solve a Quadratic Equation by Factoring

1. Write the quadratic equation in standard form,  $ax^2 + bx + c = 0$ .
2. Factor the quadratic expression.
3. Use the Zero Product Property.
4. Solve the linear equations.
5. Check. Substitute each solution separately into the original equation.

Before we factor, we must make sure the quadratic equation is in standard form.

Solving quadratic equations by factoring will make use of all the factoring techniques you have learned in this chapter! Do you recognize the special product pattern in the next example?

### Example 1.5.3

Solve:  $169q^2 = 49$ .

#### **Solution**

**Step 1: Write the quadratic equation in standard form.**

$$169q^2 - 49 = 0$$

**Step 2: Factor.**

It is a difference of squares.

$$(13q - 7)(13q + 7) = 0$$

**Step 3: Use the Zero Product Property to set each factor to 0.**

$$13q - 7 = 0 \quad 13q + 7 = 0$$

**Step 4: Solve each equation.**

$$\begin{aligned} 13q &= 7 \\ q &= \frac{7}{13} \end{aligned}$$

$$\begin{aligned} 13q &= -7 \\ q &= -\frac{7}{13} \end{aligned}$$

**Step 5: Check.**

We leave the check up to you.



**Try It**

8) Solve:  $25p^2 = 49$ .

**Solution**

$$p = \frac{7}{5}, p = -\frac{7}{5}$$

**Try It**

9) Solve:  $36x^2 = 121$ .

**Solution**

$$x = \frac{11}{6}, x = -\frac{11}{6}$$

In the next example, the left side of the equation is factored, but the right side is not zero. In order to use the Zero Product Property, one side of the equation must be zero. We'll multiply the factors and then write the equation in standard form.

### Example 1.5.4

Solve:  $(3x - 8)(x - 1) = 3x$ .

**Solution**

**Step 1: Multiply the binomials.**

$$3x^2 - 11x + 8 = 3x$$

**Step 2: Write the quadratic equation in standard form.**

$$3x^2 - 14x + 8 = 0$$

**Step 3: Factor the trinomial.**

$$(3x - 2)(x - 4) = 0$$

**Step 4: Use the Zero Product Property to set each factor to 0.**

$$3x - 2 = 0 \quad x - 4 = 0$$

**Step 5: Solve each equation.**

$$\begin{aligned} 3x &= 2 & x &= 4 \\ x &= \frac{2}{3} \end{aligned}$$

**Step 6: Check your answers.**

The check is left to you.

### Try It

10) Solve:  $(2m + 1)(m + 3) = 12m$ .

**Solution**

$$m = 1, m = \frac{3}{2}$$

**Try It**

11) Solve:  $(k + 1)(k - 1) = 8$ .

**Solution**

$$k = 3, k = -3$$

In the next example, when we factor the quadratic equation we will get three factors. However the first factor is a constant. We know that factor cannot equal 0.

**Example 1.5.5**

Solve:  $3x^2 = 12x + 63$ .

**Solution**

**Step 1:** Write the quadratic equation in standard form.

$$3x^2 - 12x - 63 = 0$$

**Step 2:** Factor the greatest common factor first.

$$3(x^2 - 4x - 21) = 0$$

**Step 3:** Factor the trinomial.

$$3(x - 7)(x + 3) = 0$$

**Step 4: Use the Zero Product Property to set each factor to 0.**

$$3 \neq 0 \quad x - 7 = 0 \quad x + 3 = 0$$

**Step 5: Solve each equation.**

$$3 \neq 0 \quad x = 7 \quad x = -3$$

**Step 6: Check your answers.**

The check is left to you.

## Try It

12) Solve:  $18a^2 - 30 = -33a$ .

**Solution**

$$a = -\frac{5}{2}, \quad a = \frac{2}{3}$$

## Try It

13) Solve:  $123b = -6 - 60b^2$ .

**Solution**

$$b = -2, \quad b = -\frac{1}{20}$$

The Zero Product Property also applies to the product of three or more factors. If the product is zero, at least one of the factors must be zero. We can solve some equations of degree greater than two by using the Zero Product Property, just like we solved quadratic equations.

### Example 1.5.6

Solve:  $9m^3 + 100m = 60m^2$ .

#### Solution

**Step 1: Bring all the terms to one side so that the other side is zero.**

$$9m^3 - 60m^2 + 100m = 0$$

**Step 2: Factor the greatest common factor first.**

$$m(9m^2 - 60m + 100) = 0$$

**Step 3: Factor the trinomial.**

$$m(3m - 10)(3m - 10) = 0$$

**Step 4: Use the Zero Product Property to set each factor to 0.**

$$m = 0 \quad 3m - 10 = 0 \quad 3m - 10 = 0$$

**Step 5: Solve each equation.**

$$m = 0 \quad m = \frac{10}{3} \quad m = \frac{10}{3}$$

**Step 6: Check your answers.**

The check is left to you.

**Try It**

14) Solve:  $8x^3 = 24x^2 - 18x$ .

**Solution**

$$x = 0, x = \frac{3}{2}$$

**Try It**

15) Solve:  $16y^2 = 32y^3 + 2y$ .

**Solution**

$$y = 0, y = \frac{1}{4}$$

## Solve Equations with Polynomial Functions

As our study of polynomial functions continues, it will often be important to know when the function will have a certain value or what points lie on the graph of the function. Our work with the Zero Product Property will help us find these answers.

## Example 1.5.7

For the function  $f(x) = x^2 + 2x - 2$ ,

- find  $x$  when  $f(x) = 6$
- find two points that lie on the graph of the function.

### Solution

a.

**Step 1: Substitute 6 for  $f(x)$ .**

$$6 = x^2 + 2x - 2$$

**Step 2: Put the quadratic in standard form.**

$$x^2 + 2x - 8 = 0$$

**Step 3: Factor the trinomial.**

$$(x + 4)(x - 2) = 0$$

**Step 4: Use the Zero Product Property.**

$$x + 4 = 0 \quad x - 2 = 0$$

**Step 5: Solve.**

$$x = -4 \quad x = 2$$

**Step 6: Check.**

$f(x) = x^2 + 2x - 2$	$f(x) = x^2 + 2x - 2$
$f(-4) = (-4)^2 + 2(-4) - 2$	$f(2) = 2^2 + 2 \cdot 2 - 2$
$f(-4) = 16 - 8 - 2$	$f(2) = 4 + 4 - 2$
$f(-4) = 6\checkmark$	$f(2) = 6\checkmark$

---

b.

**Step 1: Use  $f(-4) = 6$  and  $f(2) = 6$  to find two points that lie on the graph of the function.**

Since  $f(-4) = 6$  and  $f(2) = 6$ , the points  $(-4, 6)$  and  $(2, 6)$  lie on the graph of the function.

## Try It

16) For the function  $f(x) = x^2 - 2x - 8$ ,

- Find  $x$  when  $f(x) = 7$
- Find two points that lie on the graph of the function.

### Solution

- $x = -3$  or  $x = 5$
- $(-3, 7)$   $(5, 7)$

## Try It

17) For the function  $f(x) = x^2 - 8x + 3$ ,

- find  $x$  when  $f(x) = -4$
- Find two points that lie on the graph of the function.

### Solution



- a.  $x = 1$  or  $x = 7$   
 b.  $(1, -4)$   $(7, -4)$

The Zero Product Property also helps us determine where the function is zero. A value of  $x$  where the function is 0, is called a **zero of the function**.

### Zero of a Function

For any function  $f$ , if  $f(x) = 0$ , then  $x$  is a **zero of the function**.

When  $f(x) = 0$ , the point  $(x, 0)$  is a point on the graph. This point is an  $x$ -intercept of the graph. It is often important to know where the graph of a function crosses the axes. We will see some examples later.

### Example 1.5.8

For the function  $f(x) = 3x^2 + 10x - 8$ , find

- a. the zeros of the function,  
 b. any  $x$ -intercepts of the graph of the function  
 c. any  $y$ -intercepts of the graph of the function

#### Solution

a.

**Step 1: To find the zeros of the function, we need to find when the function value is 0.**

**Substitute 0 for  $f(x)$ .**

$$0 = 3x^2 + 10x - 8$$

**Step 2: Factor the trinomial.**

$$(x + 4)(3x - 2) = 0$$

**Step 3: Use the Zero Product Property.**

$$x + 4 = 0 \quad 3x - 2 = 0$$

**Step 4: Solve.**

$$x = -4 \quad x = \frac{2}{3}$$


---

b.

**Step 1: An  $x$ -intercept occurs when  $y = 0$ .**

Since  $f(-4) = 0$  and  $f(\frac{2}{3}) = 0$ , the points  $(-4, 0)$  and  $(\frac{2}{3}, 0)$  lie on the graph. These points are  $x$ -intercepts of the function.

---

c.

**Step 1: A  $y$ -intercept occurs when  $x = 0$ . To find the  $y$ -intercepts we need to find  $f(0)$ .**

Find  $f(0)$  by substituting 0 for  $x$ .

$$f(0) = 3 \cdot 0^2 + 10 \cdot 0 - 8$$

**Step 2: Simplify.**

$$f(0) = -8$$

Since  $f(0) = -8$ , the point  $(0, -8)$  lies on the graph. This point is the  $y$ -intercept of the function.

## Try It

18) For the function  $f(x) = 2x^2 - 7x + 5$ , find

- the zeros of the function
- any  $x$ -intercepts of the graph of the function
- any  $y$ -intercepts of the graph of the function.

### Solution

- $x = 1$  or  $x = \frac{5}{2}$
- $(1, 0)$ ,  $(\frac{5}{2}, 0)$
- $(0, 5)$

## Try It

19) For the function  $f(x) = 6x^2 + 13x - 15$ , find

- the zeros of the function
- any  $x$ -intercepts of the graph of the function
- any  $y$ -intercepts of the graph of the function.

### Solution

- $x = -3$  or  $x = \frac{5}{6}$
- $(-3, 0)$ ,  $(\frac{5}{6}, 0)$
- $(0, -15)$

## Solve Applications Modelled by Polynomial Equations

The problem-solving strategy we used earlier for applications that translate to linear equations will work just as well for applications that translate to polynomial equations. We will copy the problem-solving strategy here so we can use it for reference.

### HOW TO

#### Use a Problem Solving Strategy to Solve Word Problems

1. Read the problem. Make sure all the words and ideas are understood.
2. Identify what we are looking for.
3. Name what we are looking for. Choose a variable to represent that quantity.
4. Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. Solve the equation using appropriate algebra techniques.
6. Check the answer in the problem and make sure it makes sense.
7. Answer the question with a complete sentence.

We will start with a number problem to get practice translating words into a polynomial equation.

### Example 1.5.9

The product of two consecutive odd integers is **323**. Find the integers.

#### **Solution**

**Step 1: Read the problem.**

**Step 2: Identify what we are looking for.**

We are looking for two consecutive integers.

**Step 3: Name what we are looking for.**

Let  $n$  = the first integer

$n + 2$  = next consecutive odd integer

**Step 4: Translate into an equation. Restate the problem in a sentence.**

The product of the two consecutive odd integers is **323**.

**Step 5: Solve the equation.**

$$n(n + 2) = 323$$

$$n^2 + 2n = 323$$

**Step 6: Bring all the terms to one side.**

$$n^2 + 2n - 323 = 0$$

**Step 7: Factor the trinomial.**

$$(n - 17)(n + 19) = 0$$

**Step 8: Use the Zero Product Property.**

$$n - 17 = 0 \quad n + 19 = 0$$

**Step 9: Solve the equations.**

$$n = 17 \quad n = -19$$

**Step 10: There are two values for  $n$  that are solutions to this problem. So there are two sets of consecutive odd integers that will work.**

If the first integer is  $n = 17$ , then the next odd integer is  $n + 2$ .

$$n + 2 = 17 + 2$$

$$n + 2 = 19$$

$$(17, 19)$$

If the first integer is  $n = -19$ , then the next odd integer is  $n + 2$ .

$$n + 2 = -19 + 2$$

$$n + 2 = -17$$

$$(-19, -17)$$

**Step 11: Check the answer.**

The results are consecutive odd integers  $(17, 19)$  and  $(-19, -17)$ .

$$17 \cdot 19 = 323\checkmark \quad -19(-17) = 323\checkmark$$

Both pairs of consecutive integers are solutions.

**Step 12: Answer the question.**

The consecutive integers are  $(17, 19)$  and  $(-19, -17)$ .

## Try It

20) The product of two consecutive odd integers is **255**. Find the integers.

**Solution**

$-15, -17$  and  $15, 17$

## Try It

21) The product of two consecutive odd integers is **483**. Find the integers.

**Solution**

$-23, -21$  and  $21, 23$

Were you surprised by the pair of negative integers that is one of the solutions to the previous example? The product of the two positive integers and the product of the two negative integers both give positive results.

In some applications, negative solutions will result from the algebra, but will not be realistic for the situation.

### Example 1.5.10

A rectangular bedroom has an area **117** square feet. The length of the bedroom is four feet more than the width. Find the length and width of the bedroom.

#### Solution

##### Step 1: Read the problem.

In problems involving geometric figures, a sketch can help you visualize the situation.



Figure 1.5.1

##### Step 2: Identify what you are looking for.

We are looking for the length and width.

##### Step 3: Name what you are looking for.

The length is four feet more than the width.

Let  $w$  = the width of the bedroom

$w + 4$  = the length of the garden

##### Step 4: Translate into an equation. Restate the important information in a sentence.

The area of the bedroom is **117** square feet.

Use the formula for the area of a rectangle.

$$A = l \cdot w$$

Substitute in the variables.

$$117 = (w + 4)w$$

##### Step 5: Solve the equation.

Distribute first.

$$117 = w^2 + 4w$$

Get zero on one side.

$$0 = w^2 + 4w - 117$$

Factor the trinomial.

$$0 = (w + 13)(w - 9)$$

Use the Zero Product Property.

$$0 = w + 13 \quad 0 = w - 9$$

Solve each equation.

Since  $w$  is the width of the bedroom, it does not make sense for it to be negative. We eliminate that value for  $w$ .

$$\cancel{-13 = w} \quad 9 = w$$

$$w = 9$$

Width is **9** feet.

Find the value of the length.

$$w + 4 = 9 + 4$$

$$w + 4 = 13$$

$$l = 13$$

Length is **13** feet.

**Step 6: Check the answer. Does the answer make sense?**

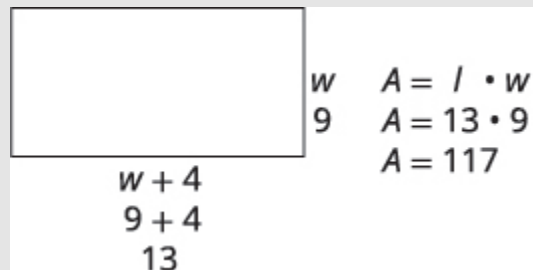


Figure 1.5.2

Yes, this makes sense.

**Step 7: Answer the question.**



The width of the bedroom is **9** feet and the length is **13** feet.

### Try It

22) A rectangular sign has area **30** square feet. The length of the sign is one foot more than the width. Find the length and width of the sign.

**Solution**

The width is **5** feet and length is **6** feet.

### Try It

23) A rectangular patio has area **180** square feet. The width of the patio is three feet less than the length. Find the length and width of the patio.

**Solution**

The length of the patio is **12** feet and the width **15** feet.

In the next example, we will use the Pythagorean Theorem ( $a^2 + b^2 = c^2$ ). This formula gives the relation between the legs and the hypotenuse of a right triangle.

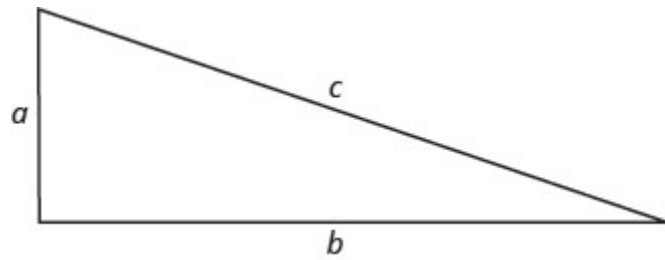


Figure 1.5.3

We will use this formula to in the next example.

### Example 1.5.11

A boat's sail is in the shape of a right triangle as shown. The hypotenuse will be **17** feet long. The length of one side will be **7** feet less than the length of the other side. Find the lengths of the sides of the sail.



Figure 1.5.4

#### Solution

**Step 1: Read the problem.**

**Step 2: Identify what you are looking for.**

We are looking for the lengths of the sides of the sail.

**Step 3: Name what you are looking for.**

One side is **7** less than the other.

Let  $x$  = length of a side of the sail

$x - 7$  = length of other side

**Step 4: Translate into an equation. Since this is a right triangle we can use the Pythagorean Theorem.**

$$a^2 + b^2 = c^2$$

Substitute in the variables.

$$x^2 + (x - 7)^2 = 17^2$$

**Step 5: Solve the equation.**

Simplify.

$$2x^2 - 14x + 49 = 289$$

It is a quadratic equation, so get zero on one side.

$$2x^2 - 14x - 240 = 0$$

Factor the greatest common factor.

$$2(x^2 - 7x - 120) = 0$$

Factor the trinomial.

$$2(x - 15)(x + 8) = 0$$

Use the Zero Product Property.

$$2 \neq 0 \quad x - 15 = 0 \quad x + 8 = 0$$

Solve.

$$2 \neq 0 \quad x = 15 \quad x = -8$$

Since  $x$  is a side of the triangle,  $x = -8$  does not make sense.

$$2 \neq 0 \quad x = 15 \quad \cancel{x = -8}$$

Find the length of the other side.

If the length of one side is  $x = 15$ , then the length of the other side is:

$$x - 7 = 15 - 7$$

$$x - 7 = 8$$

The length of the other side is 8 feet.

**Step 6: Check the answer in the problem. Do these numbers make sense?**

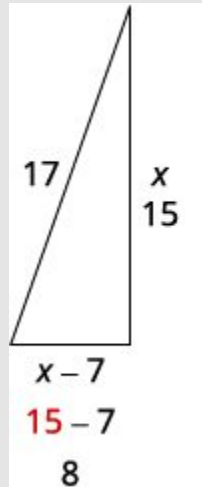


Figure 1.5.5

$$\begin{aligned}
 a^2 + b^2 &= c^2 \\
 15^2 + 8^2 &\stackrel{?}{=} 17^2 \\
 225 + 64 &\stackrel{?}{=} 289 \\
 289 &= 289 \checkmark
 \end{aligned}$$

Yes, these numbers make sense.

**Step 7: Answer the question.**

The sides of the sail are 8, 15, and 17 feet.

## Try It

24) Justine wants to put a deck in the corner of her backyard in the shape of a right triangle. The length of one side of the deck is 7 feet more than the other side. The hypotenuse is 13. Find the lengths of the two sides of the deck.

### Solution

5 feet and 12 feet

## Try It

25) A meditation garden is in the shape of a right triangle, with one leg 7 feet. The length of the hypotenuse is one more than the length of the other leg. Find the lengths of the hypotenuse and the other leg.

### Solution

24 feet and 25 feet

The next example uses the function that gives the height of an object as a function of time when it is thrown from 80 feet above the ground.

## Example 1.5.12

Dennis is going to throw his rubber band ball upward from the top of a campus building. When he throws the rubber band ball from 80 feet above the ground, the function  $h(t) = -16t^2 + 64t + 80$  models the height,  $h$ , of the ball above the ground as a function of time,  $t$ . Find:

- the zeros of this function which tell us when the ball hits the ground
- when the ball will be 80 feet above the ground
- the height of the ball at  $t = 2$  seconds.

### Solution

a.

**Step 1: The zeros of this function are found by solving  $h(t) = 0$ . This will tell us when the ball will hit the ground.**

Substitute in the polynomial for  $h(t)$ .

$$-16t^2 + 64t + 80 = 0$$

**Step 2: Factor the GCF,  $-16$ .**

$$-16(t^2 - 4t - 5) = 0$$

**Step 3: Factor the trinomial.**

$$-16(t - 5)(t + 1) = 0$$

**Step 4: Use the Zero Product Property.**

$$t - 5 = 0 \quad t + 1 = 0$$

**Step 5: Solve.**

$$t = 5 \quad t = -1$$

The result  $t = 5$  tells us the ball will hit the ground 5 seconds after it is thrown. Since time cannot be negative, the result  $t = -1$  is discarded.

---

b.

**Step 1: The ball will be 80 feet above the ground when  $h(t) = 80$ .**

Substitute in the polynomial for  $h(t)$ .

$$-16t^2 + 64t + 80 = 80$$

**Step 2: Subtract 80 from both sides.**

$$-16t^2 + 64t = 0$$

**Step 3: Factor the GCF,  $-16t$ .**

$$-16t(t - 4) = 0$$

**Step 4: Use the Zero Product Property.**

$$-16t = 0 \quad t - 4 = 0$$

**Step 5: Solve.**

$$t = 0 \quad t = 4$$

The ball will be at **80** feet the moment Dennis tosses the ball and then **4** seconds later, when the ball is falling.

---

c.

**Step 1: To find the height ball at  $t = 2$  seconds we find  $h(2)$ .**

To find  $h(2)$ , substitute **2** for  $t$ .

$$h(2) = -16(2)^2 + 64 \cdot 2 + 80$$

**Step 2: Simplify.**

$$h(2) = 144$$

After **2** seconds, the ball will be at **144** feet.

**Try It**

26) Genevieve is going to throw a rock from the top a trail overlooking the ocean. When she throws the rock upward from **160** feet above the ocean, the function  $h(t) = -16t^2 + 48t + 160$  models the height,  $h$ , of the rock above the ocean as a function of time,  $t$ . Find:

- the zeros of this function which tell us when the rock will hit the ocean
- when the rock will be **160** feet above the ocean.
- the height of the rock at  $t = 1.5$  seconds.

**Solution**

- 5

- b. 0, 3
- c. 196

## Try It

27) Calib is going to throw his lucky penny from his balcony on a cruise ship. When he throws the penny upward from **128** feet above the ground, the function  $h(t) = -16t^2 + 32t + 128$  models the height,  $h$ , of the penny above the ocean as a function of time,  $t$ . Find:

- a. the zeros of this function which is when the penny will hit the ocean
- b. when the penny will be **128** feet above the ocean.
- c. the height the penny will be at  $t = 1$  seconds which is when the penny will be at its highest point.

### Solution

- a. 4
- b. 0, 2
- c. 144

Access this online resource for additional instruction and practice with quadratic equations.

- [Beginning Algebra & Solving Quadratics with the Zero Property](#)



## Key Concepts

- **Polynomial Equation:** A polynomial equation is an equation that contains a polynomial expression. The degree of the polynomial equation is the degree of the polynomial.
- **Quadratic Equation:** An equation of the form  $ax^2 + bx + c = 0$  is called a quadratic equation.  
 $a$ ,  $b$ , and  $c$  are real numbers and  $a \neq 0$ .
- **Zero Product Property:** If  $a \cdot b = 0$ , then either  $a = 0$  or  $b = 0$  or both.
- **How to use the Zero Product Property**
  1. Set each factor equal to zero.
  2. Solve the linear equations.
  3. Check.
- **How to solve a quadratic equation by factoring.**
  1. Write the quadratic equation in standard form,  $ax^2 + bx + c = 0$ .
  2. Factor the quadratic expression.
  3. Use the Zero Product Property.
  4. Solve the linear equations.
  5. Check. Substitute each solution separately into the original equation.
- **Zero of a Function:** For any function  $f$ , if  $f(x) = 0$ , then  $x$  is a zero of the function.
- **How to use a problem solving strategy to solve word problems.**
  1. Read the problem. Make sure all the words and ideas are understood.
  2. Identify what we are looking for.
  3. Name what we are looking for. Choose a variable to represent that quantity.
  4. Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
  5. Solve the equation using appropriate algebra techniques.
  6. Check the answer in the problem and make sure it makes sense.
  7. Answer the question with a complete sentence.

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=193#h5p-14>

b) Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

## Glossary

### **degree of the polynomial equation**

The degree of the polynomial equation is the degree of the polynomial.

### **polynomial equation**

A polynomial equation is an equation that contains a polynomial expression.

### **quadratic equation**

Polynomial equations of degree two are called quadratic equations.

### **zero of the function**

A value of  $x$  where the function is 0, is called a zero of the function.

### **Zero Product Property**

The Zero Product Property says that if the product of two quantities is zero, then at least one of the quantities is zero.

# 1.6 UNIT 1 SOURCES

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**1.0** “Introduction to Factoring” from Intermediate Algebra by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted.

**1.1** “6.1 Greatest Common Factor and Factor by Grouping” from Intermediate Algebra by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted.

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**1.4** “6.4 General Strategy for Factoring Polynomials” from Intermediate Algebra by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted.

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# UNIT 2: RATIONAL EXPRESSIONS AND FUNCTIONS

## Chapter Outline

- 2.0 Introduction
- 2.1 Multiply and Divide Rational Expressions
- 2.2 Add and Subtract Rational Expressions
- 2.3 Simplify Complex Rational Expressions
- 2.4 Solve Rational Equations
- 2.5 Solve Applications with Rational Equations
- 2.6 Unit 2 Sources



## 2.0 INTRODUCTION

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Photo by OpenStax CC BY

American football is the most watched spectator sport in the United States. People around the country are constantly tracking statistics for football and other sports.

Twelve goals last season. Fifteen home runs. Nine touchdowns. Whatever the statistics, sports analysts know it. Their jobs depend on it. Compiling and analyzing sports data not only help fans appreciate their teams but also help owners and coaches decide which players to recruit, how to best use them in games, how much they should be paid, and which players to trade. Understanding this kind of data requires a knowledge of specific types of expressions and functions. In this chapter, you will work with rational expressions and perform operations on them. And you will use rational expressions and inequalities to solve real-world problems.

## 2.1 MULTIPLY AND DIVIDE RATIONAL EXPRESSIONS

---

### Learning Objectives

By the end of this section, you will be able to:

- Determine the values for which a rational expression is undefined
- Simplify rational expressions
- Multiply rational expressions
- Divide rational expressions
- Multiply and divide rational functions

### Try It

Before you get started, take this readiness quiz.

1) Simplify:  $\frac{90y}{15y^2}$

2) Multiply:  $\frac{14}{15} \cdot \frac{6}{35}$

3) Divide:  $\frac{12}{10} \div \frac{8}{25}$



We previously reviewed the properties of fractions and their operations. We introduced rational numbers, which are just fractions where the numerators and denominators are integers. In this chapter, we will work with fractions whose numerators and denominators are polynomials. We call this kind of expression a **rational expression**.

## Rational Expression

A rational expression is an expression of the form  $\frac{p}{q}$ , where  $p$  and  $q$  are polynomials and  $q \neq 0$ .

Here are some examples of rational expressions:

$$-\frac{24}{56}$$

$$\frac{5x}{12y}$$

$$\frac{4x+1}{x^2-9}$$

$$\frac{4x^2+3x-1}{2x-8}$$

Notice that the first rational expression listed above,  $-\frac{24}{56}$ , is just a fraction. Since a constant is a polynomial with degree zero, the ratio of two constants is a rational expression, provided the denominator is not zero.

We will do the same operations with rational expressions that we did with fractions. We will simplify, add, subtract, multiply, divide and use them in applications.

## Determine the Values for Which a Rational Expression is Undefined

If the denominator is zero, the rational expression is undefined. The numerator of a rational expression may be 0—but not the denominator.

When we work with a numerical fraction, it is easy to avoid dividing by zero because we can see the number in the denominator. In order to avoid dividing by zero in a rational expression, we must not allow values of the variable that will make the denominator be zero.

So before we begin any operation with a rational expression, we examine it first to find the values that would make the denominator zero. That way, when we solve a rational equation for example, we will know whether the algebraic solutions we find are allowed or not.

## HOW TO

Determine the values for which a rational expression is undefined.

1. Set the denominator equal to zero.
2. Solve the equation.

### Examples 2.1.1

Determine the value for which each rational expression is undefined:

a.  $\frac{8a^2b}{3c}$

b.  $\frac{4b - 3}{2b + 5}$

c.  $\frac{x + 4}{x^2 + 5x + 6}$

The expression will be undefined when the denominator is zero.

#### Solution

a.

**Step 1: Set the denominator equal to zero and solve for the variable.**

$$3c = 0$$

$$c = 0$$

$$\frac{8a^2b}{3c} \text{ is undefined for } c = 0.$$


---

b.

**Step 1: Set the denominator equal to zero and solve for the variable.**

$$2b + 5 = 0$$

$$2b = -5$$

$$b = -\frac{5}{2}$$

$$\frac{4b - 3}{2b + 5} \text{ is undefined for } b = -\frac{5}{2}$$


---

c.

**Step 1: Set the denominator equal to zero and solve for the variable.**

$$x^2 + 5x + 6 = 0$$

$$(x + 2)(x + 3) = 0$$

$$x + 2 = 0 \text{ or } x + 3 = 0$$

$$x = -2 \text{ or } x = -3$$

$$\frac{x + 4}{x^2 + 5x + 6} \text{ is undefined for } x = -2 \text{ or } x = -3$$

## Try It

4) Determine the value for which each rational expression is undefined.

a.  $\frac{3y^2}{8x}$

b.  $\frac{8n - 5}{3n + 1}$

c.  $\frac{a + 10}{a^2 + 4a + 3}$

**Solution**

a.  $x = 0$

b.  $n = -\frac{1}{3}$

c.  $a = -1, a = -3$

**Try It**

5) Determine the value for which each rational expression is undefined.

a.  $\frac{4p}{5q}$

b.  $\frac{y-1}{3y+2}$

c.  $\frac{m-5}{m^2+m-6}$

**Solution**

a.  $q = 0$

b.  $y = -\frac{2}{3}$

c.  $m = 2, m = -3$

## Simplify Rational Expressions

A fraction is considered simplified if there are no common factors, other than **1**, in its numerator and denominator. Similarly, a **simplified rational expression** has no common factors, other than **1**, in its numerator and denominator.

## Simplified Rational Expression

A rational expression is considered simplified if there are no common factors in its numerator and denominator.

For example,

$\frac{x+2}{x+3}$  is simplified because there are no common factors of  $x + 2$  and  $x + 3$ .

$\frac{2x}{3x}$  is not simplified because  $x$  is a common factor of  $2x$  and  $3x$ .

We use the Equivalent Fractions Property to simplify numerical fractions. We restate it here as we will also use it to simplify rational expressions.

### Equivalent Fractions Property

If  $a$ ,  $b$ , and  $c$  are numbers where  $b \neq 0$ ,  $c \neq 0$ ,

$$\text{then } \frac{a}{b} = \frac{a \cdot c}{b \cdot c} \text{ and } \frac{a \cdot c}{b \cdot c} = \frac{a}{b}.$$

Notice that in the Equivalent Fractions Property, the values that would make the denominators zero are specifically disallowed. We see  $b \neq 0$ ,  $c \neq 0$  clearly stated.

To simplify rational expressions, we first write the numerator and denominator in factored form. Then we remove the common factors using the Equivalent Fractions Property.

Be very careful as you remove common factors. Factors are multiplied to make a product. You can remove a factor from a product. You cannot remove a term from a sum.

$$\frac{2 \cdot \cancel{3} \cdot \cancel{7}}{\cancel{3} \cdot 5 \cdot \cancel{7}}$$

$$\frac{2}{5}$$

We removed the common factors **3** and **7**. They are **factors of the product**.

$$\frac{3x \cancel{(x-9)}}{5 \cancel{(x-9)}} \text{ where } x \neq 9$$

$$\frac{3x}{5}$$

We removed the common factor  $(x - 9)$ . It is a **factor of the product**.

$$\frac{x + 5}{x}$$

**NO COMMON FACTORS**

While there is an  $x$  in both the numerator and denominator, **the *rgb* 1.0, 0.0, 0.0  $x$  in the numerator is a term of a sum!**

Removing the  $x$ 's from  $\frac{x + 5}{x}$  would be like cancelling the **2**'s in the fraction  $\frac{2 + 5}{2}$ !

## Examples 2.1.2

Simplify:  $\frac{x^2 + 5x + 6}{x^2 + 8x + 12}$

### Solution

**Step 1: Factor the numerator and denominator completely.**

Factor  $x^2 + 5x + 6$  and  $x^2 + 8x + 12$

$$\frac{x^2 + 5x + 6}{x^2 + 8x + 12}$$

$$\frac{(x + 2)(x + 3)}{(x + 2)(x + 6)}$$

**Step 2: Simplify by dividing out common factors.**

Remove the common factor  $x + 2$  from the numerator and the denominator.

$$\frac{\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x+6)}$$

$$\frac{(x+3)}{(x+6)}$$

$$x \neq -2 \quad x \neq -6$$

### Try It

6) Simplify:  $\frac{x^2 - x - 2}{x^2 - 3x + 2}$

#### Solution

$$\frac{x+1}{x-1}, x \neq 2, x \neq 1$$

### Try It

7) Simplify:  $\frac{x^2 - 3x - 10}{x^2 + x - 2}$

#### Solution

$$\frac{x - 5}{x - 1}, x \neq -2, x \neq 1$$

We now summarize the steps you should follow to simplify rational expressions.

## HOW TO

### Simplify a rational expression.

1. Factor the numerator and denominator completely.
2. Simplify by dividing out common factors.

Usually, we leave the simplified rational expression in factored form. This way, it is easy to check that we have removed *all* the common factors.

We'll use the methods we have learned to factor the polynomials in the numerators and denominators in the following examples.

Every time we write a rational expression, we should make a statement disallowing values that would make a denominator zero. However, to let us focus on the work at hand, we will omit writing it in the examples.

## Examples 2.1.3

Simplify:  $\frac{3a^2 - 12ab + 12b^2}{6a^2 - 24b^2}$

### Solution

**Step 1:** Factor the numerator and denominator, first factoring out the GCF.



$$\frac{3(a^2 - 4ab + 4b^2)}{6(a^2 - 4b^2)}$$

$$\frac{3(a-2b)(a-2b)}{6(a+2b)(a-2b)}$$

**Step 2: Remove the common factors of  $a - 2ab$  and 3.**

$$\frac{\cancel{3}(a-2b) \cancel{(a-2b)}}{\cancel{3} \cdot 2(a+2b) \cancel{(a-2b)}}$$

$$\frac{a-2b}{2(a+2b)}$$

## Try It

8) Simplify:  $\frac{2x^2 - 12xy + 18y^2}{3x^2 - 27y^2}$

**Solution**

$$\frac{2(x - 3y)}{3(x + 3y)}$$

## Try It

9) Simplify:  $\frac{5x^2 - 30xy + 25y^2}{2x^2 - 50y^2}$

### Solution

$$\frac{5(x - y)}{2(x + 5y)}$$

Now we will see how to simplify a rational expression whose numerator and denominator have opposite factors. We previously introduced opposite notation: the opposite of  $a$  is  $-a$  and  $-a = -1 \cdot a$ .

The numerical fraction, say  $\frac{7}{-7}$  simplifies to  $-1$ . We also recognize that the numerator and denominator are opposites.

The fraction  $\frac{a}{-a}$ , whose numerator and denominator are opposites also simplifies to  $-1$ .

Let's look at the expression  $b - a$ .

Rewrite.

Factor out  $-1$ .

$$b - a$$

$$-a + b$$

$$-1(a - b)$$

This tells us that  $b - a$  is the opposite of  $a - b$ .

In general, we could write the opposite of  $a - b$  as  $b - a$ . So the rational expression  $\frac{a - b}{b - a}$  simplifies to  $-1$ .

## Opposites in a Rational Expression

The opposite of  $a - b$  is  $b - a$ .

$$\frac{a - b}{b - a} = -1 \quad a \neq b$$

An expression and its opposite divide to  $-1$ .

We will use this property to simplify rational expressions that contain opposites in their numerators and denominators. Be careful not to treat  $a + b$  and  $b + a$  as opposites. Recall that in addition, order doesn't matter so  $a + b = b + a$ . So if  $a \neq -b$ , then  $\frac{a + b}{b + a} = 1$ .

### Examples 2.1.4

Simplify:  $\frac{x^2 - 4x - 32}{64 - x^2}$

**Solution**

**Step 1: Factor the numerator and the denominator.**

$$\frac{(x - 8)(x + 4)}{(8 - x)(8 + x)}$$

**Step 2: Recognize the factors that are opposites.**

$$\begin{aligned} \text{Simplify.} &= \frac{(x - 8)(x + 4)}{(8 - x)(8 + x)} \\ &= \frac{\cancel{(x - 8)}(x + 4)}{\cancel{(8 - x)}(8 + x)} \\ &= -\frac{x + 4}{x + 8} \end{aligned}$$

**Try It**

10) Simplify:  $\frac{x^2 - 4x - 5}{25 - x^2}$

**Solution**

$$-\frac{x + 1}{x + 5}$$

**Try It**

11) Simplify:  $\frac{x^2 + x - 2}{1 - x^2}$

**Solution**

$$-\frac{x + 2}{x + 1}$$

## Multiply Rational Expressions

To multiply rational expressions, we do just what we did with numerical fractions. We multiply the numerators and multiply the denominators. Then, if there are any common factors, we remove them to simplify the result.

## Multiplication of Rational Expressions

If  $p$ ,  $q$ ,  $r$ , and  $s$  are polynomials where  $q \neq 0$ ,  $s \neq 0$ , then

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

To multiply rational expressions, multiply the numerators and multiply the denominators.

Remember, throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example,  $x \neq 0$ ,  $x \neq 3$ , and  $x \neq 4$ .

### Examples 2.1.5

Simplify:  $\frac{2x}{x^2 - 7x + 12} \cdot \frac{x^2 - 9}{6x^2}$

#### Solution

**Step 1: Factor each numerator and denominator completely.**

Factor  $x^2 - 9$  and  $x^2 - 7x + 12$

$$\frac{2x}{x^2 - 7x + 12} \cdot \frac{x^2 - 9}{6x^2}$$

$$\frac{2x}{(x-3)(x-4)} \cdot \frac{(x-3)(x+3)}{6x^2}$$

**Step 2: Multiply the numerators and denominators.**

Multiply the numerators and denominators. It is helpful to write the monomials first.

$$\frac{2x(x-3)(x+3)}{6x^2(x-3)(x-4)}$$

**Step 3: Simplify by dividing out common factors.**

Divide out the common factors.

Leave the denominator in factored form.

$$\frac{\cancel{2} \cancel{2} (\cancel{x-3})(x+3)}{\cancel{2} \cdot 3 \cdot \cancel{2} \cdot x (\cancel{x-3})(x-4)}$$

$$\frac{(x+3)}{3x(x-4)}$$

**Try It**

12) Simplify:  $\frac{5x}{x^2 + 5x + 6} \cdot \frac{x^2 - 4}{10x}$

**Solution**

$$\frac{x - 2}{2(x + 3)}$$

**Try It**

13) Simplify:  $\frac{9x^2}{x^2 + 11x + 30} \cdot \frac{x^2 - 36}{3x^2}$

**Solution**

$$\frac{3(x-6)}{x+5}$$

## HOW TO

### Multiply rational expressions

1. Factor each numerator and denominator completely.
2. Multiply the numerators and denominators.
3. Simplify by dividing out common factors.

## Examples 2.1.6

Multiply:  $\frac{3a^2 - 8a - 3}{a^2 - 25} \cdot \frac{a^2 + 10a + 25}{3a^2 - 14a - 5}$

### Solution

**Step 1: Factor the numerators and denominators and then multiply.**

$$\frac{(3a+1)(a-3)(a+5)(a+5)}{(a-5)(a+5)(3a+1)(a-5)}$$

**Step 2: Simplify by dividing out common factors.**

$$\frac{\cancel{(3a+1)}(a-3)\cancel{(a+5)}(a+5)}{(a-5)\cancel{(a+5)}\cancel{(3a+1)}(a-5)}$$

**Step 3: Simplify.**

$$\frac{(a - 3)(a + 5)}{(a - 5)(a - 5)}$$

**Step 4: Rewrite  $(a - 5)(a - 5)$  using an exponent.**

$$\frac{(a - 3)(a + 5)}{(a - 5)^2}$$

### Try It

14) Simplify:  $\frac{2x^2 + 5x - 12}{x^2 - 16} \cdot \frac{x^2 - 8x + 16}{2x^2 - 13x + 15}$

**Solution**

$$\frac{x - 4}{x - 5}$$

### Try It

15) Simplify:  $\frac{4b^2 + 7b - 2}{1 - b^2} \cdot \frac{b^2 - 2b + 1}{4b^2 + 15b - 4}$

**Solution**

$$-\frac{(b + 2)(b - 1)}{(1 + b)(b + 4)}$$



## Divide Rational Expressions

Just like we did for numerical fractions, to divide rational expressions, we multiply the first fraction by the reciprocal of the second.

### Division of Rational Expressions

If  $p$ ,  $q$ ,  $r$ , and  $s$  are polynomials where  $q \neq 0$ ,  $r \neq 0$ ,  $s \neq 0$ , then

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$

To divide rational expressions, multiply the first fraction by the reciprocal of the second.

Once we rewrite the division as multiplication of the first expression by the reciprocal of the second, we then factor everything and look for common factors.

### Examples 2.1.7

Divide:  $\frac{p^3 + q^3}{2p^2 + 2pq + 2q^2} \div \frac{p^2 - q^2}{6}$

#### Solution

**Step 1: Rewrite the division as the product of the first rational expression and the reciprocal of the second.**

“Flip” the second fraction and change the division sign to multiplication.

$$\frac{p^3 + q^3}{2p^2 + 2pq + 2q^2} \div \frac{p^2 - q^2}{6}$$

$$\frac{p^3 + q^3}{2p^2 + 2pq + 2q^2} \times \frac{6}{p^2 - q^2}$$

**Step 2: Factor the numerators and denominators completely.**

Factor the numerators and denominators.

$$\frac{(p+q)(p^2-pq+q^2)}{2(p^2+pq+q^2)} \times \frac{2 \times 3}{(p-q)(p+q)}$$

**Step 3: Multiply the numerators and denominators.**

Multiply the numerators and multiply the denominators.

$$\frac{(p+q)(p^2-pq+q^2)2 \times 3}{2(p^2+pq+q^2)(p-q)(p+q)}$$

**Step 4: Simplify by dividing out common factors.**

Divide out the common factors.

$$\frac{\cancel{(p+q)}(p^2-pq+q^2)\cancel{2} \times 3}{\cancel{2}(p^2+pq+q^2)(p-q)\cancel{(p+q)}}$$

$$\frac{3(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$$

**Try It**

16) Simplify:  $\frac{x^3 - 8}{3x^2 - 6x + 12} \div \frac{x^2 - 4}{6}$

**Solution**

$$\frac{2(x^2 + 2x + 4)}{(x+2)(x^2 - 2x + 4)}$$

**Try It**

17) Simplify:  $\frac{2z^2}{z^2 - 1} \div \frac{z^3 - z^2 + z}{z^3 + 1}$

**Solution**

$$\frac{2z}{z - 1}$$

**HOW TO****Divide Rational Expressions.**

1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.
2. Factor the numerators and denominators completely.
3. Multiply the numerators and denominators together.
4. Simplify by dividing out common factors.

Recall from Use the Language of Algebra that a complex fraction is a fraction that contains a fraction in the numerator, the denominator or both. Also, remember a fraction bar means division. A complex fraction is another way of writing division of two fractions.

## Examples 2.1.8

$$\text{Divide: } \frac{\frac{6x^2 - 7x + 2}{4x - 8}}{\frac{2x^2 - 7x + 3}{x^2 - 5x + 6}}$$

### Solution

**Step 1: Rewrite with a division sign.**

$$\frac{6x^2 - 7x + 2}{4x - 8} \div \frac{2x^2 - 7x + 3}{x^2 - 5x + 6}$$

**Step 2: Rewrite as product of first times reciprocal of second.**

$$\frac{6x^2 - 7x + 2}{4x - 8} \times \frac{x^2 - 5x + 6}{2x^2 - 7x + 3}$$

**Step 3: Factor the numerators and the denominators, and then multiply.**

$$\frac{(2x - 1)(3x - 2)(x - 2)(x - 3)}{4(x - 2)(2x - 1)(x - 3)}$$

**Step 4: Simplify by dividing out common factors.**

$$\frac{\cancel{(2x-1)}(3x-2)\cancel{(x-2)}\cancel{(x-3)}}{4\cancel{(x-2)}\cancel{(2x-1)}\cancel{(x-3)}}$$

Simplify.  $\frac{3x-2}{4}$

## Try It

18) Simplify:  $\frac{\frac{3x^2+7x+2}{4x+24}}{\frac{3x^2-14x-5}{x^2+x-30}}$

**Solution**

$$\frac{x+2}{4}$$

## Try It

19) Simplify:  $\frac{\frac{y^2-36}{2y^2+11y-6}}{\frac{2y^2-2y-60}{8y-4}}$

**Solution**

$$\frac{2}{y+5}$$

If we have more than two rational expressions to work with, we still follow the same procedure. The first step will be to rewrite any division as multiplication by the reciprocal. Then, we factor and multiply.

## Examples 2.1.9

Perform the indicated operations:  $\frac{3x - 6}{4x - 4} \cdot \frac{x^2 + 2x - 3}{x^2 - 3x - 10} \div \frac{2x + 12}{8x + 16}$

### Solution

**Step 1: Rewrite the division as multiplication by the reciprocal.**

$$\frac{3x - 6}{4x - 4} \times \frac{x^2 + 2x - 3}{x^2 - 3x - 10} \times \frac{rgb]1.0, 0.0, 0.08x + 16}{rgb]1.0, 0.0, 0.02x + 12}$$

**Step 2: Factor the numerators and the denominators.**

$$\frac{3(x - 2)}{4(x - 1)} \times \frac{(x + 3)(x - 1)}{(x + 2)(x - 5)} \times \frac{8(x + 2)}{2(x + 6)}$$

**Step 3: Multiply the fractions.**

Bringing the constants to the front will help when removing common factors.  
Simplify by dividing out common factors.

$$\frac{3 \times \cancel{8} (x-2)(x+3) (\cancel{x-1}) (\cancel{x+2})}{\cancel{4} \times \cancel{2} (\cancel{x-1}) (\cancel{x+2}) (x-5)(x+6)}$$

Simplify.  $\frac{3(x-2)(x+3)}{(x-5)(x+6)}$

**Try It**

20) Perform the indicated operations:  $\frac{4m + 4}{3m - 15} \times \frac{m^2 - 3m - 10}{m^2 - 4m - 32} \div \frac{12m - 36}{6m - 48}$

**Solution**

$$\frac{2(m + 1)(m + 2)}{3(m + 4)(m - 3)}$$

**Try It**

21) Perform the indicated operations:  $\frac{2n^2 + 10n}{n - 1} \div \frac{n^2 + 10n + 24}{n^2 + 8n - 9} \times \frac{n + 4}{8n^2 + 12n}$

**Solution**

$$\frac{(n + 5)(n + 9)}{2(n + 6)(2n + 3)}$$

## Multiply and Divide Rational Functions

We started this section stating that a rational expression is an expression of the form  $\frac{p}{q}$ , where  $p$  and  $q$  are polynomials and  $q \neq 0$ . Similarly, we define a **rational function** as a function of the form  $R(x) = \frac{p(x)}{q(x)}$  where  $p(x)$  and  $q(x)$  are polynomial functions and  $q(x)$  is not zero.

## Rational Function

A rational function is a function of the form

$$R(x) = \frac{p(x)}{q(x)}$$

where  $p(x)$  and  $q(x)$  are polynomial functions and  $q(x)$  is not zero.

The domain of a rational function is all real numbers except for those values that would cause division by zero. We must eliminate any values that make  $q(x) = 0$ .

## HOW TO

### Determine the domain of a rational function.

1. Set the denominator equal to zero.
2. Solve the equation.
3. The domain is all real numbers excluding the values found in Step 2:

## Examples 2.1.10

Find the domain of  $R(x) = \frac{2x^2 - 14x}{4x^2 - 16x - 48}$

The domain will be all real numbers except those values that make the denominator zero. We will



set the denominator equal to zero, solve that equation, and then exclude those values from the domain.

**Solution**

**Step 1: Set the denominator to zero.**

$$4x^2 - 16x - 48 = 0$$

**Step 2: Factor, first factor out the GCF.**

$$4(x^2 - 4x - 12) = 0$$

$$4(x - 6)(x + 2) = 0$$

**Step 3: Use the Zero Product Property.**

$$4 \neq 0 \quad x - 6 = 0 \quad x + 2 = 0$$

**Step 4: Solve.**

$$x = 6 \quad x = -2$$

The domain of  $R(x)$  is all real numbers where  $x \neq 6$  and  $x \neq -2$ .

## Try It

22) Find the domain of  $R(x) = \frac{2x^2 - 10x}{4x^2 - 16x - 20}$

**Solution**

The domain of  $R(x)$  is all real numbers where  $x \neq 5$  and  $x \neq -1$ .

## Try It

23) Find the domain of  $R(x) = \frac{4x^2 - 16x}{8x^2 - 16x - 64}$

### Solution

The domain of  $R(x)$  is all real numbers where  $x \neq 4$  and  $x \neq -2$ .

To multiply rational functions, we multiply the resulting rational expressions on the right side of the equation using the same techniques we used to multiply rational expressions.

## Examples 2.1.11

Find  $R(x) = f(x) \times g(x)$  where  $f(x) = \frac{2x - 6}{x^2 - 8x + 15}$  and  $g(x) = \frac{x^2 - 25}{2x + 10}$ .

### Solution

$$R(x) = f(x) \times g(x)$$

$$R(x) = \frac{2x - 6}{x^2 - 8x + 15} \times \frac{x^2 - 25}{2x + 10}$$

**Step 1: Factor each numerator and denominator.**

$$R(x) = \frac{2(x - 3)}{(x - 3)(x - 5)} \times \frac{(x - 5)(x + 5)}{2(x + 5)}$$

**Step 2: Multiply the numerators and denominators.**

$$R(x) = \frac{2(x - 3)(x - 5)(x + 5)}{2(x - 3)(x - 5)(x + 5)}$$

**Step 3: Remove common factors.**

$$R(x) = \frac{\cancel{2} \cancel{(x-3)} \cancel{(x-5)} \cancel{(x+5)}}{\cancel{2} \cancel{(x-3)} \cancel{(x-5)} \cancel{(x+5)}}$$

Simplify.  $R(x) = 1$

### Try It

24) Find  $R(x) = f(x) \times g(x)$  where  $f(x) = \frac{3x - 21}{x^2 - 9x + 14}$  and  $g(x) = \frac{2x^2 - 8}{3x + 6}$

**Solution**

$$R(x) = 2$$

### Try It

25) Find  $R(x) = f(x) \times g(x)$  where  $f(x) = \frac{x^2 - x}{3x^2 + 27x - 30}$  and

$$g(x) = \frac{x^2 - 100}{x^2 - 10x}$$

**Solution**

$$R(x) = \frac{1}{3}$$

To divide rational functions, we divide the resulting rational expressions on the right side of the equation using the same techniques we used to divide rational expressions.

## Examples 2.1.12

Find  $R(x) = \frac{f(x)}{g(x)}$  where  $f(x) = \frac{3x^2}{x^2 - 4x}$  and  $g(x) = \frac{9x^2 - 45x}{x^2 - 7x + 10}$

### Solution

**Step 1: Substitute in the functions  $f(x)$ ,  $g(x)$ .**

$$R(x) = \frac{f(x)}{g(x)}$$

$$R(x) = \frac{\frac{3x^2}{x^2 - 4x}}{\frac{9x^2 - 45x}{x^2 - 7x + 10}}$$

**Step 2: Rewrite the division as the product of  $f(x)$  and the reciprocal of  $g(x)$ .**

$$R(x) = \frac{3x^2}{x^2 - 4x} \times \frac{x^2 - 7x + 10}{9x^2 - 45x}$$

**Step 3: Factor the numerators and denominators and then multiply.**

$$R(x) = \frac{3 \cdot x \cdot x \cdot (x - 5)(x - 2)}{x(x - 4) \cdot 3 \cdot 3 \cdot x \cdot (x - 5)}$$

**Step 4: Simplify by dividing out common factors.**

$$R(x) = \frac{\cancel{3} \cdot \cancel{x} \cdot \cancel{x} \cdot (\cancel{x - 5})(x - 2)}{\cancel{x} (x - 4) \cdot \cancel{3} \cdot 3 \cdot \cancel{x} (\cancel{x - 5})}$$

$$R(x) = \frac{x - 2}{3(x - 4)}$$

## Try It

26) Find  $R(x) = \frac{f(x)}{g(x)}$  where  $f(x) = \frac{2x^2}{x^2 - 8x}$  and  $g(x) = \frac{8x^2 + 24x}{x^2 + x - 6}$ .

### Solution

$$R(x) = \frac{x - 2}{4(x - 8)}$$

## Try It

27) Find  $R(x) = \frac{f(x)}{g(x)}$  where  $f(x) = \frac{15x^2}{3x^2 + 33x}$  and  $g(x) = \frac{5x - 5}{x^2 + 9x - 22}$ .

### Solution

$$R(x) = \frac{x(x - 2)}{x - 1}$$

## Key Concepts

- **Determine the values for which a rational expression is undefined.**

1. Set the denominator equal to zero.
2. Solve the equation.

- **Equivalent Fractions Property**

If  $a$ ,  $b$ , and  $c$  are numbers where  $b \neq 0$ ,  $c \neq 0$ , then  $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$  and  $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$ .

- **How to simplify a rational expression.**

1. Factor the numerator and denominator completely.
2. Simplify by dividing out common factors.

- **Opposites in a Rational Expression**

The opposite of  $a - b$  is  $b - a$ .  $\frac{a - b}{b - a} = -1$   $a \neq b$

An expression and its opposite divide to  $-1$ .

- **Multiplication of Rational Expressions**

If  $p$ ,  $q$ ,  $r$ , and  $s$  are polynomials where  $q \neq 0$ ,  $s \neq 0$ , then

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

- **How to multiply rational expressions.**

1. Factor each numerator and denominator completely.
2. Multiply the numerators and denominators.
3. Simplify by dividing out common factors.

- **Division of Rational Expressions**

If  $p$ ,  $q$ ,  $r$ , and  $s$  are polynomials where  $q \neq 0$ ,  $r \neq 0$ ,  $s \neq 0$ , then

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$

- **How to divide rational expressions.**

1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.
2. Factor the numerators and denominators completely.
3. Multiply the numerators and denominators together.
4. Simplify by dividing out common factors.

- **How to determine the domain of a rational function.**

1. Set the denominator equal to zero.
2. Solve the equation.
3. The domain is all real numbers excluding the values found in Step 2:

## Self Check

After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

*<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=217#h5p-15>*

## Glossary

### rational expression

A rational expression is an expression of the form  $\frac{p}{q}$ , where  $p$  and  $q$  are polynomials and  $q \neq 0$ .

### simplified rational expression

A simplified rational expression has no common factors, other than  $1$ , in its numerator and denominator.

### rational function

A rational function is a function of the form  $R(x) = \frac{p(x)}{q(x)}$  where  $p(x)$  and  $q(x)$  are polynomial functions and  $q(x)$  is not zero.



## 2.2 ADD AND SUBTRACT RATIONAL EXPRESSIONS

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### Learning Objectives

By the end of this section, you will be able to:

- Add and subtract rational expressions with a common denominator
- Add and subtract rational expressions whose denominators are opposites
- Find the least common denominator of rational expressions
- Add and subtract rational expressions with unlike denominators
- Add and subtract rational functions

### Try It

Before you get started, take this readiness quiz:

1) Add:  $\frac{7}{10} + \frac{8}{15}$

2) Subtract:  $\frac{3x}{4} - \frac{8}{9}$

3) Subtract:  $6(2x + 1) - 4(x - 5)$

## Add and Subtract Rational Expressions with a Common Denominator

What is the first step you take when you add numerical fractions? You check if they have a common denominator. If they do, you add the numerators and place the sum over the common denominator. If they do not have a common denominator, you find one before you add.

It is the same with rational expressions. To add rational expressions, they must have a common denominator. When the denominators are the same, you add the numerators and place the sum over the common denominator.

### Rational Expression Addition and Subtraction

If  $p$ ,  $q$ , and  $r$  are polynomials where  $r \neq 0$ , then

$$\frac{p}{r} + \frac{q}{r} = \frac{p+q}{r} \quad \text{and} \quad \frac{p}{r} - \frac{q}{r} = \frac{p-q}{r}$$

To add or subtract rational expressions with a common denominator, add or subtract the numerators and place the result over the common denominator.

We always simplify rational expressions. Be sure to factor, if possible, after you subtract the numerators so you can identify any common factors.

Remember, too, we do not allow values that would make the denominator zero. What value of  $x$  should be excluded in the next example?

#### Example 2.2.1

Add:  $\frac{11x + 28}{x + 4} + \frac{x^2}{x + 4}$ .

**Solution**

Since the denominator is  $x + 4$ , we must exclude the value  $x = -4$ .

$$\frac{11x + 28}{x + 4} + \frac{x^2}{x + 4}, x \neq -4$$

**Step 1:** The fractions have a common denominator, so add the numerators and place the sum over the common denominator.

$$\frac{11x + 28 + x^2}{x + 4}$$

**Step 2:** Write the degrees in descending order.

$$\frac{x^2 + 11x + 28}{x + 4}$$

**Step 3:** Factor the numerator.

$$\frac{(x + 4)(x + 7)}{x + 4}$$

**Step 4:** Simplify by removing common factors.

$$\frac{\cancel{(x + 4)}(x + 7)}{\cancel{x + 4}}$$

**Step 5:** Simplify.

$$x + 7$$

The expression simplifies to  $x + 7$  but the original expression had a denominator of  $x + 4$  so  $x \neq -4$ .

**Try It**

4) Simplify:  $\frac{9x + 14}{x + 7} + \frac{x^2}{x + 7}$ .

**Solution**

$x + 2$

**Try It**

5) Simplify:  $\frac{x^2 + 8x}{x + 5} + \frac{15}{x + 5}$ .

**Solution**

$x + 3$

To subtract rational expressions, they must also have a common denominator. When the denominators are the same, you subtract the numerators and place the difference over the common denominator. Be careful of the signs when you subtract a binomial or trinomial.

## Example 2.2.2

Subtract:  $\frac{5x^2 - 7x + 3}{x^2 - 3x + 18} - \frac{4x^2 + x - 9}{x^2 - 3x + 18}$ .

### Solution

**Step 1:** Subtract the numerators and place the difference over the common denominator.

$$\frac{5x^2 - 7x + 3 - (4x^2 + x - 9)}{x^2 - 3x + 18}$$

**Step 2:** Distribute the sign in the numerator.

$$\frac{5x^2 - 7x + 3 - 4x^2 - x + 9}{x^2 - 3x - 18}$$

**Step 3:** Combine like terms.

$$\frac{x^2 - 8x + 12}{x^2 - 3x - 18}$$

**Step 4:** Factor the numerator and the denominator.

$$\frac{(x - 2)(x - 6)}{(x + 3)(x - 6)}$$

**Step 5:** Simplify by removing common factors.

$$\frac{(x - 2) \cancel{(x - 6)}}{(x + 3) \cancel{(x - 6)}}$$

$$\frac{(x - 2)}{(x + 3)}$$

**Try It**

6) Subtract:  $\frac{4x^2 - 11x + 8}{x^2 - 3x + 2} - \frac{3x^2 + x - 3}{x^2 - 3x + 2}$ .

**Solution**

$$\frac{x - 11}{x - 2}$$

**Try It**

7) Subtract:  $\frac{6x^2 - x + 20}{x^2 - 81} - \frac{5x^2 + 11x - 7}{x^2 - 81}$ .

**Solution**

$$\frac{x - 3}{x + 9}$$

## Add and Subtract Rational Expressions Whose Denominators are Opposites

When the denominators of two rational expressions are opposites, it is easy to get a common denominator. We just have to multiply one of the fractions by  $\frac{-1}{-1}$ .

Let's see how this works.

$$\begin{aligned}
 &= \frac{7}{d} + \frac{5}{-d} \\
 \text{Multiply the second fraction by } \frac{-1}{-1} &= \frac{7}{d} + \frac{rgb]1.0, 0.0, 0.0(-1)5}{rgb]1.0, 0.0, 0.0(-1)(-d)} \\
 \text{The denominators are the same.} &= \frac{7}{d} + \frac{-5}{d} \\
 \text{Simplify.} &= \frac{2}{d}
 \end{aligned}$$

Be careful with the signs as you work with the opposites when the fractions are being subtracted.

### Example 2.2.3

Subtract:  $\frac{m^2 - 6m}{m^2 - 1} - \frac{3m + 2}{1 - m^2}$ .

#### Solution

**Step 1:** The denominators are opposites, so multiply the second fraction by  $\frac{-1}{-1}$ .

$$\frac{m^2 - 6m}{m^2 - 1} - \frac{-1(3m + 2)}{-1(1 - m^2)}$$

**Step 2:** Simplify the second fraction.

$$\frac{m^2 - 6m}{m^2 - 1} - \frac{-3m + 2}{m^2 - 1}$$

**Step 3:** The denominators are the same. Subtract the numerators.

$$\frac{m^2 - 6m - (-3m - 2)}{m^2 - 1}$$

**Step 4:** Distribute.

$$\frac{m^2 - 6m + 3m + 2}{m^2 - 1}$$

**Step 5: Combine like terms.**

$$\frac{m^2 - 3m + 2}{m^2 - 1}$$

**Step 6: Factor the numerator and denominator.**

$$\frac{(m - 1)(m - 2)}{(m - 1)(m + 1)}$$

**Step 7: Simplify by removing common factors.**

$$\frac{\cancel{(m - 1)}(m - 2)}{\cancel{(m - 1)}(m + 1)}$$

**Step 8: Simplify.**

$$\frac{m - 2}{m + 1}$$

## Try It

8) Subtract:  $\frac{y^2 - 5y}{y^2 - 4} - \frac{6y - 6}{4 - y^2}$ .

**Solution**

$$\frac{y + 3}{y + 2}$$



## Try It

9) Subtract:  $\frac{2n^2 + 8n - 1}{n^2 - 1} - \frac{n^2 - 7n - 1}{1 - n^2}$ .

### Solution

$$\frac{3n - 2}{n - 1}$$

## Find the Least Common Denominator of Rational Expressions

When we add or subtract rational expressions with unlike denominators, we will need to get common denominators. If we review the procedure we used with numerical fractions, we will know what to do with rational expressions.

Let's look at this example:  $\frac{7}{12} + \frac{5}{18}$ . Since the denominators are not the same, the first step was to find the least common denominator (LCD).

To find the LCD of the fractions, we factored 12 and 18 into primes, lining up any common primes in columns. Then we “brought down” one prime from each column. Finally, we multiplied the factors to find the LCD.

When we add numerical fractions, once we found the LCD, we rewrote each fraction as an equivalent fraction with the LCD by multiplying the numerator and denominator by the same number. We are now ready to add.

$$\begin{aligned} &= \frac{7}{12} + \frac{5}{18} \\ &= \frac{7 \cdot \text{rgb}]1.0, 0.0, 0.03}{12 \cdot \text{rgb}]1.0, 0.0, 0.03} + \frac{5 \cdot \text{rgb}]1.0, 0.0, 0.02}{18 \cdot \text{rgb}]1.0, 0.0, 0.02} \\ &= \frac{21}{26} + \frac{10}{36} \end{aligned}$$

$$\begin{array}{r}
 12 = 2 \cdot 2 \cdot 3 \\
 18 = 2 \cdot \phantom{2} \cdot 3 \cdot 3 \\
 \hline
 LCD = 2 \cdot 2 \cdot 3 \cdot 3 \\
 LCD = 36
 \end{array}$$

We do the same thing for rational expressions. However, we leave the LCD in factored form.

## HOW TO

### Find the least common denominator of rational expressions.

1. Factor each denominator completely.
2. List the factors of each denominator. Match factors vertically when possible.
3. Bring down the columns by including all factors, but do not include common factors twice.
4. Write the LCD as the product of the factors.

Remember, we always exclude values that would make the denominator zero. What values of  $x$  should we exclude in this next example?

### Example 2.2.4

a. Find the LCD for the expressions  $\frac{8}{x^2 - 2x - 3}$ ,  $\frac{3x}{x^2 + 4x + 3}$  and

b. Rewrite them as equivalent rational expressions with the lowest common denominator.

#### Solution

a.

**Step 1: Find the LCD for**  $\frac{8}{x^2 - 2x - 3}$ ,  $\frac{3x}{x^2 + 4x + 3}$ .

**Step 2: Factor each denominator completely, lining up common factors.**

$$\begin{array}{r} x^2 - 2x - 3 = (x + 1)(x - 3) \\ x^2 + 4x + 3 = (x + 1)(x + 3) \end{array}$$

**Step 3: Bring down the columns.**

$$\begin{array}{r} x^2 - 2x - 3 = (x + 1)(x - 3) \\ x^2 + 4x + 3 = (x + 1)(x + 3) \\ \hline LCD = (x + 1)(x - 3)(x + 3) \end{array}$$

**Step 4: Write the LCD as the product of the factors.**

$$\text{The LCD is } (x + 1)(x - 3)(x + 3).$$

b.

$$\frac{8}{x^2 - 2x - 3}, \frac{3x}{x^2 + 4x + 3}$$

**Step 1: Factor each denominator.**

$$\frac{8}{(x + 1)(x - 3)}, \frac{3x}{(x + 1)(x + 3)}$$

**Step 2: Multiply each denominator by the 'missing' LCD factor and multiply each numerator by the same factor.**

$$\frac{8(x + 3)}{(x + 1)(x - 3)(x + 3)}, \frac{3x(x - 3)}{(x + 1)(x + 3)(x - 3)}$$

**Step 3: Simplify the numerators.**

$$\frac{8x + 24}{(x + 1)(x - 3)(x + 3)}, \frac{3x^2 - 9x}{(x + 1)(x + 3)(x - 3)}$$

## Try It

10) Complete the following:

a. Find the LCD for the expressions  $\frac{2}{x^2 - x - 12}$ ,  $\frac{1}{x^2 - 16}$

b. Rewrite them as equivalent rational expressions with the lowest common denominator.

### Solution

a.  $(x - 4)(x + 3)(x + 4)$

b.  $\frac{2x + 8}{(x - 4)(x + 3)(x + 4)}$ ,  $\frac{x + 3}{(x - 4)(x + 3)(x + 4)}$

## Try It

11) Complete the following:

a. Find the LCD for the expressions  $\frac{3x}{x^2 - 3x + 10}$ ,  $\frac{5}{x^2 + 3x + 2}$

b. Rewrite them as equivalent rational expressions with the lowest common denominator.

### Solution

a.  $(x + 2)(x - 5)(x + 1)$

b.  $\frac{3x^2 + 3x}{(x + 2)(x - 5)(x + 1)}$ ,  $\frac{5x - 25}{(x + 2)(x - 5)(x + 1)}$

## Add and Subtract Rational Expressions with Unlike Denominators

Now we have all the steps we need to add or subtract rational expressions with unlike denominators.

### Example 2.2.5

Add:  $\frac{3}{x-3} + \frac{2}{x-2}$ .

#### Solution

##### **Step 1: Determine if the expressions have a common denominator.**

Yes — go to Step 2:

No — Rewrite each rational expression with the LCD.

Find the LCD.

Rewrite each rational expression as an equivalent rational expression with the LCD.

No.

$$\begin{array}{l} x - 3 : (x - 3)(\quad) \\ x - 2(\quad) : (x - 2) \\ \hline \text{LCD} : (x - 3)(x - 2) \end{array}$$

Find the LCD of  $(x - 3)$  and  $(x - 2)$ .

$$\frac{3}{x-3} + \frac{2}{x-2}$$

Change into equivalent rational expressions with the LCD,  $(x - 3)$  and  $(x - 2)$ .

$$\frac{3(x-2)}{(x-3)(x-2)} + \frac{2(x-3)}{(x-2)(x-3)}$$

Keep the denominators factored!

$$\frac{3x-6}{(x-3)(x-2)} + \frac{2x-6}{(x-2)(x-3)}$$

**Step 2: Add or subtract the rational expressions.**

Add the numerators and place the sum over the common denominator.

$$\frac{3x - 6 + 2x - 6}{(x - 3)(x - 2)}$$

**Step 3: Simplify, if possible.**

Because  $5x - 12$  cannot be factored, the answer is simplified.

$$\frac{5x - 12}{(x - 3)(x - 2)}$$

**Try It**

12) Add:  $\frac{2}{x - 2} + \frac{5}{x + 3}$ .

**Solution**

$$\frac{7x - 4}{(x - 2)(x + 3)}$$

**Try It**

13) Add:  $\frac{4}{m + 3} + \frac{3}{m + 4}$ .

**Solution**

$$\frac{7m + 25}{(m + 3)(m + 4)}$$

The steps used to add rational expressions are summarized here.

## HOW TO

### Add or subtract rational expressions.

1. Determine if the expressions have a common denominator.
  - **Yes** – go to step 2.
  - **No** – Rewrite each rational expression with the LCD.
    - Find the LCD.
    - Rewrite each rational expression as an equivalent rational expression with the LCD.
2. Add or subtract the rational expressions.
3. Simplify, if possible.

Avoid the temptation to simplify too soon. In the example above, we must leave the first rational expression as  $\frac{3x - 6}{(x - 3)(x - 2)}$  to be able to add it to  $\frac{2x - 6}{(x - 2)(x - 3)}$ . Simplify *only* after you have combined the numerators.

### Example 2.2.6

Add:  $\frac{8}{x^2 - 2x - 3} + \frac{3x}{x^2 + 4x + 3}$ .

#### Solution

**Step 1: Do the expressions have a common denominator?**

No.

**Step 2: Rewrite each expression with the LCD.**

Find the LCD.

$$\begin{aligned}x^2 - 2x - 3 &= (x + 1)(x - 3) \\x^2 + 4x + 3 &= (x + 1)(x + 3) \\LCD &= (x + 1)(x - 3)(x + 3)\end{aligned}$$

**Step 3: Rewrite each rational expression as an equivalent rational expression with the LCD.**

$$\frac{8(x + 3)}{(x + 1)(x - 3)(x + 3)} + \frac{3x(x - 3)}{(x + 1)(x + 3)(x - 3)}$$

**Step 4: Simplify the numerators.**

$$\frac{8x + 24}{(x + 1)(x - 3)(x + 3)} + \frac{3x^2 - 9x}{(x + 1)(x + 3)(x - 3)}$$

**Step 5: Add the rational expressions.**

$$\frac{8x + 24 + 3x^2 - 9x}{(x + 1)(x - 3)(x + 3)}$$

**Step 6: Simplify the numerator.**

$$\frac{3x^2 - x + 24}{(x + 1)(x - 3)(x + 3)}$$

The numerator is prime, so there are no common factors.



## Try It

14) Add:  $\frac{1}{m^2 - m - 2} + \frac{5m}{m^2 + 3m + 2}$ .

### Solution

$$\frac{5m^2 - 9m + 2}{(m + 1)(m - 2)(m + 2)}$$

## Try It

15) Add:  $\frac{2n}{n^2 - 3n - 10} + \frac{6}{n^2 + 5n + 6}$ .

### Solution

$$\frac{2n^2 + 12n - 30}{(n + 2)(n - 5)(n + 3)}$$

The process we use to subtract rational expressions with different denominators is the same as for addition. We just have to be very careful of the signs when subtracting the numerators.

**Example 2.2.7**

Subtract:  $\frac{8y}{y^2 - 16} - \frac{4}{y - 4}$ .

**Solution**

$$\frac{8y}{y^2 - 16} - \frac{4}{y - 4}$$

**Step 1: Do the expressions have a common denominator?**

No.

**Step 2: Rewrite each expression with the LCD.**

Find the LCD.

$$\begin{aligned}y^2 - 16 &= (y - 4)(y + 4) \\y - 4 &= y - 4 \\LCD &= (y - 4)(y + 4)\end{aligned}$$

**Step 3: Rewrite each rational expression as an equivalent rational expression with the LCD.**

$$\frac{8y}{(y - 4)(y + 4)} - \frac{4(y + 4)}{y - 4(y + 4)}$$

**Step 4: Simplify the numerators.**

$$\frac{8y}{(y - 4)(y + 4)} - \frac{4y + 16}{y - 4(y + 4)}$$

**Step 5: Subtract the rational expressions.**

$$\frac{8y - 4y - 16}{(y - 4)(y + 4)}$$

**Step 6: Simplify the numerator.**

$$\frac{4y - 16}{(y - 4)(y + 4)}$$

**Step 7: Factor the numerator to look for common factors.**

$$\frac{4(y - 4)}{(y - 4)(y + 4)}$$

**Step 8: Remove common factors.**

$$\frac{4 \cancel{(y - 4)}}{\cancel{(y - 4)}(y + 4)}$$

**Step 9: Simplify.**

$$\frac{4}{(y + 4)}$$

## Try It

16) Subtract:  $\frac{2x}{x^2 - 4} - \frac{1}{x + 2}$ .

**Solution**

$$\frac{1}{x - 2}$$

## Try It

17) Subtract:  $\frac{3}{z+3} - \frac{6z}{z^2-9}$ .

**Solution**

$$\frac{-3}{z-3}$$

There are lots of negative signs in the next example. Be extra careful.

## Example 2.2.8

Subtract:  $\frac{-3n-9}{n^2+n-6} - \frac{n+3}{2-n}$ .

**Solution**

**Step 1: Factor the denominator.**  $\frac{-3n-9}{(n-2)(n+3)} - \frac{n+3}{2-n}$

**Step 2: Since  $n-2$  and  $2-n$  are opposites, we will multiply the second rational expression by  $\frac{-1}{-1}$ .**

$$\frac{-3n-9}{(n-2)(n+3)} - \frac{(-1)(n+3)}{(-1)(2-n)}$$

**Step 3: Write  $(-1)(2-n)$  as  $n-2$**

$$\frac{-3n - 9}{(n - 2)(n + 3)} - \frac{(-1)n + 3}{(n - 2)}$$

**Step 4: Simplify. Remember,  $a - (-b) = a + b$ .**

$$\frac{-3n - 9}{(n - 2)(n + 3)} + \frac{(n + 3)}{(n - 2)}$$

**Step 5: Do the rational expressions have a common denominator?**

No.

**Step 6: Find the LCD.**

$$\begin{aligned} n^2 + n - 6 &= (n - 2)(n + 3) \\ n - 2 &= (n - 2) \\ LCD &= (n - 2)(n + 3) \end{aligned}$$

**Step 7: Rewrite each rational expression as an equivalent rational expression with the LCD.**

$$\frac{-3n - 9}{(n - 2)(n + 3)} + \frac{(n + 3)(n + 3)}{(n - 2)(n + 3)}$$

**Step 8: Simplify the numerators.**

$$\frac{-3n - 9}{(n - 2)(n + 3)} + \frac{n^2 + 6n + 9}{(n - 2)(n + 3)}$$

**Step 9: Add the rational expressions.**

$$\frac{-3n - 9 + n^2 + 6n + 9}{(n - 2)(n + 3)}$$

**Step 10: Simplify the numerator.**

$$\frac{n^2 + 3n}{(n - 2)(n + 3)}$$

**Step 11: Factor the numerator to look for common factors.**

$$\frac{n \cancel{(n + 3)}}{(n - 2) \cancel{(n + 3)}}$$

**Step 12: Simplify.**

$$\frac{n}{(n-2)}$$

### Try It

18) Subtract:  $\frac{3x-1}{x^2-5x-6} - \frac{2}{6-x}$ .

#### Solution

$$\frac{5x+1}{(x-6)(x+1)}$$

### Try It

19) Subtract:  $\frac{-2y-2}{y^2+2y-8} - \frac{y-1}{2-y}$ .

#### Solution

$$\frac{y+3}{y+4}$$

Things can get very messy when both fractions must be multiplied by a binomial to get the common denominator.

### Example 2.2.9

Subtract:  $\frac{4}{a^2 + 6a + 5} - \frac{3}{a^2 + 7a + 10}$ .

#### Solution

**Step 1: Factor the denominators.**

$$\frac{4}{(a + 1)(a + 5)} - \frac{3}{(a + 2)(a + 5)}$$

**Step 2: Do the rational expressions have a common denominator?**

No.

**Step 3: Find the LCD.**

$$\begin{aligned} a^2 + 6a + 5 &= (a + 1)(a + 5) \\ a^2 + 7a + 10 &= (a + 5)(a + 2) \\ LCD &= (a + 1)(a + 5)(a + 2) \end{aligned}$$

**Step 4: Rewrite each rational expression as an equivalent rational expression with the LCD.**

$$\frac{4(a + 2)}{(a + 1)(a + 5)(a + 2)} - \frac{3(a + 1)}{(a + 2)(a + 5)(a + 1)}$$

**Step 5: Simplify the numerators.**

$$\frac{4a + 8}{(a + 1)(a + 5)(a + 2)} - \frac{3a + 3}{(a + 2)(a + 5)(a + 1)}$$

**Step 6: Subtract the rational expressions.**

$$\frac{4a + 8 - (3a + 3)}{(a + 1)(a + 5)(a + 2)}$$

**Step 7: Simplify the numerator.**

$$\frac{4a + 8 - 3a - 3}{(a + 1)(a + 5)(a + 2)}$$

$$\frac{a + 5}{(a + 1)(a + 5)(a + 2)}$$

**Step 8: Look for common factors.**

$$\frac{\cancel{(a + 5)}}{(a + 1) \cancel{(a + 5)} (a + 2)}$$

**Step 9: Simplify.**

$$\frac{1}{(a + 1)(a + 2)}$$

## Try It

20) Subtract:  $\frac{3}{b^2 - 4b - 5} - \frac{2}{b^2 - 6b + 5}$ .

**Solution**

$$\frac{1}{(b + 1)(b - 1)}$$



## Try It

21) Subtract:  $\frac{4}{x^2 - 4} - \frac{3}{x^2 - x - 2}$ .

**Solution**

$$\frac{1}{(x + 2)(x + 1)}$$

We follow the same steps as before to find the LCD when we have more than two rational expressions. In the next example, we will start by factoring all three denominators to find their LCD.

## Example 2.2.10

Simplify:  $\frac{2u}{u - 1} + \frac{1}{u} - \frac{2u - 1}{u^2 - u}$ .

**Solution**

**Step 1: Do the expressions have a common denominator?**

No.

**Step 2: Rewrite each expression with the LCD.**

Find the LCD.

$$\begin{aligned} u - 1 &= (u - 1) \\ u &= u \\ u^2 - u &= u(u - 1) \\ LCD &= u(u - 1) \end{aligned}$$

**Step 3:** Rewrite each rational expression as an equivalent rational expression with the LCD.

$$\frac{2u \cdot u}{(u-1)u} + \frac{1 \cdot (u-1)}{u \cdot (u-1)} - \frac{2u-1}{u(u-1)}$$

$$\frac{2u^2}{(u-1)u} + \frac{u-1}{u \cdot (u-1)} - \frac{2u-1}{u(u-1)}$$

**Step 4:** Write as one rational expression.

$$\frac{2u^2 + u - 1 - 2u + 1}{u(u-1)}$$

**Step 5:** Simplify.

$$\frac{2u^2 - u}{u(u-1)}$$

**Step 6:** Factor the numerator, and remove common factors.

$$\frac{\cancel{u}(2u-1)}{\cancel{u}(u-1)}$$

**Step 7:** Simplify.

$$\frac{2u-1}{u-1}$$

## Try It

22) Simplify:  $\frac{v}{v+1} + \frac{3}{v-1} - \frac{6}{v^2-1}$ .

**Solution**

$$\frac{v+3}{v+1}$$

## Try It

23) Simplify:  $\frac{3w}{w+2} + \frac{2}{w+7} - \frac{17w+4}{w^2+9w+14}$ .

### Solution

$$\frac{3w}{w+7}$$

## Add and Subtract Rational Functions

To add or subtract rational functions, we use the same techniques we used to add or subtract polynomial functions.

### Example 2.2.11

Find  $R(x) = f(x) - g(x)$  where  $f(x) = \frac{x+5}{x-2}$  and  $g(x) = \frac{5x+18}{x^2-4}$ .

### Solution

**Step 1:** Substitute in the functions  $f(x), g(x)$ .

$$R(x) = \frac{x + 5}{x - 2} - \frac{5x + 18}{x^2 - 4}$$

**Step 2: Factor the denominators.**

$$R(x) = \frac{x + 5}{x - 2} - \frac{5x + 18}{(x - 2)(x + 2)}$$

**Step 3: Do the expressions have a common denominator?**

No.

**Step 4: Rewrite each expression with the LCD.**

Find the LCD.

$$\begin{aligned}x - 2 &= (x - 2) \\x^2 - 4 &= (x - 2)(x + 2) \\LCD &= (x - 2)(x + 2)\end{aligned}$$

**Step 5: Rewrite each rational expression as an equivalent rational expression with the LCD.**

$$R(x) = \frac{(x + 5)(x + 2)}{(x - 2)(x + 2)} - \frac{5x + 18}{(x - 2)(x + 2)}$$

**Step 6: Write as one rational expression.**

$$R(x) = \frac{(x + 5)(x + 2) - (5x + 18)}{(x - 2)(x + 2)}$$

**Step 7: Simplify.**

$$\begin{aligned}R(x) &= \frac{x^2 + 7x + 10 - 5x - 18}{(x - 2)(x + 2)} \\R(x) &= \frac{x^2 + 2x - 8}{(x - 2)(x + 2)}\end{aligned}$$

**Step 8: Factor the numerator, and remove common factors.**

$$R(x) = \frac{(x + 4) \cancel{(x - 2)}}{\cancel{(x - 2)}(x + 2)}$$

**Step 9: Simplify.**

$$R(x) = \frac{(x + 4)}{(x + 2)}$$

### Try It

24) Find  $R(x) = f(x) - g(x)$  where  $f(x) = \frac{x + 1}{x + 3}$  and  $g(x) = \frac{x + 17}{x^2 - x - 12}$ .

#### Solution

$$\frac{x - 7}{x - 4}$$

### Try It

25) Find  $R(x) = f(x) + g(x)$  where  $f(x) = \frac{x - 4}{x + 3}$  and  $g(x) = \frac{4x + 6}{x^2 - 9}$ .

#### Solution

$$\frac{x^2 - 3x + 18}{(x + 3)(x - 3)}$$

Access this online resource for additional instruction and practice with adding and subtracting rational expressions.

- Add and Subtract Rational Expressions- Unlike Denominators

## Key Concepts

- **Rational Expression Addition and Subtraction**

If  $p$ ,  $q$ , and  $r$  are polynomials where  $r \neq 0$ , then

$$\frac{p}{r} + \frac{q}{r} = \frac{p+q}{r} \text{ and } \frac{p}{r} - \frac{q}{r} = \frac{p-q}{r}$$

- **How to find the least common denominator of rational expressions.**

1. Factor each expression completely.
2. List the factors of each expression. Match factors vertically when possible.
3. Bring down the columns.
4. Write the LCD as the product of the factors.

- **How to add or subtract rational expressions.**

1. Determine if the expressions have a common denominator.
  - Yes – go to step 2.
  - No – Rewrite each rational expression with the LCD.
    - Find the LCD.
    - Rewrite each rational expression as an equivalent rational expression with the LCD.
2. Add or subtract the rational expressions.
3. Simplify, if possible.

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

*<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=292#h5p-16>*

b) After reviewing this checklist, what will you do to become confident for all objectives?

## 2.3 SIMPLIFY COMPLEX RATIONAL EXPRESSIONS

### Learning Objectives

By the end of this section, you will be able to:

- Simplify a complex rational expression by writing it as division
- Simplify a complex rational expression by using the LCD

### Try It

Before you get started, take this readiness quiz:

1) Simplify:  $\frac{\frac{3}{5}}{\frac{9}{10}}$

2) Simplify:  $\frac{1 - \frac{1}{3}}{4^2 + 4 \cdot 5}$

3) Solve:  $\frac{1}{2x} + \frac{1}{4} = \frac{1}{8}$

**Simplify a Complex Rational Expression by Writing it as**



## Division

Complex fractions are fractions in which the numerator or denominator contains a fraction. We previously simplified complex fractions like these:

$$\frac{\frac{3}{4}}{\frac{5}{8}} \qquad \frac{\frac{x}{2}}{\frac{xy}{6}}$$

In this section, we will simplify complex rational expressions, which are rational expressions with rational expressions in the numerator or denominator.

## Complex Rational Expression

A **complex rational expression** is a rational expression in which the numerator and/or the denominator contains a rational expression.

Here are a few complex rational expressions:

$$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}} \qquad \frac{\frac{1}{x} + \frac{1}{y}}{\frac{x}{y} - \frac{y}{x}} \qquad \frac{\frac{2}{x+6}}{\frac{4}{x-6} - \frac{4}{x^2-36}}$$

Remember, we always exclude values that would make any denominator zero.

We will use two methods to simplify complex rational expressions.

We have already seen this complex rational expression earlier in this chapter.

$$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-8x+3}{x^2-5x+6}}$$

We noted that fraction bars tell us to divide, so rewrote it as the division problem:

$$\left( \frac{6x^2 - 7x + 2}{4x - 8} \right) \div \left( \frac{2x^2 - 8x + 3}{x^2 - 5x + 6} \right).$$

Then, we multiplied the first rational expression by the reciprocal of the second, just like we do when we divide two fractions.

This is one method to simplify complex rational expressions. We make sure the complex rational expression is of the form where one fraction is over one fraction. We then write it as if we were dividing two fractions.

### Example 2.3.1

Simplify the complex rational expression by writing it as division:  $\frac{\frac{6}{x-4}}{\frac{3}{x^2-16}}$ .

#### Solution

**Step 1: Rewrite the complex fraction as division.**

$$\frac{6}{x-4} \div \frac{3}{x^2-16}$$

**Step 2: Rewrite as the product of first times the reciprocal of the second.**

$$\frac{6}{x-4} \cdot \frac{x^2-16}{3}$$

**Step 3: Factor.**

$$\frac{3 \cdot 2}{x-4} \cdot \frac{(x-4)(x+4)}{3}$$

**Step 4: Multiply.**

$$\frac{3 \cdot 2(x-4)(x+4)}{3(x-4)}$$

**Step 5: Remove common factors.**

$$\frac{\cancel{3} \cdot 2 \cancel{(x-4)} (x+4)}{\cancel{3} \cancel{(x-4)}}$$

**Step 6: Simplify.**

$$2(x+4)$$

Are there any values of  $x$  that should not be allowed?

The original complex rational expression had denominators of  $x - 4$  and  $x^2 - 16$ .

This expression would be undefined if  $x = 4$  or  $x = -4$ .

## Try It

4) Simplify the complex rational expression by writing it as division:  $\frac{\frac{2}{x^2-1}}{\frac{3}{x+1}}$ .

### Solution

$$\frac{2}{3(x-1)}$$

## Try It

5) Simplify the complex rational expression by writing it as division:  $\frac{\frac{1}{x^2-7x+12}}{\frac{2}{x-4}}$ .

### Solution

$$\frac{1}{2(x-3)}$$

Fraction bars act as grouping symbols. So to follow the Order of Operations, we simplify the numerator and denominator as much as possible before we can do the division.

## Example 2.3.2

Simplify the complex rational expression by writing it as division:  $\frac{\frac{1}{3} + \frac{1}{6}}{\frac{1}{2} - \frac{1}{3}}$ .

### Solution

#### Step 1: Simplify the numerator and denominator.

- a: Find the LCD and add the fractions in the numerator.
- b: Find the LCD and subtract the fractions in the denominator.

$$\frac{\frac{1 \cdot 2}{3 \cdot 2} + \frac{1}{6}}{\frac{1 \cdot 3}{2 \cdot 3} - \frac{1 \cdot 2}{3 \cdot 2}}$$

#### Step 2: Simplify the numerator and denominator.

$$\frac{\frac{2}{6} + \frac{1}{6}}{\frac{3}{6} - \frac{2}{6}}$$

#### Step 3: Rewrite the complex rational expression as a division problem.

$$\frac{3}{6} \div \frac{1}{6}$$

#### Step 4: Multiply the first by the reciprocal of the second.

$$\frac{3}{6} \cdot \frac{6}{1}$$

#### Step 5: Simplify.

$$3$$

## Try It

6) Simplify the complex rational expression by writing it as division:  $\frac{\frac{1}{2} + \frac{2}{3}}{\frac{5}{6} + \frac{1}{12}}$ .

**Solution**

$$\frac{14}{11}$$

## Try It

7) Simplify the complex rational expression by writing it as division:  $\frac{\frac{3}{4} - \frac{1}{3}}{\frac{1}{8} + \frac{5}{6}}$ .

**Solution**

$$\frac{10}{23}$$

We follow the same procedure when the complex rational expression contains variables.

### Example 2.3.3

Simplify the complex rational expression by writing it as division:  $\frac{\frac{1}{x} + \frac{1}{y}}{\frac{x}{y} - \frac{y}{x}}$ .

#### Solution

##### Step 1: Simplify the numerator and denominator.

We will simplify the sum in the numerator and the difference in the denominator.

$$\frac{\frac{1}{x} + \frac{1}{y}}{\frac{x}{y} - \frac{y}{x}}$$

Find a common denominator and add the fractions in the numerator

$$\frac{\frac{1 \cdot y}{x \cdot y} + \frac{1 \cdot x}{y \cdot x}}{\frac{x \cdot x}{y \cdot x} - \frac{y \cdot y}{x \cdot y}}$$

$$\frac{\frac{y}{xy} + \frac{x}{xy}}{\frac{x^2}{xy} - \frac{y^2}{xy}}$$

Find a common denominator and subtract the fractions in the denominator.

We now have just one rational expression in the numerator and one in the denominator.

$$\frac{\frac{y+x}{xy}}{\frac{x^2-y^2}{xy}}$$

##### Step 2: Rewrite the complex rational expression as a division problem.

We write the numerator divided by the denominator.

$$\left(\frac{y+x}{xy}\right) \div \left(\frac{x^2-y^2}{xy}\right)$$

##### Step 3: Divide the expressions.

Multiply the first by the reciprocal of the second.

$$\left(\frac{y+x}{xy}\right) \div \left(\frac{x^2-y^2}{xy}\right)$$

Factor any expressions if possible.

$$\frac{xy(y+x)}{xy(x-y)(x+y)}$$

Remove common factors.

$$\frac{\cancel{x} \cancel{y} (y+x)}{\cancel{x} \cancel{y} (x-y) \cancel{(y+x)}}$$

Simplify.

$$\frac{1}{x-y}$$

## Try It

8) Simplify the complex rational expression by writing it as division:  $\frac{\frac{1}{x} + \frac{1}{y}}{\frac{1}{x} - \frac{1}{y}}$ .

**Solution**

$$\frac{y+x}{y-x}$$

## Try It

9) Simplify the complex rational expression by writing it as division:  $\frac{\frac{1}{a} + \frac{1}{b}}{\frac{1}{a^2} - \frac{1}{b^2}}$ .

### Solution

$$\frac{ab}{b - a}$$

We summarize the steps here.

## HOW TO

### Simplify a complex rational expression by writing it as division.

1. Simplify the numerator and denominator.
2. Rewrite the complex rational expression as a division problem.
3. Divide the expressions.



### Example 2.3.4

Simplify the complex rational expression by writing it as division:  $\frac{n - \frac{4n}{n+5}}{\frac{1}{n+5} + \frac{1}{n-5}}$ .

#### Solution

##### Step 1: Simplify the numerator and denominator.

Find common denominators for the numerator and denominator.

$$\frac{\frac{n(n+5)}{1(n+5)} - \frac{4n}{n+5}}{\frac{1(n-5)}{(n+5)(n-5)} + \frac{1(n+5)}{(n-5)(n+5)}}$$

##### Step 2: Simplify the numerators.

$$\frac{\frac{n^2+5n}{n+5} - \frac{4n}{n+5}}{\frac{n-5}{(n+5)(n-5)} + \frac{(n+5)}{(n-5)(n+5)}}$$

##### Step 3: Subtract the rational expressions in the numerator and add in the denominator.

$$\frac{\frac{n^2+5n-4n}{n+5}}{\frac{n-5+n+5}{(n+5)(n-5)}}$$

##### Step 4: Simplify. (We now have one rational expression over one rational expression.)

$$\frac{\frac{n^2+n}{n+5}}{\frac{2n}{(n+5)(n-5)}}$$

##### Step 5: Rewrite as fraction division.

$$\frac{n^2 + n}{n + 5} \div \frac{2n}{(n + 5)(n - 5)}$$

##### Step 6: Multiply the first times the reciprocal of the second.

$$\frac{n^2 + n}{n + 5} \cdot \frac{(n + 5)(n - 5)}{2n}$$

**Step 7: Factor any expressions if possible.**

$$\frac{n(n + 1)(n + 5)(n - 5)}{(n + 5)2n}$$

**Step 8: Remove common factors.**

$$\frac{\cancel{n}(n + 1) \cancel{(n + 5)}(n - 5)}{\cancel{(n + 5)} 2 \cancel{n}}$$

**Step 9: Simplify.**

$$\frac{(n + 1)(n - 5)}{2}$$

## Try It

10) Simplify the complex rational expression by writing it as division:  $\frac{b - \frac{3b}{b+5}}{\frac{2}{b+5} + \frac{1}{b-5}}$ .

**Solution**

$$\frac{b(b + 2)(b - 5)}{3b - 5}$$

**Try It**

11) Simplify the complex rational expression by writing it as division:  $\frac{1 - \frac{3}{c+4}}{\frac{1}{c+4} + \frac{c}{3}}$ .

**Solution**

$$\frac{3}{c+3}$$

## Simplify a Complex Rational Expression by Using the LCD

We “cleared” the fractions by multiplying by the LCD when we solved equations with fractions. We can use that strategy here to simplify complex rational expressions. We will multiply the numerator and denominator by the LCD of all the rational expressions.

Let’s look at the complex rational expression we simplified one way in Example 2.3.2. We will simplify it here by multiplying the numerator and denominator by the LCD. When we multiply by  $\frac{\text{LCD}}{\text{LCD}}$  we are multiplying by 1, so the value stays the same.

**Example 2.3.5**

Simplify the complex rational expression by using the LCD:  $\frac{\frac{1}{3} + \frac{1}{6}}{\frac{1}{2} - \frac{1}{3}}$ .

**Solution**

The LCD of all the fractions in the whole expression is 6.

**Step 1: Clear the fractions by multiplying the numerator and denominator by that LCD.**

$$\begin{aligned}
 &= \frac{rgb]1.0, 0.0, 0.06 \cdot \left(\frac{1}{3} + \frac{1}{6}\right)}{rgb]1.0, 0.0, 0.06 \cdot \left(\frac{1}{2} - \frac{1}{3}\right)} \\
 \text{Distribute.} \quad &= \frac{rgb]1.0, 0.0, 0.06 \cdot \frac{1}{3} + rgb]1.0, 0.0, 0.06 \cdot \frac{1}{6}}{rgb]1.0, 0.0, 0.06 \cdot \frac{1}{2} - rgb]1.0, 0.0, 0.06 \cdot \frac{1}{3}} \\
 \text{Simplify.} \quad &= \frac{2 + 1}{3 - 2} \\
 &= \frac{3}{1} \\
 &= 3
 \end{aligned}$$

## Try It

12) Simplify the complex rational expression by using the LCD:  $\frac{\frac{1}{2} + \frac{1}{5}}{\frac{1}{10} + \frac{1}{5}}$ .

**Solution**

$$\frac{7}{3}$$

## Try It

13) Simplify the complex rational expression by using the LCD:  $\frac{\frac{1}{4} + \frac{3}{8}}{\frac{1}{2} - \frac{5}{16}}$ .

### Solution

$$\frac{10}{3}$$

We will use the same example as in Example 2.3.3. Decide which method works better for you.

## Example 2.3.6

Simplify the complex rational expression by using the LCD:  $\frac{\frac{1}{x} + \frac{1}{y}}{\frac{x}{y} - \frac{y}{x}}$ .

### Solution

**Step 1: Find the LCD of all fractions in the complex rational expression.**

The LCD of all the fractions is  $xy$ .

$$\frac{\frac{1}{x} + \frac{1}{y}}{\frac{x}{y} - \frac{y}{x}}$$

**Step 2: Multiply the numerator and denominator by the LCD.**

Multiply both the numerator and denominator by  $xy$ .

$$\frac{xy \cdot \left( \frac{1}{x} + \frac{x}{y} \right)}{xy \cdot \left( \frac{x}{y} - \frac{y}{x} \right)}$$

**Step 3: Simplify the expression.**

Distribute.

$$\frac{xy \cdot \frac{1}{x} + xy \cdot \frac{x}{y}}{xy \cdot \frac{x}{y} - xy \cdot \frac{y}{x}}$$

$$\frac{y+x}{x^2-y^2}$$

Simplify.

$$\frac{\cancel{(y+x)}}{(x-y)\cancel{(x+y)}}$$

Remove common factors.

$$\frac{1}{x-y}$$

## Try It

14) Simplify the complex rational expression by using the LCD:  $\frac{\frac{1}{a} + \frac{1}{b}}{\frac{a}{b} + \frac{b}{a}}$ .

**Solution**

$$\frac{b+a}{a^2+b^2}$$

## Try It

15) Simplify the complex rational expression by using the LCD:  $\frac{\frac{1}{x^2} - \frac{1}{y^2}}{\frac{1}{x} + \frac{1}{y}}$ .

### Solution

$$\frac{y - x}{xy}$$

## HOW TO

### Simplify a complex rational expression by using the LCD.

1. Find the LCD of all fractions in the complex rational expression.
2. Multiply the numerator and denominator by the LCD.
3. Simplify the expression.

Be sure to start by factoring all the denominators so you can find the LCD.

### Example 2.3.7

Simplify the complex rational expression by using the LCD:  $\frac{\frac{2}{x+6}}{\frac{4}{x-6} - \frac{4}{x^2-36}}$ .

#### Solution

**Step 1: Find the LCD of all fractions in the complex rational expression.**

The LCD is  $x^2 - 36 = (x + 6)(x - 6)$ .

**Step 2: Multiply the numerator and denominator by the LCD.**

$$\frac{(x + 6)(x - 6) \frac{2}{x+6}}{(x + 6)(x - 6) \left( \frac{4}{x-6} - \frac{4}{(x+6)(x-6)} \right)}$$

**Step 3: Simplify the expression.**

Formula does not parse

**Step 4: To simplify the denominator, distribute and combine like terms.**

$$\frac{2(x - 6)}{4x + 20}$$

**Step 5: Factor the denominator.**

$$\frac{2(x - 6)}{4(x + 5)}$$

**Step 6: Remove common factors.**

$$\frac{\cancel{2}(x - 6)}{\cancel{2} \cdot 2(x + 5)}$$

**Step 7: Simplify.**

$$\frac{(x - 6)}{2(x + 5)}$$



Notice that there are no more factors common to the numerator and denominator.

### Try It

16) Simplify the complex rational expression by using the LCD:  $\frac{\frac{3}{x+2}}{\frac{5}{x-2} - \frac{3}{x^2-4}}$ .

#### Solution

$$\frac{3(x-2)}{5x+7}$$

### Try It

17) Simplify the complex rational expression by using the LCD:  $\frac{\frac{2}{x-7} - \frac{1}{x+7}}{\frac{6}{x+7} - \frac{1}{x^2-49}}$ .

#### Solution

$$\frac{x+21}{6x-43}$$

Be sure to factor the denominators first. Proceed carefully as the math can get messy!

### Example 2.3.8

Simplify the complex rational expression by using the LCD:  $\frac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3} - \frac{2}{m-4}}$ .

#### Solution

**Step 1: Find the LCD of all fractions in the complex rational expression.**

The LCD is  $(m - 3)(m - 4)$ .

**Step 2: Multiply the numerator and denominator by the LCD.**

$$\frac{(m - 3)(m - 4) \frac{4}{(m-3)(m-4)}}{(m - 3)(m - 4) \left( \frac{3}{m-3} - \frac{2}{m-4} \right)}$$

**Step 3: Simplify.**

Formula does not parse

**Step 4: Distribute.**

$$\frac{4}{3m - 12 - 2m + 6}$$

**Step 5: Combine like terms.**

$$\frac{4}{m - 6}$$

## Try It

18) Simplify the complex rational expression by using the LCD:  $\frac{\frac{3}{x^2+7x+10}}{\frac{4}{x+2} + \frac{1}{x+5}}$ .

**Solution**

$$\frac{3}{5x + 22}$$

## Try It

19) Simplify the complex rational expression by using the LCD:  $\frac{\frac{4y}{y+5} + \frac{2}{y+6}}{\frac{3y}{y^2+11y+30}}$ .

**Solution**

$$\frac{2(2y^2 + 13y + 5)}{3y}$$

### Example 2.3.9

Simplify the complex rational expression by using the LCD:  $\frac{\frac{y}{y+1}}{1 + \frac{1}{y-1}}$ .

#### Solution

**Step 1: Find the LCD of all fractions in the complex rational expression.**

The LCD is  $(y + 1)(y - 1)$ .

**Step 2: Multiply the numerator and denominator by the LCD.**

$$\frac{(y + 1)(y - 1) \frac{y}{y+1}}{(y + 1)(y - 1) \left(1 + \frac{1}{y-1}\right)}$$

**Step 3: Distribute in the denominator and simplify.**

Formula does not parse

**Step 5: Simplify the denominator and leave the numerator factored.**

$$\frac{y(y - 1)}{y^2 - 1 + y + 1}$$

$$\frac{y(y - 1)}{y^2 + y}$$

**Step 6: Factor the denominator and remove factors common with the numerator.**

$$\frac{\cancel{y}(y-1)}{\cancel{y}(y+1)}$$

Simplify.  $\frac{y-1}{y+1}$

**Try It**

20) Simplify the complex rational expression by using the LCD:  $\frac{\frac{x}{x+3}}{1 + \frac{1}{x+3}}$ .

**Solution**

$$\frac{x}{x+4}$$

**Try It**

21) Simplify the complex rational expression by using the LCD:  $\frac{1 + \frac{1}{x-1}}{\frac{3}{x+1}}$ .

**Solution**

$$\frac{x(x+1)}{3(x-1)}$$

Access this online resource for additional instruction and practice with complex fractions.

- Complex Fractions

## Key Concepts

- **How to simplify a complex rational expression by writing it as division.**
  1. Simplify the numerator and denominator.
  2. Rewrite the complex rational expression as a division problem.
  3. Divide the expressions.
- **How to simplify a complex rational expression by using the LCD.**
  1. Find the LCD of all fractions in the complex rational expression.
  2. Multiply the numerator and denominator by the LCD.
  3. Simplify the expression.

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=340#h5p-17>

b) After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

## Glossary

**complex rational expression**

A complex rational expression is a rational expression in which the numerator and/or denominator contains a rational expression.

## 2.4 SOLVE RATIONAL EQUATIONS

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### Learning Objectives

By the end of this section, you will be able to:

- Solve rational equations
- Use rational functions
- Solve a rational equation for a specific variable

### Try It

Before you get started, take this readiness quiz:

1) Solve:  $\frac{1}{6}x + \frac{1}{2} = \frac{1}{3}$ .

2) Solve:  $n^2 - 5n - 36 = 0$ .

3) Solve the formula  $5x + 2y = 10$  for  $y$ .

After defining the terms ‘expression’ and ‘equation’ earlier, we have used them throughout this book. We have *simplified* many kinds of *expressions* and *solved* many kinds of *equations*. We have simplified many rational expressions so far in this chapter. Now we will *solve* a **rational equation**.



## Rational Equation

A **rational equation** is an equation that contains a rational expression.

You must make sure to know the difference between rational expressions and rational equations. The equation contains an equal sign.

Rational Expression	Rational Equation
$\frac{1}{8}x + \frac{1}{2}$	$\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$
$\frac{y+6}{y^2-36}$	$\frac{y+6}{y^2-36} = y+1$
$\frac{1}{n-3} - \frac{1}{n+4}$	$\frac{1}{n-3} - \frac{1}{n+4} = \frac{15}{n^2+n-12}$

## Solve Rational Equations

We have already solved linear equations that contained fractions. We found the LCD of all the fractions in the equation and then multiplied both sides of the equation by the LCD to “clear” the fractions.

We will use the same strategy to solve rational equations. We will multiply both sides of the equation by the LCD. Then, we will have an equation that does not contain rational expressions and thus is much easier for us to solve. But because the original equation may have a variable in a denominator, we must be careful that we don’t end up with a solution that would make a denominator equal to zero.

So before we begin solving a rational equation, we examine it first to find the values that would make any denominators zero. That way, when we solve a rational equation we will know if there are any algebraic solutions we must discard.

An algebraic solution to a rational equation that would cause any of the rational expressions to be undefined is called an **extraneous solution to a rational equation**.

## Extraneous Solution to a Rational Equation

An **extraneous solution to a rational equation** is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

We note any possible extraneous solutions,  $c$ , by writing  $x \neq c$  next to the equation.

**Example 2.4.1**

Solve:  $\frac{1}{x} + \frac{1}{3} = \frac{5}{6}$

**Solution**

**Step 1: Note any value of the variable that would make any denominator zero.**

If  $x = 0$ , then  $\frac{1}{x}$  is undefined.

So we'll write  $x \neq 0$  next to the equation.

$$\frac{1}{x} + \frac{1}{3} = \frac{5}{6}, x \neq 0$$

**Step 2: Find the least common denominator of all denominators in the equation.**

Find the LCD of  $\frac{1}{x}$ ,  $\frac{1}{3}$ , and  $\frac{5}{6}$ .

The LCD is  $6x$

**Step 3: Clear the fractions by multiplying both sides of the equation by the LCD.**

Multiply both sides of the equation by the LCD,  $6x$ .

$$6x \cdot \left( \frac{1}{x} + \frac{1}{3} \right) = 6x \cdot \left( \frac{5}{6} \right)$$

Use the Distributive Property.

$$6x \cdot \frac{1}{x} + 6x \cdot \frac{1}{3} = 6x \cdot \left( \frac{5}{6} \right)$$

Simplify – and notice, no more fractions!

$$6 + 2x = 5x$$

**Step 4: Solve the resulting equation.**

$$\begin{array}{ll} \text{Simplify.} & 6 = 3x \\ & 2 = x \end{array}$$

**Step 5: Check.**

If any values found in Step 1 are algebraic solutions, discard them.  
Check any remaining solutions in the original equation.

We did not get  $0$  as an algebraic solution.

We substitute  $x = 2$  into the original equation.

$$\frac{1}{x} + \frac{1}{3} = \frac{5}{6}$$

$$\frac{1}{2} + \frac{1}{3} \stackrel{?}{=} \frac{5}{6}$$

$$\frac{3}{6} + \frac{2}{6} \stackrel{?}{=} \frac{5}{6}$$

$$\frac{5}{6} = \frac{5}{6} \checkmark$$

## Try It

4) Solve:  $\frac{1}{y} + \frac{2}{3} = \frac{1}{5}$ .

**Solution**

$$y = -\frac{7}{15}$$

## Try It

5) Solve:  $\frac{2}{3} + \frac{1}{5} = \frac{1}{x}$ .

**Solution**

$$x = \frac{13}{15}$$

The steps of this method are shown.

## HOW TO

### Solve equations with rational expressions.

1. Note any value of the variable that would make any denominator zero.
2. Find the least common denominator of *all* denominators in the equation.
3. Clear the fractions by multiplying both sides of the equation by the LCD.
4. Solve the resulting equation.
5. Check:
  - If any values found in Step 1 are algebraic solutions, discard them.
  - Check any remaining solutions in the original equation.

We always start by noting the values that would cause any denominators to be zero.

## Example 2.4.2

Solve:  $1 - \frac{5}{y} = -\frac{6}{y^2}$ .

### Solution

**Step 1:** Note any value of the variable that would make any denominator zero.

$$1 - \frac{5}{y} = -\frac{6}{y^2}, y \neq 0$$

**Step 2:** Find the least common denominator of all denominators in the equation.

The LCD is  $y^2$ .

**Step 3:** Clear the fractions by multiplying both sides of the equation by the LCD.

$$y^2 \left( 1 - \frac{5}{y} \right) = y^2 \left( -\frac{6}{y^2} \right)$$

**Step 4:** Distribute.

$$y^2 \cdot 1 - y^2 \left( \frac{5}{y} \right) = y^2 \left( -\frac{6}{y^2} \right)$$

**Step 5:** Multiply.

$$y^2 - 5y = -6$$

**Step 6:** Solve the resulting equation. First write the quadratic equation in standard form.

$$y^2 - 5y + 6 = 0$$

**Step 7:** Factor.

$$(y - 2)(y - 3) = 0$$

**Step 8:** Use the Zero Product Property.

$$y - 2 = 0 \text{ or } y - 3 = 0$$

**Step 9:** Solve.

$$y = 2 \text{ or } y = 3$$

**Step 10: Check.**

We did not get 0 as an algebraic solution.

Check  $y = 2$  and  $y = 3$  in the original equation.

$$1 - \frac{5}{y} = -\frac{6}{y^2}$$

$$1 - \frac{5}{2} \stackrel{?}{=} -\frac{6}{2^2}$$

$$1 - \frac{5}{2} \stackrel{?}{=} -\frac{6}{4}$$

$$\frac{2}{2} - \frac{5}{2} \stackrel{?}{=} -\frac{6}{4}$$

$$-\frac{3}{2} \stackrel{?}{=} -\frac{6}{4}$$

$$-\frac{3}{2} = -\frac{3}{2} \checkmark$$

$$1 - \frac{5}{y} = -\frac{6}{y^2}$$

$$1 - \frac{5}{3} \stackrel{?}{=} -\frac{6}{3^2}$$

$$1 - \frac{5}{3} \stackrel{?}{=} -\frac{6}{9}$$

$$\frac{3}{3} - \frac{5}{3} \stackrel{?}{=} -\frac{6}{9}$$

$$-\frac{2}{3} \stackrel{?}{=} -\frac{6}{9}$$

$$-\frac{2}{3} = -\frac{2}{3} \checkmark$$

The solution is  $y = 2, y = 3$ .

**Try It**

6) Solve:  $1 - \frac{2}{x} = \frac{15}{x^2}$ .

**Solution**

$$x = -3, x = 5$$

## Try It

7) Solve:  $1 - \frac{4}{y} = \frac{12}{y^2}$ .

### Solution

$$y = -2, y = 6$$

In the next example, the last denominator is a difference of squares. Remember to factor it first to find the LCD.

## Example 2.4.3

Solve:  $\frac{2}{x+2} + \frac{4}{x-2} = \frac{x-1}{x^2-4}$ .

### Solution

**Step 1:** Note any value of the variable that would make any denominator zero.

$$\frac{2}{x+2} + \frac{4}{x-2} = \frac{x-1}{(x+2)(x-2)}, x \neq -2, x \neq 2$$

**Step 2:** Find the least common denominator of all denominators in the equation.

$$\text{The LCD is } (x+2)(x-2).$$

**Step 3:** Clear the fractions by multiplying both sides of the equation by then LCD.

$$(x+2)(x-2) \left( \frac{2}{x+2} + \frac{4}{x-2} \right) = (x+2)(x-2) \left( \frac{x-1}{x^2-4} \right)$$

**Step 4:** Distribute.

$$(x+2)(x-2)\frac{2}{x+2} + (x+2)(x-2)\frac{4}{x-2} = (x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$$

**Step 5: Remove common factors.**

$$\cancel{(x+2)}\cancel{(x-2)}\frac{2}{\cancel{x+2}} + \cancel{(x+2)}\cancel{(x-2)}\frac{4}{\cancel{x-2}} = \cancel{(x+2)}\cancel{(x-2)}\left(\frac{x-1}{\cancel{x^2-4}}\right)$$

**Step 6: Simplify.**

$$2(x-2) + 4(x+2) = x-1$$

**Step 7: Distribute.**

$$2x - 4 + 4x + 8 = x - 1$$

**Step 8: Solve.**

$$6x + 4 = x - 1$$

$$5x = -5$$

$$x = -1$$

**Step 9: Check:**

We did not get 2 or -2 as algebraic solutions.

Check  $x = -1$  in the original equation.

$$\begin{aligned} \frac{2}{x+2} + \frac{4}{x-2} &= \frac{x-1}{x^2-4} \\ \frac{2}{(-1)+2} + \frac{4}{(-1)-2} &\stackrel{?}{=} \frac{(-1)-1}{(-1)^2-4} \\ \frac{2}{1} + \frac{4}{-3} &\stackrel{?}{=} \frac{-2}{-3} \\ \frac{6}{3} - \frac{4}{3} &\stackrel{?}{=} \frac{2}{3} \\ \frac{2}{3} &= \frac{2}{3} \checkmark \end{aligned}$$

The solution is  $x = -1$ .



**Try It**

8) Solve:  $\frac{2}{x+1} + \frac{1}{x-1} = \frac{1}{x^2-1}$ .

**Solution**

$$x = \frac{2}{3}$$

**Try It**

9) Solve:  $\frac{5}{y+3} + \frac{2}{y-3} = \frac{5}{y^2-9}$ .

**Solution**

$$y = 2$$

In the next example, the first denominator is a trinomial. Remember to factor it first to find the LCD.

### Example 2.4.4

Solve:  $\frac{m + 11}{m^2 - 5m + 4} = \frac{5}{m - 4} - \frac{3}{m - 1}$ .

#### Solution

**Step 1: Note any value of the variable that would make any denominator zero.**

Use the factored form of the quadratic denominator.

$$\frac{m + 11}{(m - 4)(m - 1)} = \frac{5}{m - 4} - \frac{3}{m - 1}, m \neq 4, m \neq 1$$

**Step 2: Find the least common denominator of all denominators in the equation.**

The LCD is  $(m - 4)(m - 1)$ .

**Step 3: Clear the fractions by multiplying both sides of the equation by the LCD.**

$$(m - 4)(m - 1) \left( \frac{m + 11}{(m - 4)(m - 1)} \right) = (m - 4)(m - 1) \left( \frac{5}{m - 4} - \frac{3}{m - 1} \right)$$

**Step 4: Distribute.**

$$(m - 4)(m - 1) \left( \frac{m + 11}{(m - 4)(m - 1)} \right) = (m - 4)(m - 1) \frac{5}{m - 4} - (m - 4)(m - 1) \frac{3}{m - 1}$$

**Step 5: Remove common factors.**

$$\cancel{(m - 4)} \cancel{(m - 1)} \left( \frac{m + 11}{\cancel{(m - 4)} \cancel{(m - 1)}} \right) = \cancel{(m - 4)} (m - 1) \frac{5}{\cancel{m - 4}} - (m - 4) \cancel{(m - 1)} \frac{3}{\cancel{m - 1}}$$

**Step 6: Simplify.**

$$m + 11 = 5(m - 1) - 3(m - 4)$$

**Step 7: Solve the resulting equation.**

$$\begin{aligned} m + 11 &= 5m - 5 - 3m + 12 \\ 4 &= m \end{aligned}$$

**Step 8: Check.**

The only algebraic solution was  $4$ , but we said that  $4$  would make a denominator equal to zero. The algebraic solution is an extraneous solution.

There is no solution to this equation.

### Try It

10) Solve:  $\frac{x + 13}{x^2 - 7x + 10} = \frac{6}{x - 5} - \frac{4}{x - 2}$ .

#### Solution

There is no solution.

### Try It

11) Solve:  $\frac{y - 6}{y^2 + 3y - 4} = \frac{2}{y + 4} + \frac{7}{y - 1}$ .

#### Solution

There is no solution.

The equation we solved in the previous example had only one algebraic solution, but it was an extraneous solution. That left us with no solution to the equation. In the next example we get two algebraic solutions. Here one or both could be extraneous solutions.

### Example 2.4.5

Solve:  $\frac{y}{y+6} = \frac{72}{y^2-36} + 4$ .

#### Solution

**Step 1: Factor all the denominators, so we can note any value of the variable that would make any denominator zero.**

$$\frac{y}{y+6} = \frac{72}{(y-6)(y+6)} + 4, y \neq 6, y \neq -6$$

**Step 2: Find the least common denominator.**

The LCD is  $(y-6)(y+6)$ .

**Step 3: Clear the fractions.**

$$(y-6)(y+6) \left( \frac{y}{y+6} \right) = (y-6)(y+6) \left( \frac{72}{(y-6)(y+6)} + 4 \right)$$

**Step 4: Simplify.**

$$\begin{aligned} (y-6) \cdot y &= 72 + (y-6)(y+6) \cdot 4 \\ \text{Simplify.} \quad y(y-6) &= 72 + 4(y^2-36) \end{aligned}$$

**Step 5: Solve the resulting equation.**

$$\begin{aligned} y^2 - 6y &= 72 + 4y^2 - 144 \\ 0 &= 3y^2 + 6y - 72 \\ 0 &= 3(y^2 + 2y - 24) \\ 0 &= 3(y+6)(y-4) \\ y &= -6, y = 4 \end{aligned}$$

**Step 6: Check.**

$y = -6$  is an extraneous solution.

Check  $y = 4$  in the original equation.

$$\begin{aligned} \frac{y}{y+6} &= \frac{72}{y^2-36} + 4 \\ \frac{4}{4+6} &\stackrel{?}{=} \frac{72}{4^2-36} + 4 \\ \frac{4}{10} &\stackrel{?}{=} \frac{72}{-20} + 4 \\ \frac{4}{10} &\stackrel{?}{=} -\frac{36}{10} + \frac{40}{10} \\ \frac{4}{10} &= \frac{4}{10} \checkmark \end{aligned}$$

The solution is  $y = 4$ .

### Try It

12) Solve:  $\frac{x}{x+4} = \frac{32}{x^2-16} + 5$ .

**Solution**

$$x = 3$$

### Try It

13) Solve:  $\frac{y}{y+8} = \frac{128}{y^2-64} + 9$ .

**Solution**

$$y = 7$$

In some cases, all the algebraic solutions are extraneous.

**Example 2.4.6**

$$\text{Solve: } \frac{x}{2x-2} - \frac{2}{3x+3} = \frac{5x^2 - 2x + 9}{12x^2 - 12}.$$

**Solution**

**Step 1:** We will start by factoring all denominators, to make it easier to identify extraneous solutions and the LCD.

$$\frac{x}{2(x-1)} - \frac{2}{3(x+1)} = \frac{5x^2 - 2x + 9}{12(x-1)(x+1)}$$

**Step 2:** Note any value of the variable that would make any denominator zero.

$$\frac{x}{2(x-1)} - \frac{2}{3(x+1)} = \frac{5x^2 - 2x + 9}{12(x-1)(x+1)}, x \neq 1, x \neq -1$$

**Step 3:** Find the least common denominator.

$$\text{The LCD is } 12(x-1)(x+1).$$

**Step 4:** Clear the fractions.

$$12(x-1)(x+1) \left( \frac{x}{2(x-1)} - \frac{2}{3(x+1)} \right) = 12(x-1)(x+1) \left( \frac{5x^2 - 2x + 9}{12(x-1)(x+1)} \right)$$

**Step 5:** Simplify.

$$\begin{aligned} 6(x+1) \cdot x - 4(x-1) \cdot 2 &= 5x^2 - 2x + 9 \\ \text{Simplify. } 6x(x+1) - 4 \cdot 2(x-1) &= 5x^2 - 2x + 9 \end{aligned}$$

**Step 6:** Solve the resulting equation.

$$\begin{aligned}
 6x^2 + 6x - 8x + 8 &= 5x^2 - 2x + 9 \\
 x^2 - 1 &= 0 \\
 (x - 1)(x + 1) &= 0 \\
 x = 1 \text{ or } x = -1
 \end{aligned}$$

**Step 7: Check.**

$x = 1$  and  $x = -1$  are extraneous solutions.

The equation has no solution.

**Try It**

14) Solve:  $\frac{y}{5y - 10} - \frac{5}{3y + 6} = \frac{2y^2 - 19y + 54}{15y^2 - 60}$ .

**Solution**

There is no solution.

**Try It**

15) Solve:  $\frac{z}{2z + 8} - \frac{3}{4z - 8} = \frac{3z^2 - 16z - 16}{8z^2 + 2z - 64}$ .

**Solution**

There is no solution.

### Example 2.4.7

Solve:  $\frac{4}{3x^2 - 10x + 3} + \frac{3}{3x^2 + 2x - 1} = \frac{2}{x^2 - 2x - 3}$ .

#### Solution

**Step 1: Factor all the denominators, so we can note any value of the variable that would make any denominator zero.**

$$\frac{4}{(3x - 1)(x - 3)} + \frac{3}{(3x - 1)(x + 1)} = \frac{2}{(x - 3)(x + 1)}, x \neq -1, x \neq \frac{1}{3}, x \neq 3$$

**Step 2: Find the least common denominator.**

$$\text{The LCD is } (3x - 1)(x + 1)(x - 3).$$

**Step 3: Clear the fractions.**

$$(3x - 1)(x + 1)(x - 3) \left( \frac{4}{(3x - 1)(x - 3)} + \frac{3}{(3x - 1)(x + 1)} \right) = (3x - 1)(x + 1)(x - 3) \left( \frac{2}{(x - 3)(x + 1)} \right)$$

**Step 4: Simplify.**

$$4(x + 1) + 3(x - 3) = 2(3x - 1)$$

**Step 5: Distribute.**

$$4x + 4 + 3x - 9 = 6x - 2$$

**Step 6: Simplify.**

$$\begin{aligned} 7x - 5 &= 6x - 2 \\ x &= 3 \end{aligned}$$

The only algebraic solution was  $x = 3$ , but we said that  $x = 3$  would make a denominator equal to zero. The algebraic solution is an extraneous solution.

There is no solution to this equation.



**Try It**

16) Solve:  $\frac{15}{x^2 + x - 6} - \frac{3}{x - 2} = \frac{2}{x + 3}$ .

**Solution**

There is no solution.

**Try It**

17) Solve:  $\frac{5}{x^2 + 2x - 3} - \frac{3}{x^2 + x - 2} = \frac{1}{x^2 + 5x + 6}$ .

**Solution**

There is no solution.

## Use Rational Functions

Working with functions that are defined by rational expressions often lead to rational equations. Again, we use the same techniques to solve them.

### Example 2.4.8

For rational function,  $f(x) = \frac{2x - 6}{x^2 - 8x + 15}$ :

- Find the domain of the function
- Solve  $f(x) = 1$
- Find the points on the graph at this function value

#### Solution

a. The domain of a rational function is all real numbers except those that make the rational expression undefined. So to find them, we will set the denominator equal to zero and solve.

$$x^2 - 8x + 15 = 0$$

**Step 1: Factor the trinomial.**

$$(x - 3)(x - 5) = 0$$

**Step 2: Use the Zero Product Property.**

$$x - 3 = 0, x - 5 = 0$$

**Step 3: Solve.**

$$x = 3, x = 5$$

The domain is all real numbers except  $x \neq 3, x \neq 5$ .

b.

**Step 1: Substitute in the rational expression.**

$$\frac{2x - 6}{x^2 - 8x + 15} = 1$$

**Step 2: Factor the denominator.**

$$\frac{2x - 6}{(x - 3)(x - 5)} = 1$$

**Step 3: Multiply both sides by the LCD,  $(x - 3)(x - 5)$ .**

$$(x - 3)(x - 5) \left( \frac{2x - 6}{(x - 3)(x - 5)} \right) = (x - 3)(x - 5)(1)$$

**Step 4: Simplify.**

$$2x - 6 = x^2 - 8x + 15$$

**Step 5: Solve.**

$$0 = x^2 - 10x + 21$$

**Step 6: Factor.**

$$0 = (x - 7)(x - 3)$$

**Step 7: Use the Zero Product Property.**

$$x - 7 = 0, x - 3 = 0$$

**Step 8: Solve.**

$$x = 7, x = 3$$

c. The value of the function is **1** when  $x = 7$ ,  $x = 3$ . So the points on the graph of this function when  $f(x) = 1$ , will be  $(7, 1)$ ,  $(3, 1)$ .

## Try It

18) For rational function,  $f(x) = \frac{8 - x}{x^2 - 7x + 12}$ ,

- Find the domain of the function
- Solve  $f(x) = 3$
- Find the points on the graph at this function value.

**Solution**

- a. The domain is all real numbers except  $x \neq 3$  and  $x \neq 4$
- b.  $x = 2, x = \frac{14}{3}$
- c.  $(2, 3), \left(\frac{14}{3}, 3\right)$

**Try It**

19) For rational function,  $f(x) = \frac{x - 1}{x^2 - 6x + 5}$ ,

- a. Find the domain of the function
- b. Solve  $f(x) = 4$
- c. Find the points on the graph at this function value.

**Solution**

- a. The domain is all real numbers except  $x \neq 1$  and  $x \neq 5$
- b.  $x = \frac{21}{4}$
- c.  $\left(\frac{21}{4}, 4\right)$

## Solve a Rational Equation for a Specific Variable

When we solved linear equations, we learned how to solve a formula for a specific variable. Many formulas used in business, science, economics, and other fields use rational equations to model the relation between two or more variables. We will now see how to solve a rational equation for a specific variable.

When we developed the point-slope formula from our slope formula, we cleared the fractions by multiplying by the LCD.

$$m = \frac{y - y_1}{x - x_1}$$

Multiply both sides of the equation by  $x - x_1$ .

$$m(x - x_1) = \left( \frac{y - y_1}{x - x_1} \right) (x - x_1)$$

Simplify.

$$m(x - x_1) = y - y_1$$

Rewrite the equation with the  $y$  terms on the left.

$$y - y_1 = m(x - x_1)$$

In the next example, we will use the same technique with the formula for slope that we used to get the point-slope form of an equation of a line through the point  $(2, 3)$ . We will add one more step to solve for  $y$ .

### Example 2.4.9

Solve:  $m = \frac{y - 2}{x - 3}$  for  $y$ .

#### Solution

**Step 1:** Note any value of the variable that would make any denominator zero.

$$m = \frac{y - 2}{x - 3}, x \neq 3$$

**Step 2:** Clear the fractions by multiplying both sides of the equation by the LCD,  $x - 3$ .

$$(x - 3)m = (x - 3) \left( \frac{y - 2}{x - 3} \right)$$

**Step 3:** Simplify.

$$xm - 3m = y - 2$$

**Step 4:** Isolate the term with  $y$ .

$$xm - 3m + 2 = y$$

**Try It**

20) Solve:  $m = \frac{y - 5}{x - 4}$  for  $y$ .

**Solution**

$$y = mx - 4m + 5$$

**Try It**

21) Solve:  $m = \frac{y - 1}{x + 5}$  for  $y$ .

**Solution**

$$y = mx + 5m + 1$$

Remember to multiply both sides by the LCD in the next example.

**Example 2.4.10**

Solve:  $\frac{1}{c} + \frac{1}{m} = 1$  for  $c$ .

**Solution****Step 1:** Note any value of the variable that would make any denominator zero.

$$\frac{1}{c} + \frac{1}{m} = 1, c \neq 0, m \neq 0$$

**Step 2:** Clear the fractions by multiplying both sides of the equations by the LCD,  $cm$ .

$$cm \left( \frac{1}{c} + \frac{1}{m} \right) = cm(1)$$

**Step 3:** Distribute.

$$cm \left( \frac{1}{c} \right) + cm \frac{1}{m} = cm(1)$$

Simplify.

$$m + c = cm$$

**Step 4:** Collect the terms with  $c$  to the right.

$$m = cm - c$$

**Step 5:** Factor the expression on the right.

$$m = c(m - 1)$$

**Step 6:** To isolate  $c$ , divide both sides by  $m - 1$ .

$$\frac{m}{m - 1} = \frac{c(m - 1)}{m - 1}$$

**Step 7:** Simplify by removing common factors.

$$\frac{m}{m - 1} = c$$

Notice that even though we excluded  $c = 0$ ,  $m = 0$  from the original equation, we must also now state that  $m \neq 1$ .

## Try It

22) Solve:  $\frac{1}{a} + \frac{1}{b} = c$  for  $a$ .

**Solution**

$$a = \frac{b}{cb - 1}$$

## Try It

23) Solve:  $\frac{2}{x} + \frac{1}{3} = \frac{1}{y}$  for  $y$ .

**Solution**

$$y = \frac{3x}{x + 6}$$

Access this online resource for additional instruction and practice with equations with rational expressions.

- Equations with Rational Expressions



## Key Concepts

- **How to solve equations with rational expressions.**

1. Note any value of the variable that would make any denominator zero.
2. Find the least common denominator of all denominators in the equation.
3. Clear the fractions by multiplying both sides of the equation by the LCD.
4. Solve the resulting equation.
5. Check:
  - If any values found in Step 1 are algebraic solutions, discard them.
  - Check any remaining solutions in the original equation.

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=427#h5p-18>

b) On a scale of **1–10**, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

## Glossary

**extraneous solution to a rational equation**

An extraneous solution to a rational equation is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

**rational equation**

A rational equation is an equation that contains a rational expression.

## 2.5 SOLVE APPLICATIONS WITH RATIONAL EQUATIONS

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### Learning Objectives

By the end of this section, you will be able to:

- Solve proportions
- Solve similar figure applications
- Solve uniform motion applications
- Solve work applications
- Solve direct variation problems
- Solve inverse variation problems

### Try It

Before you get started, take this readiness quiz:

1) Solve:  $2(n - 1) - 4 = -10$ .

2) An express train and a charter bus leave Chicago to travel to Champaign. The express train can make the trip in two hours and the bus takes five hours for the trip. The speed of the express train is **42** miles per hour faster than the speed of the bus. Find the speed of the

bus.

$$3) \text{ Solve } \frac{1}{3}x + \frac{1}{4}x = \frac{5}{6}.$$

## Solve Proportions

When two rational expressions are equal, the equation relating them is called a **proportion**.

### Proportion

A **proportion** is an equation of the form  $\frac{a}{b} = \frac{c}{d}$ , where  $b \neq 0, d \neq 0$ .

The proportion is read “ $a$  is to  $b$  as  $c$  is to  $d$ .”

The equation  $\frac{1}{2} = \frac{4}{8}$  is a proportion because the two fractions are equal. The proportion  $\frac{1}{2} = \frac{4}{8}$  is read “1 is to 2 as 4 is to 8.”

Since a proportion is an equation with rational expressions, we will solve proportions the same way we solved rational equations. We’ll multiply both sides of the equation by the LCD to clear the fractions and then solve the resulting equation.

### Example 2.5.1

$$\text{Solve: } \frac{n}{n+14} = \frac{5}{7}.$$

**Solution**

**Step 1: Multiply both sides by LCD.**

$$7(n + 14) \left( \frac{n}{n + 14} \right) = 7(n + 14) \left( \frac{5}{7} \right)$$

**Step 2: Remove common factors on each side.**

$$7n = 5(n + 14)$$

**Step 3: Simplify.**

$$7n = 5n + 70$$

**Step 4: Solve for  $n$ .**

$$2n = 70$$

$$n = 35$$

**Step 5: Check.**

$$\frac{n}{n + 14} = \frac{5}{7}$$

Substitute  $n = 35$        $\frac{35}{35 + 14} \stackrel{?}{=} \frac{5}{7}$

Simplify.       $\frac{35}{49} \stackrel{?}{=} \frac{5}{7}$

Show common factors.       $\frac{5 \cdot 7}{7 \cdot 7} \stackrel{?}{=} \frac{5}{7}$

Simplify.       $\frac{5}{7} = \frac{5}{7} \checkmark$

### Try It

4) Solve the proportion:  $\frac{y}{y + 55} = \frac{3}{8}$ .

**Solution**

$y = 33$

### Try It

5) Solve the proportion:  $\frac{z}{z - 84} = -\frac{1}{5}$ .

**Solution**

$z = 14$

Notice in the last example that when we cleared the fractions by multiplying by the LCD, the result is the same as if we had cross-multiplied.

$$\frac{a}{n+14} = \frac{5}{7} \qquad \frac{a}{n+14} = \frac{5}{7} \qquad \frac{a}{n+14} = \frac{5}{7}$$

$\left(\frac{a}{n+14}\right) = \left(\frac{5}{7}\right) \qquad \left(\frac{a}{n+14}\right) = \left(\frac{5}{7}\right) \qquad \left(\frac{a}{n+14}\right) = \left(\frac{5}{7}\right)$   
 $7a = 5(n+14) \qquad 7a = 5(n+14) \qquad 7a = 5(n+14)$

For any proportion,  $\frac{a}{b} = \frac{c}{d}$ , we get the same result when we clear the fractions by multiplying by the LCD as when we cross-multiply.

$$\frac{a}{b} = \frac{c}{d} \qquad \frac{a}{b} = \frac{c}{d} \qquad \frac{a}{b} = \frac{c}{d}$$

$\left(\frac{a}{b} = \frac{c}{d}\right) \qquad \left(\frac{a}{b} = \frac{c}{d}\right) \qquad \left(\frac{a}{b} = \frac{c}{d}\right)$   
 $ad = bc \qquad ad = bc \qquad ad = bc$

To solve applications with proportions, we will follow our usual strategy for solving applications. But when we set up the proportion, we must make sure to have the units correct—the units in the numerators must match each other and the units in the denominators must also match each other.

## Example 2.5.2

When pediatricians prescribe acetaminophen to children, they prescribe 5 millilitres (ml) of acetaminophen for every 25 pounds of the child's weight. If Zoe weighs 80 pounds, how many millilitres of acetaminophen will her doctor prescribe?

### Solution

**Step 1: Identify what we are asked to find, and choose a variable to represent it.**

How many ml of acetaminophen will the doctor prescribe?

Let  $a =$  ml of acetaminophen.

**Step 2: Write a sentence that gives the information to find it.**

If 5 ml is prescribed for every 25 pounds, how much will be prescribed for 80 pounds?

**Step 3: Translate into a proportion—be careful of the units.**

$$\frac{\text{ml}}{\text{pounds}} = \frac{\text{ml}}{\text{pounds}}$$

$$\frac{5}{25} = \frac{a}{80}$$

**Step 4: Multiply both sides by the LCD, 400.**

$$400 \left( \frac{5}{25} \right) = 400 \left( \frac{a}{80} \right)$$

**Step 5: Remove common factors on each side.**

Formula does not parse

**Step 6: Simplify, but don't multiply on the left. Notice what the next step will be.**

$$16 \cdot 5 = 5a$$

**Step 7: Solve for  $a$ .**

$$\frac{16 \cdot 5}{5} = \frac{5a}{5}$$

$$16 = a$$

**Step 8: Check.**

Is the answer reasonable?

Yes, since **80** is about **3** times **25**, the medicine should be about **3** times **5**. So **16** ml makes sense. Substitute  $a = 16$  in the original proportion.

$$\frac{5}{25} = \frac{a}{80}$$

$$\frac{5}{25} \stackrel{?}{=} \frac{16}{80}$$

$$\frac{1}{5} = \frac{1}{5} \checkmark$$

**Step 9: Write a complete sentence.**

The pediatrician would prescribe **16** ml of acetaminophen to Zoe.

**Try It**

6) Pediatricians prescribe **5** millilitres (ml) of acetaminophen for every **25** pounds of a child's weight. How many millilitres of acetaminophen will the doctor prescribe for Emilia, who weighs **60** pounds?

**Solution**

The pediatrician will prescribe **12** ml of acetaminophen to Emilia.



## Try It

7) For every 1 kilogram (kg) of a child's weight, pediatricians prescribe 15 milligrams (mg) of a fever reducer. If Isabella weighs 12 kg, how many milligrams of the fever reducer will the pediatrician prescribe?

### Solution

The pediatrician will prescribe 180 mg of fever reducer to Isabella.

## Solve Similar Figure Applications

When you shrink or enlarge a photo on a phone or tablet, figure out a distance on a map, or use a pattern to build a bookcase or sew a dress, you are working with **similar figures**. If two figures have exactly the same shape, but different sizes, they are said to be similar. One is a scale model of the other. All their corresponding angles have the same measures and their corresponding sides have the same ratio.

### Similar Figures

Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides have the same ratio.

For example, the two triangles in Figure 2.5.1 are similar. Each side of  $\triangle ABC$  is four times the length of the corresponding side of  $\triangle XYZ$ .

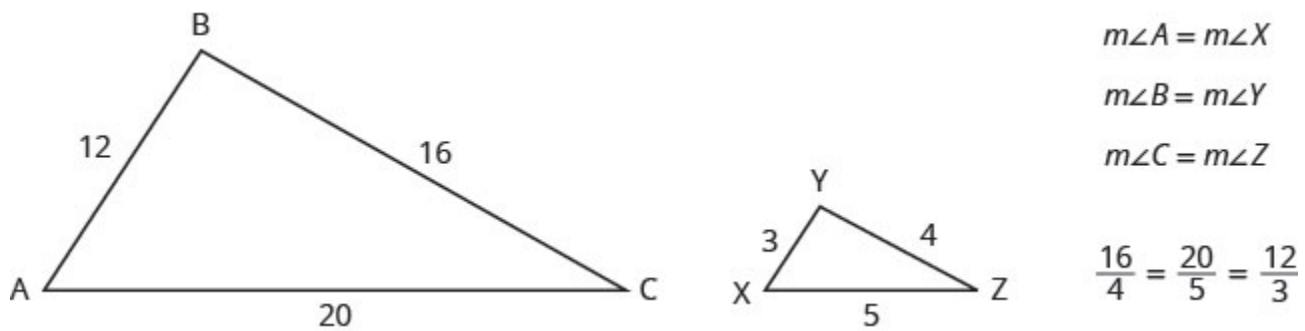


Figure 2.5.1

This is summed up in the Property of Similar Triangles.

### Property of Similar Triangles

If  $\triangle ABC$  is similar to  $\triangle XYZ$ , then their corresponding angle measure are equal and their corresponding sides have the same ratio.

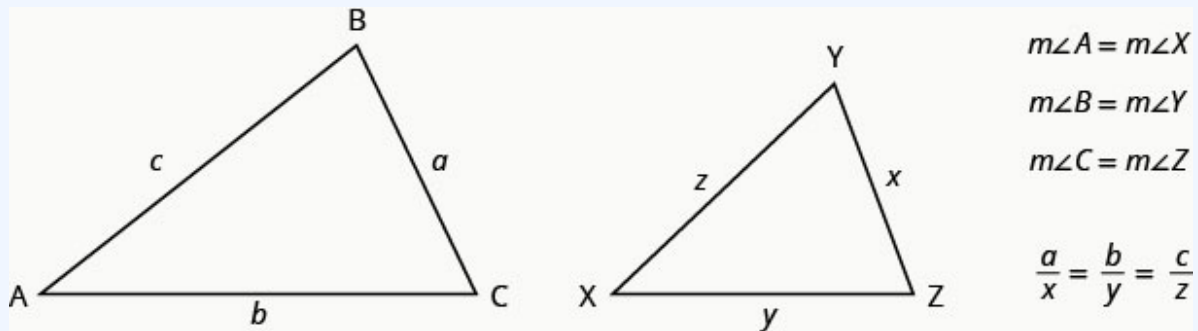


Figure 2.5.2

To solve applications with similar figures we will follow the Problem-Solving Strategy for Geometry Applications we used earlier.

### Example 2.5.3

On a map, San Francisco, Las Vegas, and Los Angeles form a triangle. The distance between the cities is measured in inches. The figure on the left below represents the triangle formed by the cities on the map. If the actual distance from Los Angeles to Las Vegas is **270** miles, find the distance from Los Angeles to San Francisco.

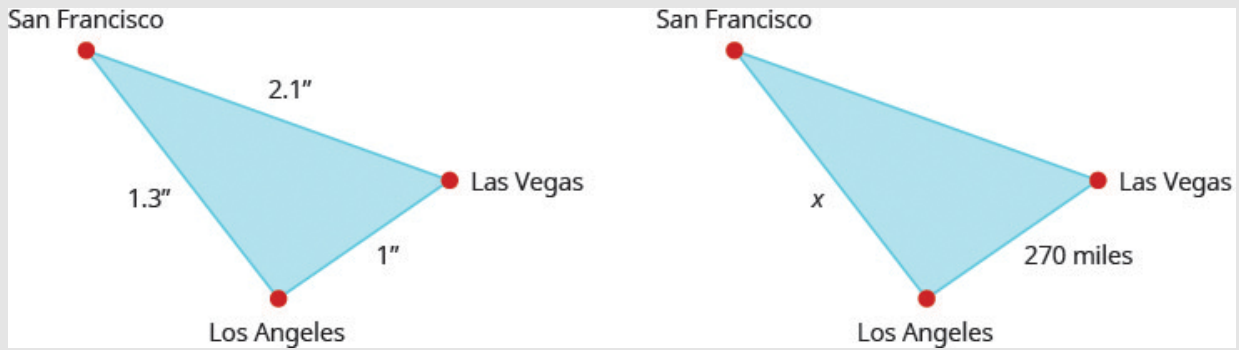


Figure 2.5.3

**Solution**

Since the triangles are similar, the corresponding sides are proportional.

**Step 1: Read the problem. Draw the figures and label it with the given information.**

The figures are shown above.

**Step 2: Identify what we are looking for.**

The actual distance from Los Angeles to San Francisco

**Step 3: Name the variables.**

Let  $x =$  distance from Los Angeles to San Francisco.

**Step 4: Translate into an equation.**

Since the triangles are similar, the corresponding sides are proportional. We'll make the numerators "miles" and the denominators "inches".

$$\frac{x \text{ miles}}{1.3 \text{ inches}} = \frac{270 \text{ miles}}{1 \text{ inch}}$$

**Step 5: Solve the equation.**

$$1.3 \left( \frac{x}{1.3} \right) = 1.3 \left( \frac{270}{1} \right)$$

$$x = 351$$

**Step 6: Check.**

On the map, the distance from Los Angeles to San Francisco is more than the distance from Los Angeles to Las Vegas. Since **351** is more than **270** the answer makes sense.

Check  $x =$  **351** in the original proportion.

Use a calculator.

$$\frac{x \text{ miles}}{1.3 \text{ inches}} = \frac{270 \text{ miles}}{1 \text{ inch}}$$

$$\frac{351 \text{ miles}}{1.3 \text{ inches}} \stackrel{?}{=} \frac{270 \text{ miles}}{1 \text{ inch}}$$

$$\frac{270 \text{ miles}}{1 \text{ inch}} = \frac{270 \text{ miles}}{1 \text{ inch}} \checkmark$$

**Step 7: Answer the question.**

The distance from Los Angeles to San Francisco is **351** miles.

On the map, Seattle, Portland, and Boise form a triangle. The distance between the cities is measured in inches. The figure on the left below represents the triangle formed by the cities on the map. The actual distance from Seattle to Boise is 400 miles.

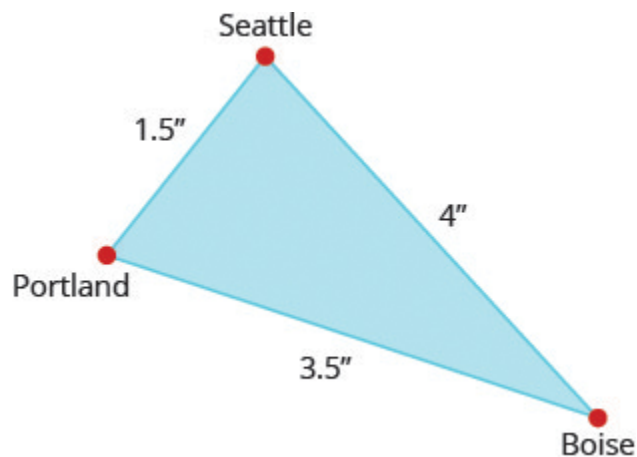


Figure 2.5.4

**Try It**

8) Find the actual distance from Seattle to Portland.

**Solution**

The distance is **150** miles.

**Try It**

9) Find the actual distance from Portland to Boise.

**Solution**

The distance is **350** miles.

We can use similar figures to find heights that we cannot directly measure.

**Example 2.5.4**

Tyler is **6** feet tall. Late one afternoon, his shadow was **8** feet long. At the same time, the shadow of a tree was **24** feet long. Find the height of the tree.

**Solution**

**Step 1: Read the problem and draw a figure.**

We are looking for  $h$ , the height of the tree.

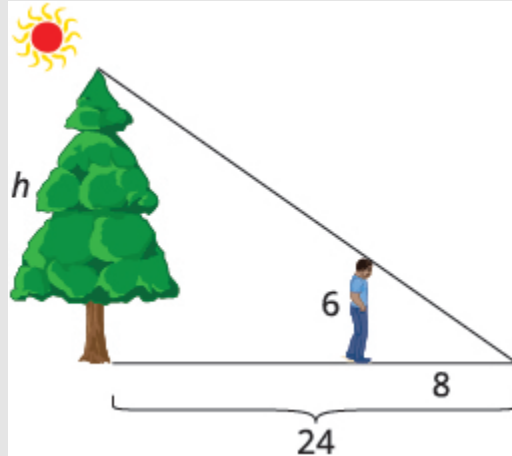


Figure 2.5.5

**Step 2: We will use similar triangles to write an equation.**

The small triangle is similar to the large triangle.

$$\frac{h}{24} = \frac{6}{8}$$

**Step 3: Solve the proportion.**

$$24 \left( \frac{6}{8} \right) = 24 \left( \frac{h}{24} \right)$$

**Step 4: Simplify.**

$$18 = h$$

**Step 5: Check.**

Tyler's height is less than his shadow's length so it makes sense that the tree's height is less than the length of its shadow. Check  $h = 18$  in the original proportion.

$$\frac{6}{8} = \frac{h}{24}$$

$$\frac{6}{8} \stackrel{?}{=} \frac{18}{24}$$

$$\frac{3}{4} = \frac{3}{4} \checkmark$$

**Try It**

10) A telephone pole casts a shadow that is **50** feet long. Nearby, an **8** foot tall traffic sign casts a shadow that is **10** feet long. How tall is the telephone pole?

**Solution**

The telephone pole is **40** feet tall.

**Try It**

11) A pine tree casts a shadow of **80** feet next to a **30** foot tall building which casts a **40** feet shadow. How tall is the pine tree?

**Solution**

The pine tree is **60** feet tall.

## Solve Uniform Motion Applications

We have solved uniform motion problems using the formula  $D = rt$  in previous chapters. We used a table like the one below to organize the information and lead us to the equation.

	Rate	•	Time	=	Distance

Figure 2.5.6

The formula  $D = rt$  assumes we know  $r$  and  $t$  and use them to find  $D$ . If we know  $D$  and  $r$  and need to find  $t$ , we would solve the equation for  $t$  and get the formula  $t = \frac{D}{r}$ .

We have also explained how flying with or against the wind affects the speed of a plane. We will revisit that idea in the next example.

### Example 2.5.5

An airplane can fly 200 miles into a 30 mph headwind in the same amount of time it takes to fly 300 miles with a 30 mph tailwind. What is the speed of the airplane?

#### Solution

This is a uniform motion situation. A diagram will help us visualize the situation.

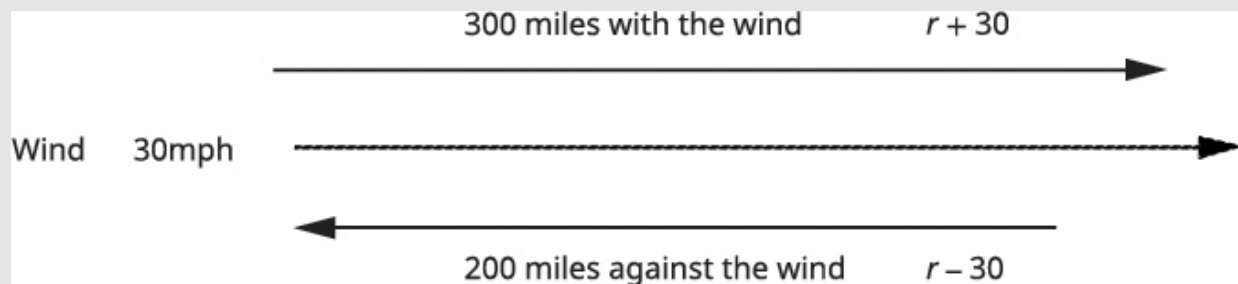


Figure 2.5.7

We fill in the chart to organize the information.

#### Step 1: We are looking for the speed of the airplane.

Let  $r =$  the speed of the airplane.

When the plane flies with the wind, the wind increases its speed and so the rate is  $r + 30$ .

When the plane flies against the wind, the wind decreases its speed and the rate is  $r - 30$ .

#### Step 2: Write in the rates.

#### Step 3: Write in the distances.

Since  $D = r \cdot t$ , we solve for  $t$  and get  $t = \frac{D}{r}$ .



**Step 4:** We divide the distance by the rate in each row, and place the expression in the time column.

	Rate • Time = Distance		
Headwind	$r - 30$	$\frac{200}{r - 30}$	200
Tailwind	$r + 30$	$\frac{300}{r + 30}$	300

**Step 5:** We know the times are equal and so we write our equation.

$$\frac{200}{r - 30} = \frac{300}{r + 30}$$

**Step 6:** We multiply both sides by the LCD.

$$(r + 30)(r - 30) \left( \frac{200}{r - 30} \right) = (r + 30)(r - 30) \left( \frac{300}{r + 30} \right)$$

**Step 7:** Simplify.

$$(r + 30)(200) = (r - 30)300$$

$$200r + 6000 = 300r - 9000$$

Solve.

$$15000 = 100r$$

**Step 8:** Check.

Is 150 mph a reasonable speed for an airplane? Yes.

If the plane is travelling 150 mph and the wind is 30 mph,

Tailwind	$150 + 30 = 180\text{mph}$	$\frac{300}{180} = \frac{5}{3}\text{hours}$
Headwind	$150 - 30 = 120\text{mph}$	$\frac{200}{120} = \frac{5}{3}\text{hours}$

The times are equal, so it checks.

The plane was travelling 150 mph.

### Try It

12) Link can ride his bike **20** miles into a **3** mph headwind in the same amount of time he can ride **30** miles with a **3** mph tailwind. What is Link's biking speed?

**Solution**

Link's biking speed is **15** mph.

### Try It

13) Danica can sail her boat **5** miles into a **7** mph headwind in the same amount of time she can sail **12** miles with a **7** mph tailwind. What is the speed of Danica's boat without a wind?

**Solution**

The speed of Danica's boat is **17** mph.

In the next example, we will know the total time resulting from travelling different distances at different speeds.

### Example 2.5.6

Jazmine trained for **3** hours on Saturday. She ran **8** miles and then biked **24** miles. Her biking speed is **4** mph faster than her running speed. What is her running speed?

**Solution**

This is a uniform motion situation. A diagram will help us visualize the situation.



Figure 2.5.8

We fill in the chart to organize the information.

**Step 1: We are looking for Jazmine's running speed.**

Let  $r =$  Jazmine's running speed.

**Step 2: Her biking speed is 4 miles faster than her running speed.**

$r + 4 =$  her biking speed

**Step 3: The distances are given, enter them into the chart.**

Since  $D = r \cdot t$ , we solve for  $t$  and get  $t = \frac{D}{r}$ .

**Step 4: We divide the distance by the rate in each row, and place the expression in the time column.**

	Rate • Time = Distance		
Run	$r$	$\frac{8}{r}$	8
Bike	$r + 4$	$\frac{24}{r + 4}$	24
		3	

**Step 5: Write a word sentence.**

Her time plus the time biking is 3 hours.

**Step 6: Translate the sentence to get the equation.**

$$\frac{8}{r} + \frac{24}{r + 4} = 3$$

**Step 7: Solve.**

$$\begin{aligned}
 r(r+4) \left( \frac{8}{r} + \frac{24}{r+24} \right) &= 3 \cdot r(r+4) \\
 8(r+4) + 24r &= 3r(r+4) \\
 8r + 32 + 24r &= 3r^2 + 12r \\
 32 + 32r &= 3r^2 + 12r \\
 0 &= 3r^2 - 20r - 32 \\
 0 &= (3r+4)(r-8) \\
 (3r+4) = 0, (r-8) = 0
 \end{aligned}$$

$$\cancel{r = -\frac{4}{3}}, r = 8$$

**Step 8: Check.**

A negative speed does not make sense in this problem, so  $r = 8$  is the solution.

Is 8 mph a reasonable running speed? Yes.

If Jazmine's running rate is 4, then her biking rate,

$$r + 4, \text{ which is } 8 + 4 = 12.$$

<b>Run 8 mph</b>	$\frac{8 \text{ miles}}{12 \text{ mph}} = 1 \text{ hour}$
<b>Bike 12 mph</b>	$\frac{24 \text{ miles}}{12 \text{ mph}} = 2 \text{ hours}$

Total **3** hours.

Jazmine's running speed is 8 mph.

## Try It

14) Dennis went cross-country skiing for 6 hours on Saturday. He skied 20 mile uphill and then 20 miles back downhill, returning to his starting point. His uphill speed was 5 mph slower than his downhill speed. What was Dennis' speed going uphill and his speed going downhill?

### Solution

Dennis's uphill speed was 10 mph and his downhill speed was 5 mph.

## Try It

15) Joon drove 4 hours to his home, driving 208 miles on the interstate and 40 miles on country roads. If he drove 15 mph faster on the interstate than on the country roads, what was his rate on the country roads?

### Solution

Joon's rate on the country roads is 50 mph.

Once again, we will use the uniform motion formula solved for the variable  $t$ .

### Example 2.5.7

Hamilton rode his bike downhill **12** miles on the river trail from his house to the ocean and then rode uphill to return home. His uphill speed was **8** miles per hour slower than his downhill speed. It took him **2** hours longer to get home than it took him to get to the ocean. Find Hamilton's downhill speed.

#### Solution

This is a uniform motion situation. A diagram will help us visualize the situation.

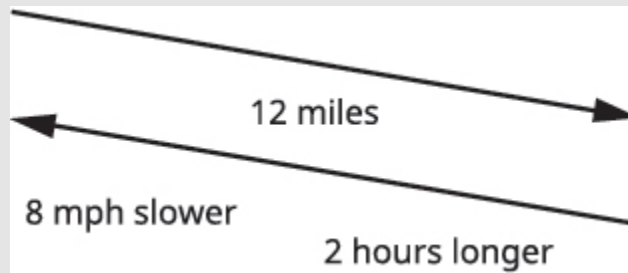


Figure 2.5.9

We fill in the chart to organize the information.

#### Step 1: We are looking for Hamilton's downhill speed.

Let  $h$  = Hamilton's downhill speed.

#### Step 2: His uphill speed is 8 miles per hour slower.

$h - 8$  = Hamilton's uphill speed

#### Step 3: Enter the rates into the chart.

The distance is the same in both directions.

12 miles.

#### Step 4: Since $D = r \cdot t$ , we solve for $t$ and get $t = \frac{D}{r}$ .

#### Step 5: We divide the distance by the rate in each row, and place the expression in the time column.

	Rate • Time = Distance		
Downhill	$h$	$\frac{12}{h}$	12
Uphill	$h - 8$	$\frac{12}{h - 8}$	12

**Step 6: Write a word sentence about the line.**

He took 2 hours longer uphill than downhill.

The uphill time is 2 more than the downhill time.

**Step 7: Translate the sentence to get the equation.**

$$\frac{12}{h - 8} = \frac{12}{h} + 2$$

**Step 8: Solve.**

$$\begin{aligned} \frac{12}{h-8} &= \frac{12}{h} + 2 \\ h(h-8) \left( \frac{12}{h-8} \right) &= h(h-8) \left( \frac{12}{h} + 2 \right) \\ 12h &= 12(h-8) + 2h(h-8) \\ 12h &= 12h - 96 + 2h^2 - 16h \\ 0 &= 2h^2 - 16h - 96 \\ 0 &= 2(h^2 - 8h - 48) \\ 0 &= 2(h-12)(h+4) \\ h-12=0, h+4=0 \\ h=12, h &= \cancel{-4} \end{aligned}$$

**Step 9: Check.**

Is 12 mph a reasonable speed for biking downhill? Yes.

<b>Downhill</b>	12 mph	$\frac{12 \text{ miles}}{12 \text{ mph}} = 1 \text{ hour}$
<b>Uphill</b>	$12 - 8 = 4 \text{ mph}$	$\frac{12 \text{ miles}}{4 \text{ mph}} = 3 \text{ hours}$

The uphill time is **2** hours more than the downhill time.

Hamilton's downhill speed is **12** mph.

## Try It

16) Kayla rode her bike **75** miles home from college one weekend and then rode the bus back to college. It took her **2** hours less to ride back to college on the bus than it took her to ride home on her bike, and the average speed of the bus was **10** miles per hour faster than Kayla's biking speed. Find Kayla's biking speed.

### Solution

Kayla's biking speed was **15** mph.

## Try It

17) Victoria jogs **12** miles to the park along a flat trail and then returns by jogging on an **20** mile



hilly trail. She jogs 1 mile per hour slower on the hilly trail than on the flat trail, and her return trip takes her two hours longer. Find her rate of jogging on the flat trail.

### Solution

Victoria jogged 6 mph on the flat trail.

## Solve Work Applications

The weekly gossip magazine has a big story about the Princess' baby and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 6 hours to do the job and Press #2 takes 12 hours to do the job. How long will it take the printer to get the magazine printed with both presses running together?

This is a typical 'work' application. There are three quantities involved here—the time it would take each of the two presses to do the job alone and the time it would take for them to do the job together.

If Press #1 can complete the job in 6 hours, in one hour it would complete  $\frac{1}{6}$  of the job.

If Press #2 can complete the job in 12 hours, in one hour it would complete  $\frac{1}{12}$  of the job.

We will let  $t$  be the number of hours it would take the presses to print the magazines with both presses running together. So in 1 hour working together they have completed  $\frac{1}{t}$  of the job.

We can model this with the word equation and then translate to a rational equation. To find the time it would take the presses to complete the job if they worked together, we solve for  $t$ .

A chart will help us organize the information. We are looking for how many hours it would take to complete the job with both presses running together.

**Step 1:** Let  $t$  = the number of hours needed to complete the job together.

**Step 2:** Enter the hours per job for Press #1, Press #2, and when they work together.

If a job on Press #1 takes 6 hours, then in 1 hour  $\frac{1}{6}$  of the job is completed.

Similarly find the part of the job completed/hours for Press #2 and when they're both running together.

	Number of hours to complete the job.	Part of the job completed/hour
Press #1	6	$\frac{1}{6}$
Press #2	12	$\frac{1}{12}$
Together	$t$	$\frac{1}{t}$

**Step 3: Write a word sentence.**

The part completed by Press #1 plus the part completed by Press #2 equals the amount completed together.

**Step 4: Translate into an equation.**

Work completed by

**Formula does not parse**

**Step 5: Solve.**

$$\frac{1}{6} + \frac{1}{12} = \frac{1}{t}$$

**Step 6: Multiply by the LCD,  $12t$**

$$12t \left( \frac{1}{6} + \frac{1}{12} \right) = 12t \left( \frac{1}{t} \right)$$

**Step 7: Simplify.**

$$\begin{aligned} 2t + t &= 12 \\ 3t &= 12 \\ t &= 4 \end{aligned}$$

When both presses are running it takes **4** hours to do the job.

Keep in mind, it should take less time for two presses to complete a job working together than for either press to do it alone.

### Example 2.5.8

Suppose Pete can paint a room in **10** hours. If he works at a steady pace, in **1** hour he would paint  $\frac{1}{10}$  of the room. If Alicia would take **8** hours to paint the same room, then in **1** hour she would paint  $\frac{1}{8}$  of the room. How long would it take Pete and Alicia to paint the room if they worked together (and didn't interfere with each other's progress)?

#### Solution

This is a 'work' application. A chart will help us organize the information. We are looking for the numbers of hours it will take them to paint the room together.

In one hour Pete did  $\frac{1}{10}$  of the job. Alicia did  $\frac{1}{8}$  of the job. And together they did  $\frac{1}{t}$  of the job.

**Step 1: Let  $t$  be the number of hours needed to paint the room together.**

**Step 2: Enter the hours per job for Pete, Alicia, and when they work together.**

In **1** hour working together, they have completed  $\frac{1}{t}$  of the job.

Similarly, find the part of the job completed/hour by Pete and then by Alicia.

	Number of complete hours to the job.	Part of job completed/hour
Pete	10	$\frac{1}{10}$
Alicia	8	$\frac{1}{8}$
Together	$t$	$\frac{1}{t}$

**Step 3: Write a word sentence.**

The work completed by Pete plus the work completed by Alicia equals the total work completed.

Work completed by:

**Formula does not parse**

**Step 4: Multiply by the LCD,  $40t$ .**

$$40t \left( \frac{1}{10} + \frac{1}{8} \right) = 40t \left( \frac{1}{t} \right)$$

**Step 5: Distribute.**

$$40t \cdot \frac{1}{10} + 40t \cdot \frac{1}{8} = 40t \left( \frac{1}{t} \right)$$

**Step 6: Simplify and solve.**

$$\begin{aligned} 4t + 5t &= 40 \\ 9t &= 40 \\ t &= \frac{40}{9} \end{aligned}$$

**Step 7: We'll write as a mixed number so that we can convert it to hours and minutes.**

$$t = 4\frac{4}{9} \text{ hours}$$

**Step 8: Remember, 1 hour = 60 minutes.**

$$t = 4 \text{ hours} = \frac{4}{9} (60 \text{ minutes})$$

**Step 9: Multiply, and then round to the nearest minute.**

$$t = 4 \text{ hours} + 27 \text{ minutes}$$

It would take Pete and Alicia about **4** hours and **27** minutes to paint the room.

## Try It

18) One gardener can mow a golf course in **4** hours, while another gardener can mow the same golf course in **6** hours. How long would it take if the two gardeners worked together to mow the golf course?

### **Solution**

When the two gardeners work together it takes **2** hours and **24** minutes.

## Try It

19) Daria can weed the garden in **7** hours, while her mother can do it in **3**. How long will it take the two of them working together?

### **Solution**

When Daria and her mother work together it takes **2** hours and **6** minutes.

### Example 2.5.9

Ra'shon can clean the house in **7** hours. When his sister helps him it takes **3** hours. How long does it take his sister when she cleans the house alone?

#### Solution

This is a work problem. A chart will help us organize the information.

We are looking for how many hours it would take Ra'shon's sister to complete the job by herself.

**Step 1: Let  $s$  be the number of hours Ra'shon's sister takes to clean the house alone.**

**Step 2: Enter the hours per job for Ra'shon, his sister, and when they work together.**

If Ra'shon takes **7** hours, then in **1** hour  $\frac{1}{7}$  of the job is completed.

If Ra'shon's sister takes  $s$  hours, then in **1** hour  $\frac{1}{s}$  of the job is completed.

	Number of hours to clean the house	Part of the job completed/hour
Ra'shon	7	$\frac{1}{7}$
His sister	$s$	$\frac{1}{s}$
Together	3	$\frac{1}{3}$

**Step 3: Write a word sentence.**

The part completed by Ra'shon plus the part by his sister equals the amount completed together.

**Step 4: Translate to an equation.**

Work completed by:

Formula does not parse

**Step 5: Solve.**

$$\frac{1}{7} + \frac{1}{s} = \frac{1}{3}$$

**Step 6: Multiply by the LCD, 21.**

$$21s \left( \frac{1}{7} + \frac{1}{s} \right) = \left( \frac{1}{3} \right) 21s$$

**Step 7: Simplify.**

$$\begin{aligned} -4s &= -21 \\ s &= \frac{-21}{-4} \\ &= \frac{21}{4} \end{aligned}$$

**Step 8: Write as a mixed number to convert it to hours and minutes.**

$$s = 5\frac{1}{4} \text{ hours}$$

There are **60** minutes in **1** hour.

$$s = 5 \text{ hours} + \frac{1}{4}(60 \text{ minutes})$$

$$s = 5 \text{ hours} + 15 \text{ minutes}$$

It would take Ra'shon's sister **5** hours and **15** minutes to clean the house alone.

## Try It

20) Alice can paint a room in **6** hours. If Kristina helps her it takes them **4** hours to paint the room. How long would it take Kristina to paint the room by herself?

### Solution

Kristina can paint the room in **12** hours.

## Try It

21) Tracy can lay a slab of concrete in **3** hours, with Jordan's help they can do it in **2** hours. If Jordan works alone, how long will it take?

### Solution

It will take Jordan **6** hours.

## Solve Direct Variation Problems

When two quantities are related by a proportion, we say they are *proportional* to each other. Another way to express this relation is to talk about the *variation* of the two quantities. We will discuss direct variation and inverse variation in this section.

Lindsay gets paid \$15 per hour at her job. If we let  $s$  be her salary and  $h$  be the number of hours she has worked, we could model this situation with the equation  $s = 15h$

Lindsay's salary is the product of a constant, 15, and the number of hours she works. We say that Lindsay's salary *varies directly* with the number of hours she works. Two variables vary directly if one is the product of a constant and the other.

## Direct Variation

For any two variables  $x$  and  $y$ ,  $y$  varies directly with  $x$  if  $y = kx$ , where  $k \neq 0$

The constant  $k$  is called the constant of variation.

In applications using direct variation, generally we will know values of one pair of the variables and will be asked to find the equation that relates  $x$  and  $y$ . Then we can use that equation to find values of  $y$  for other values of  $x$ .

We'll list the steps here.



## HOW TO

### Solve direct variation problems.

1. Write the formula for direct variation.
2. Substitute the given values for the variables.
3. Solve for the constant of variation.
4. Write the equation that relates  $x$  and  $y$  using the constant of variation.

Now we'll solve an application of direct variation.

### Example 2.5.10

When Raoul runs on the treadmill at the gym, the number of calories,  $c$ , he burns varies directly with the number of minutes,  $m$ , he uses the treadmill. He burned **315** calories when he used the treadmill for **18** minutes.

- a. Write the equation that relates  $c$  and  $m$ .
- b. How many calories would he burn if he ran on the treadmill for **25** minutes?

#### Solution

- a. The number of calories,  $c$ , varies directly with the number of minutes,  $m$ , on the treadmill, and  $c = 315$  when  $m = 18$ .

**Step 1: Write the formula for direct variation.**

$$y = kx$$

**Step 2: We will use  $c$  in place of  $y$  and  $m$  in place of  $x$ .**

$$c = km$$

**Step 2: Substitute the given values for the variables.**

$$315 = k \cdot 18$$

**Step 3: Solve for the constant of variation.**

$$\frac{315}{18} = \frac{k \cdot 18}{18}$$

$$17.5 = k$$

**Step 4: Write the equation that relates  $c$  and  $m$ .**

$$c = km$$

**Step 5: Substitute in the constant of variation.**

$$c = 17.5m$$


---

b. Find  $c$  when  $m = 25$ .

**Step 1: Write the equation that relates  $c$  and  $m$ .**

$$c = 17.5m$$

**Step 2: Substitute the given value for  $m$ .**

$$c = 17.5(25)$$

**Step 3: Simplify.**

$$c = 437.5$$

Raoul would burn **437.5** calories if he used the treadmill for **25** minutes.

## Try It

22) The number of calories,  $c$ , burned varies directly with the amount of time,  $t$ , spent exercising. Arnold burned **312** calories in **65** minutes exercising.

- a. Write the equation that relates  $c$  and  $t$ .  
 b. How many calories would he burn if he exercises for **90** minutes?

**Solution**

- a.  $c = 4.8t$   
 b. He would burn **432** calories.

**Try It**

23) The distance a moving body travels,  $d$ , varies directly with time,  $t$ , it moves. A train travels **100** miles in **2** hours

- a. Write the equation that relates  $d$  and  $t$ .  
 b. How many miles would it travel in **5** hours?

**Solution**

- a.  $d = 50t$   
 b. It would travel **250** miles.

## Solve Inverse Variation Problems

Many applications involve two variable that *vary inversely*. As one variable increases, the other decreases. The equation that relates them is  $y = \frac{k}{x}$ .

### Inverse Variation

For any two variables  $x$  and  $y$ ,  $y$  varies inversely with  $x$  if  $y = \frac{k}{x}$ , where  $k \neq 0$

The constant  $k$  is called the constant of variation.

The word ‘inverse’ in inverse variation refers to the multiplicative inverse. The multiplicative inverse of  $x$  is  $\frac{1}{x}$ .

We solve inverse variation problems in the same way we solved direct variation problems. Only the general form of the equation has changed. We will copy the procedure box here and just change ‘direct’ to ‘inverse’.

## HOW TO

### Solve inverse variation problems.

1. Write the formula for inverse variation.
2. Substitute the given values for the variables.
3. Solve for the constant of variation.
4. Write the equation that relates  $x$  and  $y$  using the constant of variation.

### Example 2.5.11

The frequency of a guitar string varies inversely with its length. A **26** in.-long string has a frequency of **440** vibrations per second.

- a. Write the equation of variation.
- b. How many vibrations per second will there be if the string’s length is reduced to **20** inches by putting a finger on a fret?

#### **Solution**

- a. The frequency varies inversely with the length.

#### **Step 1: Name the variables.**

Let  $f$  = frequency.

Let  $L = \text{length}$ .

**Step 2: Write the formula for inverse variation.**

$$y = \frac{k}{x}$$

**Step 3: We will use  $f$  in place of  $y$  and  $L$  in place of  $x$ .**

$$f = \frac{k}{L}$$

**Step 4: Substitute the given values for the variables.**

$$f = 440 \text{ when } L = 26$$

$$440 = \frac{k}{26}$$

**Step 5: Solve for the constant of variation**

$$26(440) = 26 \left( \frac{k}{26} \right)$$

$$11,440 = k$$

**Step 6: Write the equation that relates  $f$  and  $L$ .**

$$f = \frac{k}{L}$$

**Step 7: Substitute the constant of variation**

$$f = \frac{11,440}{L}$$

b. Find  $f$  when  $L = 20$ .

**Step 1: Write the equation that relates  $f$  and  $L$ .**

$$f = \frac{11,440}{L}$$

**Step 2: Substitute the given value for  $L$ .**

$$f = \frac{11,440}{20}$$

**Step 3: Simplify.**

$$f = 572$$

A 20-inch guitar string has a frequency of 572 vibrations per second.

## Try It

24) The number of hours it takes for ice to melt varies inversely with the air temperature. Suppose a block of ice melts in 2 hours when the temperature is 65 degrees Celsius.

- Write the equation of variation.
- How many hours would it take for the same block of ice to melt if the temperature was 78 degrees?

### Solution

- $h = \frac{130}{t}$
- $1\frac{2}{3}$  hours

## Try It

25) Xander's new business found that the daily demand for its product was inversely proportional to the price,  $p$ . When the price is \$5, the demand is 700 units.

- Write the equation of variation.
- What is the demand if the price is raised to \$7?

### Solution

a.  $x = \frac{3500}{p}$

b. 500 units

Access this online resource for additional instruction and practice with applications of rational expressions

- Applications of Rational Expressions

## Key Concepts

- A proportion is an equation of the form  $\frac{a}{b} = \frac{c}{d}$ , where  $b \neq 0, d \neq 0$ . The proportion is read “ $a$  is to  $b$  as  $c$  is to  $d$ .”

- **Property of Similar Triangles**

If  $\triangle ABC$  is similar to  $\triangle XYZ$ ,

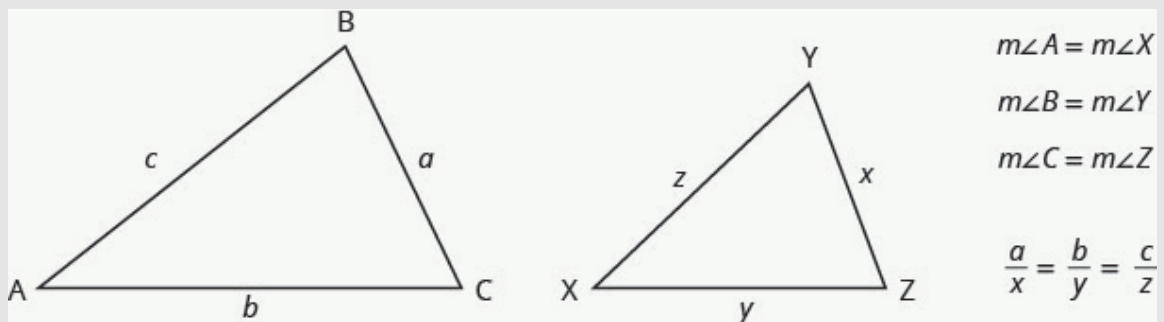


Figure 2.5.11

- then their corresponding angle measure are equal and their corresponding sides have the

same ratio.

- **Direct Variation**

- For any two variables  $x$  and  $y$ ,  $y$  varies directly with  $x$  if  $y = kx$ , where  $k \neq 0$ . The constant  $k$  is called the constant of variation.
- How to solve direct variation problems.
  1. Write the formula for direct variation.
  2. Substitute the given values for the variables.
  3. Solve for the constant of variation.
  4. Write the equation that relates  $x$  and  $y$ .

- **Inverse Variation**

- For any two variables  $x$  and  $y$ ,  $y$  varies inversely with  $x$  if  $y = \frac{k}{x}$ , where  $k \neq 0$ . The constant  $k$  is called the constant of variation.
- How to solve inverse variation problems.
  1. Write the formula for inverse variation.
  2. Substitute the given values for the variables.
  3. Solve for the constant of variation.
  4. Write the equation that relates  $x$  and  $y$ .

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



An interactive H5P element has been excluded from this version of the text. You can view it



online here:

<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=515#h5p-19>

b) After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

## Glossary

### **proportion**

When two rational expressions are equal, the equation relating them is called a proportion.

### **similar figures**

Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides have the same ratio.

## 2.6 UNIT 2 SOURCES

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**2.0** “Rational Expressions and Functions Introduction” from Intermediate Algebra by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted.

**2.1** “7.1 Multiply and Divide Rational Expressions” from Intermediate Algebra by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted.

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# UNIT 3: ROOTS AND RADICALS

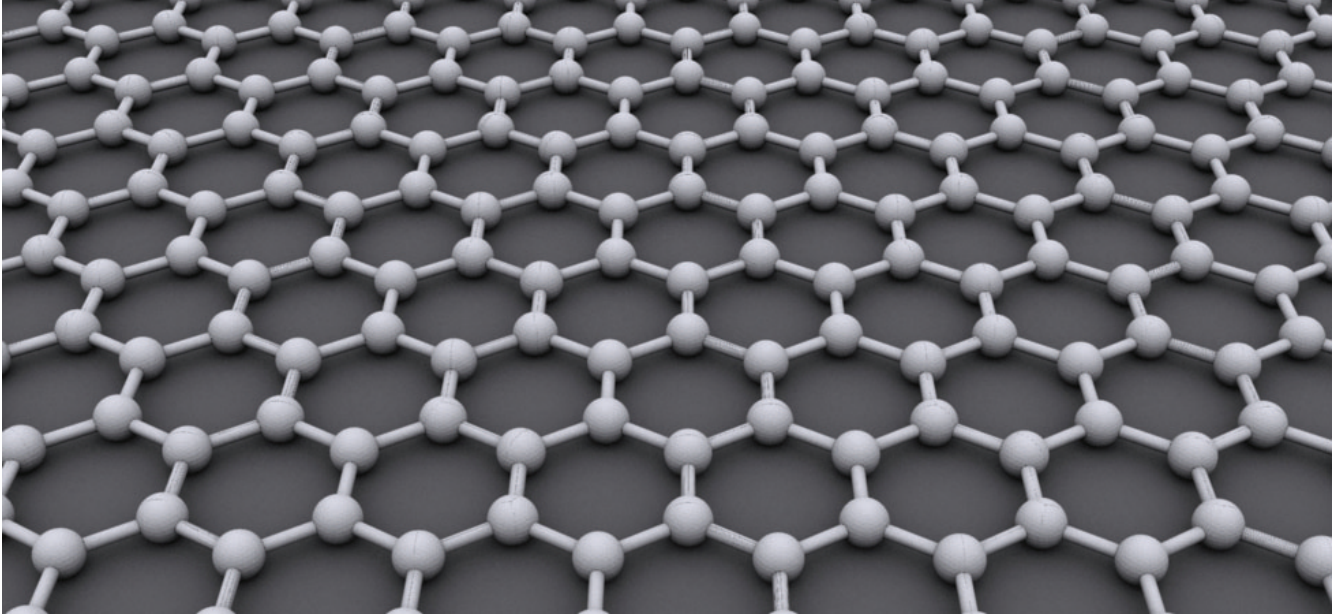
## Chapter Outline

- 3.0 Introduction
- 3.1 Simplify Expressions with Roots
- 3.2 Simplify Radical Expressions
- 3.3 Simplify Rational Exponents
- 3.4 Add, Subtract, and Multiply Radical Expressions
- 3.5 Divide Radical Expressions
- 3.6 Solve Radical Equations
- 3.7 Use the Complex Number System
- 3.8 Unit 3 Sources



## 3.0 INTRODUCTION

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“Graphene” by AlexanderAIUS CC BY-SA 3.0

Graphene is an incredibly strong and flexible material made from carbon. It can also conduct electricity. Notice the hexagonal grid pattern.

Imagine charging your cell phone is less than five seconds. Consider cleaning radioactive waste from contaminated water. Think about filtering salt from ocean water to make an endless supply of drinking water. Ponder the idea of bionic devices that can repair spinal injuries. These are just a few of the many possible uses of a material called graphene. Materials scientists are developing a material made up of a single layer of carbon atoms that is stronger than any other material, completely flexible, and conducts electricity better than most metals. Research into this type of material requires a solid background in mathematics, including understanding roots and radicals. In this chapter, you will learn to simplify expressions containing roots and radicals, perform operations on radical expressions and equations, and evaluate radical functions.

# 3.1 SIMPLIFY EXPRESSIONS WITH ROOTS

## Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with roots
- Estimate and approximate roots
- Simplify variable expressions with roots

## Try It

Before you get started, take this readiness quiz:

1) Simplify:

a.  $(-9)^2$

b.  $-9^2$

c.  $(-9)^3$

2) Round **3.846** to the nearest hundredth.

3) Simplify:

a.  $x^3 \cdot x^3$

b.  $y^2 \cdot y^2 \cdot y^2$

c.  $z^3 \cdot z^3 \cdot z^3 \cdot z^3$

## Simplify Expressions with Roots

In Foundations, we briefly looked at square roots. Remember that when a real number  $n$  is multiplied by itself, we write  $n^2$  and read it ‘ $n$  squared’. This number is called the **square** of  $n$ , and  $n$  is called the **square root**. For example,

$$\begin{aligned} 13^2 &\text{ is read “13 squared”} \\ 169 &\text{ is called the square of 13, since } 13^2 = 169 \\ 13 &\text{ is a square root of 169} \end{aligned}$$

### Square and Square Root of a number

#### Square

If  $n^2 = m$ , then  $m$  is the square of  $n$ .

#### Square Root

If  $n^2 = m$ , then  $n$  is a square root of  $m$ .

Notice  $(-13)^2 = 169$  also, so  $-13$  is also a square root of  $169$ . Therefore, both  $13$  and  $-13$  are square roots of  $169$ .

So, every positive number has two square roots—one positive and one negative. What if we only wanted the positive square root of a positive number? We use a radical sign, and write,  $\sqrt{m}$ , which denotes the positive square root of  $m$ . The positive square root is also called the principal square root.

We also use the radical sign for the square root of zero. Because  $0^2 = 0$ ,  $\sqrt{0} = 0$ . Notice that zero has only one square root.

## Square Root Notation

$\sqrt{m}$ ; \text{is read “the square root of”};  $m$  \\ \text{if } n^2 = m, \text{ then } n = \sqrt{m} \text{ for } n \geq 0.

$\sqrt{169} = 13$        $-\sqrt{169} = -13$

We know that every positive number has two square roots and the radical sign indicates the positive one. We write  $\sqrt{169} = 13$ . If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example,  $-\sqrt{169} = -13$ .

### Example 3.1.1

Simplify:

- $\sqrt{144}$
- $-\sqrt{289}$

#### Solution

a.

**Step 1:** Since  $12^2 = 144$

$$12$$

b.

**Step 1:** Since  $17^2 = 289$  and the negative is in front of the radical sign.

$$-17$$



## Try It

4) Simplify:

a.  $-\sqrt{64}$

b.  $\sqrt{225}$

### Solution

a.  $-8$

b.  $15$

## Try It

5) Simplify:

a.  $\sqrt{100}$

b.  $-\sqrt{121}$

### Solution

a.  $10$

b.  $-11$

Can we simplify  $\sqrt{-49}$ ? Is there a number whose square is  $-49$ ?

$$()^2 = -49$$

Any positive number squared is positive. Any negative number squared is positive. There is no real number equal to  $\sqrt{-49}$ . The square root of a negative number is not a real number.

### Example 3.1.2

Simplify:

a.  $\sqrt{-196}$

b.  $-\sqrt{64}$

#### Solution

a.

**Step 1:** There is no real number whose square is  $-196$ .

$\sqrt{-196}$  is not a real number.

---

b.

**Step 1:** The negative is in front of the radical.

$-8$

### Try It

6) Simplify:

a.  $\sqrt{-169}$

b.  $-\sqrt{81}$

#### Solution

- a. Not a real number
- b.  $-9$

## Try It

7) Simplify:

- a.  $-\sqrt{49}$
- b.  $\sqrt{-121}$

### Solution

- a.  $-7$
- b. Not a real number

So far we have only talked about squares and square roots. Let's now extend our work to include higher powers and higher roots.

Let's review some vocabulary first.

We write:

$$n^2$$

$$n^3$$

$$n^4$$

$$n^5$$

We say:

$n$  squared

$n$  cubed

$n$  to the fourth power

$n$  to the fifth power

The terms 'squared' and 'cubed' come from the formulas for area of a square and volume of a cube. It will be helpful to have a table of the powers of the integers from  $-5$  to  $5$ .

Table 3.1.1

Number	Square	Cube	Fourth Power	Fifth Power
$n$	$n^2$	$n^3$	$n^4$	$n^5$
1	1	1	1	1
2	4	8	16	32
3	9	27	81	243
4	16	64	256	1024
5	25	125	625	3125
$x$	$x^2$	$x^3$	$x^4$	$x^5$
$x^2$	$x^4$	$x^6$	$x^8$	$x^{10}$

Table 3.1.2

Number	Square	Cube	Fourth Power	Fifth Power
$n$	$n^2$	$n^3$	$n^4$	$n^5$
-1	1	-1	1	-1
-2	4	-8	16	-32
-3	9	-27	81	-243
-4	16	-64	256	-1024
-5	25	-125	625	-3125

Notice the signs in the table. All powers of positive numbers are positive, of course. But when we have a negative number, the *even* powers are positive and the *odd* powers are negative. We'll copy the row with the powers of  $-2$  to help you see this.

Table 3.1.3

<small><math>rgb[1.0, 0.0, 0.0]</math></small>	<small><math>rgb[0.0, 0.0, 1.0]</math></small>	<small><math>rgb[1.0, 0.0, 0.0]</math></small>	<small><math>rgb[0.0, 0.0, 1.0]</math></small>	<small><math>rgb[1.0, 0.0, 0.0]</math></small>
-2	4	-8	16	-32
<b>Even</b> power. <b>Positive</b> result.			<b>Odd</b> power. <b>Negative</b> result.	

We will now extend the square root definition to higher roots.

## $n^{\text{th}}$ Root of a Number

If  $b^n = a$ , then  $b$  is an  $n^{\text{th}}$  root of  $a$ . The principal  $n^{\text{th}}$  root of  $a$  is written as  $\sqrt[n]{a}$  and is called the  $n^{\text{th}}$  index of the radical.

Just like we use the word ‘cubed’ for  $b^3$ , we use the term ‘cube root’ for  $\sqrt[3]{a}$ .

We can refer to Table 3.1.1 to help find higher roots.

$$\begin{array}{lcl} 4^3 & = & 64 & \sqrt[3]{64} & = & 4 \\ 3^4 & = & 81 & \sqrt[4]{81} & = & 3 \\ (-2)^5 & = & -32 & \sqrt[5]{-32} & = & -2 \end{array}$$

Could we have an even root of a negative number? We know that the square root of a negative number is not a real number. The same is true for any even root. *Even* roots of negative numbers are not real numbers. *Odd* roots of negative numbers are real numbers.

## Properties of $\sqrt[n]{a}$

When  $n$  is an even number and

- $a \geq 0$ , then  $\sqrt[n]{a}$  is a real number.
- $a < 0$ , then  $\sqrt[n]{a}$  is not a real number.

When  $n$  is an odd number,  $\sqrt[n]{a}$  is a real number for all values of  $a$ .

We will apply these properties in the next two examples.

### Example 3.1.3

Simplify:

a.  $\sqrt[3]{64}$

b.  $\sqrt[4]{81}$

c.  $\sqrt[5]{32}$

**Solution**

a.

**Step 1:** Since  $4^3 = 64$ .

4

b.

**Step 1:** Since  $3^4 = 81$ .

3

c.

**Step 1:** Since  $2^5 = 32$ .

2

**Try It**

8) Simplify:

a.  $\sqrt[3]{27}$

b.  $\sqrt[4]{256}$

c.  $\sqrt[5]{243}$

**Solution**

a. 3

- b. 4
- c. 3

## Try It

9) Simplify:

- a.  $\sqrt[3]{1000}$
- b.  $\sqrt[4]{16}$
- c.  $\sqrt[5]{243}$

### Solution

- a. 10
- b. 2
- c. 3

In this example be alert for the negative signs as well as even and odd powers.

## Example 3.1.4

Simplify:

- a.  $\sqrt[3]{-125}$
- b.  $\sqrt[4]{16}$
- c.  $\sqrt[5]{-243}$

### Solution

a.

**Step 1: Since**  $(-5)^3 = -125$ . $-5$ 

b.

**Step 1: Think,**  $(?)^4 = -16$ .

No real number raised to the fourth power is negative.

Not a real number.

c.

**Step 1: Since**  $(-3)^5 = -243$ . $-3$ 

## Try It

10) Simplify:

a.  $\sqrt[3]{-27}$

b.  $\sqrt[4]{-256}$

c.  $\sqrt[5]{-32}$

**Solution**

a.  $-3$

b. Not real.

c.  $-2$



## Try It

11) Simplify:

a.  $\sqrt[3]{-216}$

b.  $\sqrt[4]{-81}$

c.  $\sqrt[5]{-1024}$

### Solution

a.  $-6$

b. Not real.

c.  $-4$

## Estimate and Approximate Roots

When we see a number with a radical sign, we often don't think about its numerical value. While we probably know that the  $\sqrt{4} = 2$ , what is the value of  $\sqrt{21}$  or  $\sqrt[3]{50}$ ? In some situations a quick estimate is meaningful and in others it is convenient to have a decimal approximation.

To get a numerical estimate of a square root, we look for perfect square numbers closest to the radicand. To find an estimate of  $\sqrt{11}$ , we see 11 is between perfect square numbers 9 and 16, *closer* to 9. Its square root then will be between 3 and 4, but closer to 3.

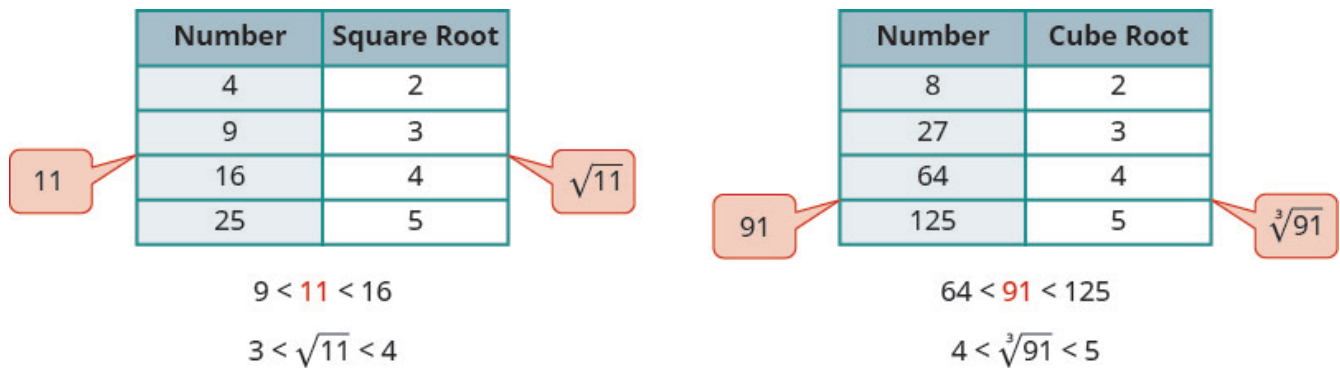


Figure 3.1

Similarly, to estimate  $\sqrt[3]{91}$ , we see 91 is between perfect cube numbers 64 and 125. The cube root then will be between 4 and 5.

### Example 3.1.5

Estimate each root between two consecutive whole numbers:

a.  $\sqrt{105}$

b.  $\sqrt[3]{43}$

#### Solution

a.

**Step 1: Think of the perfect square numbers closest to 105.**

Make a small table of these perfect squares and their square roots.

$\sqrt{105}$

Number	Square Root
81	9
100	10
121	11
144	12

105       $\sqrt{105}$

Figure 3.1.2

**Step 2: Locate 105 between two consecutive perfect squares.**

$$100 < 105 < 121$$

$\sqrt{105}$  is between their square roots.

$$10 < \sqrt{105} < 11$$

b.

**Step 1: Similarly we locate 43 between two perfect cube numbers.**

$$\sqrt[3]{43}$$

	Number	Cube Root
	8	2
	27	3
43	64	4
	125	5

Figure 3.1.3

**Step 2: Locate 43 between two consecutive perfect cubes.**

$$27 < 43 < 64$$

$\sqrt[3]{43}$  is between their cube roots.

$$3 < \sqrt[3]{43} < 4$$

## Try It

12) Estimate each root between two consecutive whole numbers:

a.  $\sqrt{38}$

b.  $\sqrt[3]{93}$

### Solution

a.  $6 < \sqrt{38} < 7$

b.  $4 < \sqrt[3]{93} < 5$

## Try It

13) Estimate each root between two consecutive whole numbers:

a.  $\sqrt{84}$

b.  $\sqrt[3]{152}$

### Solution

a.  $9 < \sqrt{84} < 10$

b.  $5 < \sqrt[3]{152} < 6$

There are mathematical methods to approximate square roots, but nowadays most people use a calculator to find square roots. To find a square root you will use the  $\sqrt{x}$  key on your calculator. To find a cube root, or any root with higher index, you will use the  $\sqrt[y]{x}$  key.

When you use these keys, you get an approximate value. It is an approximation, accurate to the number of digits shown on your calculator's display. The symbol for an approximation is  $\approx$  and it is read 'approximately'.

Suppose your calculator has a 10 digit display. You would see that

$$\sqrt{5} \approx 2.236067978 \text{ rounded to two decimal places is } \sqrt{5} \approx 2.24$$

$$\sqrt[4]{93} \approx 3.105422799 \text{ rounded to two decimal places is } \sqrt[4]{93} \approx 3.11$$

How do we know these values are approximations and not the exact values? Look at what happens when we square them:

$$(2.236067978)^2 = 5.000000002 \quad (3.105422799)^4 = 92.999999991$$

$$(2.24)^2 = 5.0176 \quad (3.11)^4 = 93.54951841$$

Their squares are close to 5, but are not exactly equal to 5. The fourth powers are close to 93, but not equal to 93.

### Example 3.1.6

Round to two decimal places:

a.  $\sqrt{17}$

b.  $\sqrt[3]{49}$

c.  $\sqrt[4]{51}$

#### Solution

a.

**Step 1:** Use the calculator square root key.

$$4.123105626\dots$$

**Step 2:** Round to two decimal places.

$$4.12$$


---

b.

**Step 1:** Use the calculator  $\sqrt[y]{x}$  key.

$$3.659305710\dots$$

**Step 2:** Round to two decimal places.

$$3.66$$

$$\sqrt[3]{49} \approx 3.66$$


---

c.

**Step 1:** Use the calculator  $\sqrt[y]{x}$  key.

$$2.6723451177\dots$$

**Step 2:** Round to two decimal places.

$$2.67$$
$$\sqrt[4]{51} \approx 2.67$$

## Try It

14) Round to two decimal places:

- a.  $\sqrt{11}$
- b.  $\sqrt[3]{71}$
- c.  $\sqrt[4]{127}$

### Solution

- a.  $\approx 3.32$
- b.  $\approx 4.14$
- c.  $\approx 3.36$

## Try It

15) Round to two decimal places:

- a.  $\sqrt{13}$
- b.  $\sqrt[3]{84}$
- c.  $\sqrt[4]{98}$

### Solution

- a.  $\approx 3.61$   
 b.  $\approx 4.38$   
 c.  $\approx 3.15$

## Simplify Variable Expressions with Roots

The odd root of a number can be either positive or negative. For example,

$$\begin{aligned} & \sqrt[3]{64} \text{ and } \sqrt[3]{-64} \\ & \sqrt[3]{64} = 4 \text{ and } \sqrt[3]{-64} = -4 \end{aligned}$$

In either case, when  $n$  is odd,  $\sqrt[n]{a} = \sqrt[n]{a}$ .

But what about an even root? We want the principal root, so  $\sqrt[4]{625} = 5$ .

But notice,

$$\begin{aligned} & \sqrt[4]{625} \text{ and } \sqrt[4]{625} \\ & \sqrt[4]{625} = 5 \text{ and } \sqrt[4]{625} = 5 \end{aligned}$$

In either case, when  $n$  is even,  $\sqrt[n]{a} = \sqrt[n]{a}$ .

How can we make sure the fourth root of  $-5$  raised to the fourth power is  $5$ ? We can use the absolute value.  $|-5| = 5$ . So we say that when  $n$  is even  $\sqrt[n]{a^n} = |a|$ . This guarantees the principal root is positive.

## Simplifying Odd and Even Roots

For any integer  $n \geq 2$ ,

when the index  $n$  is odd  $\sqrt[n]{a^n} = a$

when the index  $n$  is even  $\sqrt[n]{a^n} = |a|$

We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

### Example 3.1.7

Simplify:

a.  $\sqrt{x^2}$

b.  $\sqrt[3]{n^3}$

c.  $\sqrt[4]{p^4}$

d.  $\sqrt[5]{y^5}$

#### Solution

a. We use the absolute value to be sure to get the positive root.

**Step 1: Since the index  $n$  is even,**  $\sqrt[n]{a^n} = |a|$   
 $|x|$

---

b. This is an odd indexed root so there is no need for an absolute value sign.

**Step 1: Since the index  $n$  is odd,**  $\sqrt[n]{a^n} = a$   
 $m$

---

c.

**Step 1: Since the index  $n$  is even**  $\sqrt[n]{a^n} = |a|$   
 $|p|$



---

d.

**Step 1:** Since the index  $n$  is odd  $\sqrt[n]{a^n} = a$

$y$

## Try It

16) Simplify:

a.  $\sqrt{b^2}$

b.  $\sqrt[3]{w^3}$

c.  $\sqrt[4]{m^4}$

d.  $\sqrt[5]{q^5}$

**Solution**

a.  $|b|$

b.  $w$

c.  $|m|$

d.  $q$

## Try It

17) Simplify:

a.  $\sqrt{y^2}$

b.  $\sqrt[3]{p^3}$

c.  $\sqrt[4]{z^4}$

d.  $\sqrt[5]{q^5}$

### Solution

a.  $|y|$

b.  $p$

c.  $|z|$

d.  $q$

What about square roots of higher powers of variables? The Power Property of Exponents says  $(a^m)^n = a^{m \cdot n}$ . So if we square  $a^m$ , the exponent will become  $2m$ .

$$(a^m)^2 = a^{2m}$$

Looking now at the square root,

$$\text{Since } (a^m)^2 = a^{2m} \quad \begin{array}{l} \sqrt{a^{2m}} \\ \sqrt{(a^m)^2} \end{array}$$

$$\text{Since } n \text{ is even } \sqrt[n]{a^n} = |a| \quad |a^m|$$

$$\text{So } \sqrt{a^{2m}} = |a^m|$$

We apply this concept in the next example.

### Example 3.1.8

Simplify:

a.  $\sqrt{x^6}$

b.  $\sqrt{y^{16}}$

**Solution**

a.

**Step 1:** Since  $(x^3)^2 = x^6$

$$\sqrt{(x^3)^2}$$

**Step 2:** Since the index  $n$  is even  $\sqrt{a^n} = |a|$

$$|x^3|$$


---

b.

**Step 1:** Since  $(y^8)^2 = y^{16}$ .

$$\sqrt{(y^8)^2}$$

**Step 2:** Since the index  $n$  is even  $\sqrt[n]{a^n} = |a|$ .

$$y^8$$

In this case the absolute value sign is not needed as  $y^8$  is positive.

## Try It

18) Simplify:

a.  $\sqrt{y^{18}}$

b.  $\sqrt{z^{12}}$

**Solution**

a.  $|y^9|$

b.  $z^6$

## Try It

19) Simplify:

a.  $\sqrt{m^4}$

b.  $\sqrt{b^{10}}$

**Solution**

a.  $m^2$

b.  $|b^5|$

The next example uses the same idea for higher roots.

### Example 3.1.9

Simplify:

a.  $\sqrt[3]{y^{18}}$

b.  $\sqrt[4]{z^8}$

#### Solution

a.

**Step 1:** Since  $(y^6)^3 = y^{18}$

$$\sqrt[3]{(y^6)^3}$$

**Step 2:** Since  $n$  is odd  $\sqrt[n]{a^n} = a$

$$y^6$$


---

b.

**Step 1:** Since  $(z^2)^4 = z^8$

$$\sqrt[4]{(z^2)^4}$$

**Step 2:** Since  $z^2$  is positive, we do not need an absolute value sign.

$$z^2$$

## Try It

20) Simplify:

a.  $\sqrt[4]{u^{12}}$

b.  $\sqrt[3]{v^{15}}$

**Solution**

a.  $|u^3|$

b.  $v^5$

## Try It

21) Simplify:

a.  $\sqrt[5]{c^{20}}$

b.  $\sqrt[6]{d^{24}}$

**Solution**

a.  $c^4$

b.  $d^4$

In the next example, we now have a coefficient in front of the variable. The concept  $\sqrt{a^{2m}} = |a^m|$  works in much the same way.

$$\sqrt{16r^{22}} = 4|r^{11}| \text{ because } (4r^{11})^2 = 16r^{22}$$

But notice  $\sqrt{25u^8} = 5u^4$  and no absolute value sign is needed as  $u^4$  is always positive.

### Example 3.1.10

Simplify:

a.  $\sqrt{16n^2}$

b.  $-\sqrt{81c^2}$

#### Solution

a.

**Step 1:** Since  $(4n)^2 = 16n^2$

$$\sqrt{(4n)^2}$$

**Step 2:** Since the index  $n$  is even  $\sqrt[n]{a^n} = |a|$

$$4|n|$$


---

b.

**Step 1:** Since  $(9c)^2 = 81c^2$

$$-\sqrt{(9c)^2}$$

**Step 2:** Since the index  $n$  is even  $\sqrt[n]{a^n} = |a|$

$$-9|c|$$

## Try It

22) Simplify:

a.  $\sqrt{64x^2}$

b.  $-\sqrt{100p^2}$

### Solution

a.  $8|x|$

b.  $-10|p|$

## Try It

23) Simplify:

a.  $\sqrt{169y^2}$

b.  $-\sqrt{121y^2}$

### Solution

a.  $13|y|$

b.  $-11|y|$

This example just takes the idea farther as it has roots of higher index.



### Example 3.1.11

Simplify:

a.  $\sqrt[3]{64p^6}$

b.  $\sqrt[4]{16q^{12}}$

#### Solution

a.

**Step 1: Rewrite  $64p^6$  as  $(4p^2)^3$**

$$\sqrt[3]{(4p^2)^3}$$

**Step 2: Take the cube root.**

$$4p^2$$


---

b.

**Step 1: Rewrite the radicand as a fourth power.**

$$\sqrt[4]{(2q^3)^4}$$

**Step 2: Take the fourth root.**

$$2|q^3|$$

## Try It

24) Simplify:

a.  $\sqrt[3]{27x^{27}}$

b.  $\sqrt[4]{81q^{28}}$

### Solution

a.  $3x^9$

b.  $3|q^7|$

## Try It

25) Simplify:

a.  $\sqrt[3]{125q^9}$

b.  $\sqrt[5]{243q^{25}}$

### Solution

a.  $5p^3$

b.  $3q^5$

The next examples have two variables.

### Example 3.1.12

Simplify:

a.  $\sqrt{36x^2y^2}$

b.  $\sqrt{121a^6b^8}$

c.  $\sqrt[3]{64p^{63}q^9}$

**Solution**

a.

**Step 1:** Since  $(6xy)^2 = 36x^2y^2$

$$\sqrt{(6xy)^2}$$

**Step 2:** Take the square root.

$$6|xy|$$


---

b.

**Step 1:** Since  $(11a^3b^4)^2 = 121a^6b^8$

$$\sqrt{(11a^3b^4)^2}$$

**Step 2:** Take the square root.

$$11|a^3b^4|$$


---

c.

**Step 1:** Since  $(4p^{21}q^3)^3 = 64p^{63}q^9$

$$x\sqrt[3]{(4p^{21}q^3)^3}$$

**Step 2: Take the cube root.**

$$4p^{21}q^3$$

## Try It

26) Simplify:

a.  $\sqrt{100a^2b^2}$

b.  $\sqrt{144p^{12}q^{20}}$

c.  $\sqrt[3]{8x^{30}y^{12}}$

**Solution**

a.  $10|ab|$

b.  $12p^6q^{10}$

c.  $2x^{10}y^4$

## Try It

27) Simplify:

a.  $\sqrt{225m^2n^2}$

b.  $\sqrt{169x^{10}y^{14}}$

c.  $\sqrt[3]{27w^{36}z^{15}}$

**Solution**

a.  $15|mn|$

b.  $13|x^5y^7|$

c.  $3w^{12}z^5$

Access this online resource for additional instruction and practice with simplifying expressions with roots.

- Simplifying Variables Exponents with Roots using Absolute Values

## Key Concepts

- **Square Root Notation**

- $\sqrt{m}$  is read 'the square root of  $m$ '
- If  $n^2 = m$ , then  $n = \sqrt{m}$ , for  $n \geq 0$ .  
 $rgb]0.0, 0.0, 1.0 \rightarrow rgb]0.0, 0.0, 1.0\sqrt{rgb]1.0, 0.0, 0.0m} \quad rgb]1.0, 0.0, 0.0 \leftarrow$
- $rgb]0.0, 0.0, 1.0$ Radical  $rgb]1.0, 0.0, 0.0$ Radican
- The square root of  $m$ ,  $\sqrt{m}$ , is a positive number whose square is  $m$ .

- **$n^{\text{th}}$  Root of a Number**

- If  $b^n = a$ , then  $b$  is an  $n^{\text{th}}$  root of  $a$ .

- The principal  $n^{\text{th}}$  root of  $a$  is written  $\sqrt[n]{a}$ .
- $n$  is called the *index* of the radical.
- **Properties of  $\sqrt[n]{a}$** 
  - When  $n$  is an even number and
    - $a \geq 0$ , then  $\sqrt[n]{a}$  is a real number
    - $a < 0$ , then  $\sqrt[n]{a}$  is not a real number
  - When  $n$  is an odd number,  $\sqrt[n]{a}$  is a real number for all values of  $a$ .
- **Simplifying Odd and Even Roots**
  - For any integer  $n \geq 2$ ,
    - when  $n$  is odd  $\sqrt[n]{a^n} = a$
    - when  $n$  is even  $\sqrt[n]{a^n} = |a|$
  - We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



An interactive H5P element has been excluded from this version of the text. You can view it online here:

<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=555#h5p-20>

## Glossary

### square of a number

If  $n^2 = m$ , then  $m$  is the square of  $n$ .

### square root of a number

If  $n^2 = m$ , then  $n$  is a square root of  $m$ .

## 3.2 SIMPLIFY RADICAL EXPRESSIONS

---

### Learning Objectives

By the end of this section, you will be able to:

- Use the Product Property to simplify radical expressions
- Use the Quotient Property to simplify radical expressions

### Try It

Before you get started, take this readiness quiz:

1) Simplify:  $\frac{x^9}{x^4}$

2) Simplify:  $\frac{y^3}{y^{11}}$

3) Simplify:  $(n^2)^6$

### Use the Product Property to Simplify Radical Expressions

We will simplify radical expressions in a way similar to how we simplified fractions. A fraction is simplified



if there are no common factors in the numerator and denominator. To simplify a fraction, we look for any common factors in the numerator and denominator.

A radical expression,  $\sqrt[n]{a}$ , is considered simplified if it has no factors of  $m^n$ . So, to simplify a radical expression, we look for any factors in the radicand that are powers of the index.

### Simplified Radical Expression

For real numbers  $a$  and  $m$ , and  $n \geq 2$ ,

$\sqrt[n]{a}$  is considered simplified if  $a$  has no factors of  $m^n$

For example,  $\sqrt{5}$  is considered simplified because there are no perfect square factors in 5. But  $\sqrt{12}$  is not simplified because 12 has a perfect square factor of 4.

Similarly,  $\sqrt[3]{4}$  is simplified because there are no perfect cube factors in 4. But  $\sqrt[3]{24}$  is not simplified because 24 has a perfect cube factor of 8.

To simplify radical expressions, we will also use some properties of roots. The properties we will use to simplify radical expressions are similar to the properties of exponents. We know that  $(ab)^n = a^n b^n$ . The corresponding Product Property of Roots says that  $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

### Product Property of $n^{\text{th}}$ Roots

If  $\sqrt[n]{a}$  and  $\sqrt[n]{b}$  are real numbers, and  $n \geq 2$  is an integer, then

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \text{ and } \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

We use the Product Property of Roots to remove all perfect square factors from a square root.

### Example 3.2.1

Simplify:  $\sqrt{98}$

#### Solution

**Step 2: Find the largest factor in the radicand that is a perfect power of the index.**

Rewrite the radicand as a product of two factors, using that factor.

We see that **49** is the largest factor of **98** that has a power of **2**.

In other words, **49** is the largest perfect square factor of **98**.

$$98 = 49 \times 2$$

Always write the perfect square factor first.

$$\begin{aligned} &\sqrt{98} \\ &\sqrt{49 \times 2} \end{aligned}$$

**Step 2: Use the product rule to rewrite the radical as the product of two radicals.**

$$\sqrt{49} \times \sqrt{2}$$

**Step 3: Simplify the root of the perfect power.**

$$7\sqrt{2}$$

### Try It

4) Simplify:  $\sqrt{48}$

#### Solution

$$4\sqrt{3}$$

## Try It

5) Simplify:  $\sqrt{45}$

**Solution**

$$3\sqrt{5}$$

Notice in the previous example that the simplified form of  $\sqrt{98}$  is  $7\sqrt{2}$ , which is the product of an integer and a square root. We always write the integer in front of the square root.

Be careful to write your integer so that it is not confused with the index. The expression  $7\sqrt{2}$  is very different from  $\sqrt[7]{2}$ .

## HOW TO

### Simplify a radical expression using the Product Property.

1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.
2. Use the product rule to rewrite the radical as the product of two radicals.
3. Simplify the root of the perfect power.

We will apply this method in the next example. It may be helpful to have a table of perfect squares, cubes, and fourth powers.

## Example 3.2.2

Simplify:

a.  $\sqrt{500}$

b.  $\sqrt[3]{16}$

c.  $\sqrt[4]{243}$

### Solution

a.

**Step 1:** Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{100 \cdot 5}$$

**Step 2:** Rewrite the radical as the product of two radicals.

$$\sqrt{100} \cdot \sqrt{5}$$

**Step 3:** Simplify.

$$10\sqrt{5}$$


---

b.

**Step 1:** Rewrite the radicand as a product using the largest perfect cube factor.  $2^3 = 8$

$$\sqrt[3]{8 \cdot 2}$$

**Step 2:** Rewrite the radical as the product of two radicals.

$$\sqrt[3]{8} \cdot \sqrt[3]{2}$$

**Step 3:** Simplify.

$$2\sqrt[3]{2}$$


---

c.

**Step 1: Rewrite the radicand as a product using the greatest perfect fourth power factor.**

$$3^4 = 81$$

$$\sqrt[4]{81 \cdot 3}$$

**Step 2: Rewrite the radical as the product of two radicals.**

$$\sqrt[4]{81} \cdot \sqrt[4]{3}$$

**Step 3: Simplify.**

$$\sqrt[4]{3}$$

## Try It

6) Simplify:

a.  $\sqrt{288}$

b.  $\sqrt[3]{81}$

c.  $\sqrt[4]{64}$

**Solution**

a.  $12\sqrt{2}$

b.  $3\sqrt[3]{3}$

c.  $2\sqrt[4]{4}$

## Try It

7) Simplify:

a.  $\sqrt{432}$

b.  $\sqrt[3]{625}$

c.  $\sqrt[4]{729}$

### Solution

a.  $12\sqrt{3}$

b.  $5\sqrt[3]{5}$

c.  $3\sqrt[4]{9}$

The next example is much like the previous examples, but with variables. Don't forget to use the absolute value signs when taking an even root of an expression with a variable in the radical.

## Example 3.2.3

Simplify:

a.  $\sqrt{x^3}$

b.  $\sqrt[3]{x^4}$

c.  $\sqrt[4]{x^7}$

### Solution

a.

**Step 1:** Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{x^2 \cdot x}$$

**Step 2: Rewrite the radical as the product of two radicals.**

$$\sqrt{x^2} \cdot \sqrt{x}$$

**Step 3: Simplify.**

$$|x| \sqrt{x}$$


---

b.

**Step 1: Rewrite the radicand as a product using the largest perfect cube factor.**

$$\sqrt[3]{x^3 \cdot x}$$

**Step 2: Rewrite the radical as the product of two radicals.**

$$\sqrt[3]{x^3} \cdot \sqrt[3]{x}$$

**Step 3: Simplify.**

$$x \sqrt[3]{x}$$


---

c.

**Step 1: Rewrite the radicand as a product using the greatest perfect fourth power factor.**

$$\sqrt[4]{x^4 \cdot x^3}$$

**Step 2: Rewrite the radical as the product of two radicals.**

$$\sqrt[4]{x^4} \cdot \sqrt[4]{x^3}$$

**Step 3: Simplify.**

$$|x| \sqrt[4]{x^3}$$

## Try It

8) Simplify:

a.  $\sqrt{b^5}$

b.  $\sqrt[4]{y^6}$

c.  $\sqrt[3]{z^5}$

### Solution

a.  $b^2\sqrt{b}$

b.  $|y|\sqrt[4]{y^2}$

c.  $z\sqrt[3]{z^2}$

## Try It

9) Simplify:

a.  $\sqrt{p^9}$

b.  $\sqrt[5]{y^8}$

c.  $\sqrt[6]{q^{13}}$

### Solution

a.  $p^4\sqrt{p}$



b.  $p\sqrt[5]{p^3}$

c.  $q^2\sqrt[6]{q}$

We follow the same procedure when there is a coefficient in the radicand. In the next example, both the constant and the variable have perfect square factors.

### Example 3.2.4

Simplify:

a.  $\sqrt{72n^7}$

b.  $\sqrt[3]{24x^7}$

c.  $\sqrt[4]{80y^{14}}$

#### Solution

a.

**Step 1:** Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{36n^6 \cdot 2n}$$

**Step 2:** Rewrite the radical as the product of two radicals.

$$\sqrt{36n^6} \cdot \sqrt{2n}$$

**Step 3:** Simplify.

$$6|n^3| \sqrt{2n}$$


---

b.

**Step 1:** Rewrite the radicand as a product using the perfect cube factors.

$$\sqrt[3]{8x^6 \cdot 3x}$$

**Step 2: Rewrite the radical as the product of two radicals.**

$$\sqrt[3]{8x^6} \cdot \sqrt[3]{3x}$$

**Step 3: Rewrite the first radicand as  $(2x^2)^3$**

$$\sqrt[3]{(2x^2)^3} \cdot \sqrt[3]{3x}$$

**Step 4: Simplify.**

$$2x^2 \sqrt[3]{3x}$$


---

C.

**Step 1: Rewrite the radicand as a product using perfect fourth power factors.**

$$\sqrt[4]{16y^{12} \cdot 5y^2}$$

**Step 2: Rewrite the radical as the product of two radicals.**

$$\sqrt[4]{16y^{12}} \cdot \sqrt[4]{5y^2}$$

**Step 3: Rewrite the first radicand as  $(2y^3)^4$**

$$\sqrt[4]{(2y^3)^4} \cdot \sqrt[4]{5y^2}$$

**Step 4: Simplify.**

$$2|y^3| \sqrt[4]{5y^2}$$

## Try It

10) Simplify:

a.  $\sqrt{32y^5}$

b.  $\sqrt[3]{54p^{10}}$

c.  $\sqrt[4]{64q^{10}}$

### Solution

a.  $4y^2 \sqrt{2y}$

b.  $3p^3 \sqrt[3]{2p}$

c.  $2q^2 \sqrt[4]{4q^2}$

## Try It

11) Simplify:

a.  $\sqrt{75a^9}$

b.  $\sqrt[3]{128m^{11}}$

c.  $\sqrt[4]{162n^7}$

### Solution

a.  $5a^4 \sqrt{3a}$

b.  $4m^3 \sqrt[3]{2m^2}$

c.  $3|n| \sqrt[4]{2n^3}$

In the next example, we continue to use the same methods even though there are more than one variable under the radical.

### Example 3.2.5

Simplify:

a.  $\sqrt{63u^3v^5}$

b.  $\sqrt[3]{40x^4y^5}$

c.  $\sqrt[4]{48x^4y^7}$

#### Solution

a.

**Step 1:** Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{9u^2v^4 \cdot 7uv}$$

**Step 2:** Rewrite the radical as the product of two radicals.

$$\sqrt{9u^2v^4} \cdot \sqrt{7uv}$$

**Step 3:** Rewrite the first radicand as  $(3uv^2)^2$

$$\sqrt{(3uv^2)^2} \cdot \sqrt{7uv}$$

**Step 4:** Simplify.

$$3|u|v^2 \sqrt{7uv}$$

b.

**Step 1:** Rewrite the radicand as a product using the largest perfect cube factor.

$$\sqrt[3]{8x^3y^3 \cdot 5xy^2}$$

**Step 2: Rewrite the radical as the product of two radicals.**

$$\sqrt[3]{8x^3y^3} \cdot \sqrt[3]{5xy^2}$$

**Step 3: Rewrite the first radicand as  $(2xy)^3$**

$$\sqrt[3]{(2xy)^3} \cdot \sqrt[3]{5xy^2}$$

**Step 4: Simplify.**

$$2xy \sqrt[3]{5xy^2}$$


---

C.

**Step 1: Rewrite the radicand as a product using the largest perfect fourth power factor.**

$$\sqrt[4]{16x^4y^4 \cdot 3y^3}$$

**Step 2: Rewrite the radical as the product of two radicals.**

$$\sqrt[4]{16x^4y^4} \cdot \sqrt[4]{3y^3}$$

**Step 3: Rewrite the first radicand as  $(2xy)^4$**

$$\sqrt[4]{(2xy)^4} \cdot \sqrt[4]{3y^3}$$

**Step 4: Simplify.**

$$2|xy| \sqrt[4]{3y^3}$$

## Try It

12) Simplify:

a.  $\sqrt{98a^7b^5}$

b.  $\sqrt[3]{56x^5y^4}$

c.  $\sqrt[4]{32x^5y^8}$

### Solution

a.  $7|a^3|b^2|\sqrt{2ab}$

b.  $2xy\sqrt[3]{7x^2y}$

c.  $2|x|y^2\sqrt[4]{2x}$

## Try It

13) Simplify:

a.  $\sqrt{180m^9n^{11}}$

b.  $\sqrt[3]{72x^6y^5}$

c.  $\sqrt[4]{80x^7y^4}$

### Solution

a.  $6m^4|n^5|\sqrt{5mn}$

b.  $2x^2y\sqrt[3]{9y^2}$

c.  $2|xy|\sqrt[4]{5x^3}$

### Example 3.2.6

Simplify:

a.  $\sqrt[3]{-27}$

b.  $\sqrt[4]{-16}$

#### Solution

a.

**Step 1:** Rewrite the radicand as a product using perfect cube factors.

$$\sqrt[3]{(-3)^3}$$

**Step 2:** Take the cube root.

$$-3$$

b.

There is no real number  $n$  where  $n^4 = -16$ .

## Try It

14) Simplify:

a.  $\sqrt[3]{-64}$

b.  $\sqrt[4]{-81}$

### Solution

a.  $-4$

b. No real number.

## Try It

15) Simplify:

a.  $\sqrt[3]{-625}$

b.  $\sqrt[4]{-324}$

### Solution

a.  $-5\sqrt[3]{5}$

b. No real number.

We have seen how to use the order of operations to simplify some expressions with radicals. In the next example, we have the sum of an integer and a square root. We simplify the square root but cannot add the resulting expression to the integer since one term contains a radical and the other does not. The next example also includes a fraction with a radical in the numerator. Remember that in order to simplify a fraction you need a common factor in the numerator and denominator.



### Example 3.2.7

Simplify:

a.  $3 + \sqrt{32}$

b.  $\frac{4 - \sqrt{48}}{2}$

#### Solution

a.

**Step 1: Rewrite the radicand as a product using the largest perfect square factor.**

$$3 + \sqrt{16 \cdot 2}$$

**Step 2: Rewrite the radical as the product of two radicals.**

$$3 + \sqrt{16} \cdot \sqrt{2}$$

**Step 3: Simplify.**

$$3 + 4\sqrt{2}$$

The terms cannot be added as one has a radical and the other does not. Trying to add an integer and a radical is like trying to add an integer and a variable. They are not like terms!

---

b.

**Step 1: Rewrite the radicand as a product using the largest perfect square factor.**

$$\frac{4 - \sqrt{16 \cdot 3}}{2}$$

**Step 2: Rewrite the radical as the product of two radicals.**

$$\frac{4 - \sqrt{16} \cdot \sqrt{3}}{2}$$

**Step 3: Simplify.**

$$\frac{4 - 4\sqrt{3}}{2}$$

**Step 4: Factor the common factor from the numerator.**

$$\frac{4(1 - \sqrt{3})}{2}$$

**Step 5: Remove the common factor, 2, from the numerator and denominator.**

$$\frac{\cancel{2} \cdot 2(1 - \sqrt{3})}{\cancel{2}}$$

**Step 6: Simplify.**

$$2(1 - \sqrt{3})$$

## Try It

16) Simplify:

a.  $5 + \sqrt{75}$

b.  $\frac{10 - \sqrt{75}}{5}$

**Solution**

a.  $5 + 5\sqrt{3}$

b.  $2 - \sqrt{3}$

## Try It

17) Simplify:

a.  $2 + \sqrt{98}$

b.  $\frac{6 - \sqrt{45}}{3}$

### Solution

a.  $2 + 7\sqrt{2}$

b.  $2 - \sqrt{5}$

## Use the Quotient Property to Simplify Radical Expressions

Whenever you have to simplify a radical expression, the first step you should take is to determine whether the radicand is a perfect power of the index. If not, check the numerator and denominator for any common factors, and remove them. You may find a fraction in which both the numerator and the denominator are perfect powers of the index.

### Example 3.2.8

Simplify:

a.  $\sqrt{\frac{45}{80}}$

b.  $\sqrt[3]{\frac{16}{54}}$

c.  $\sqrt[4]{\frac{5}{80}}$

**Solution**

a.

**Step 1: Simplify inside the radical first.****Step 2: Rewrite showing the common factors of the numerator and denominator.**

$$\sqrt{\frac{5 \cdot 9}{5 \cdot 16}}$$

**Step 3: Simplify the fraction by removing common factors.**

$$\sqrt{\frac{9}{16}}$$

**Step 4: Simplify. Note  $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$** 

$$\frac{3}{4}$$


---

b.

**Step 1: Simplify inside the radical first.****Step 2: Rewrite showing the common factors of the numerator and denominator.**

$$\sqrt[3]{\frac{2 \cdot 8}{2 \cdot 27}}$$

**Step 3: Simplify the fraction by removing common factors.**

$$\sqrt[3]{\frac{8}{27}}$$

**Step 4: Simplify. Note  $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$**

$$\frac{2}{3}$$


---

c.

**Step 1: Simplify inside the radical first.**

**Step 2: Rewrite showing the common factors of the numerator and denominator.**

$$\sqrt[4]{\frac{5 \cdot 1}{5 \cdot 16}}$$

**Step 3: Simplify the fraction by removing common factors.**

$$\sqrt[4]{\frac{1}{16}}$$

**Step 4: Simplify. Note  $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$**

$$\frac{1}{2}$$

## Try It

18) Simplify:

a.  $\sqrt{\frac{75}{48}}$

b.  $\sqrt[3]{\frac{54}{250}}$

c.  $\sqrt[4]{\frac{32}{162}}$

**Solution**

- a.  $\frac{5}{4}$   
 b.  $\frac{3}{5}$   
 c.  $\frac{2}{3}$

**Try It**

19) Simplify:

- a.  $\sqrt{\frac{98}{162}}$   
 b.  $\sqrt[3]{\frac{24}{375}}$   
 c.  $\sqrt[4]{\frac{4}{324}}$

**Solution**

- a.  $\frac{7}{9}$   
 b.  $\frac{2}{5}$   
 c.  $\frac{1}{3}$

In the last example, our first step was to simplify the fraction under the radical by removing common factors. In the next example we will use the Quotient Property to simplify under the radical. We divide the like bases by subtracting their exponents,

$$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$$

### Example 3.2.9

Simplify:

a.  $\sqrt{\frac{m^6}{m^4}}$

b.  $\sqrt[3]{\frac{a^8}{a^5}}$

c.  $\sqrt[4]{\frac{a^{10}}{a^2}}$

#### Solution

a.

**Step 1: Simplify the fraction inside the radical first.**

**Step 2: Divide the like bases by subtracting the exponents.**

$$\sqrt{m^2}$$

**Step 3: Simplify.**

$$|m|$$


---

b.

**Step 1: Use the Quotient Property of exponents to simplify the fraction under the radical first.**

$$\sqrt[3]{a^3}$$

**Step 2: Simplify.**

$$a$$


---

c.

**Step 1:** Use the Quotient Property of exponents to simplify the fraction under the radical first.

$$\sqrt[4]{a^8}$$

**Step 2:** Rewrite the radicand using perfect fourth power factors.

$$\sqrt[4]{(a^2)^4}$$

**Step 3:** Simplify.

$$a^2$$

## Try It

20) Simplify:

a.  $\sqrt{\frac{a^8}{a^6}}$

b.  $\sqrt[4]{\frac{x^7}{x^3}}$

c.  $\sqrt[4]{\frac{y^{17}}{y^5}}$

### Solution

a.  $|a|$

b.  $|x|$

c.  $y^3$



## Try It

21) Simplify:

a.  $\sqrt{\frac{x^{14}}{x^{10}}}$

b.  $\sqrt[3]{\frac{m^{13}}{m^7}}$

c.  $\sqrt[5]{\frac{n^{12}}{n^2}}$

### Solution

a.  $x^2$

b.  $m^2$

c.  $n^2$

Remember the Quotient to a Power Property? It said we could raise a fraction to a power by raising the numerator and denominator to the power separately.

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$$

We can use a similar property to simplify a root of a fraction. After removing all common factors from the numerator and denominator, if the fraction is not a perfect power of the index, we simplify the numerator and denominator separately.

## Quotient Property of Radical Expressions

If  $\sqrt[n]{a}$  and  $\sqrt[n]{b}$  are real numbers,  $b \neq 0$ , and for any integer  $n \geq 2$  then,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

**Example 3.2.10**

Simplify:  $\sqrt{\frac{27m^3}{196}}$

**Solution**

**Step 2: Simplify the fraction in the radicand, if possible.**

$\frac{27m^3}{196}$ ; \text{cannot be simplified}

$$\sqrt{\frac{27m^3}{196}}$$

**Step 2: Use the Quotient Property to rewrite the radical as the quotient of two radicals.**

We rewrite  $\sqrt{\frac{27m^3}{196}}$  as the quotient of  $\sqrt{27m^3}$  and  $\sqrt{196}$

$$\frac{\sqrt{27m^3}}{\sqrt{196}}$$

**Step 3: Simplify the radicals in the numerator and the denominator.**

$9m^2$  and  $196$  are perfect squares.

$$\begin{aligned} &= \frac{\sqrt{9m^2} \times \sqrt{3m}}{\sqrt{196}} \\ &= \frac{3m\sqrt{3m}}{14} \end{aligned}$$

**Try It**

22) Simplify:  $\sqrt{\frac{24p^3}{49}}$

**Solution**

$$\frac{2|p|\sqrt{6p}}{7}$$

**Try It**

23) Simplify:  $\sqrt{\frac{48x^5}{100}}$

**Solution**

$$\frac{2x^2\sqrt{3x}}{5}$$

**HOW TO**

**Simplify a square root using the Quotient Property.**

1. Simplify the fraction in the radicand, if possible.

2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.
3. Simplify the radicals in the numerator and the denominator.

### Example 3.2.11

Simplify:

a.  $\sqrt{\frac{45x^5}{y^4}}$

b.  $\sqrt[3]{\frac{24x^7}{y^3}}$

c.  $\sqrt[4]{\frac{48x^{10}}{y^8}}$

#### Solution

a.

**Step 1:** We cannot simplify the fraction in the radicand. Rewrite using the Quotient Property.

$$\frac{\sqrt{45x^5}}{\sqrt{y^4}}$$

**Step 2:** Simplify the radicals in the numerator and the denominator.

$$\frac{\sqrt{9x^4} \cdot \sqrt{5x}}{y^2}$$

**Step 3:** Simplify.

$$\frac{3x^2 \sqrt{5x}}{y^2}$$


---

b.

**Step 1:** The fraction in the radicand cannot be simplified. Use the Quotient Property to write as two radicals.

$$\frac{\sqrt[3]{24x^7}}{\sqrt[3]{y^3}}$$

**Step 2:** Rewrite each radicand as a product using perfect cube factors.

$$\frac{\sqrt[3]{8x^6 \cdot 3x}}{\sqrt[3]{y^3}}$$

**Step 3:** Rewrite the numerator as the product of two radicals.

$$\frac{\sqrt[3]{(2x^2)^3} \cdot \sqrt[3]{3x}}{\sqrt[3]{y^3}}$$

**Step 4:** Simplify.

$$\frac{2x^2 \sqrt[3]{3x}}{y}$$


---

c.

**Step 1:** The fraction in the radicand cannot be simplified.

$$\frac{\sqrt[4]{48x^{10}}}{\sqrt[4]{y^8}}$$

**Step 2:** Use the Quotient Property to write as two radicals. Rewrite each radicand as a product using perfect fourth power factors.

$$\frac{\sqrt[4]{16x^8 \cdot 3x^2}}{\sqrt[4]{y^8}}$$

**Step 3:** Rewrite the numerator as the product of two radicals.

$$\frac{\sqrt[4]{(2x^2)^4} \cdot \sqrt[4]{3x^2}}{\sqrt[4]{(y^2)^4}}$$

**Step 4:** Simplify.

$$\frac{2x^2 \sqrt[4]{3x^2}}{y^2}$$

## Try It

24) Simplify:

a.  $\sqrt{\frac{80m^3}{n^6}}$

b.  $\sqrt[3]{\frac{108c^{10}}{d^6}}$

c.  $\sqrt[4]{\frac{80x^{10}}{y^4}}$

**Solution**

a.  $\frac{4|m|\sqrt{5m}}{|n^3|}$

$$\text{b. } \frac{3c^3 \sqrt[3]{4c}}{d^2}$$

$$\text{c. } \frac{2x^2 \sqrt[4]{5x^2}}{|y|}$$

## Try It

25) Simplify:

$$\text{a. } \sqrt{\frac{54u^7}{v^8}}$$

$$\text{b. } \sqrt[3]{\frac{40r^3}{s^6}}$$

$$\text{c. } \sqrt[4]{\frac{162m^{14}}{n^{12}}}$$

### Solution

$$\text{a. } \frac{3u^3 \sqrt{6u}}{v^4}$$

$$\text{b. } \frac{2r^3 \sqrt{5}}{s^2}$$

$$\text{c. } \frac{3|m^3| \sqrt[4]{2m^2}}{|n^3|}$$

Be sure to simplify the fraction in the radicand first, if possible.

### Example 3.2.12

Simplify:

a.  $\sqrt{\frac{18p^5q^7}{32pq^2}}$

b.  $\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}}$

c.  $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$

#### Solution

a.

**Step 1: Simplify the fraction in the radicand, if possible.**

$$\sqrt{\frac{9p^4q^5}{16}}$$

**Step 2: Rewrite using the Quotient Property.**

$$\frac{\sqrt{9p^4q^5}}{\sqrt{16}}$$

**Step 3: Simplify the radicals in the numerator and the denominator.**

$$\frac{\sqrt{9p^4q^4} \cdot \sqrt{q}}{4}$$

**Step 4: Simplify.**

$$\frac{3p^2q^2\sqrt{q}}{4}$$


---



b.

**Step 1: Simplify the fraction in the radicand, if possible.**

$$\sqrt[3]{\frac{8x^3y^5}{27}}$$

**Step 2: Rewrite using the Quotient Property.**

$$\frac{\sqrt[3]{8x^3y^5}}{\sqrt[3]{27}}$$

**Step 3: Simplify the radicals in the numerator and the denominator.**

$$\frac{\sqrt[3]{8x^3y^3} \cdot \sqrt[3]{y^2}}{\sqrt[3]{27}}$$

**Step 4: Simplify.**

$$\frac{2xy \sqrt[3]{y^2}}{3}$$


---

c.

**Step 1: Simplify the fraction in the radicand, if possible.**

$$\sqrt[4]{\frac{a^5b^4}{16}}$$

**Step 2: Rewrite using the Quotient Property.**

$$\frac{\sqrt[4]{a^5b^4}}{\sqrt[4]{16}}$$

**Step 3: Simplify the radicals in the numerator and the denominator.**

$$\frac{\sqrt[4]{a^4b^4} \cdot \sqrt[4]{a}}{\sqrt[4]{16}}$$

**Step 4: Simplify.**

$$\frac{|ab|\sqrt[4]{a}}{2}$$

## Try It

26) Simplify:

a.  $\sqrt{\frac{50x^5y^3}{72x^4y}}$

b.  $\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}}$

c.  $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$

### Solution

a.  $\frac{5|y|\sqrt{x}}{6}$

b.  $\frac{2xy\sqrt[3]{y^2}}{3}$

c.  $\frac{|ab|\sqrt[4]{a}}{2}$

## Try It

27) Simplify:

a.  $\sqrt{\frac{48m^7n^2}{100m^5n^8}}$

b.  $\sqrt[3]{\frac{54x^7y^5}{250x^2y^2}}$

c.  $\sqrt[4]{\frac{32a^9b^7}{162a^3b^3}}$

### Solution

a.  $\frac{2|m|\sqrt{3}}{5|n^3|}$

b.  $\frac{3xy\sqrt[3]{x^2}}{5}$

c.  $\frac{2|ab|\sqrt[4]{a^2}}{3}$

In the next example, there is nothing to simplify in the denominators. Since the index on the radicals is the same, we can use the Quotient Property again, to combine them into one radical. We will then look to see if we can simplify the expression.

### Example 3.2.13

Simplify:

a.  $\frac{\sqrt{48a^7}}{\sqrt{3a}}$

b.  $\frac{\sqrt[3]{-108}}{\sqrt[3]{2}}$

c.  $\frac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$

**Solution**

a.

**Step 1:** The denominator cannot be simplified, so use the Quotient Property to write as one radical.

$$\sqrt{\frac{48a^7}{3a}}$$

**Step 2:** Simplify the fraction under the radical.

$$\sqrt{16a^6}$$

**Step 3:** Simplify.

$$4|a^3|$$


---

b.

**Step 1:** The denominator cannot be simplified, so use the Quotient Property to write as one radical.

$$\sqrt[3]{\frac{-108}{2}}$$

**Step 2:** Simplify the fraction under the radical.

$$\sqrt[3]{-54}$$

**Step 3:** Rewrite the radicand as a product using perfect cube factors.

$$\sqrt[3]{(-3)^3 \cdot 2}$$

**Step 4:** Rewrite the radical as the product of two radicals.

$$\sqrt[3]{(-3)^3} \cdot \sqrt[3]{2}$$

**Step 5: Simplify.**

$$-3 \sqrt[3]{2}$$


---

C.

**Step 1:** The denominator cannot be simplified, so use the Quotient Property to write as one radical.

$$\sqrt[4]{\frac{96x^7}{3x^2}}$$

**Step 2:** Simplify the fraction under the radical.

$$\sqrt[4]{32x^5}$$

**Step 3:** Rewrite the radicand as a product using perfect fourth power factors.

$$\sqrt[4]{16x^4} \cdot \sqrt[4]{2x}$$

**Step 4:** Rewrite the radicals as the product of two radicals.

$$\sqrt[4]{(2x)^4} \cdot \sqrt[4]{2x}$$

**Step 5:** Simplify.

$$2|x| \sqrt[4]{2x}$$

## Try It

28) Simplify:

$$\text{a. } \frac{\sqrt{98z^5}}{\sqrt{2z}}$$

$$\text{b. } \frac{\sqrt[3]{-500}}{\sqrt[3]{2}}$$

$$\text{c. } \frac{\sqrt[4]{486m^{11}}}{\sqrt[4]{3m^5}}$$

**Solution**

$$\text{a. } 7z^2$$

$$\text{b. } -5\sqrt[3]{2}$$

$$\text{c. } 3|m|\sqrt[4]{2m^2}$$

**Try It**

29) Simplify:

$$\text{a. } \frac{\sqrt{128m^9}}{\sqrt{2m}}$$

$$\text{b. } \frac{\sqrt[3]{-192}}{\sqrt[3]{3}}$$

$$\text{c. } \frac{\sqrt[4]{324n^7}}{\sqrt[4]{2n^3}}$$

**Solution**

$$\text{a. } 8m^4$$

$$\text{b. } -4$$

$$\text{c. } 3|n|\sqrt[4]{2}$$

Access these online resources for additional instruction and practice with simplifying radical expressions.

- Simplifying Square Root and Cube Root with Variables
- Express a Radical in Simplified Form-Square and Cube Roots with Variables and Exponents
- Simplifying Cube Roots

## Key Concepts

- **Simplified Radical Expression**

- For real numbers  $a$ ,  $m$  and  $n \geq 2$ ,  $\sqrt[n]{a}$  is considered simplified if  $a$  has no factors of  $m^n$

- **Product Property of  $n^{\text{th}}$  Roots**

- For any real numbers,  $\sqrt[n]{a}$  and  $\sqrt[n]{b}$ , and for any integer  $n \geq 2$ ,  $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$  and  $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

- **How to simplify a radical expression using the Product Property**

1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.
2. Use the product rule to rewrite the radical as the product of two radicals.
3. Simplify the root of the perfect power.

- **Quotient Property of Radical Expressions**

- If  $\sqrt[n]{a}$  and  $\sqrt[n]{b}$  are real numbers,  $b \neq 0$ , and for any integer  $n \geq 2$  then,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \text{ and } \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

• **How to simplify a radical expression using the Quotient Property.**

1. Simplify the fraction in the radicand, if possible.
2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.
3. Simplify the radicals in the numerator and the denominator.

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

*<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=563#h5p-21>*

b) After reviewing this checklist, what will you do to become confident for all objectives?



## 3.3 SIMPLIFY RATIONAL EXPONENTS

---

### Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with  $a^{\frac{1}{n}}$
- Simplify expressions with  $a^{\frac{m}{n}}$
- Use the properties of exponents to simplify expressions with rational exponents

### Try It

Before you get started, take this readiness quiz:

- 1) Add:  $\frac{7}{15} + \frac{5}{12}$
- 2) Simplify:  $(4x^2y^5)^3$
- 3) Simplify:  $5^{-3}$

## Simplify Expressions with $a^{\frac{1}{n}}$

Rational exponents are another way of writing expressions with radicals. When we use rational exponents, we can apply the properties of exponents to simplify expressions.

The Power Property for Exponents says that  $(a^m)^n = a^{m \cdot n}$  when  $m$  and  $n$  are whole numbers. Let's assume we are now not limited to whole numbers.

Suppose we want to find a number  $p$  such that  $(8^p)^3 = 8$ . We will use the Power Property of Exponents to find the value of  $p$ .

Multiply the exponents on the left.

Write the exponent 1 on the right.

Since the bases are the same, the exponents must be equal.

Solve for  $p$ .

$$(8^p)^3 = 8$$

$$8^{3p} = 8$$

$$8^{3p} = 8^1$$

$$3p = 1$$

$$p = \frac{1}{3}$$

So  $\left(8^{\frac{1}{3}}\right)^3 = 8$ . But we know also  $(\sqrt[3]{8})^3 = 8$ . Then it must be that  $8^{\frac{1}{3}} = \sqrt[3]{8}$ .

This same logic can be used for any positive integer exponent  $n$  to show that  $a^{\frac{1}{n}} = \sqrt[n]{a}$ .

### Rational Exponent $a^{\frac{1}{n}}$

If  $\sqrt[n]{a}$  is a real number and  $n \geq 2$ , then

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

The denominator of the rational exponent is the index of the radical.

There will be times when working with expressions will be easier if you use rational exponents and times when it will be easier if you use radicals. In the first few examples, you'll practice converting expressions between these two notations.

### Example 3.3.1

Write as a radical expression:

a.  $x^{\frac{1}{2}}$

b.  $y^{\frac{1}{3}}$

c.  $z^{\frac{1}{4}}$

We want to write each expression in the form  $\sqrt[n]{a}$

#### Solution

a.

**Step 1: The denominator of the rational exponent is 2, so the index of the radical is 2.**

We do not show the index when it is 2.

$$\sqrt{x}$$


---

b.

**Step 1: The denominator of the exponent is 3, so the index is 3.**

$$\sqrt[3]{y}$$


---

c.

**Step 1: The denominator of the exponent is 4, so the index is 4.**

$$\sqrt[4]{z}$$

## Try It

4) Write as a radical expression:

a.  $t^{\frac{1}{2}}$

b.  $m^{\frac{1}{3}}$

c.  $r^{\frac{1}{4}}$

### Solution

a.  $\sqrt{t}$

b.  $\sqrt[3]{m}$

c.  $\sqrt[4]{r}$

## Try It

5) Write as a radical expression:

a.  $b^{\frac{1}{6}}$

b.  $z^{\frac{1}{5}}$

c.  $p^{\frac{1}{4}}$

### Solution

a.  $\sqrt[6]{b}$

b.  $\sqrt[5]{z}$

c.  $\sqrt[4]{p}$

In the next example, we will write each radical using a rational exponent. It is important to use parentheses around the entire expression in the radicand since the entire expression is raised to the rational power.

### Example 3.3.2

Write with a rational exponent:

- a.  $\sqrt{5y}$
- b.  $\sqrt[3]{4x}$
- c.  $3\sqrt[4]{5z}$

We want to write each radical in the form  $a^{\frac{1}{n}}$

#### Solution

a.

**Step 1: No index is shown, so it is 2.**

**Step 2: The denominator of the exponent will be 2.**

Put parentheses around the entire expression  $5y$ .

$$(5y)^{\frac{1}{2}}$$


---

b.

**Step 1: The index is 3, so the denominator of the exponent is 3.**

Include parentheses  $(4x)$ .

$$(4x)^{\frac{1}{3}}$$


---

c.

**Step 1: The index is 4, so the denominator of the exponent is 4.**

Put parentheses only around the  $5z$  since  $3$  is not under the radical sign.

$$3(5z)^{\frac{1}{4}}$$

## Try It

6) Write with a rational exponent:

a.  $\sqrt{10m}$

b.  $\sqrt[5]{3n}$

c.  $3\sqrt[4]{6y}$

### Solution

a.  $(10m)^{\frac{1}{2}}$

b.  $(3n)^{\frac{1}{5}}$

c.  $3(6y)^{\frac{1}{4}}$

## Try It

7) Write with a rational exponent:

a.  $\sqrt[7]{3k}$

b.  $\sqrt[4]{5j}$

c.  $8\sqrt[3]{2a}$

**Solution**

a.  $(3k)^{\frac{1}{7}}$

b.  $(5j)^{\frac{1}{4}}$

c.  $8(2a)^{\frac{1}{3}}$

In the next example, you may find it easier to simplify the expressions if you rewrite them as radicals first.

### Example 3.3.3

Simplify:

a.  $25^{\frac{1}{2}}$

b.  $64^{\frac{1}{3}}$

c.  $256^{\frac{1}{4}}$

**Solution**

a.

**Step 1: Rewrite as a square root.**

$$\sqrt{25}$$

**Step 2: Simplify.**

$$5$$


---

b.

**Step 1: Rewrite as a cube root.**

$$\sqrt[3]{64}$$

**Step 2: Recognize 64 is a perfect cube.**

$$\sqrt[3]{4^3}$$

**Step 3: Simplify.**

$$4$$


---

c.

**Step 1: Rewrite as a fourth root.**

$$\sqrt[4]{256}$$

**Step 2: Recognize 256 is a perfect fourth power.**

$$\sqrt[4]{4^4}$$

**Step 3: Simplify.**

$$4$$

## Try It

8) Simplify:

a.  $36^{\frac{1}{2}}$

b.  $8^{\frac{1}{3}}$

c.  $16^{\frac{1}{4}}$

**Solution**

a. 6



b. 2

c. 2

## Try It

9) Simplify:

a.  $100^{\frac{1}{2}}$

b.  $27^{\frac{1}{3}}$

c.  $81^{\frac{1}{4}}$

**Solution**

a. 10

b. 3

c. 3

Be careful of the placement of the negative signs in the next example. We will need to use the property

$$a^{-n} = \frac{1}{a^n} \text{ in one case.}$$

## Example 3.3.4

Simplify:

a.  $(-16)^{\frac{1}{4}}$

b.  $-16^{\frac{1}{4}}$

c.  $(16)^{-\frac{1}{4}}$

**Solution**

a.

**Step 1: Rewrite as a fourth root.**

$$\sqrt[4]{-16}$$

$$\sqrt[4]{(-2)^4}$$

**Step 2: Simplify.**

No real solution

b.

**Step 1: The exponent only applies to the 16. Rewrite as a fourth root.**

$$\sqrt[4]{16}$$

**Step 2: Rewrite 16 as  $2^4$** 

$$\sqrt[4]{2^4}$$

**Step 3: Simplify.**

$$-2$$

c.

**Step 1: Rewrite using the property  $a^{-n} = \frac{1}{a^n}$** 

$$\frac{1}{(16)^{\frac{1}{4}}}$$

**Step 2: Rewrite as a fourth root.**

$$\frac{1}{\sqrt[4]{16}}$$

**Step 3: Rewrite 16 as  $2^4$**

$$\frac{1}{\sqrt[4]{2^4}}$$

**Step 4: Simplify.**

$$\frac{1}{2}$$

## Try It

10) Simplify:

a.  $(-64)^{-\frac{1}{2}}$

b.  $-64^{\frac{1}{2}}$

c.  $(64)^{-\frac{1}{2}}$

### Solution

a. No real solution

b.  $-8$

c.  $\frac{1}{8}$

## Try It

11) Simplify:

a.  $(-256)^{\frac{1}{4}}$

b.  $-256^{\frac{1}{4}}$

c.  $(256)^{-\frac{1}{4}}$

### Solution

a. No real solution

b.  $-4$

c.  $\frac{1}{4}$

## Simplify Expressions with $a^{\frac{m}{n}}$

We can look at  $a^{\frac{m}{n}}$  in two ways. Remember the Power Property tells us to multiply the exponents and so  $\left(a^{\frac{1}{n}}\right)^m$  and  $(a^m)^{\frac{1}{n}}$  both equal  $a^{\frac{m}{n}}$ . If we write these expressions in radical form, we get

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (\sqrt[n]{a})^m \quad \text{and} \quad a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$$

This leads us to the following definition.

### Rational Exponent $a^{\frac{m}{n}}$

For any positive integers  $m$  and  $n$ ,

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m \quad \text{and} \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

Which form do we use to simplify an expression? We usually take the root first—that way we keep the numbers in the radicand smaller, before raising it to the power indicated.

### Example 3.3.5

Write with a rational exponent:

a.  $\sqrt{y^3}$

b.  $(\sqrt[3]{2x})^4$

c.  $\sqrt{\left(\frac{3a}{4b}\right)^3}$

We want to use  $a^{\frac{m}{n}} = \sqrt[n]{a^m}$  to write each radical in the form  $a^{\frac{m}{n}}$

#### Solution

a.

The numerator of the exponent is the exponent,  $rgb]1.0, 0.0, 0.03$ .

$$y^{\frac{rgb]1.0,0.0,0.03}{rgb]0.0,0.0,1.02}}$$

The denominator of the exponent is the index of the radical,  $rgb]0.0, 0.0, 1.02$ .

---

b.

The numerator of the exponent is the exponent,  $rgb]1.0, 0.0, 0.04$ .

$$(2x)^{\frac{rgb]1.0,0.0,0.04}{rgb]0.0,0.0,1.03}}$$

The denominator of the exponent is the index of the radical,  $rgb]0.0, 0.0, 1.03$ .

$$(\sqrt[3]{2x})^4$$


---

c.

The numerator of the exponent is the exponent,  $rgb]1.0, 0.0, 0.03$ .

The denominator of the exponent is the index of the radical,  $rgb]0.0, 0.0, 1.02$ .

$$\left(\frac{3a}{4b}\right)^{\frac{rgb]1.0,0.0,0.03}{rgb]0.0,0.0,1.02}}$$

$$\sqrt{\left(\frac{3a}{4b}\right)^3}$$

## Try It

12) Write with a rational exponent:

a.  $\sqrt{x^5}$

b.  $(\sqrt[4]{3y})^3$

c.  $\sqrt{\left(\frac{2m}{3n}\right)^5}$

### Solution

a.  $x^{\frac{5}{2}}$

b.  $(3y)^{\frac{3}{4}}$

c.  $\left(\frac{2m}{3n}\right)^{\frac{5}{2}}$

## Try It

13) Write with a rational exponent:

- a.  $\sqrt[5]{a^2}$   
 b.  $(\sqrt[3]{5ab})^5$   
 c.  $\sqrt{\left(\frac{7xy}{z}\right)^3}$

**Solution**

- a.  $a^{\frac{2}{5}}$   
 b.  $(5ab)^{\frac{5}{3}}$   
 c.  $\left(\frac{7xy}{z}\right)^{\frac{3}{2}}$

Remember that  $a^{-n} = \frac{1}{a^n}$ . The negative sign in the exponent does not change the sign of the expression.

**Example 3.3.6**

Simplify:

- a.  $125^{\frac{2}{3}}$   
 b.  $16^{-\frac{3}{2}}$   
 c.  $32^{-\frac{2}{5}}$

We will rewrite the expression as a radical first using the definition,  $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ . This form lets us take the root first and so we keep the numbers in the radicand smaller than if we used the other form.

**Solution**

a.

**Step 1:** The power of the radical is the numerator of the exponent, 2.

**Step 2: The index of the radical is the denominator of the exponent, 3.**

$$(\sqrt[3]{125})^2$$

**Step 3: Simplify.**

$$\frac{(5)^2}{25}$$


---

b.

We will rewrite each expression first using  $a^{-n} = \frac{1}{a^n}$  and then change to radical form.

**Step 1: Rewrite using  $a^{-n} = \frac{1}{a^n}$ .**

$$\frac{1}{16^{\frac{3}{2}}}$$

**Step 2: Change to radical form.**

The power of the radical is the numerator of the exponent, 3.

The index is the denominator of the exponent, 2.

$$\frac{1}{(\sqrt{16})^3}$$

**Step 3: Simplify.**

$$\frac{1}{4^3}$$

$$\frac{1}{64}$$


---

c.

**Step 1: Rewrite using  $a^{-n} = \frac{1}{a^n}$ .**



$$\frac{1}{32^{\frac{2}{5}}}$$

**Step 2: Change to radical form.**

$$\frac{1}{(\sqrt[5]{32})^2}$$

**Step 3: Rewrite the radicand as a power.**

$$\frac{1}{(\sqrt[5]{2^5})^2}$$

**Step 4: Simplify.**

$$\frac{1}{2^2}$$

$$\frac{1}{4}$$

## Try It

14) Simplify:

a.  $27^{\frac{2}{3}}$

b.  $81^{-\frac{3}{2}}$

c.  $16^{-\frac{3}{4}}$

**Solution**

a. 9

b.  $\frac{1}{729}$

c.  $\frac{1}{8}$

## Try It

15) Simplify:

a.  $4^{\frac{3}{2}}$

b.  $27^{-\frac{2}{3}}$

c.  $625^{-\frac{3}{4}}$

### Solution

a. 8

b.  $\frac{1}{9}$

c.  $\frac{1}{125}$

## Example 3.3.7

Simplify:

a.  $-25^{\frac{3}{2}}$

b.  $-25^{-\frac{3}{2}}$

c.  $(-25)^{\frac{3}{2}}$

**Solution**

a.

**Step 1: Rewrite in radical form.**

$$(\sqrt{25})^3$$

**Step 2: Simplify the radical.**

$$-(5)^3$$

**Step 3: Simplify.**

$$-125$$

---

b.

**Step 1: Rewrite using  $a^{-n} = \frac{1}{a^n}$ .**

$$-\left(\frac{1}{25^{\frac{3}{2}}}\right)$$

**Step 2: Rewrite in radical form.**

$$-\left(\frac{1}{(\sqrt{25})^3}\right)$$

**Step 3: Simplify the radical.**

$$-\left(\frac{1}{(5)^3}\right)$$

**Step 4: Simplify.**

$$-\frac{1}{125}$$

---

c.

**Step 1: Rewrite in radical form.**

$$(\sqrt{-25})^3$$

There is no real number whose square root is  $-25$ .

Not a real number.

## Try It

16) Simplify:

a.  $-16^{\frac{3}{2}}$

b.  $-16^{-\frac{3}{2}}$

c.  $(-16)^{-\frac{3}{2}}$

**Solution**

a.  $-64$

b.  $-\frac{1}{64}$

c. Not a real number.

## Try It

17) Simplify:

- a.  $-81^{\frac{3}{2}}$   
 b.  $-81^{-\frac{3}{2}}$   
 c.  $(-81)^{-\frac{3}{2}}$

**Solution**

- a.  $-729$   
 b.  $-\frac{1}{729}$   
 c. Not a real number.

## Use the Properties of Exponents to Simplify Expressions with Rational Exponents

The same properties of exponents that we have already used also apply to rational exponents. We will list the Properties of Exponents here to have them for reference as we simplify expressions.

### Properties of Exponents

If  $a$  and  $b$  are real numbers and  $m$  and  $n$  are rational numbers, then

**Product Property**

$$a^m \cdot a^n = a^{m+n}$$

**Power Property**

$$(a^m)^n = a^{m \cdot n}$$

**Product to a Power**

$$(ab)^m = a^m b^m$$

**Quotient Property**

$$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$$

**Zero Exponent Definition**

$$a^0 = 1, a \neq 0$$

**Quotient to a Power Property**

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$$

**Negative Exponent Property**

$$a^{-n} = \frac{1}{a^n}, a \neq 0$$

We will apply these properties in the next example.

### Example 3.3.8

Simplify:

a.  $x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$

b.  $(z^9)^{\frac{2}{3}}$

c.  $\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$

#### Solution

a.

The Product Property tells us that when we multiply the same base, we add the exponents.

**Step 1: The bases are the same, so we add the exponents.**

$$x^{\frac{1}{2} + \frac{5}{6}}$$

**Step 2: Add the fractions.**

$$x^{\frac{8}{6}}$$

**Step 3: Simplify the exponent.**

$$x^{\frac{4}{3}}$$


---

b.

The Power Property tells us that when we raise a power to a power, we multiply the exponents.

**Step 1: To raise a power to a power, we multiply the exponents.**

$$z^{9 \cdot \frac{2}{3}}$$

**Step 2: Simplify.**

$$z^6$$


---

c.

The Quotient Property tells us that when we divide with the same base, we subtract the exponents.

**Step 1: To divide with the same base, we subtract the exponents.**

$$\frac{1}{x^{\frac{5}{3} - \frac{1}{3}}}$$

**Step 2: Simplify.**

$$\frac{1}{x^{\frac{4}{3}}}$$

## Try It

18) Simplify:

a.  $x^{\frac{1}{6}} \cdot x^{\frac{4}{3}}$

b.  $(x^6)^{\frac{4}{3}}$

c.  $\frac{x^{\frac{2}{3}}}{x^{\frac{5}{3}}}$

**Solution**

a.  $x^{\frac{3}{2}}$

b.  $x^8$

c.  $\frac{1}{x}$

## Try It

19) Simplify:

a.  $y^{\frac{3}{4}} \cdot y^{\frac{5}{8}}$

b.  $(m^9)^{\frac{2}{9}}$

c.  $\frac{d^{\frac{1}{5}}}{d^{\frac{6}{5}}}$

### Solution

a.  $y^{\frac{11}{8}}$

b.  $m^2$

c.  $\frac{1}{d}$

Sometimes we need to use more than one property. In the next example, we will use both the Product to a Power Property and then the Power Property.

## Example 3.3.9

Simplify:

a.  $\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$

b.  $\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}}$



**Solution**

a.

**Step 1: First we use the Product to a Power Property.**

$$(27)^{\frac{2}{3}} \left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$$

**Step 2: Rewrite 27 as a power of 3.**

$$(3^3)^{\frac{2}{3}} \left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$$

**Step 3: To raise a power to a power, we multiply the exponents.**

$$(3^2) \left(u^{\frac{1}{3}}\right)$$

**Step 4: Simplify.**

$$9u^{\frac{1}{3}}$$


---

b.

**Step 1: First we use the Product to a Power Property.**

$$\left(m^{\frac{2}{3}}\right)^{\frac{3}{2}} \left(n^{\frac{1}{2}}\right)^{\frac{3}{2}}$$

**Step 2: To raise a power to a power, we multiply the exponents.**

$$mn^{\frac{3}{4}}$$

## Try It

20) Simplify:

a.  $\left(32x^{\frac{1}{3}}\right)^{\frac{3}{5}}$

b.  $\left(x^{\frac{3}{4}}y^{\frac{1}{2}}\right)^{\frac{2}{3}}$

### Solution

a.  $8x^{\frac{1}{5}}$

b.  $x^{\frac{1}{2}}y^{\frac{1}{3}}$

## Try It

21) Simplify:

a.  $\left(81n^{\frac{2}{5}}\right)^{\frac{3}{2}}$

b.  $\left(a^{\frac{3}{2}}b^{\frac{1}{2}}\right)^{\frac{4}{3}}$

### Solution

a.  $729n^{\frac{3}{5}}$

b.  $a^2b^{\frac{2}{3}}$

We will use both the Product Property and the Quotient Property in the next example.

### Example 3.3.10

Simplify:

a.  $\frac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$

b.  $\left( \frac{16x^{\frac{4}{3}}y^{-\frac{5}{6}}}{x^{-\frac{2}{3}}y^{\frac{1}{6}}} \right)^{\frac{1}{2}}$

#### Solution

a.

**Step 1: Use the Product Property in the numerator, add the exponents.**

$$\frac{x^{\frac{2}{4}}}{x^{-\frac{6}{4}}}$$

**Step 2: Use the Quotient Property, subtract the exponents.**

$$x^{\frac{8}{4}}$$

**Step 3: Simplify.**

$$x^2$$

b.

Follow the order of operations to simplify inside the parentheses first.

**Step 1: Use the Quotient Property, subtract the exponents.**

$$\left( \frac{16x^{\frac{6}{3}}}{y^{\frac{6}{6}}} \right)^{\frac{1}{2}}$$

**Step 2: Simplify.**

$$\left( \frac{16x^2}{y} \right)^{\frac{1}{2}}$$

**Step 3: Use the Product to a Power Property, multiply the exponents.**

$$\frac{4x}{y^{\frac{1}{2}}}$$

## Try It

22) Simplify:

a.  $\frac{m^{\frac{2}{3}} \cdot m^{-\frac{1}{3}}}{m^{-\frac{5}{3}}}$

b.  $\left( \frac{25m^{\frac{1}{6}} n^{\frac{11}{6}}}{m^{\frac{2}{3}} n^{-\frac{1}{6}}} \right)^{\frac{1}{2}}$

**Solution**

a.  $m^2$

b.  $\frac{5n}{m^{\frac{1}{4}}}$

## Try It

23) Simplify:

a. 
$$\frac{u^{\frac{4}{5}} \cdot u^{-\frac{2}{5}}}{u^{-\frac{13}{5}}}$$

b. 
$$\left( \frac{27x^{\frac{4}{5}}y^{\frac{1}{6}}}{x^{\frac{1}{5}}y^{-\frac{5}{6}}} \right)^{\frac{1}{3}}$$

**Solution**

a.  $u^3$

b.  $3x^{\frac{1}{5}}y^{\frac{1}{3}}$

Access these online resources for additional instruction and practice with simplifying rational exponents.

- Review-Rational Exponents
- Using Laws of Exponents on Radicals: Properties of Rational Exponents

## Key Concepts

- **Rational Exponent**  $a^{\frac{1}{n}}$ 
  - If  $\sqrt[n]{a}$  is a real number and  $n \geq 2$ , then  $a^{\frac{1}{n}} = \sqrt[n]{a}$
- **Rational Exponent**  $a^{\frac{m}{n}}$ 
  - For any positive integers  $m$  and  $n$ ,  $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$  and  $a^{\frac{m}{n}} = \sqrt[n]{a^m}$
- **Properties of Exponents**
  - If  $a, b$  are real numbers and  $m, n$  are rational numbers, then
    - **Product Property**  $a^m \cdot a^n = a^{m+n}$
    - **Power Property**  $(a^m)^n = a^{m \cdot n}$
    - **Product to a Power**  $(ab)^m = a^m b^m$
    - **Quotient Property**  $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
    - **Zero Exponent Definition**  $a^0 = 1, a \neq 0$
    - **Quotient to a Power Property**  $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$
    - **Negative Exponent Property**  $a^{-n} = \frac{1}{a^n}, a \neq 0$

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

*<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=568#h5p-22>*

b) What does this checklist tell you about your mastery of this section? What steps will you take to improve?

# 3.4 ADD, SUBTRACT, AND MULTIPLY RADICAL EXPRESSIONS

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## Learning Objectives

By the end of this section, you will be able to:

- Add and subtract radical expressions
- Multiply radical expressions
- Use polynomial multiplication to multiply radical expressions

## Try It

Before you get started, take this readiness quiz:

- 1) Add:  $3x^2 + 9x - 5 - (x^2 - 2x + 3)$
- 2) Simplify:  $(2 + a)(4 - a)$
- 3) Simplify:  $(9 - 5y)^2$



## Add and Subtract Radical Expressions

Adding radical expressions with the same index and the same radicand is just like adding like terms. We call radicals with the same index and the same radicand **like radicals** to remind us they work the same as like terms.

### Like Radicals

Like radicals are radical expressions with the same index and the same radicand.

We add and subtract like radicals in the same way we add and subtract like terms. We know that  $3x + 8x$  is  $11x$ . Similarly we add  $3\sqrt{x} + 8\sqrt{x}$  and the result is  $11\sqrt{x}$ .

Think about adding like terms with variables as you do the next few examples. When you have like radicals, you just add or subtract the coefficients. When the radicals are not like, you cannot combine the terms.

### Example 3.4.1

Simplify:

a.  $2\sqrt{2} - 7\sqrt{2}$

b.  $5\sqrt[3]{y} + 4\sqrt[3]{y}$

c.  $7\sqrt[4]{x} - 2\sqrt[4]{y}$

#### Solution

a.

**Step 1:** Since the radicals are like, we subtract the coefficients.

$$-5\sqrt{2}$$


---

b.

**Step 1: Since the radicals are like, we add the coefficients.**

$$9\sqrt[3]{y}$$

---

c.

The indices are the same but the radicals are different. These are not like radicals. Since the radicals are not like, we cannot subtract them.

## Try It

4) Simplify:

a.  $8\sqrt{2} - 9\sqrt{2}$

b.  $4\sqrt[3]{x} + 7\sqrt[3]{x}$

c.  $3\sqrt[4]{x} - 5\sqrt[4]{y}$

**Solution**

a.  $-\sqrt{2}$

b.  $11\sqrt[3]{x}$

c.  $3\sqrt[4]{x} - 5\sqrt[4]{y}$

## Try It

5) Simplify:

a.  $5\sqrt{3} - 9\sqrt{3}$

b.  $5\sqrt[3]{y} + 3\sqrt[3]{y}$

c.  $5\sqrt[4]{m} - 2\sqrt[3]{m}$

### Solution

a.  $-4\sqrt{3}$

b.  $8\sqrt[3]{y}$

c.  $5\sqrt[4]{m} - 2\sqrt[3]{m}$

For radicals to be like, they must have the same index and radicand. When the radicands contain more than one variable, as long as all the variables and their exponents are identical, the radicands are the same.

## Example 3.4.2

Simplify:

a.  $2\sqrt{5n} - 6\sqrt{5n} + 4\sqrt{5n}$

b.  $\sqrt[4]{3xy} + 5\sqrt[4]{3xy} - 4\sqrt[4]{3xy}$

### Solution

a.

**Step 1:** Since the radicals are like, we combine them.

$$0\sqrt{5n}$$

**Step 2: Simplify.**

$$0$$


---

b.

**Step 1: Since the radicals are like, we combine them.**

$$2 \sqrt[4]{3xy}$$

## Try It

6) Simplify:

a.  $\sqrt{7x} - 7\sqrt{7x} + 4\sqrt{7x}$

b.  $4\sqrt[4]{5xy} + 2\sqrt[4]{5xy} - 7\sqrt[4]{5xy}$

**Solution**

a.  $-2\sqrt{7x}$

b.  $-\sqrt[4]{5xy}$

## Try It

7) Simplify:

- a.  $4\sqrt{3y} - 7\sqrt{3y} + 2\sqrt{3y}$   
 b.  $6\sqrt[3]{7mn} + \sqrt[3]{7mn} - 4\sqrt[3]{7mn}$

**Solution**

- a.  $-\sqrt{3y}$   
 b.  $3\sqrt[3]{7mn}$

Remember that we always simplify radicals by removing the largest factor from the radicand that is a power of the index. Once each radical is simplified, we can then decide if they are like radicals.

**Example 3.4.3**

Simplify:

- a.  $\sqrt{20} + 3\sqrt{5}$   
 b.  $\sqrt[3]{24} - \sqrt[3]{375}$   
 c.  $\frac{1}{2}\sqrt[4]{48} - \frac{2}{3}\sqrt[4]{243}$

**Solution**

a.

**Step 1: Simplify the radicals, when possible.**

$$\begin{aligned} &\sqrt{4} \cdot \sqrt{5} + 3\sqrt{5} \\ &2\sqrt{5} + 3\sqrt{5} \end{aligned}$$

**Step 2: Combine the like radicals.**

$$5\sqrt{5}$$

b.

**Step 1: Simplify the radicals, when possible.**

$$\begin{aligned} & \sqrt[3]{8} \cdot \sqrt[3]{3} - \sqrt[3]{125} \cdot \sqrt[3]{3} \\ & 2 \sqrt[3]{3} - 5 \sqrt[3]{3} \end{aligned}$$

**Step 2: Combine the like radicals.**

$$-3\sqrt[3]{3}$$


---

c.

**Step 1: Simplify the radicals.**

$$\begin{aligned} & \frac{1}{2} \sqrt[4]{16} \cdot \sqrt[4]{3} - \frac{2}{3} \sqrt[4]{81} \cdot \sqrt[4]{3} \\ & \frac{1}{2} \cdot 2 \cdot \sqrt[4]{3} - \frac{2}{3} \cdot 3 \cdot \sqrt[4]{3} \\ & \sqrt[4]{3} - 2 \sqrt[4]{3} \end{aligned}$$

**Step 2: Combine the like radicals.**

$$-\sqrt[4]{3}$$

## Try It

8) Simplify:

a.  $\sqrt{18} + 6\sqrt{2}$

b.  $6\sqrt[3]{16} - 2\sqrt[3]{250}$

c.  $\frac{2}{3}\sqrt[3]{81} - \frac{1}{2}\sqrt[3]{24}$

**Solution**

- a.  $9\sqrt{2}$
- b.  $2\sqrt[3]{2}$
- c.  $\sqrt[3]{3}$

## Try It

9) Simplify:

- a.  $\sqrt{27} + 4\sqrt{3}$
- b.  $4\sqrt[3]{5} - 7\sqrt[3]{40}$
- c.  $\frac{1}{2}\sqrt[3]{128} - \frac{5}{3}\sqrt[3]{54}$

### Solution

- a.  $7\sqrt{3}$
- b.  $-10\sqrt[3]{5}$
- c.  $-3\sqrt[3]{2}$

In the next example, we will remove both constant and variable factors from the radicals. Now that we have practised taking both the even and odd roots of variables, it is common practice at this point for us to assume all variables are greater than or equal to zero so that absolute values are not needed. We will use this assumption throughout the rest of this chapter.

### Example 3.4.4

Simplify:

a.  $9\sqrt{50m^2} - 6\sqrt{48m^2}$

b.  $\sqrt[3]{54n^5} - \sqrt[3]{16n^5}$

#### Solution

a.

**Step 1: Simplify the radicals.**

$$\begin{aligned} & 9\sqrt{25m^2} \cdot \sqrt{2} - 6\sqrt{16m^2} \cdot \sqrt{3} \\ & 9 \cdot 5m \cdot \sqrt{2} - 6 \cdot 4m \cdot \sqrt{3} \\ & 45m\sqrt{2} - 24m\sqrt{3} \end{aligned}$$

**Step 2: The radicals are not like and so cannot be combined.**

---

b.

**Step 1: Simplify the radicals.**

$$\begin{aligned} & \sqrt[3]{27n^3} \cdot \sqrt[3]{2n^2} - \sqrt[3]{8n^3} \cdot \sqrt[3]{2n^2} \\ & 3n\sqrt[3]{2n^2} - 2n\sqrt[3]{2n^2} \end{aligned}$$

**Step 2: Combine the like radicals.**

$$\sqrt[3]{2n^2}$$



## Try It

10) Simplify:

a.  $\sqrt{32m^7} - \sqrt{50m^7}$

b.  $\sqrt[3]{135x^7} - \sqrt[3]{40x^7}$

**Solution**

a.  $-m^3\sqrt{2m}$

b.  $x^2\sqrt[3]{5x}$

## Try It

11) Simplify:

a.  $\sqrt{27p^3} - \sqrt{48p^3}$

b.  $\sqrt[3]{256y^5} - \sqrt[3]{32n^5}$

**Solution**

a.  $-p\sqrt{3p}$

b.  $4y\sqrt[3]{4y^2} - 2n\sqrt[3]{4n^2}$

## Multiply Radical Expressions

We have used the Product Property of Roots to simplify square roots by removing the perfect square factors.

We can use the Product Property of Roots ‘in reverse’ to multiply square roots. Remember, we assume all variables are greater than or equal to zero.

We will rewrite the Product Property of Roots so we see both ways together.

### Product Property of Roots

For any real numbers,  $\sqrt[n]{a}$  and  $\sqrt[n]{b}$ , and for any integer  $n \geq 2$

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \text{ and } \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

When we multiply two radicals they must have the same index. Once we multiply the radicals, we then look for factors that are a power of the index and simplify the radical whenever possible.

Multiplying radicals with coefficients is much like multiplying variables with coefficients. To multiply  $4x \cdot 3y$  we multiply the coefficients together and then the variables. The result is  $12xy$ . Keep this in mind as you do these examples.

### Example 3.4.5

Simplify:

a.  $(6\sqrt{2})(3\sqrt{10})$

b.  $(-5\sqrt[3]{4})(-4\sqrt[3]{6})$

#### Solution

a.

**Step 1: Multiply using the Product Property.**

$$18\sqrt{20}$$

**Step 2: Simplify the radical.**

$$18\sqrt{4} \cdot \sqrt{5}$$

**Step 3: Simplify.**

$$\begin{aligned} & 18 \cdot 2 \cdot \sqrt{5} \\ & 36 \sqrt{5} \end{aligned}$$


---

b.

**Step 1: Multiply using the Product Property.**

$$20 \sqrt[3]{24}$$

**Step 2: Simplify the radical.**

$$20 \sqrt[3]{8} \cdot \sqrt[3]{3}$$

**Step 3: Simplify.**

$$\begin{aligned} & 20 \cdot 2 \cdot \sqrt[3]{3} \\ & 40 \sqrt[3]{3} \end{aligned}$$

## Try It

12) Simplify:

a.  $(3\sqrt{2})(2\sqrt{30})$

b.  $(2\sqrt[3]{18})(-3\sqrt[3]{6})$

**Solution**

a.  $12\sqrt{15}$

b.  $-18\sqrt[3]{2}$

## Try It

13) Simplify:

a.  $(3\sqrt{3})(3\sqrt{6})$

b.  $(-4\sqrt[3]{9})(3\sqrt[3]{6})$

### Solution

a.  $27\sqrt{2}$

b.  $-36\sqrt[3]{2}$

We follow the same procedures when there are variables in the radicands.

## Example 3.4.6

Simplify:

a.  $(10\sqrt{6p^3})(4\sqrt{3p})$

b.  $(2\sqrt[4]{20y^2})(3\sqrt[4]{28y^3})$

### Solution

a.

**Step 1: Multiply.**

$$40\sqrt{18p^4}$$

**Step 2: Simplify the radical.**

$$40\sqrt{9p^4} \cdot \sqrt{2}$$

**Step 3: Simplify.**

$$\begin{aligned} &40 \cdot 3p^2 \cdot \sqrt{2} \\ &120p^2\sqrt{2} \end{aligned}$$


---

b.

When the radicands involve large numbers, it is often advantageous to factor them in order to find the perfect powers.

**Step 1: Multiply.**

$$6 \sqrt[4]{4 \cdot 5 \cdot 4 \cdot 7y^5}$$

**Step 2: Simplify the radical.**

$$6 \sqrt[4]{16y^4} \cdot \sqrt[4]{35y}$$

**Step 3: Simplify.**

$$6 \cdot 2y \sqrt[4]{35y}$$

**Step 4: Multiply.**

$$12y \sqrt[4]{35y}$$

## Try It

14) Simplify:

$$\text{a. } (6\sqrt{6x^2}) (8\sqrt{30x^4})$$

$$\text{b. } (-4\sqrt[4]{12y^3}) (-\sqrt[4]{8y^3})$$

**Solution**

$$\text{a. } 36x^3\sqrt{5}$$

$$\text{b. } 8y\sqrt[4]{3y^2}$$

**Try It**

15) Simplify:

$$\text{a. } (2\sqrt{6y^4}) (12\sqrt{30y})$$

$$\text{b. } (-4\sqrt[4]{9a^3}) (3\sqrt[4]{27a^2})$$

**Solution**

$$\text{a. } 144y^2\sqrt{5y}$$

$$\text{b. } -36\sqrt[4]{3a}$$

## Use Polynomial Multiplication to Multiply Radical Expressions

In the next few examples, we will use the Distributive Property to multiply expressions with radicals. First we will distribute and then simplify the radicals when possible.

### Example 3.4.7

Simplify:

a.  $\sqrt{6} (\sqrt{2} + \sqrt{18})$

b.  $\sqrt[3]{9} (5 - \sqrt[3]{18})$

#### Solution

a.

**Step 1: Multiply.**

$$\sqrt{12} + \sqrt{108}$$

**Step 2: Simplify.**

$$\begin{aligned} \sqrt{4} \cdot \sqrt{3} + \sqrt{36} \cdot \sqrt{3} \\ 2\sqrt{3} + 6\sqrt{3} \end{aligned}$$

**Step 3: Combine like radicals.**

$$8\sqrt{3}$$

b.

**Step 1: Distribute.**

$$5\sqrt[3]{9} - \sqrt[3]{162}$$

**Step 2: Simplify.**

$$\begin{aligned} 5\sqrt[3]{9} - \sqrt[3]{27} \cdot \sqrt[3]{6} \\ 5\sqrt[3]{9} - 3\sqrt[3]{6} \end{aligned}$$

## Try It

16) Simplify:

a.  $\sqrt{6} (1 + 3\sqrt{6})$

b.  $\sqrt[3]{4} (-2 - \sqrt[3]{6})$

### Solution

a.  $18 + \sqrt{6}$

b.  $-2\sqrt[3]{4} - 2\sqrt[3]{3}$

## Try It

17) Simplify:

a.  $\sqrt{8} (2 - 5\sqrt{8})$

b.  $\sqrt[3]{3} (-\sqrt[3]{9} - \sqrt[3]{6})$

### Solution

a.  $-40 + 4\sqrt{2}$

b.  $-3 - \sqrt[3]{18}$

When we worked with polynomials, we multiplied binomials by binomials. Remember, this gave us four products before we combined any like terms. To be sure to get all four products, we organized our work—usually by the FOIL method.



### Example 3.4.8

Simplify:

a.  $(3 - 2\sqrt{7})(4 - 2\sqrt{7})$

b.  $(\sqrt[3]{x} - 2)(\sqrt[3]{x} + 4)$

#### Solution

a.

**Step 1: Multiply.**

$$12 - 6\sqrt{7} - 8\sqrt{7} + 4 \cdot 7$$

**Step 2: Simplify.**

$$12 - 6\sqrt{7} - 8\sqrt{7} + 28$$

**Step 3: Combine like terms.**

$$40 - 14\sqrt{7}$$

b.

**Step 1: Multiply.**

$$\sqrt[3]{x^2} + 4\sqrt[3]{x} - 2\sqrt[3]{x} - 8$$

**Step 2: Combine like terms.**

$$\sqrt[3]{x^2} + 2\sqrt[3]{x} - 8$$

## Try It

18) Simplify:

a.  $(6 - 3\sqrt{7})(3 + 4\sqrt{7})$

b.  $(\sqrt[3]{x} - 2)(\sqrt[3]{x} - 3)$

### Solution

a.  $-66 + 15\sqrt{7}$

b.  $\sqrt[3]{x^2} - 5\sqrt[3]{x} + 6$

## Try It

19) Simplify:

a.  $(2 - 3\sqrt{11})(4 - \sqrt{11})$

b.  $(\sqrt[3]{x} + 1)(\sqrt[3]{x} + 3)$

### Solution

a.  $41 - 14\sqrt{11}$

b.  $\sqrt[3]{x^2} + 4\sqrt[3]{x} + 3$

### Example 3.4.9

Simplify:  $(3\sqrt{2} - \sqrt{5})(\sqrt{2} + 4\sqrt{5})$

**Solution**

**Step 1: Multiply.**

$$3 \cdot 2 + 12\sqrt{10} - \sqrt{10} - 4 \cdot 5$$

**Step 2: Simplify.**

$$6 + 12\sqrt{10} - \sqrt{10} - 20$$

**Step 3: Combine like terms.**

$$-14 + 11\sqrt{10}$$

### Try It

20) Simplify:  $(5\sqrt{3} - \sqrt{7})(\sqrt{3} + 2\sqrt{7})$

**Solution**

$$1 + 9\sqrt{21}$$

## Try It

21) Simplify:  $(\sqrt{6} - 3\sqrt{8})(2\sqrt{6} + \sqrt{8})$

**Solution**

$$-12 - 20\sqrt{3}$$

Recognizing some special products made our work easier when we multiplied binomials earlier. This is true when we multiply radicals, too. The special product formulas we used are shown here.

## Special Products

### Binomial Squares

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

### Product of Conjugates

$$(a + b)(a - b) = a^2 - b^2$$

We will use the special product formulas in the next few examples. We will start with the Product of Binomial Squares Pattern.

## Example 3.4.10

Simplify:

- a.  $(2 + \sqrt{3})^2$   
 b.  $(4 - 2\sqrt{5})^2$

Be sure to include the  $2ab$  term when squaring a binomial.

### Solution

a.

$$\begin{aligned} & \text{rgb}[1.0, 0.0, 0.0] (a + b) \text{rgb}[1.0, 0.0, 0.02] \\ & (2 + \sqrt{3})^2 \end{aligned}$$

**Step 1: Multiply, using the Product of Binomial Squares Pattern.**

$$2^2 + 2 \cdot 2 \cdot \sqrt{3} + (\sqrt{3})^2$$

**Step 2: Simplify.**

$$4 + 4\sqrt{3} + 3$$

**Step 3: Combine like terms.**

$$7 + 4\sqrt{3}$$

b.

$$\begin{aligned} & \text{rgb}[1.0, 0.0, 0.0] (a - b) \text{rgb}[1.0, 0.0, 0.02] \\ & (4 - 2\sqrt{5})^2 \end{aligned}$$

**Step 1: Multiply, using the Product of Binomial Squares Pattern.**

$$4^2 - 2 \cdot 4 \cdot 2\sqrt{5} + (2\sqrt{5})^2$$

**Step 2: Simplify.**

$$16 - 16\sqrt{5} + 4 \cdot 5$$
$$16 - 16\sqrt{5} + 20$$

**Step 3: Combine like terms.**

$$36 - 16\sqrt{5}$$

## Try It

22) Simplify:

a.  $(10 + \sqrt{2})^2$

b.  $(1 + 3\sqrt{6})^2$

**Solution**

a.  $102 + 20\sqrt{2}$

b.  $55 + 6\sqrt{6}$

## Try It

23) Simplify:

a.  $(6 - \sqrt{5})^2$

b.  $(9 - 2\sqrt{10})^2$

**Solution**

- a.  $41 - 12\sqrt{5}$   
 b.  $121 - 36\sqrt{10}$

In the next example, we will use the Product of Conjugates Pattern. Notice that the final product has no radical.

### Example 3.4.11

Simplify:  $(5 - 2\sqrt{3})(5 + 2\sqrt{3})$

**Solution**

$$(a - b)(a + b)$$

$$(5 - 2\sqrt{3})(5 + 2\sqrt{3})$$

**Step 1: Multiply, using the Product of Conjugates Pattern.**

$$a^2 - (2\sqrt{3})^2$$

**Step 2: Simplify.**

$$25 - 4 \cdot 3$$

$$= 13$$

### Try It

24) Simplify:  $(3 - 2\sqrt{5})(3 + 2\sqrt{5})$

**Solution**

−11

**Try It**25) Simplify:  $(4 + 5\sqrt{7})(4 - 5\sqrt{7})$ **Solution**

−159

Access these online resources for additional instruction and practice with adding, subtracting, and multiplying radical expressions.

- Multiplying Adding Subtracting Radicals
- Multiplying Special Products: Square Binomials Containing Square Roots
- Multiplying Conjugates

**Key Concepts**

- **Product Property of Roots**



- For any real numbers,  $\sqrt[n]{a}$  and  $\sqrt[n]{b}$ , and for any integer  $n \geq 2$   
 $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$  and  $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

- Special Products**

**Binomial Squares**

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

**Product of Conjugates**

$$(a + b)(a - b) = a^2 - b^2$$

## Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=583#h5p-23>

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

## Glossary

### like radicals

Like radicals are radical expressions with the same index and the same radicand.

## 3.5 DIVIDE RADICAL EXPRESSIONS

---

### Learning Objectives

By the end of this section, you will be able to:

- Divide radical expressions
- Rationalize a one term denominator
- Rationalize a two term denominator

### Try It

Before you get started, take this readiness quiz:

1) Simplify:  $\frac{30}{48}$

2) Simplify:  $x^2 \cdot x^4$

3) Multiply:  $(7 + 3x)(7 - 3x)$

### Divide Radical Expressions

We have used the Quotient Property of Radical Expressions to simplify roots of fractions. We will need to use this property 'in reverse' to simplify a fraction with radicals.

We give the Quotient Property of Radical Expressions again for easy reference. Remember, we assume all variables are greater than or equal to zero so that no absolute value bars are needed.

### Quotient Property of Radical Expressions

If  $\sqrt[n]{a}$  and  $\sqrt[n]{b}$  are real numbers,  $b \neq 0$ , and for any integer  $n \geq 2$  then,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

We will use the Quotient Property of Radical Expressions when the fraction we start with is the quotient of two radicals, and neither radicand is a perfect power of the index. When we write the fraction in a single radical, we may find common factors in the numerator and denominator.

#### Example 3.5.1

Simplify:

a.  $\frac{\sqrt{72x^3}}{\sqrt{162x}}$

b.  $\frac{\sqrt[3]{32x^2}}{\sqrt[3]{4x^5}}$

#### Solution

a.

**Step 1:** Rewrite using the quotient property,  $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$

$$\sqrt{\frac{72x^3}{162x}}$$

**Step 2: Remove common factors.**

$$\sqrt{\frac{\overline{)18 \cdot 4 \cdot x^2 \cdot )x}}{\overline{)18 \cdot 9 \cdot )x}}}$$

**Step 3: Simplify.**

$$\sqrt{\frac{4x^2}{9}}$$

**Step 4: Simplify the radical.**

$$\frac{2x}{3}$$


---

b.

**Step 1: Rewrite using the quotient property,**  $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$

$$\sqrt[3]{\frac{32x^2}{4x^5}}$$

**Step 2: Simplify the fraction under the radical.**

$$\sqrt[3]{\frac{8}{x^3}}$$

**Step 3: Simplify the radical.**

$$\frac{2}{x}$$

## Try It

4) Simplify:

a.  $\frac{\sqrt{50s^3}}{\sqrt{128s}}$

b.  $\frac{\sqrt[3]{56a}}{\sqrt[3]{7a^4}}$

**Solution**

a.  $\frac{5s}{8}$

b.  $\frac{2}{a}$

## Try It

5) Simplify:

a.  $\frac{\sqrt{75q^5}}{\sqrt{108q}}$

b.  $\frac{\sqrt[3]{72b^2}}{\sqrt[3]{9b^5}}$

**Solution**

a.  $\frac{5q^2}{6}$   
 b.  $\frac{2}{b}$

### Example 3.5.2

Simplify:

a.  $\frac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$   
 b.  $\frac{\sqrt[3]{-250mn^{-2}}}{\sqrt[3]{2m^{-2}n^4}}$

#### Solution

a.

**Step 1:** Rewrite using the quotient property.

$$\sqrt{\frac{147ab^8}{3a^3b^4}}$$

**Step 2:** Remove common factors in the fraction.

$$\sqrt{\frac{49b^4}{a^2}}$$

**Step 3:** Simplify the radical.

$$\frac{7b^2}{a}$$


---

b.

**Step 1: Rewrite using the quotient property.**

$$\sqrt[3]{\frac{-250mn^{-2}}{2m^{-2}n^4}}$$

**Step 2: Simplify the fraction under the radical.**

$$\sqrt[3]{\frac{-125m^3}{n^6}}$$

**Step 3: Simplify the radical.**

$$-\frac{5m}{n^2}$$

## Try It

6) Simplify:

$$\text{a. } \frac{\sqrt{162x^{10}y^2}}{\sqrt{2x^6y^6}}$$

$$\text{b. } \frac{\sqrt[3]{-128x^2y^{-1}}}{\sqrt[3]{2x^{-1}y^2}}$$

**Solution**

$$\text{a. } \frac{9x^2}{y^2}$$

$$\text{b. } \frac{-4x}{y}$$

## Try It

7) Simplify:

a.  $\frac{\sqrt{300m^3n^7}}{\sqrt{3m^5n}}$

b.  $\frac{\sqrt[3]{-81pq^{-1}}}{\sqrt[3]{3p^{-2}q^5}}$

**Solution**

a.  $\frac{10n^3}{m}$

b.  $\frac{-3p}{q^2}$

## Example 3.5.3

Simplify:  $\frac{\sqrt{54x^5y^3}}{\sqrt{3x^2y}}$

**Solution**

**Step 1:** Rewrite using the quotient property.

$$\sqrt{\frac{54x^5y^3}{3x^2y}}$$

**Step 2:** Remove common factors in the fraction.



$$\sqrt{18x^3y^2}$$

**Step 3:** Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{9x^2y^2 \cdot 2x}$$

**Step 4:** Rewrite the radical as the product of two radicals.

$$\sqrt{9x^2y^2} \cdot \sqrt{2x}$$

**Step 5:** Simplify.

$$3xy\sqrt{2x}$$

## Try It

8) Simplify:  $\frac{\sqrt{64x^4y^5}}{\sqrt{2xy^3}}$

**Solution**

$$4xy\sqrt{2x}$$

## Try It

9) Simplify:  $\frac{\sqrt{96a^5b^4}}{\sqrt{2a^3b}}$

**Solution**

$$4ab\sqrt{3b}$$

## Rationalize a One Term Denominator

Before the calculator became a tool of everyday life, approximating the value of a fraction with a radical in the denominator was a very cumbersome process!

For this reason, a process called **rationalizing the denominator** was developed. A fraction with a radical in the denominator is converted to an equivalent fraction whose denominator is an integer. Square roots of numbers that are not perfect squares are irrational numbers. When we rationalize the denominator, we write an equivalent fraction with a rational number in the denominator.

This process is still used today, and is useful in other areas of mathematics, too.

### Rationalizing the Denominator

Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

Even though we have calculators available nearly everywhere, a fraction with a radical in the denominator still must be rationalized. It is not considered simplified if the denominator contains a radical.

Similarly, a radical expression is not considered simplified if the radicand contains a fraction.

## Simplified Radical Expressions

A radical expression is considered simplified if there are

- no factors in the radicand have perfect powers of the index
- no fractions in the radicand
- no radicals in the denominator of a fraction

To rationalize a denominator with a square root, we use the property that  $(\sqrt{a})^2 = a$ . If we square an irrational square root, we get a rational number.

We will use this property to rationalize the denominator in the next example.

### Example 3.5.4

Simplify:

a.  $\frac{4}{\sqrt{3}}$

b.  $\sqrt{\frac{3}{20}}$

c.  $\frac{3}{\sqrt{6x}}$

#### Solution

a.

To rationalize a denominator with one term, we can multiply a square root by itself. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

**Step 1: Multiply both the numerator and denominator by  $\sqrt{3}$**

$$\frac{4 \cdot \text{rgb}]1.0, 0.0, 0.0\sqrt{3}}{\sqrt{3} \cdot \text{rgb}]1.0, 0.0, 0.0\sqrt{3}}$$

**Step 2: Simplify.**

$$\frac{4 \cdot \text{rgb}]0.1, 0.1, 0.1\sqrt{3}}{3}$$


---

b.

We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

**Step 1: The fraction is not a perfect square, so rewrite using the Quotient Property.**

$$\frac{\sqrt{3}}{\sqrt{20}}$$

**Step 2: Simplify the denominator.**

$$\frac{\sqrt{3}}{2\sqrt{5}}$$

**Step 3: Multiply the numerator and denominator by  $\sqrt{5}$**

$$\frac{\sqrt{3} \cdot \text{rgb}]1.0, 0.0, 0.0\sqrt{5}}{2\sqrt{5} \cdot \text{rgb}]1.0, 0.0, 0.0\sqrt{5}}$$

**Step 4: Simplify.**

$$\frac{\sqrt{15}}{2 \cdot 5}$$

$$\frac{\sqrt{15}}{10}$$


---

c.

**Step 1: Multiply the numerator and denominator by  $\sqrt{6x}$**

$$\frac{3 \cdot rgb]1.0, 0.0, 0.0\sqrt{6x}}{\sqrt{6x} \cdot rgb]1.0, 0.0, 0.0\sqrt{6x}}$$

**Step 2: Simplify.**

$$\frac{3 \, rgb]0.1, 0.1, 0.1\sqrt{6x}}{6x}$$

$$\frac{rgb]0.1, 0.1, 0.1\sqrt{6x}}{2x}$$

## Try It

10) Simplify:

a.  $\frac{5}{\sqrt{3}}$

b.  $\sqrt{\frac{3}{32}}$

c.  $\frac{2}{\sqrt{2x}}$

**Solution**

a.  $\frac{5\sqrt{3}}{3}$

b.  $\frac{\sqrt{6}}{8}$

c.  $\frac{\sqrt{2x}}{x}$

## Try It

11) Simplify:

a.  $\frac{6}{\sqrt{5}}$

b.  $\sqrt{\frac{7}{18}}$

c.  $\frac{5}{\sqrt{5x}}$

### Solution

a.  $\frac{6\sqrt{5}}{5}$

b.  $\frac{\sqrt{14}}{6}$

c.  $\frac{\sqrt{5x}}{x}$

When we rationalized a square root, we multiplied the numerator and denominator by a square root that would give us a perfect square under the radical in the denominator. When we took the square root, the denominator no longer had a radical.

We will follow a similar process to rationalize higher roots. To rationalize a denominator with a higher index radical, we multiply the numerator and denominator by a radical that would give us a radicand that is a perfect power of the index. When we simplify the new radical, the denominator will no longer have a radical.

For example,

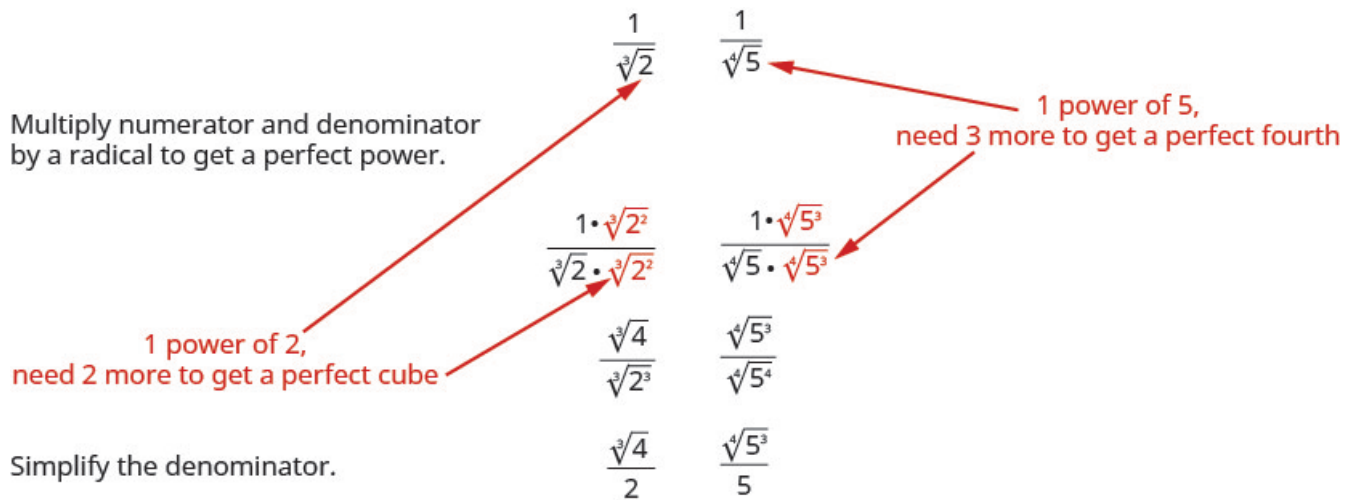


Figure 3.5.1

We will use this technique in the next examples.

### Example 3.5.5

Simplify

a.  $\frac{1}{\sqrt[3]{6}}$

b.  $\sqrt[3]{\frac{7}{24}}$

c.  $\frac{3}{\sqrt[3]{4x}}$

#### Solution

a.

To rationalize a denominator with a cube root, we can multiply by a cube root that will give us a perfect cube in the radicand in the denominator. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

**Step 1: The radical in the denominator has one factor of 6.**

**Step 2: Multiply both the numerator and denominator by  $\sqrt[3]{6^2}$ , which gives us 2 more factors of 6.**

$$\frac{1 \cdot \text{rgb}]1.0, 0.0, 0.0\sqrt[3]{6^2}}{\sqrt[3]{6} \cdot \text{rgb}]1.0, 0.0, 0.0\sqrt[3]{6^2}}$$

**Step 3: Multiply. Notice the radicand in the denominator has 3 powers of 6.**

$$\frac{\sqrt[3]{6^2}}{\sqrt[3]{6^3}}$$

**Step 4: Simplify the cube root in the denominator.**

$$\frac{\sqrt[3]{36}}{6}$$


---

b.

We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

**Step 1: The fraction is not a perfect cube, so rewrite using the Quotient Property.**

$$\frac{\sqrt[3]{7}}{\sqrt[3]{24}}$$

**Step 2: Simplify the denominator.**

$$\frac{\sqrt[3]{7}}{2\sqrt[3]{3}}$$

**Step 3: Multiply the numerator and denominator by  $\sqrt[3]{3^2}$**

This will give us 3 factors of 3.

$$\frac{\sqrt[3]{7} \cdot \text{rgb}]1.0, 0.0, 0.0\sqrt[3]{3^2}}{2\sqrt[3]{3} \cdot \text{rgb}]1.0, 0.0, 0.0\sqrt[3]{3^2}}$$

**Step 4: Simplify.**



Remember,  $\sqrt[3]{3^3} = 3$

$$\frac{\sqrt[3]{63}}{2\sqrt[3]{3^3}}$$

$$\frac{\sqrt[3]{63}}{2 \cdot 3}$$

**Step 5: Simplify.**

$$\frac{\sqrt[3]{63}}{6}$$


---

C.

**Step 1: Rewrite the radicand to show the factors.**

$$\frac{3}{\sqrt[3]{2^2 \cdot x}}$$

**Step 2: Multiply the numerator and denominator by  $\sqrt[3]{2 \cdot x^2}$**

**Step 3: This will get us 3 factors of 2 and 3 factors of  $x$ .**

$$\frac{3 \cdot \sqrt[3]{2 \cdot x^2}}{\sqrt[3]{2^2 x} \cdot \sqrt[3]{2 \cdot x^2}}$$

**Step 4: Simplify.**

$$\frac{3 \sqrt[3]{2x^2}}{\sqrt[3]{2^3 x^3}}$$

**Step 5: Simplify the radical in the denominator.**

$$\frac{3 \sqrt[3]{2x^2}}{2x}$$

## Try It

12) Simplify:

a.  $\frac{1}{\sqrt[3]{7}}$

b.  $\sqrt[3]{\frac{5}{12}}$

c.  $\frac{5}{\sqrt[3]{9y}}$

**Solution**

a.  $\frac{\sqrt[3]{49}}{7}$

b.  $\frac{\sqrt[3]{90}}{6}$

c.  $\frac{5\sqrt[3]{3y^2}}{3y}$

## Try It

13) Simplify:

a.  $\frac{1}{\sqrt[3]{2}}$

$$\text{b. } \frac{\sqrt[3]{\frac{3}{20}}}{2}$$

$$\text{c. } \frac{\sqrt[3]{25n}}{\sqrt[3]{25n}}$$

**Solution**

$$\text{a. } \frac{\sqrt[3]{4}}{2}$$

$$\text{b. } \frac{\sqrt[3]{150}}{10}$$

$$\text{c. } \frac{2\sqrt[3]{5n^2}}{5n}$$

**Example 3.5.6**

Simplify:

$$\text{a. } \frac{1}{\sqrt[4]{2}}$$

$$\text{b. } \frac{\sqrt[4]{5}}{\sqrt[4]{64}}$$

$$\text{c. } \frac{2}{\sqrt[4]{8x}}$$

**Solution**

a.

To rationalize a denominator with a fourth root, we can multiply by a fourth root that will give us a perfect fourth power in the radicand in the denominator. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

**Step 1: The radical in the denominator has one factor of 2.**

**Step 2: Multiply both the numerator and denominator by  $\sqrt[4]{2^3}$ , which gives us 3 more factors of 2.**

$$\frac{1 \cdot rgb]1.0, 0.0, 0.0\sqrt[4]{2^3}}{\sqrt[4]{2} \cdot rgb]1.0, 0.0, 0.0\sqrt[4]{2^3}}$$

**Step 3: Multiply. Notice the radicand in the denominator has 4 powers of 2.**

$$\frac{\sqrt[4]{8}}{\sqrt[4]{2^4}}$$

**Step 4: Simplify the fourth root in the denominator.**

$$\frac{\sqrt[4]{8}}{2}$$


---

b.

We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

**Step 1: The fraction is not a perfect fourth power, so rewrite using the Quotient Property.**

$$\frac{\sqrt[4]{5}}{\sqrt[4]{64}}$$

**Step 2: Rewrite the radicand in the denominator to show the factors.**

$$\frac{\sqrt[4]{5}}{\sqrt[4]{2^6}}$$

**Step 3: Simplify the denominator.**

$$\frac{\sqrt[4]{5}}{2\sqrt[4]{2^2}}$$

**Step 4: Multiply the numerator and denominator by  $\sqrt[4]{2^2}$**

This will give us 4 factors of 2.

$$\frac{\sqrt[4]{5} \cdot \sqrt[4]{2^2}}{2\sqrt[4]{2^2}}$$

**Step 5: Simplify.**

$$\frac{\sqrt[4]{5} \cdot \sqrt[4]{4}}{2\sqrt[4]{2^4}}$$

Remember,  $\sqrt[4]{2^4} = 2$

$$\frac{\sqrt[4]{20}}{2 \cdot 2}$$

**Step 6: Simplify.**

$$\frac{\sqrt[4]{20}}{4}$$

C.

**Step 1: Rewrite the radicand to show the factors.**

$$\frac{2}{\sqrt[4]{2^3 \cdot x}}$$

**Step 2: Multiply the numerator and denominator by  $\sqrt[4]{2 \cdot x^3}$**

This will get us 4 factors of 2 and 4 factors of  $x$ .

$$\frac{2 \cdot \sqrt[4]{2 \cdot x^3}}{\sqrt[4]{2^3 x} \cdot \sqrt[4]{2 \cdot x^3}}$$

**Step 3: Simplify.**

$$\frac{2 \sqrt[4]{2x^3}}{\sqrt[4]{2^4 x^4}}$$

**Step 4: Simplify the radical in the denominator.**

$$\frac{2 \sqrt[4]{2x^3}}{2^4 x^4}$$

**Step 5: Simplify the fraction.**

$$\frac{\sqrt[4]{2x^3}}{x}$$

## Try It

14) Simplify:

a.  $\frac{1}{\sqrt[4]{3}}$

b.  $\sqrt[4]{\frac{3}{64}}$

c.  $\frac{3}{\sqrt[4]{125x}}$

**Solution**

a.  $\frac{\sqrt[4]{27}}{3}$

b.  $\frac{\sqrt[4]{12}}{4}$

c.  $\frac{3\sqrt[4]{5x^3}}{5x}$

## Try It

15) Simplify:

a.  $\frac{1}{\sqrt[4]{5}}$

b.  $\sqrt[4]{\frac{7}{128}}$

c.  $\frac{4}{\sqrt[4]{4x}}$

**Solution**

a.  $\frac{\sqrt[4]{125}}{5}$

b.  $\frac{\sqrt[4]{224}}{8}$

c.  $\frac{\sqrt[4]{64x^3}}{x}$

## Rationalize a Two Term Denominator

When the denominator of a fraction is a sum or difference with square roots, we use the Product of Conjugates Pattern to rationalize the denominator.

$$(a - b)(a + b)$$

$$a^2 - b^2$$

$$(2 - \sqrt{5})(2 + \sqrt{5})$$

$$2^2 - (\sqrt{5})^2$$

$$4 - 5$$

$$-1$$

When we multiply a binomial that includes a square root by its conjugate, the product has no square roots.

### Example 3.5.7

Simplify:  $\frac{5}{2 - \sqrt{3}}$

**Solution**

**Step 1: Multiply the numerator and denominator by the conjugate of the denominator.**

$$\frac{5 (2 + \sqrt{3})}{(2 - \sqrt{3})(2 + \sqrt{3})}$$

**Step 2: Multiply the conjugates in the denominator.**

$$\frac{5 (2 + \sqrt{3})}{2^2 - (\sqrt{3})^2}$$

**Step 3: Simplify the denominator.**

$$\frac{5 (2 + \sqrt{3})}{4 - 3}$$

**Step 4: Simplify the denominator.**

$$\frac{5 (2 + \sqrt{3})}{1}$$

**Step 5: Simplify.**

$$5 (2 + \sqrt{3})$$



## Try It

16) Simplify:  $\frac{3}{1 - \sqrt{5}}$

**Solution**

$$-\frac{3(1 + \sqrt{5})}{4}$$

## Try It

17) Simplify:  $\frac{2}{4 - \sqrt{6}}$

**Solution**

$$\frac{4 + \sqrt{6}}{5}$$

Notice we did not distribute the 5 in the answer of the last example. By leaving the result factored we can see if there are any factors that may be common to both the numerator and denominator.

### Example 3.5.8

Simplify:  $\frac{\sqrt{3}}{\sqrt{u} - \sqrt{6}}$

#### Solution

**Step 1: Multiply the numerator and denominator by the conjugate of the denominator.**

$$\frac{\sqrt{3}(\sqrt{u} + \sqrt{6})}{(\sqrt{u} - \sqrt{6})(\sqrt{u} + \sqrt{6})}$$

**Step 2: Multiply the conjugates in the denominator.**

$$\frac{\sqrt{3}(\sqrt{u} + \sqrt{6})}{(\sqrt{u})^2 - (\sqrt{6})^2}$$

**Step 3: Simplify the denominator.**

$$\frac{\sqrt{3}(\sqrt{u} + \sqrt{6})}{u - 6}$$

### Try It

18) Simplify:  $\frac{\sqrt{5}}{\sqrt{x} + \sqrt{2}}$

#### Solution

$$\frac{\sqrt{5}(\sqrt{x} - \sqrt{2})}{x - 2}$$

## Try It

19) Simplify:  $\frac{\sqrt{10}}{\sqrt{y} - \sqrt{3}}$

### Solution

$$\frac{\sqrt{10}(\sqrt{y} + \sqrt{3})}{y - 3}$$

Be careful of the signs when multiplying. The numerator and denominator look very similar when you multiply by the conjugate.

## Example 3.5.9

Simplify:  $\frac{\sqrt{x} + \sqrt{7}}{\sqrt{x} - \sqrt{7}}$

### Solution

**Step 1:** Multiply the numerator and denominator by the conjugate of the denominator.

$$\frac{(\sqrt{x} + \sqrt{7}) \text{ rgb}[1.0, 0.0, 0.0] (\sqrt{x} + \sqrt{7})}{(\sqrt{x} - \sqrt{7}) \text{ rgb}[1.0, 0.0, 0.0] (\sqrt{x} + \sqrt{7})}$$

**Step 2: Multiply the conjugates in the denominator.**

$$\frac{(\sqrt{x} + \sqrt{7}) (\sqrt{x} + \sqrt{7})}{(\sqrt{x})^2 - (\sqrt{7})^2}$$

**Step 3: Simplify the denominator.**

$$\frac{(\sqrt{x} + \sqrt{7})^2}{x - 7}$$

We do not square the numerator. Leaving it in factored form, we can see there are no common factors to remove from the numerator and denominator.

## Try It

20) Simplify:  $\frac{\sqrt{p} + \sqrt{2}}{\sqrt{p} - \sqrt{2}}$

**Solution**

$$\frac{(\sqrt{p} + \sqrt{2})^2}{p - 2}$$

## Try It

21) Simplify:  $\frac{\sqrt{q} - \sqrt{10}}{\sqrt{q} + \sqrt{10}}$

### Solution

$$\frac{(\sqrt{q} - \sqrt{10})^2}{q - 10}$$

Access these online resources for additional instruction and practice with dividing radical expressions.

- Rationalize the Denominator
- Dividing Radical Expressions and Rationalizing the Denominator
- Simplifying a Radical Expression with a Conjugate
- Rationalize the Denominator of a Radical Expression

## Key Concepts

- **Quotient Property of Radical Expressions**

- If  $\sqrt[n]{a}$  and  $\sqrt[n]{b}$  are real numbers,  $b \neq 0$ , and for any integer  $n \geq 2$  then,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \text{ and } \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

### • Simplified Radical Expressions

- A radical expression is considered simplified if there are:
  - no factors in the radicand that have perfect powers of the index
  - no fractions in the radicand
  - no radicals in the denominator of a fraction

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

*<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=647#h5p-24>*

b) After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

## Glossary

### **rationalizing the denominator**

Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

## 3.6 SOLVE RADICAL EQUATIONS

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### Learning Objectives

By the end of this section, you will be able to:

- Solve radical equations
- Solve radical equations with two radicals
- Use radicals in applications

### Try It

Before you get started, take this readiness quiz:

- 1) Simplify:  $(y - 3)^2$
- 2) Solve:  $2x - 5 = 0$
- 3) Solve  $n^2 - 6n + 8 = 0$

### Solve Radical Equations

In this section we will solve equations that have a variable in the radicand of a radical expression. An equation of this type is called a **radical equation**.



## Radical Equation

An equation in which a variable is in the radicand of a radical expression is called a radical equation.

As usual, when solving these equations, what we do to one side of an equation we must do to the other side as well. Once we isolate the radical, our strategy will be to raise both sides of the equation to the power of the index. This will eliminate the radical.

Solving radical equations containing an even index by raising both sides to the power of the index may introduce an algebraic solution that would not be a solution to the original radical equation. Again, we call this an extraneous solution as we did when we solved rational equations.

In the next example, we will see how to solve a radical equation. Our strategy is based on raising a radical with index  $n$  to the  $n^{\text{th}}$  power. This will eliminate the radical.

$$\text{For } a \geq 0, (\sqrt[n]{a})^n = a$$

### Example 3.6.1

Solve:  $\sqrt{5n-4} - 9 = 0$

#### Solution

**Step 1: Isolate the radical on one side of the equation.**

To isolate the radical, add 9 to both sides.

Simplify.

$$\begin{aligned} \sqrt{5n-4} - 9 &= 0 \\ \sqrt{5n-4} - 9 + 9 &= 0 + 9 \\ \sqrt{5n-4} &= 9 \end{aligned}$$

**Step 2: Raise both sides of the equation to the power of the index.**

Since the index of a square root is 2, we square both sides.

$$(\sqrt{5n - 4})^2 = (9)^2$$

**Step 3: Solve the new equation.**

Remember,  $(\sqrt{a})^2 = a$

$$5n - 4 = 81$$

$$5n = 85$$

$$n = 17$$

**Step 4: Check the answer in the original equation.**

Check the answer.

$$\begin{aligned} \sqrt{5n - 4} - 9 &= 0 \\ \sqrt{5(17) - 4} - 9 &\stackrel{?}{=} 0 \\ \sqrt{85 - 4} - 9 &\stackrel{?}{=} 0 \\ \sqrt{81} - 9 &\stackrel{?}{=} 0 \\ 9 - 9 &\stackrel{?}{=} 0 \\ 0 &= 0 \checkmark \end{aligned}$$

The solution is  $n = 17$

## Try It

4) Solve:  $\sqrt{3m + 2} - 5 = 0$

**Solution**

$$m = \frac{23}{3}$$

## Try It

5) Solve:  $\sqrt{10z + 1} - 2 = 0$

**Solution**

$$z = \frac{3}{10}$$

## HOW TO

**Solve a radical equation with one radical.**

1. Isolate the radical on one side of the equation.
2. Raise both sides of the equation to the power of the index.
3. Solve the new equation.
4. Check the answer in the original equation.

When we use a radical sign, it indicates the principal or positive root. If an equation has a radical with an even index equal to a negative number, that equation will have no solution.

## Example 3.6.2

Solve:  $\sqrt{9k - 2} + 1 = 0$

**Solution****Step 1:** To isolate the radical, subtract 1 to both sides.

$$\sqrt{9k - 2} + 1 - 1 = -1 - 1$$

**Step 2:** Simplify.

$$\sqrt{9k - 2} = -1$$

Because the square root is equal to a negative number, the equation has no solution.

**Try It**

6) Solve:  $\sqrt{2r - 3} + 5 = 0$

**Solution**

no solution

**Try It**

7) Solve:  $\sqrt{7s - 3} + 2 = 0$

**Solution**

no solution

If one side of an equation with a square root is a binomial, we use the Product of Binomial Squares Pattern when we square it.

## Binomial Squares

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

Don't forget the middle term!

### Example 3.6.3

Solve:  $\sqrt{p-1} + 1 = p$

#### Solution

**Step 1:** To isolate the radical, subtract 1 from both sides.

$$\sqrt{p-1} + 1 - 1 = p - 1$$

**Step 2:** Simplify.

$$\sqrt{p-1} = p - 1$$

**Step 3:** Square both sides of the equation.

$$(\sqrt{p-1})^2 = (p-1)^2$$

**Step 4:** Simplify, using the Product of Binomial Squares Pattern on the right. Then solve the new equation.

$$p - 1 = p^2 - 2p + 1$$

**Step 5:** It is a quadratic equation, so get zero on one side.

$$0 = p^2 - 3p + 2$$

**Step 6:** Factor the right side.

$$0 = (p-1)(p-2)$$

**Step 7:** Use the Zero Product Property.

$$0 = p - 1 \quad 0 = p - 2$$

**Step 8: Solve each equation.**

$$p = 1 \quad p = 2$$

**Step 9: Check the answers.**

$$\begin{aligned} p &= rgb]1.0, 0.0, 0.01 \\ \sqrt{p-1} + 1 &= p \\ \sqrt{rgb]1.0, 0.0, 0.01 - 1} + 1 &\stackrel{?}{=} rgb]1.0, 0.0, 0.01 \\ \sqrt{0} + 1 &\stackrel{?}{=} rgb]1.0, 0.0, 0.01 \\ 1 &= 1\checkmark \end{aligned}$$

$$\begin{aligned} p &= rgb]1.0, 0.0, 0.02 \\ \sqrt{p-1} + 1 &= p \\ \sqrt{rgb]1.0, 0.0, 0.02 - 1} + 1 &\stackrel{?}{=} rgb]1.0, 0.0, 0.02 \\ \sqrt{1} + 1 &\stackrel{?}{=} rgb]1.0, 0.0, 0.02 \\ 2 &= 2\checkmark \end{aligned}$$

The solutions are  $p = 1$ ,  $p = 2$

## Try It

8) Solve:  $\sqrt{x-2} + 2 = x$

**Solution**

$$x = 2, x = 3$$

## Try It

9) Solve:  $\sqrt{y-5} + 5 = y$

**Solution**

$$y = 5, y = 6$$

When the index of the radical is 3, we cube both sides to remove the radical.

$$(\sqrt[3]{a})^3 = a$$

### Example 3.6.4

Solve:  $\sqrt[3]{5x + 1} + 8 = 4$

#### Solution

**Step 1:** To isolate the radical, subtract 8 from both sides.

$$\sqrt[3]{5x + 1} = -4$$

**Step 2:** Cube both sides of the equation.

$$(\sqrt[3]{5x + 1})^3 = (-4)^3$$

**Step 3:** Simplify.

$$5x + 1 = -64$$

**Step 4:** Solve the equation.

$$\begin{aligned} 5x &= -65 \\ x &= -13 \end{aligned}$$

**Step 5:** Check the answer.

$$\begin{aligned} x &= \text{rgb}[1.0, 0.0, 0.0] - 13 \\ \sqrt[3]{5x + 1} + 8 &= 4 \\ \sqrt[3]{5\text{rgb}[1.0, 0.0, 0.0](-13) + 1} + 8 &\stackrel{?}{=} 4 \\ \sqrt[3]{-64} + 8 &\stackrel{?}{=} 4 \\ -4 + 8 &\stackrel{?}{=} 4 \\ 4 &= 4\checkmark \end{aligned}$$

The solution is  $x = -13$ .

### Try It

10) Solve:  $\sqrt[3]{4x - 3} + 8 = 5$

**Solution**

$x = -6$

### Try It

11) Solve:  $\sqrt[3]{6x - 10} + 1 = -3$

**Solution**

$x = -9$

Sometimes an equation will contain rational exponents instead of a radical. We use the same techniques to solve the equation as when we have a radical. We raise each side of the equation to the power of the denominator of the rational exponent. Since  $(a^m)^n = a^{m \cdot n}$ , we have for example,

$$\left(x^{\frac{1}{2}}\right)^2 = x, \quad \left(x^{\frac{1}{3}}\right)^3 = x$$

Remember,  $x^{\frac{1}{2}} = \sqrt{x}$  and  $x^{\frac{1}{3}} = \sqrt[3]{x}$



### Example 3.6.5

Solve:  $(3x - 2)^{\frac{1}{4}} + 3 = 5$

#### Solution

**Step 1:** To isolate the term with the rational exponent, subtract 3 from both sides.

$$(3x - 2)^{\frac{1}{4}} = 2$$

**Step 2:** Raise each side of the equation to the fourth power.

$$\left( (3x - 2)^{\frac{1}{4}} \right)^4 = (2)^4$$

**Step 3:** Simplify.

$$3x - 2 = 16$$

**Step 4:** Solve the equation.

$$3x = 18$$

$$x = 6$$

**Step 5:** Check the answer.

$$x = \text{rgb}[1.0, 0.0, 0.06]$$

$$(3x - 2)^{\frac{1}{4}} + 3 = 5$$

$$(3 \cdot \text{rgb}[1.0, 0.0, 0.06] - 2)^{\frac{1}{4}} + 3 \stackrel{?}{=} 5$$

$$(16)^{\frac{1}{4}} + 3 \stackrel{?}{=} 5$$

$$2 + 3 \stackrel{?}{=} 5$$

$$5 = 5 \checkmark$$

The solution is  $x = 6$ .

**Try It**

12) Solve:  $(9x + 9)^{\frac{1}{4}} - 2 = 1$

**Solution**

$x = 8$

**Try It**

13) Solve:  $(4x - 8)^{\frac{1}{4}} + 5 = 7$

**Solution**

$x = 6$

Sometimes the solution of a radical equation results in two algebraic solutions, but one of them may be an extraneous solution!

**Example 3.6.6**

Solve:  $\sqrt{r + 4} - r + 2 = 0$

**Solution**

**Step 1: Isolate the radical.**

$$\sqrt{r+4} = r - 2$$

**Step 2: Square both sides of the equation.**

$$(\sqrt{r+4})^2 = (r-2)^2$$

**Step 3: Simplify and then solve the equation**

$$r + 4 = r^2 - 4r + 4$$

**Step 4: It is a quadratic equation, so get zero on one side.**

$$0 = r^2 - 5r$$

**Step 5: Factor the right side.**

$$0 = r(r - 5)$$

**Step 6: Use the Zero Product Property.**

$$0 = r \quad 0 = r - 5$$

**Step 7: Solve the equation.**

$$r = 0 \quad r = 5$$

**Step 8: Check your answer.**

$$\begin{array}{l} r = \text{rgb}[1.0, 0.0, 0.00] \\ \sqrt{r+4} - r + 2 = 0 \\ \sqrt{\text{rgb}[1.0, 0.0, 0.00] + 4} - \text{rgb}[1.0, 0.0, 0.00] + 2 \stackrel{?}{=} 0 \\ \sqrt{4} + 2 \stackrel{?}{=} 0 \\ 4 \neq 0 \end{array}$$

$$\begin{array}{l} r = \text{rgb}[1.0, 0.0, 0.05] \\ \sqrt{r+4} - r + 2 = 0 \\ \sqrt{\text{rgb}[1.0, 0.0, 0.05] + 4} - \text{rgb}[1.0, 0.0, 0.05] + 2 \stackrel{?}{=} 0 \\ \sqrt{9} - 3 \stackrel{?}{=} 0 \\ 0 = 0 \checkmark \end{array}$$

The solution is  $r = 5$

$r = 0$  is an extraneous solution.

## Try It

14) Solve:  $\sqrt{m+9} - m + 3 = 0$

**Solution**

$$m = 7$$

## Try It

15) Solve:  $\sqrt{n+1} - n + 1 = 0$

**Solution**

$$n = 3$$

When there is a coefficient in front of the radical, we must raise it to the power of the index, too.

## Example 3.6.7

Solve:  $3\sqrt{3x-5} - 8 = 4$

**Solution**

**Step 1: Isolate the radical term.**

$$3\sqrt{3x - 5} = 12$$

**Step 2: Isolate the radical by dividing both sides by 3.**

$$\sqrt{3x - 5} = 4$$

**Step 3: Square both sides of the equation.**

$$(\sqrt{3x - 5})^2 = (4)^2$$

**Step 4: Simplify, then solve the new equation.**

$$3x - 5 = 16$$

$$3x = 21$$

**Step 5: Solve the equation.**

$$x = 7$$

**Step 6: Check the answer.**

$$x = \text{rgb}[1.0, 0.0, 0.07]$$

$$3\sqrt{3x - 5} - 8 = 4$$

$$3\sqrt{\text{rgb}[1.0, 0.0, 0.0](7) - 5} - 8 \stackrel{?}{=} 4$$

$$3\sqrt{21 - 5} - 8 \stackrel{?}{=} 4$$

$$3\sqrt{16} - 8 \stackrel{?}{=} 4$$

$$3(4) - 8 \stackrel{?}{=} 4$$

$$4 = 4\checkmark$$

The solution is  $x = 7$ .

## Try It

16) Solve:  $2\sqrt{4a + 4} - 16 = 16$

**Solution**

$$a = 63$$

## Try It

17) Solve:  $3\sqrt{2b+3} - 25 = 50$

**Solution**

$$b = 311$$

## Solve Radical Equations with Two Radicals

If the radical equation has two radicals, we start out by isolating one of them. It often works out easiest to isolate the more complicated radical first.

In the next example, when one radical is isolated, the second radical is also isolated.

### Example 3.6.8

Solve:  $\sqrt[3]{4x-3} = \sqrt[3]{3x+2}$

**Solution**

**Step 1: The radical terms are isolated.**

$$\sqrt[3]{4x-3} = \sqrt[3]{3x+2}$$

**Step 2: Since the index is 3, cube both sides of the equation.**

$$\left(\sqrt[3]{4x - 3}\right)^3 = \left(\sqrt[3]{3x + 2}\right)^3$$

**Step 3: Simplify, then solve the new equation.**

$$4x - 3 = 3x + 2$$

$$x - 3 = 2$$

$$x = 5$$

The solution is  $x = 5$ .

**Step 4: Check the answer.**

We leave it to you to show that 5 checks!

## Try It

18) Solve:  $\sqrt[3]{5x - 4} = \sqrt[3]{2x + 5}$

**Solution**

$$x = 3$$

## Try It

19) Solve:  $\sqrt[3]{7x + 1} = \sqrt[3]{2x - 5}$

**Solution**

$$x = -\frac{6}{5}$$

Sometimes after raising both sides of an equation to a power, we still have a variable inside a radical. When that happens, we repeat Step 1 and Step 2 of our procedure. We isolate the radical and raise both sides of the equation to the power of the index again.

### Example 3.6.9

Solve:  $\sqrt{m} + 1 = \sqrt{m + 9}$

#### Solution

**Step 1: Isolate one of the radical terms on one side of the equation.**

The radical on the right is isolated.

$$\sqrt{m} + 1 = \sqrt{m + 9}$$

**Step 2: Raise both sides of the equation to the power of the index.**

We square both sides.

$$(\sqrt{m} + 1)^2 = (\sqrt{m + 9})^2$$

Simplify — be very careful as you multiply!

**Step 3: Are there any more radicals?**

If yes, repeat Step 1 and Step 2 again. If no, solve the new equation.

There is still a radical in the equation. So we must repeat the previous steps. Isolate the radical term.

$$m + 2\sqrt{m} + 1 = m + 9$$

Here, we can easily isolate the radical by dividing both sides by 2.

$$2\sqrt{m} = 8$$

$$\sqrt{m} = 4$$

Square both sides.

$$(\sqrt{m})^2 = (4)^2$$

$$m = 16$$

**Step 4: Check the answer in the original equation.**



$$\begin{aligned}\sqrt{rgb]1.0, 0.0, 0.0m} + 1 &= \sqrt{rgb]1.0, 0.0, 0.0m} + 9 \\ \sqrt{rgb]1.0, 0.0, 0.016} + 1 &\stackrel{?}{=} \sqrt{rgb]1.0, 0.0, 0.016} + 9 \\ 4 + 1 &\stackrel{?}{=} 5 \\ 5 &= 5\checkmark\end{aligned}$$

The solution is  $m = 16$

### Try It

20) Solve:  $3 - \sqrt{x} = \sqrt{x - 3}$

**Solution**

$$x = 4$$

### Try It

21) Solve:  $\sqrt{x} + 2 = \sqrt{x + 16}$

**Solution**

$$x = 9$$

We summarize the steps here. We have adjusted our previous steps to include more than one radical in the equation. This procedure will now work for any radical equations.

## HOW TO

### Solve a radical equation.

1. Isolate one of the radical terms on one side of the equation.
2. Raise both sides of the equation to the power of the index.
3. Are there any more radicals?
  - If yes, repeat Step 1 and Step 2 again.
  - If no, solve the new equation.
4. Check the answer in the original equation.

Be careful as you square binomials in the next example. Remember the pattern is  $(a + b)^2 = a^2 + 2ab + b^2$  or  $(a - b)^2 = a^2 - 2ab + b^2$

### Example 3.6.10

Solve:  $\sqrt{q-2} + 3 = \sqrt{4q+1}$

#### Solution

**Step 1: The radical on the right is isolated. Square both sides.**

$$(\sqrt{q-2} + 3)^2 = (\sqrt{4q+1})^2$$

**Step 2: Simplify.**

$$q - 2 + 6\sqrt{q-2} + 9 = 4q + 1$$

**Step 3: There is still a radical in the equation so we must repeat the previous steps. Isolate the radical.**

$$6\sqrt{q-2} = 3q - 6$$

**Step 4: Square both sides. It would not help to divide both sides by 6. Remember to square both the 6 and the  $\sqrt{q-2}$**

$$(6\sqrt{q-2})^2 = \left( \begin{array}{ccc} rgb[1.0, 0.0, 0.0]a & rgb[1.0, 0.0, 0.0]- & rgb[1.0, 0.0, 0.0]b \\ 3q & - & 6 \end{array} \right)^2$$

$$6^2 (\sqrt{q-2})^2 = \begin{array}{ccc} rgb[1.0, 0.0, 0.0]a^2 & rgb[1.0, 0.0, 0.0]- & rgb[1.0, 0.0, 0.0]2ab & rgb[1.0, 0.0, 0.0]+ & rgb[1.0, 0.0, 0.0]b^2 \\ (3q)^2 & - & 2 \cdot 3q \cdot 6 & + & 6^2 \end{array}$$

**Step 5: Simplify, then solve the new equation.**

$$36(q-2) = 9q^2 - 36q + 36$$

**Step 6: Distribute.**

$$36q - 72 = 9q^2 - 36q + 36$$

**Step 7: It is a quadratic equation, so get zero on one side.**

$$0 = 9q^2 - 72q + 108$$

**Step 8: Factor the right side.**

$$0 = 9(q^2 - 8q + 12)$$

$$0 = 9(q-6)(q-2)$$

**Step 9: Use the Zero Product Property.**

$$\begin{array}{ll} q-6=0 & q-2=0 \\ q=6 & q=2 \end{array}$$

**Step 10: The checks are left to you.**

The solutions are  $q = 6$  and  $q = 2$

## Try It

22) Solve:  $\sqrt{x-1} + 2 = \sqrt{2x+6}$

**Solution**

$$x = 5$$

## Try It

23) Solve:  $\sqrt{x} + 2 = \sqrt{3x + 4}$

**Solution**

$$x = 0 \quad x = 4$$

## Use Radicals in Applications

As you progress through your college courses, you'll encounter formulas that include radicals in many disciplines. We will modify our Problem Solving Strategy for Geometry Applications slightly to give us a plan for solving applications with formulas from any discipline.

### HOW TO

#### Use a problem solving strategy for applications with formulas.

1. Read the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
2. Identify what we are looking for.
3. Name what we are looking for by choosing a variable to represent it.
4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. Solve the equation using good algebra techniques.

6. Check the answer in the problem and make sure it makes sense.
7. Answer the question with a complete sentence.

One application of radicals has to do with the effect of gravity on falling objects. The formula allows us to determine how long it will take a fallen object to hit the ground.

## Falling Objects

On Earth, if an object is dropped from a height of  $h$  feet, the time in seconds it will take to reach the ground is found by using the formula

$$t = \frac{\sqrt{h}}{4}$$

For example, if an object is dropped from a height of **64** feet, we can find the time it takes to reach the ground by substituting  $h = 64$  into the formula.

$$t = \frac{\sqrt{h}}{4}$$

$$t = \frac{\sqrt{r gb]1.0, 0.0, 0.064}}{4}$$

Take the square root of 64.  $t = \frac{8}{4}$

Simplify the fraction.  $t = 2$

It would take **2** seconds for an object dropped from a height of **64** feet to reach the ground.

### Example 3.6.11

Marissa dropped her sunglasses from a bridge 400 feet above a river. Use the formula  $t = \frac{\sqrt{h}}{4}$  to find how many seconds it took for the sunglasses to reach the river.

#### Solution

**Step 1: Read the problem.**

**Step 2: Identify what we are looking for.**

The time it takes for the sunglasses to reach the river.

**Step 3: Name what we are looking.**

Let  $t =$  time.

**Step 4: Translate into an equation by writing the appropriate formula. Substitute in the given information.**

$$t = \frac{\sqrt{h}}{4}, \text{ and } h = 400$$

$$t = \frac{\sqrt{rgh}]{1.0, 0.0, 0.0400}}{4}$$

**Step 5: Solve the equation.**

$$t = \frac{20}{4}$$

$$t = 5$$

**Step 6: Check the answer in the problem and make sure it makes sense.**

$$5 \stackrel{?}{=} \frac{\sqrt{400}}{4}$$

$$5 \stackrel{?}{=} \frac{20}{4}$$

$$5 = 5 \checkmark$$

Does 5 seconds seem like a reasonable length of time?

Yes.

**Step 7: Answer the question.**

It will take **5** seconds for the sunglasses to reach the river.

## Try It

24) A helicopter dropped a rescue package from a height of **1,296** feet. Use the formula

$t = \frac{\sqrt{h}}{4}$  to find how many seconds it took for the package to reach the ground.

**Solution**

**9** seconds.

## Try It

25) A window washer dropped a squeegee from a platform **196** feet above the sidewalk. Use the

formula  $t = \frac{\sqrt{h}}{4}$  to find how many seconds it took for the squeegee to reach the sidewalk.

**Solution**

**3.5** seconds.

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the speed, in miles per hour, a car was going before applying the brakes.

## Skid Marks and Speed of a Car

If the length of the skid marks is  $d$  feet, then the speed,  $s$ , of the car before the brakes were applied can be found by using the formula

$$s = \sqrt{24d}$$

### Example 3.6.12

After a car accident, the skid marks for one car measured 190 feet. Use the formula  $s = \sqrt{24d}$  to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

#### Solution

**Step 1: Read the problem**

**Step 2: Identify what we are looking for.**

The speed of a car.

**Step 3: Name what we are looking for.**

Let  $s$  = the speed.

**Step 4: Translate into an equation by writing the appropriate formula. Substitute in the given information.**

$$s = \sqrt{24d}, \text{ and } d = 190$$

$$s = \sqrt{24(190)}$$

**Step 5: Solve the equation.**

$$s = \sqrt{4560}$$

$$s = 67.52777$$

Round to 1 decimal place.



$$s \approx 67.5$$

$$67.5 \stackrel{?}{\approx} \sqrt{24(190)}$$

$$67.5 \stackrel{?}{\approx} \sqrt{4560}$$

$$67.5 \approx 67.5277\checkmark$$

The speed of the car before the brakes were applied was **67.5** miles per hour.

### Try It

26) An accident investigator measured the skid marks of the car. The length of the skid marks was **76** feet. Use the formula  $s = \sqrt{24d}$  to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

#### **Solution**

**42.7** feet.

### Try It

27) The skid marks of a vehicle involved in an accident were **122** feet long. Use the formula  $s = \sqrt{24d}$  to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

#### **Solution**

**54.1** feet.

Access these online resources for additional instruction and practice with solving radical equations.

- Solving an Equation Involving a Single Radical
- Solving Equations with Radicals and Rational Exponents
- Solving Radical Equations
- Solve Radical Equations
- Radical Equation Application

## Key Concepts

- **Binomial Squares**

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

- **Solve a Radical Equation**

1. Isolate one of the radical terms on one side of the equation.
2. Raise both sides of the equation to the power of the index.
3. Are there any more radicals?
  - If yes, repeat Step 1 and Step 2 again.
  - If no, solve the new equation.
4. Check the answer in the original equation.

- **Problem Solving Strategy for Applications with Formulas**

1. Read the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
2. Identify what we are looking for.
3. Name what we are looking for by choosing a variable to represent it.

4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. Solve the equation using good algebra techniques.
6. Check the answer in the problem and make sure it makes sense.
7. Answer the question with a complete sentence.

- **Falling Objects**

- On Earth, if an object is dropped from a height of  $h$  feet, the time in seconds it will take to reach the ground is found by using the formula  $t = \frac{\sqrt{h}}{4}$

- **Skid Marks and Speed of a Car**

- If the length of the skid marks is  $d$  feet, then the speed,  $s$ , of the car before the brakes were applied can be found by using the formula  $s = \sqrt{24d}$

## Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=697#h5p-25>

b. After reviewing this checklist, what will you do to become confident for all objectives?

## Glossary

### **radical equation**

An equation in which a variable is in the radicand of a radical expression is called a radical equation.

## 3.7 USE THE COMPLEX NUMBER SYSTEM

### Learning Objectives

By the end of this section, you will be able to:

- Evaluate the square root of a negative number
- Add and subtract complex numbers
- Multiply complex numbers
- Divide complex numbers
- Simplify powers of  $i$

### Try It

Before you get started, take this readiness quiz:

1) Given the numbers  $-4$ ,  $-\sqrt{7}$ ,  $0$ ,  $\bar{5}$ ,  $\frac{7}{3}$ ,  $3$ ,  $\sqrt{81}$ , list the a) rational numbers, b)

irrational numbers, c) real numbers.

2) Multiply:  $(x - 3)(2x + 5)$

3) Rationalize the denominator:  $\frac{\sqrt{5}}{\sqrt{5} - \sqrt{3}}$

## Evaluate the Square Root of a Negative Number

Whenever we have a situation where we have a square root of a negative number we say there is no real number that equals that square root. For example, to simplify  $\sqrt{-1}$ , we are looking for a real number  $x$  so that  $x^2 = -1$ . Since all real numbers squared are positive numbers, there is no real number that equals  $-1$  when squared.

Mathematicians have often expanded their numbers systems as needed. They added  $0$  to the counting numbers to get the whole numbers. When they needed negative balances, they added negative numbers to get the integers. When they needed the idea of parts of a whole they added fractions and got the rational numbers. Adding the irrational numbers allowed numbers like  $\sqrt{5}$ . All of these together gave us the real numbers and so far in your study of mathematics, that has been sufficient.

But now we will expand the real numbers to include the square roots of negative numbers. We start by defining the **imaginary unit**  $i$  as the number whose square is  $-1$ .

### Imaginary Unit

The imaginary unit  $i$  is the number whose square is  $-1$ .

$$i^2 = -1 \text{ or } i = \sqrt{-1}$$

We will use the imaginary unit to simplify the square roots of negative numbers.

### Square Root of a Negative Number

If  $b$  is a positive real number, then

$$\sqrt{-b} = \sqrt{b} i$$

We will use this definition in the next example. Be careful that it is clear that the  $i$  is not under the radical. Sometimes you will see this written as  $\sqrt{-b} = i\sqrt{b}$  to emphasize the  $i$  is not under the radical. But the  $\sqrt{-b} = \sqrt{b} i$  is considered **standard form**.

### Example 3.7.1

Write each expression in terms of  $i$  and simplify if possible:

a.  $\sqrt{-25}$

b.  $\sqrt{-7}$

c.  $\sqrt{-12}$

#### Solution

a.

**Step 1: Use the definition of the square root of negative numbers.**

$$\sqrt{25} i$$

**Step 2: Simplify.**

$$5i$$


---

b.

**Step 1: Use the definition of the square root of negative numbers.**

$$\sqrt{7} i$$

**Step 2: Simplify.**

Be careful that it is clear that  $i$  is not under the radical sign.

---

c.

**Step 1: Use the definition of the square root of negative numbers.**

$$\sqrt{12} i$$

**Step 2: Simplify  $\sqrt{12}$ .**

$$2\sqrt{3} i$$

## Try It

4) Write each expression in terms of  $i$  and simplify if possible:

a.  $\sqrt{-81}$

b.  $\sqrt{-5}$

c.  $\sqrt{-18}$

### Solution

a.  $9i$

b.  $\sqrt{5}i$

c.  $3\sqrt{2}i$

## Try It

5) Write each expression in terms of  $i$  and simplify if possible:

a.  $\sqrt{-36}$

b.  $\sqrt{-3}$

c.  $\sqrt{-27}$

### Solution

a.  $6i$

b.  $\sqrt{3}i$

c.  $3\sqrt{3}i$

Now that we are familiar with the imaginary number  $i$ , we can expand the real numbers to include imaginary



numbers. The **complex number system** includes the real numbers and the imaginary numbers. A **complex number** is of the form  $a + bi$ , where  $a, b$  are real numbers. We call  $a$  the real part and  $b$  the imaginary part.

## Complex Number

A complex number is of the form  $a + bi$ , where  $a$  and  $b$  are real numbers.

$$\begin{array}{c} a + bi \\ \text{Real part: } a \\ \text{Imaginary part: } b \end{array}$$

A complex number is in standard form when written as  $a + bi$ , where  $a$  and  $b$  are real numbers.

If  $b = 0$ , then  $a + bi$  becomes  $a + 0 \times i = a$ , and is a real number.

If  $b \neq 0$ , then  $a + bi$  is an imaginary number.

If  $a = 0$ , then  $a + bi$  becomes  $0 + bi = bi$ , and is called a pure imaginary number.

We summarize this here.

$a + bi$		
$b = 0$	$a + 0 \times i = a$	Real number
$b \neq 0$	$a + bi$	Imaginary number
$a = 0$	$0 + bi = bi$	Pure imaginary number

The standard form of a complex number is  $a + bi$ , so this explains why the preferred form is  $\sqrt{-b} = \sqrt{b}i$  when  $b > 0$

The diagram helps us visualize the complex number system. It is made up of both the real numbers and the imaginary numbers.

$$\begin{array}{c} \text{Real Numbers} \\ \text{Complex Numbers} \\ \text{Imaginary Numbers} \end{array}$$

## Add or Subtract Complex Numbers

We are now ready to perform the operations of addition, subtraction, multiplication and division on the complex numbers—just as we did with the real numbers.

Adding and subtracting complex numbers is much like adding or subtracting like terms. We add or subtract the real parts and then add or subtract the imaginary parts. Our final result should be in standard form.

### Example 3.7.2

Add:  $\sqrt{-12} + \sqrt{-27}$

**Solution**

**Step 1: Use the definition of the square root of negative numbers.**

$$\sqrt{12}i + \sqrt{27}i$$

**Step 2: Simplify the square roots.**

$$2\sqrt{3}i + 3\sqrt{3}i$$

**Step 3: Add.**

$$5\sqrt{3}i$$

### Try It

6) Add:  $\sqrt{-8} + \sqrt{-32}$

**Solution**

$$6\sqrt{2}i$$

## Try It

7) Add:  $\sqrt{-27} + \sqrt{-48}$

**Solution**

$$7\sqrt{3}i$$

Remember to add both the real parts and the imaginary parts in this next example.

## Example 3.7.3

Simplify:

a.  $(4 - 3i) + (5 + 6i)$

b.  $(2 - 5i) - (5 - 2i)$

**Solution**

a.

**Step 1: Use the Associative Property to put the real parts and the imaginary parts together.**

$$(4 + 5) + (-3i + 6i)$$

**Step 2: Simplify.**

$$9 + 3i$$

b.

**Step 1: Distribute.**

$$2 - 5i - 5 + 2i$$

**Step 2: Use the Associative Property to put the real parts and the imaginary parts together.**

$$2 - 5 - 5i + 2i$$

**Step 3: Simplify.**

$$-3 - 3i$$

## Try It

8) Simplify:

a.  $(2 + 7i) + (4 - 2i)$

b.  $(8 - 4i) - (2 - i)$

**Solution**

a.  $6 + 5i$

b.  $6 - 3i$

## Try It

9) Simplify:

a.  $(3 - 2i) + (-5 - 4i)$

b.  $(4 + 3i) - (2 - 6i)$

**Solution**

a.  $-2 - 6i$

b.  $2 + 9i$

## Multiply Complex Numbers

Multiplying complex numbers is also much like multiplying expressions with coefficients and variables. There is only one special case we need to consider. We will look at that after we practice in the next two examples.

### Example 3.7.4

Multiply:  $2i(7 - 5i)$

**Solution**

**Step 1: Distribute.**

$$14i - 10i^2$$

**Step 2: Simplify  $i^2$ .**

$$14i - 10(-1)$$

**Step 3: Multiply.**

$$14i + 10$$

**Step 4: Write in standard form.**

$$10 + 14i$$

## Try It

10) Multiply:  $4i(5 - 3i)$

**Solution**

$$12 + 20i$$

## Try It

11) Multiply:  $-3i(2 + 4i)$

**Solution**

$$12 + 6i$$

In the next example, we multiply the binomials using the Distributive Property or FOIL.

## Example 3.7.5

Multiply:  $(3 + 2i)(4 - 3i)$

**Solution**

**Step 1: FOIL.**

$$12 - 9i + 8i - 6i^2$$

**Step 2: Simplify  $i^2$  and combine like terms.**

$$12 - i - 6(-1)$$

**Step 3: Multiply.**

$$12 - i + 6$$

**Step 4: Combine the real parts.**

$$18 - i$$

## Try It

12) Multiply:  $(5 - 3i)(-1 - 2i)$

**Solution**

$$-11 - 7i$$

## Try It

13) Multiply:  $(-4 - 3i)(2 + i)$

**Solution**

$$-5 - 10i$$

In the next example, we could use FOIL or the Product of Binomial Squares Pattern.

### Example 3.7.6

Multiply:  $(3 + 2i)^2$

**Solution**

$$\left( \begin{array}{c} rgb]1.0, 0.0, 0.0argb]1.0, 0.0, 0.0+rgb]1.0, 0.0, 0.0b \\ 3 + 2i \end{array} \right)^2$$

**Step 1: Use the Product of Binomial Squares Pattern,  $(a + b)^2 = a^2 + 2ab + b^2$**

$$rgb]1.0, 0.0, 0.0a^{rgb]1.0, 0.0, 0.02} rgb]1.0, 0.0, 0.0+rgb]1.0, 0.0, 0.02rgb]1.0, 0.0, 0.0 \times rgb]1.0, 0.0, 0.0argb]1.0, 0.0, 0.0 \times rgb]1.0, 0.0, 0.0brgb]1.0, 0.0, 0.0+rgb]1.0, 0.0, 0.0b^{rgb]1.0, 0.0, 0.02}$$

$$3^2 + 2 \times 3 \times 2i + (2i)^2$$

**Step 2: Simplify.**

$$9 + 12i + 4i^2$$

**Step 3: Simplify  $i^2$**

$$9 + 12i + 4(-1)$$

**Step 4: Simplify.**

$$5 + 12i$$

### Try It

14) Multiply using the Binomial Squares pattern:  $(-2 - 5i)^2$

**Solution**

$$-21 - 20i$$



## Try It

15) Multiply using the Binomial Squares pattern:  $(-5 + 4i)^2$

### Solution

$$9 - 40i$$

Since the square root of a negative number is not a real number, we cannot use the Product Property for Radicals. In order to multiply square roots of negative numbers we should first write them as complex numbers, using  $\sqrt{-b} = \sqrt{b}i$ . This is one place students tend to make errors, so be careful when you see multiplying with a negative square root.

## Example 3.7.7

Multiply:  $\sqrt{-36} \cdot \sqrt{-4}$

### Solution

To multiply square roots of negative numbers, we first write them as complex numbers.

**Step 1: Write as complex numbers using  $\sqrt{-b} = \sqrt{b}i$ .**

$$\sqrt{36}i \cdot \sqrt{4}i$$

**Step 2: Simplify.**

$$6i \cdot 2i$$

**Step 3: Multiply.**

$$12i^2$$

**Step 4: Simplify  $i^2$  and multiply.**

$-12$ **Try It**16) Multiply:  $\sqrt{-49} \cdot \sqrt{-4}$ **Solution** $-14$ **Try It**17) Multiply:  $\sqrt{-36} \cdot \sqrt{-81}$ **Solution** $-54$ 

In the next example, each binomial has a square root of a negative number. Before multiplying, each square root of a negative number must be written as a complex number.

### Example 3.7.8

Multiply:  $(3 - \sqrt{-12})(5 + \sqrt{-27})$

**Solution**

To multiply square roots of negative numbers, we first write them as complex numbers.

**Step 1: Write as complex numbers using  $\sqrt{-b} = \sqrt{b}i$ .**

$$(3 - 2\sqrt{3}i)(5 + 3\sqrt{3}i)$$

**Step 2: Use FOIL.**

$$15 + 9\sqrt{3}i - 10\sqrt{3}i - 6 \cdot 3i^2$$

**Step 3: Combine like terms and simplify  $i^2$ .**

$$15 - \sqrt{3}i - 6 \cdot (-3)$$

**Step 4: Multiply and combine like terms.**

$$33 - \sqrt{3}i$$

### Try It

18) Multiply:  $(4 - \sqrt{-12})(3 - \sqrt{-48})$

**Solution**

$$-12 - 22\sqrt{3}i$$

## Try It

19) Multiply:  $(-2 + \sqrt{-8})(3 - \sqrt{-18})$

### Solution

$$6 + 12\sqrt{2}i$$

We first looked at conjugate pairs when we studied polynomials. We said that a pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference is called a *conjugate pair* and is of the form  $(a - b)$ ,  $(a + b)$

A **complex conjugate pair** is very similar. For a complex number of the form  $a + bi$ , its conjugate is  $a - bi$ . Notice they have the same first term and the same last term, but one is a sum and one is a difference.

## Complex Conjugate Pair

### Complex Conjugate Pair

A *complex conjugate pair* is of the form  $a + bi$ ,  $a - bi$

We will multiply a complex conjugate pair in the next example.

### Example 3.7.9

Multiply:  $(3 - 2i)(3 + 2i)$

**Solution**

**Step 1: FOIL.**

$$9 + 6i - 6i - 4i^2$$

**Step 2: Combine like terms and simplify  $i^2$ .**

$$9 - 4(-1)$$

**Step 3: Multiply and combine like terms.**

$$13$$

### Try It

20) Multiply:  $(4 - 3i) \cdot (4 + 3i)$

**Solution**

25

## Try It

21) Multiply:  $(-2 + 5i) \cdot (-2 - 5i)$

### Solution

29

From our study of polynomials, we know the product of conjugates is always of the form  $(a - b)(a + b) = a^2 - b^2$ . The result is called a difference of squares. We can multiply a complex conjugate pair using this pattern.

The last example we used FOIL. Now we will use the Product of Conjugates Pattern.

$$\begin{aligned}
 & \left( \begin{matrix} \text{rgb}[1.0,0.0,0.0] \text{argb}[1.0,0.0,0.0] - \text{rgb}[1.0,0.0,0.0] b \\ 3 - 2i \end{matrix} \right) \left( \begin{matrix} \text{rgb}[1.0,0.0,0.0] \text{argb}[1.0,0.0,0.0] + \text{rgb}[1.0,0.0,0.0] b \\ 3 + 2i \end{matrix} \right) \\
 &= \begin{matrix} \text{rgb}[1.0,0.0,0.0] a \text{rgb}[1.0,0.0,0.0] \\ (3)^2 \end{matrix} - \begin{matrix} \text{rgb}[1.0,0.0,0.0] b \text{rgb}[1.0,0.0,0.0] \\ (2i)^2 \end{matrix} \\
 &= 9 - 4i^2 \\
 &= 9 - 4(-1) \\
 &= 13
 \end{aligned}$$

When we multiply complex conjugates, the product of the last terms will always have an  $i^2$  which simplifies to  $-1$

$$\begin{aligned}
 & (a - bi)(a + bi) \\
 &= a^2 - (bi)^2 \\
 &= a^2 - b^2 \cdot i^2 \\
 &= a^2 - b^2(-1) \\
 &= a^2 + b^2
 \end{aligned}$$

This leads us to the Product of Complex Conjugates Pattern:  $(a - bi)(a + bi) = a^2 + b^2$

## Product of Complex Conjugates

If  $a$  and  $b$  are real numbers, then

$$(a - bi)(a + bi) = a^2 + b^2$$

### Example 3.7.10

Multiply using the Product of Complex Conjugates Pattern:  $(8 - 2i)(8 + 2i)$

**Solution**

$$\begin{array}{l} rgb]1.0,0.0,0.0(a-b)rgb]1.0,0.0,0.0(a+b) \\ (8 - 2i)(8 + 2i) \end{array}$$

**Step 1: Use the Product of Complex Conjugates Pattern,  $(a - bi)(a + bi) = a^2 + b^2$**

$$\begin{array}{l} rgb]1.0,0.0,0.0a^2rgb]1.0,0.0,0.02^2+rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.0b^2 \\ 8^2 + 2^2 \end{array}$$

**Step 2: Simplify the squares.**

$$64 + 4$$

**Step 3: Add.**

$$68$$

### Try It

22) Multiply using the Product of Complex Conjugates Pattern:  $(3 - 10i)(3 + 10i)$

**Solution**

109

## Try It

23) Multiply using the Product of Complex Conjugates Pattern:  $(-5 + 4i)(-5 - 4i)$

### Solution

41

## Divide Complex Numbers

Dividing complex numbers is much like rationalizing a denominator. We want our result to be in standard form with no imaginary numbers in the denominator.

### Example 3.7.11

Divide:  $\frac{4 + 3i}{3 - 4i}$

### Solution

**Step 1: Write both the numerator and the denominator in standard form.**

They are both in standard form.

$$\frac{4 + 3i}{3 - 4i}$$

**Step 2: Multiply the numerator and denominator by the complex conjugate of the denominator.**

The complex conjugate of  $3 - 4i$  is  $3 + 4i$



$$\frac{(4 + 3i) \operatorname{rgb}[1.0, 0.0, 0.0] (3 + 4i)}{(3 - 4i) \operatorname{rgb}[1.0, 0.0, 0.0] (3 + 4i)}$$

**Step 3: Simplify and write the result in standard form.**

Use the pattern  $(a - bi)(a + bi) = a^2 + b^2$  in the denominator.

$$\frac{12 + 16i + 9i + 12i^2}{9 + 16}$$

Combine like terms.

$$\frac{12 + 25i - 12}{25}$$

Simplify.

$$\frac{25i}{25}$$

Write the result in standard form.

$$i$$

## Try It

24) Divide:  $\frac{2 + 5i}{5 - 2i}$

**Solution**

$$i$$

## Try It

25) Divide:  $\frac{1 + 6i}{6 - i}$

### Solution

$i$

We summarize the steps here.

## HOW TO

### How to divide complex numbers.

1. Write both the numerator and denominator in standard form.
2. Multiply the numerator and denominator by the complex conjugate of the denominator.
3. Simplify and write the result in standard form.

## Example 3.7.12

Divide, writing the answer in standard form:  $\frac{-3}{5 + 2i}$

### Solution

**Step 1: Multiply the numerator and denominator by the complex conjugate of the denominator.**

$$\frac{-3(5 - 2i)}{(5 + 2i)(5 - 2i)}$$

**Step 2: Multiply in the numerator and use the Product of Complex Conjugated Pattern in the denominator.**

$$\frac{-15 + 6i}{5^2 + 2^2}$$

**Step 3: Simplify.**

$$\frac{-15 + 6i}{29}$$

**Step 4: Write in standard form.**

$$-\frac{15}{29} + \frac{6}{29}i$$

## Try It

26) Divide, writing the answer in standard form:  $\frac{4}{1 - 4i}$

**Solution**

$$\frac{4}{17} + \frac{16}{17}i$$

## Try It

27) Divide, writing the answer in standard form:  $\frac{-2}{-1 + 2i}$

### Solution

$$\frac{2}{5} + \frac{4}{5}i$$

Be careful as you find the conjugate of the denominator.

## Example 3.7.13

Divide:  $\frac{5 + 3i}{4i}$

### Solution

**Step 1: Write the denominator in standard form.**

$$\frac{5 + 3i}{0 + 4i}$$

**Step 2: Multiply the numerator and denominator by the complex conjugate of the denominator.**

$$\frac{(5 + 3i)(0 - 4i)}{(0 + 4i)(0 - 4i)}$$

**Step 3: Simplify.**

$$\frac{(5 + 3i)(-4i)}{(4i)(-4i)}$$

**Step 4: Multiply.**

$$\frac{-20i - 12i^2}{-16i^2}$$

**Step 5: Simplify the  $i^2$ .**

$$\frac{-20i + 12}{16}$$

**Step 6: Rewrite in standard form.**

$$\frac{12}{16} - \frac{20}{16}$$

**Step 7: Simplify the fractions.**

$$\frac{3}{4} - \frac{5}{4}i$$

## Try It

28) Divide:  $\frac{3 + 3i}{2i}$

**Solution**

$$\frac{3}{2} - \frac{3}{2}i$$

## Try It

29) Divide:  $\frac{2 + 4i}{5i}$

### Solution

$$\frac{4}{5} - \frac{2}{5}i$$

## Simplify Powers of $i$

The powers of  $i$  make an interesting pattern that will help us simplify higher powers of  $i$ . Let's evaluate the powers of  $i$  to see the pattern.

$i^1$	$i^2$	$i^3$	$i^4$	$i^5$	$i^6$	$i^7$	$i^8$
$i$	$-1$	$i^2$ $-1 \cdot i$ $-1$	$i^2 \cdot i^2$ $(-1)(-1)$ $1$	$i^4 \cdot i$ $1 \cdot i$ $i$	$i^4 \cdot i^2$ $1 \cdot i^2$ $i^2$ $-1$	$i^4 \cdot i^3$ $1 \cdot i^3$ $i^3$ $-i$	$i^4 \cdot i^4$ $1 \cdot 1$ $1$

We summarize this now.

$$\begin{array}{ll}
 i^1 = i & i^5 = i \\
 i^2 = -1 & i^6 = -1 \\
 i^3 = -i & i^7 = -i \\
 i^4 = 1 & i^8 = 1
 \end{array}$$

If we continued, the pattern would keep repeating in blocks of four. We can use this pattern to help us simplify powers of  $i$ . Since  $i^4 = 1$ , we rewrite each power,  $i^n$ , as a product using  $i^4$  to a power and another power of  $i$ .

We rewrite it in the form  $i^n = (i^4)^q \cdot i^r$ , where the exponent,  $q$ , is the quotient of  $n$  divided by 4 and the exponent,  $r$ , is the remainder from this division. For example, to simplify  $i^{57}$ , we divide 57 by 4 and we

get 14 with a remainder of 1. In other words,  $57 = 4 \cdot 14 + 1$ . So we write  $i^{57} = (i^4)^{14} \cdot i^1$  and then simplify from there.

$$\begin{array}{r} 14 \\ 4 \overline{)57} \\ \underline{4} \\ 17 \\ \underline{16} \\ 1 \end{array} \quad \begin{array}{l} i^{57} \\ (i^4)^{14} \cdot i^1 \\ 1 \cdot i \\ i \end{array}$$

### Example 3.7.14

Simplify:  $i^{86}$

#### Solution

**Step 1: Divide 86 by 4 and rewrite  $i^{86}$  in the  $i^n = (i^4)^q \cdot i^r$  form.**

$$(i^4)^{21} \cdot i^2$$

$$\begin{array}{r} 21 \\ 4 \overline{)86} \\ \underline{8} \\ 6 \\ \underline{4} \\ 2 \end{array}$$

**Step 2: Simplify.**

$$\begin{array}{l} (i^4)^{21} \cdot (-1) \\ -1 \end{array}$$

## Try It

30) Simplify:  $i^{75}$

**Solution**

$-i$

## Try It

31) Simplify:  $i^{92}$

**Solution**

1

Access these online resources for additional instruction and practice with solving radical equations.

- Expressing Square Roots of Negative Numbers with  $i$
- Subtract and Multiply Complex Numbers
- Dividing Complex Numbers
- Rewriting Powers of  $i$



## Key Concepts

- **Square Root of a Negative Number**

- If  $b$  is a positive real number, then  $\sqrt{-b} = \sqrt{b}i$

$a + bi$		
$b = 0$	$a + 0 \times i = a$	Real number
$b \neq 0$	$a + bi$	Imaginary number
$a = 0$	$0 + bi = bi$	Pure imaginary number

- A complex number is in *standard form* when written as  $a + bi$ , where  $a, b$  are real numbers.

$\begin{array}{ccc} \xrightarrow{\boxed{\begin{array}{c} \text{Real Numbers} \\ a+bi \\ ; \\ ; \\ b=0 \end{array}}} & \& \boxed{\begin{array}{c} \text{Complex Numbers} \\ a+bi \\ \end{array}} & \xleftarrow{\boxed{\begin{array}{c} \text{Imaginary Numbers} \\ a+bi \\ ; \\ ; \\ b \neq 0 \end{array}}} \end{array}$

- **Product of Complex Conjugates**

- If  $a, b$  are real numbers, then

$$(a - bi)(a + bi) = a^2 + b^2$$

- **How to Divide Complex Numbers**

1. Write both the numerator and denominator in standard form.
2. Multiply the numerator and denominator by the complex conjugate of the denominator.
3. Simplify and write the result in standard form.

## Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



An interactive H5P element has been excluded from this version of the text. You can view it online here:

<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=740#h5p-26>

b. On a scale of **1** — **10**, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

## Glossary

### complex conjugate pair

A complex conjugate pair is of the form  $a + bi$ ,  $a - bi$ .

### complex number

A complex number is of the form  $a + bi$ , where  $a$  and  $b$  are real numbers. We call  $a$  the real part and  $b$  the imaginary part.

### complex number system

The complex number system is made up of both the real numbers and the imaginary numbers.

### imaginary unit

The imaginary unit  $i$  is the number whose square is  $-1$ .  $i^2 = -1$  or  $i = \sqrt{-1}$

### standard form

A complex number is in standard form when written as  $a + bi$ , where  $a$ ,  $b$  are real numbers.

## 3.8 UNIT 3 SOURCES

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**3.0** “Roots and Radicals Introduction” from Intermediate Algebra by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted.

**3.1** “8.1 Simplify Expressions with Roots” from Intermediate Algebra by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted.

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# UNIT 4: QUADRATIC EQUATIONS AND FUNCTIONS

## Chapter Outline

- 4.0 Introduction
- 4.1 Solve Quadratic Equations Using the Square Root Property
- 4.2 Solve Quadratic Equations by Completing the Square
- 4.3 Solve Quadratic Equations Using the Quadratic Formula
- 4.4 Solve Quadratic Equations in Quadratic Form
- 4.5 Solve Applications of Quadratic Equations
- 4.6 Graph Quadratic Functions Using Properties
- 4.7 Graph Quadratic Functions Using Transformations
- 4.8 Unit 4 Sources



## 4.0 INTRODUCTION

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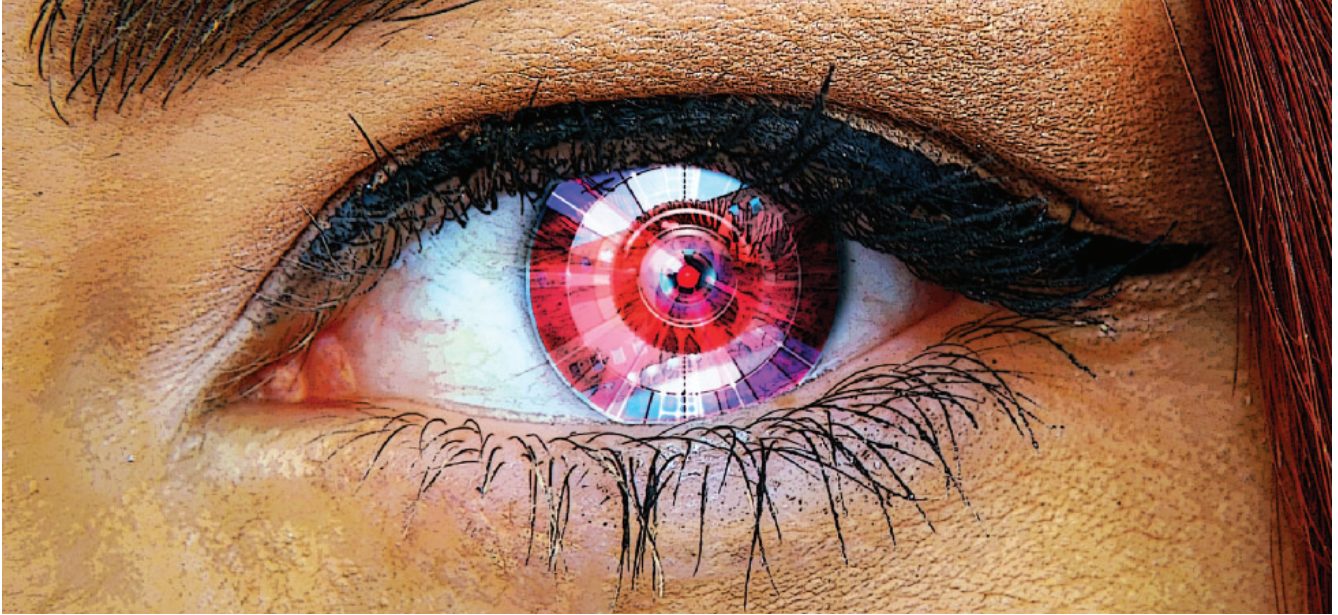


Photo by Open Stax CC BY

Several companies have patented contact lenses equipped with cameras, suggesting that they may be the future of wearable camera technology.

Blink your eyes. You've taken a photo. That's what will happen if you are wearing a contact lens with a built-in camera. Some of the same technology used to help doctors see inside the eye may someday be used to make cameras and other devices. These technologies are being developed by biomedical engineers using many mathematical principles, including an understanding of quadratic equations and functions. In this chapter, you will explore these kinds of equations and learn to solve them in different ways. Then you will solve applications modeled by quadratics, graph them, and extend your understanding to quadratic inequalities.

# 4.1 SOLVE QUADRATIC EQUATIONS USING THE SQUARE ROOT PROPERTY

---

## Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations of the form  $ax^2 = k$  using the Square Root Property
- Solve quadratic equations of the form  $a(x-h)^2 = k$  using the Square Root Property

## Try It

Before you get started, take this readiness quiz:

1) Simplify:  $\sqrt{128}$

2) Simplify:  $\sqrt{\frac{32}{5}}$ .

3) Factor:  $9x^2 - 12x + 4$ .

A quadratic equation is an equation of the form  $ax^2 + bx + c = 0$ , where  $a \neq 0$ . Quadratic equations differ from linear equations by including a quadratic term with the variable raised to the second power of the



form  $ax^2$ . We use different methods to solve quadratic equations than linear equations, because just adding, subtracting, multiplying, and dividing terms will not isolate the variable.

We have seen that some quadratic equations can be solved by factoring. In this chapter, we will learn three other methods to use in case a quadratic equation cannot be factored.

## Solve Quadratic Equations of the form $ax^2 = k$ using the Square Root Property

We have already solved some quadratic equations by factoring. Let's review how we used factoring to solve the quadratic equation  $x^2 = 9$ .

$$\begin{aligned} x^2 &= 9 \\ x^2 - 9 &= 0 \\ (x - 3)(x + 3) &= 0 \end{aligned}$$

Put the equation in standard form.

Factor the difference of squares.

Use the Zero Product Property.

$$x - 3 = 0 \quad x - 3 = 0$$

Solve each equation.

$$x = 3 \quad x = -3$$

We can easily use factoring to find the solutions of similar equations, like  $x^2 = 16$  and  $x^2 = 25$ , because 16 and 25 are perfect squares. In each case, we would get two solutions,  $x = 4$ ,  $x = -4$  and  $x = 5$ ,  $x = -5$ .

But what happens when we have an equation like  $x^2 = 7$ ? Since 7 is not a perfect square, we cannot solve the equation by factoring.

Previously we learned that since 169 is the square of 13, we can also say that 13 is a *square root* of 169. Also,  $(-13)^2 = 169$ , so  $-13$  is also a square root of 169. Therefore, both 13 and  $-13$  are square roots of 169. So, every positive number has two square roots—one positive and one negative. We earlier defined the square root of a number in this way:

If  $n^2 = m$ , then  $n$  is a square root of  $m$ .

Since these equations are all of the form  $x^2 = k$ , the square root definition tells us the solutions are the two square roots of  $k$ . This leads to the Square Root Property.

### Square Root Property

If  $x^2 = k$ , then

$$x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k} \quad \text{or} \quad x = \pm\sqrt{k}.$$

Notice that the Square Root Property gives two solutions to an equation of the form  $x^2 = k$ , the principal square root of  $k$  and its opposite. We could also write the solution as  $x = \pm\sqrt{k}$ . We read this as  $x$  equals positive or negative the square root of  $k$ .

Now we will solve the equation  $x^2 = 9$  again, this time using the Square Root Property.

$$\begin{array}{rcl} x^2 & = & 9 \\ \text{Use the Square Root Property.} & & x = \pm\sqrt{9} \\ & & x = \pm 3 \end{array}$$

$$\text{So } x = 3 \quad \text{or} \quad x = -3.$$

What happens when the constant is not a perfect square? Let's use the Square Root Property to solve the equation  $x^2 = 7$ .

$$\begin{array}{rcl} & & x^2 = 7 \\ \text{Use the Square Root Property.} & & x = \sqrt{7}, \quad x = -\sqrt{7} \\ \text{We cannot simplify } \sqrt{7}, & & \text{so we leave the answer as a radical.} \end{array}$$

### Example 4.1.1

How to solve a Quadratic Equation of the form  $ax^2 = k$  Using the Square Root Property.

Solve:  $x^2 - 50 = 0$ .

#### Solution

**Step 1: Isolate the quadratic term and make its coefficient one.**

Add 50 to both sides to get  $x^2$  by itself.

$$x^2 - 50 = 0$$

$$x^2 = 50$$

**Step 2: Use Square Root Property.**

Remember to write the  $\pm$  symbol.

$$x = \pm\sqrt{50}$$

**Step 3: Simplify the radical.**

Rewrite to show two solutions.

$$x = \pm\sqrt{25} \cdot \sqrt{2}$$

$$x = \pm 5\sqrt{2}$$

$$x = 5\sqrt{2}, x = -5\sqrt{2}$$

**Step 4: Check the solutions.**

Substitute in  $x = 5\sqrt{2}$  and  $x = -5\sqrt{2}$ .

$$x^2 - 50 = 0$$

$$(5\sqrt{2})^2 - 50 \stackrel{?}{=} 0$$

$$25 \times 2 - 50 \stackrel{?}{=} 0$$

$$0 = 0 \checkmark$$

$$x^2 - 50 = 0$$

$$(-5\sqrt{2})^2 - 50 \stackrel{?}{=} 0$$

$$25 \times 2 - 50 \stackrel{?}{=} 0$$

$$0 = 0 \checkmark$$

## Try It

4) Solve:  $x^2 - 48 = 0$ .

**Solution**

$$x = 4\sqrt{3}, x = -4\sqrt{3}$$

## Try It

5) Solve:  $y^2 - 27 = 0$ .

**Solution**

$$y = 3\sqrt{3}, y = -3\sqrt{3}$$

The steps to take to use the Square Root Property to solve a quadratic equation are listed here.

## HOW TO

**Solve a quadratic equation using the square root property.**

1. Isolate the quadratic term and make its coefficient one.
2. Use Square Root Property.
3. Simplify the radical.
4. Check the solutions.

In order to use the Square Root Property, the coefficient of the variable term must equal one. In the next example, we must divide both sides of the equation by the coefficient 3 before using the Square Root Property.

### Example 4.1.2

Solve:  $3z^2 = 108$ .

**Solution**

**Step 1: The quadratic term is isolated.**

$$3z^2 = 108$$

**Step 2: Divide by 3 to make its coefficient 1.**

$$\frac{3z^2}{3} = \frac{108}{3}$$

**Step 3: Simplify.**

$$z^2 = 36$$

**Step 4: Use the Square Root Property.**

$$z = \pm\sqrt{36}$$

**Step 5: Simplify the radical.**

$$z = \pm 6$$

**Step 6: Rewrite to show two solutions.**

$$z = 6, z = -6$$

**Step 7: Check the solutions:**

$$\begin{aligned}
 3rgb]1.0, 0.0, 0.0z^2 &= 108 \\
 3(rgb]1.0, 0.0, 0.06)^2 &= 108 \\
 3(36) &= 108 \\
 108 &= 108 \checkmark
 \end{aligned}$$

$$\begin{aligned}
 3rgb]1.0, 0.0, 0.0z^2 &= 108 \\
 3(rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.06)^2 &= 108 \\
 3(36) &= 108 \\
 108 &= 108 \checkmark
 \end{aligned}$$

### Try It

6) Solve:  $2x^2 = 98$ .

**Solution**

$$x = 7, x = -7$$

### Try It

7) Solve:  $5m^2 = 80$ .

**Solution**

$$m = 4, m = -4$$

The Square Root Property states 'If  $x^2 = k$ ,' What will happen if  $k < 0$ ? This will be the case in the next example.

### Example 4.1.3

Solve:  $x^2 + 72 = 0$ .

**Solution**

**Step 1: Isolate the quadratic term.**

$$x^2 = -72$$

**Step 2: Use the Square Root Property.**

$$x = \pm\sqrt{-72}$$

**Step 3: Simplify using complex numbers.**

$$x = \pm\sqrt{72}i$$

**Step 4: Simplify the radical.**

$$x = \pm 6\sqrt{2}i$$

**Step 5: Rewrite to show two solutions.**

$$x = 6\sqrt{2}i, x = -6\sqrt{2}i$$

**Step 6: Check the solutions.**

$$\begin{aligned}
 &rgb[1.0, 0.0, 0.0]x^2 + 72 = 0 \\
 &(rgb[1.0, 0.0, 0.0]6\sqrt{2}i)^2 + 72 \stackrel{?}{=} 0 \\
 &6^2(\sqrt{2})^2i^2 + 72 \stackrel{?}{=} 0 \\
 &36 \times 2 \times (-1) + 72 \stackrel{?}{=} 0 \\
 &0 = 0 \checkmark
 \end{aligned}$$

$$\begin{aligned}
 &rgb[1.0, 0.0, 0.0]x^2 + 72 = 0 \\
 &(rgb[1.0, 0.0, 0.0]-6\sqrt{2}i)^2 + 72 \stackrel{?}{=} 0 \\
 &(-6)^2(\sqrt{2})^2i^2 + 72 \stackrel{?}{=} 0 \\
 &36 \times 2 \times (-1) + 72 \stackrel{?}{=} 0 \\
 &0 = 0 \checkmark
 \end{aligned}$$

## Try It

8) Solve:  $c^2 + 12 = 0$ .

**Solution**

$$c = 2\sqrt{3}i, c = -2\sqrt{3}i$$



## Try It

9) Solve:  $q^2 + 24 = 0$ .

### Solution

$$c = 2\sqrt{6}i, c = -2\sqrt{6}i$$

Our method also works when fractions occur in the equation, we solve as any equation with fractions. In the next example, we first isolate the quadratic term, and then make the coefficient equal to one.

## Example 4.1.4

Solve:  $\frac{2}{3}u^2 + 5 = 17$ .

### Solution

**Step 1: Isolate the quadratic term.**

$$\frac{2}{3}u^2 = 12$$

**Step 2: Multiply by  $\frac{3}{2}$  to make the coefficient 1.**

$$\frac{3}{2} \times \frac{2}{3}u^2 = \frac{3}{2} \times 12$$

**Step 3: Simplify.**

$$u^2 = 18$$

**Step 4: Use the Square Root Property.**

$$u = \pm\sqrt{18}$$

**Step 5: Simplify the radical.**

$$u = \pm\sqrt{9 \times 2}$$

**Step 6: Simplify.**

$$u = \pm 3\sqrt{2}$$

**Step 7: Rewrite to show two solutions.**

$$u = 3\sqrt{2}, u = -3\sqrt{2}$$

**Step 8: Check.**

$$\begin{aligned} \frac{2}{3}rgb[1.0, 0.0, 0.0]u^2 + 5 &= 17 \\ \frac{2}{3}(rgb[1.0, 0.0, 0.0]3\sqrt{2})^2 + 5 &\stackrel{?}{=} 17 \\ \frac{2}{3} \times 18 + 5 &\stackrel{?}{=} 17 \\ 12 + 5 &\stackrel{?}{=} 17 \\ 17 &= 17 \checkmark \end{aligned}$$

$$\begin{aligned} \frac{2}{3}rgb[1.0, 0.0, 0.0]u^2 + 5 &= 17 \\ \frac{2}{3}(rgb[1.0, 0.0, 0.0] - 3\sqrt{2})^2 + 5 &\stackrel{?}{=} 17 \\ \frac{2}{3} \times 18 + 5 &\stackrel{?}{=} 17 \\ 12 + 5 &\stackrel{?}{=} 17 \\ 17 &= 17 \checkmark \end{aligned}$$

## Try It

10) Solve:  $\frac{1}{2}x^2 + 4 = 24$ .

**Solution**

$$x = 2\sqrt{10}, x = -2\sqrt{10}$$

## Try It

11) Solve:  $\frac{3}{4}y^2 - 3 = 18$ .

**Solution**

$$y = 2\sqrt{7}, y = -2\sqrt{7}$$

The solutions to some equations may have fractions inside the radicals. When this happens, we must rationalize the denominator.

## Example 4.1.5

Solve:  $2x^2 - 8 = 41$ .

**Solution****Step 1: Isolate the quadratic term.**

$$2x^2 = 49$$

**Step 2: Divide by 2 to make the coefficient 1.**

$$\frac{2x^2}{2} = \frac{49}{2}$$

**Step 3: Simplify.**

$$x^2 = \frac{49}{2}$$

**Step 4: Use the Square Root Property.**

$$x = \pm \sqrt{\frac{49}{2}}$$

**Step 5: Rewrite the radical as a fraction of square roots.**

$$x = \pm \frac{\sqrt{49}}{\sqrt{2}}$$

**Step 6: Rationalize the denominator.**

$$x = \pm \frac{\sqrt{49} \times \text{rgb}[1.0, 0.0, 0.0]\sqrt{2}}{\sqrt{2} \times \text{rgb}[1.0, 0.0, 0.0]\sqrt{2}}$$

**Step 7: Simplify.**

$$x = \pm \frac{7\sqrt{2}}{2}$$

**Step 8: Rewrite to show two solutions.**

$$x = \frac{7\sqrt{2}}{2}, x = -\frac{7\sqrt{2}}{2}$$

**Step 9: Check.**

We leave the check for you.

**Try It**

12) Solve:  $5r^2 - 2 = 34$ .

**Solution**

$$r = \frac{6\sqrt{5}}{5}, r = -\frac{6\sqrt{5}}{5}$$

**Try It**

13) Solve:  $3t^2 + 6 = 70$ .

**Solution**

$$t = \frac{8\sqrt{3}}{3}, t = -\frac{8\sqrt{3}}{3}$$

**Solve Quadratic Equations of the Form  $a(x - h)^2 = k$  Using the Square Root Property**

We can use the Square Root Property to solve an equation of the form  $a(x - h)^2 = k$  as well. Notice that the quadratic term,  $x$ , in the original form  $ax^2 = k$  is replaced with  $(x - h)$ .

$$ax^2 = k \quad a(x - h)^2 = k$$

The first step, like before, is to isolate the term that has the variable squared. In this case, a binomial is being squared. Once the binomial is isolated, by dividing each side by the coefficient of  $a$ , then the Square Root Property can be used on  $(x - h)^2$ .

### Example 4.1.6

Solve:  $4(y - 7)^2 = 48$ .

#### Solution

**Step 1: Divide both sides by the coefficient 4.**

$$(y - 7)^2 = 12$$

**Step 2: Use the Square Root Property on the binomial.**

$$y - 7 = \pm\sqrt{12}$$

**Step 3: Simplify the radical.**

$$y - 7 = \pm 2\sqrt{3}$$

**Step 4: Solve for  $y$ .**

$$y = 7 \pm 2\sqrt{3}$$

**Step 5: Rewrite to show two solutions.**

$$y = 7 + 2\sqrt{3}, y = 7 - 2\sqrt{3}$$

**Step 6: Check.**

$$\begin{aligned} 4(y - 7)^2 &= 48 \\ 4(7 + 2\sqrt{3} - 7)^2 &\stackrel{?}{=} 48 \\ 4(2\sqrt{3})^2 &\stackrel{?}{=} 48 \\ 4(12) &\stackrel{?}{=} 48 \\ 48 &= 48 \checkmark \end{aligned}$$

$$\begin{aligned} 4(y - 7)^2 &= 48 \\ 4(7 - 2\sqrt{3} - 7)^2 &\stackrel{?}{=} 48 \\ 4(-2\sqrt{3})^2 &\stackrel{?}{=} 48 \\ 4(12) &\stackrel{?}{=} 48 \\ 48 &= 48 \checkmark \end{aligned}$$

**Try It**

14) Solve:  $3(a - 3)^2 = 54$ .

**Solution**

$$a = 3 + 3\sqrt{2}, a = 3 - 3\sqrt{2}$$

**Try It**

15) Solve:  $2(b + 2)^2 = 80$ .

**Solution**

$$b = -2 + 2\sqrt{10}, b = -2 - 2\sqrt{10}$$

Remember when we take the square root of a fraction, we can take the square root of the numerator and denominator separately.

**Example 4.1.7**

Solve:  $\left(x - \frac{1}{3}\right)^2 = \frac{5}{9}$ .

**Solution****Step 1: Use the Square Root Property.**

$$x - \frac{1}{3} = \pm \sqrt{\frac{5}{9}}$$

**Step 2: Rewrite the radical as a fraction of square roots.**

$$x - \frac{1}{3} = \pm \frac{\sqrt{5}}{\sqrt{9}}$$

**Step 3: Simplify the radical.**

$$x - \frac{1}{3} = \pm \frac{\sqrt{5}}{3}$$

**Step 4: Solve for  $x$ .**

$$x = \frac{1}{3} \pm \frac{\sqrt{5}}{3}$$

**Step 5: Rewrite to show two solutions.**

$$x = \frac{1}{3} + \frac{\sqrt{5}}{3}, x = \frac{1}{3} - \frac{\sqrt{5}}{3}$$

**Step 6: Check.**

We leave the check for you.

**Try It**

16) Solve:  $\left(x - \frac{1}{2}\right)^2 = \frac{5}{4}$ .

**Solution**



$$x = \frac{1}{2} + \frac{\sqrt{5}}{2}, x = \frac{1}{2} - \frac{\sqrt{5}}{2}$$

## Try It

17) Solve:  $\left(y + \frac{3}{4}\right)^2 = \frac{7}{16}$ .

### Solution

$$y = -\frac{3}{4} + \frac{\sqrt{7}}{4}, y = -\frac{3}{4} - \frac{\sqrt{7}}{4}$$

We will start the solution to the next example by isolating the binomial term.

## Example 4.1.8

Solve:  $2(x - 2)^2 + 3 = 57$ .

### Solution

**Step 1:** Subtract 3 from both sides to isolate the binomial term.

$$2(x - 2)^2 = 54$$

**Step 2:** Divide both sides by 2.

$$(x - 2)^2 = 27$$

**Step 3:** Use the Square Root Property.

$$x - 2 = \pm\sqrt{27}$$

**Step 4: Simplify the radical.**

$$x - 2 = \pm 3\sqrt{3}$$

**Step 5: Solve for  $x$ .**

$$x = 2 \pm 3\sqrt{3}$$

**Step 6: Rewrite to show two solutions.**

$$x = 2 + 3\sqrt{3}, x = 2 - 3\sqrt{3}$$

**Step 7: Check.**

We leave the check for you.

## Try It

18) Solve:  $5(a - 5)^2 + 4 = 104$ .

**Solution**

$$a = 5 + 2\sqrt{5}, a = 5 - 2\sqrt{5}$$

## Try It

19) Solve:  $3(b + 3)^2 - 8 = 88$ .

**Solution**

$$b = -3 + 4\sqrt{2}, b = -3 - 4\sqrt{2}$$

Sometimes the solutions are complex numbers.

### Example 4.1.9

Solve:  $(2x - 3)^2 = -12$ .

#### Solution

**Step 1: Use the Square Root Property.**

$$2x - 3 = \pm\sqrt{-12}$$

**Step 2: Simplify the radical.**

$$2x - 3 = \pm 2\sqrt{3}i$$

**Step 3: Add 3 to both sides.**

$$2x = 3 \pm 2\sqrt{3}i$$

**Step 4: Divide both sides by 2.**

$$x = \frac{3 \pm 2\sqrt{3}i}{2}$$

**Step 5: Rewrite in standard form.**

$$x = \frac{3}{2} \pm \frac{2\sqrt{3}i}{2}$$

**Step 6: Simplify.**

$$x = \frac{3}{2} \pm \sqrt{3}i$$

**Step 7: Rewrite to show two solutions.**

$$x = \frac{3}{2} + \sqrt{3}i, x = \frac{3}{2} - \sqrt{3}i$$

**Step 8: Check.**

We leave the check for you.

**Try It**

20) Solve:  $(3r + 4)^2 = -8$ .

**Solution**

$$r = -\frac{4}{3} + \frac{2\sqrt{2}i}{3}, r = -\frac{4}{3} - \frac{2\sqrt{2}i}{3}$$

**Try It**

21) Solve:  $(2t - 8)^2 = -10$ .

**Solution**

$$t = 4 + \frac{\sqrt{10}i}{2}, t = 4 - \frac{\sqrt{10}i}{2}$$

The left sides of the equations in the next two examples do not seem to be of the form  $a(x - h)^2$ . But they are perfect square trinomials, so we will factor to put them in the form we need.

**Example 4.1.10**

Solve:  $4n^2 + 4n + 1 = 16$ .

**Solution**

We notice the left side of the equation is a perfect square trinomial. We will factor it first.

**Step 1: Factor the perfect square trinomial.**

$$(2n + 1)^2 = 16$$

**Step 2: Use the Square Root Property.**

$$2n + 1 = \pm\sqrt{16}$$

**Step 3: Simplify the radical.**

$$2n + 1 = \pm 4$$

**Step 4: Solve for  $n$ .**

$$2n = -1 \pm 4$$

**Step 5: Divide each side by 2.**

$$\begin{aligned} \frac{2n}{2} &= \frac{-1 \pm 4}{2} \\ n &= \frac{-1 \pm 4}{2} \end{aligned}$$

**Step 6: Rewrite to show two solutions.**

$$n = \frac{-1 + 4}{2}, \quad n = \frac{-1 - 4}{2}$$

**Step 7: Simplify each equation.**

$$n = \frac{3}{2}, \quad n = -\frac{5}{2}$$

**Step 8: Check.**

$$\begin{aligned}
 4rgb]1.0, 0.0, 0.0n^2 + 4rgb]1.0, 0.0, 0.0n + 1 &= 16 \\
 4\left(rgb]1.0, 0.0, 0.0\frac{3}{2}\right)^2 + 4\left(rgb]1.0, 0.0, 0.0\frac{3}{2}\right) + 1 &\stackrel{?}{=} 16 \\
 4\left(\frac{9}{4}\right) + 4\left(\frac{3}{2}\right) + 1 &\stackrel{?}{=} 16 \\
 9 + 6 + 1 &\stackrel{?}{=} 16 \\
 16 &= 16 \checkmark
 \end{aligned}$$

$$\begin{aligned}
 4rgb]1.0, 0.0, 0.0n^2 + 4rgb]1.0, 0.0, 0.0n + 1 &= 16 \\
 4\left(rgb]1.0, 0.0, 0.0 - \frac{5}{2}\right)^2 + 4\left(rgb]1.0, 0.0, 0.0 - \frac{5}{2}\right) + 1 &\stackrel{?}{=} 16 \\
 4\left(\frac{25}{4}\right) + 4\left(-\frac{5}{2}\right) + 1 &\stackrel{?}{=} 16 \\
 25 - 10 + 1 &\stackrel{?}{=} 16 \\
 16 &= 16 \checkmark
 \end{aligned}$$

## Try It

22) Solve:  $9m^2 - 12m + 4 = 25$ .

**Solution**

$$m = \frac{7}{3}, m = -1$$

## Try It

23) Solve:  $16n^2 + 40n + 25 = 4$ .

### Solution

$$n = -\frac{3}{4}, n = -\frac{7}{4}$$

Access this online resource for additional instruction and practice with using the Square Root Property to solve quadratic equations.

- Solving Quadratic Equations: The Square Root Property
- Using the Square Root Property to Solve Quadratic Equations

## Key Concepts

- **Square Root Property**

- If  $x^2 = k$ , then  $x = \sqrt{k}$  or  $x = -\sqrt{k}$  or  $x = \pm\sqrt{k}$

- **How to solve a quadratic equation using the square root property.**

1. Isolate the quadratic term and make its coefficient one.

2. Use Square Root Property.
3. Simplify the radical.
4. Check the solutions.

## Self Check

After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

*<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=771#h5p-27>*



## 4.2 SOLVE QUADRATIC EQUATIONS BY COMPLETING THE SQUARE

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### Learning Objectives

By the end of this section, you will be able to:

- Complete the square of a binomial expression
- Solve quadratic equations of the form  $x^2 + bx + c = 0$  by completing the square
- Solve quadratic equations of the form  $ax^2 + bx + c = 0$  by completing the square

### Try It

Before you get started, take this readiness quiz:

- 1) Expand:  $(x + 9)^2$ .
- 2) Factor  $y^2 - 14y + 49$ .
- 3) Factor  $5n^2 + 40n + 80$ .

So far we have solved quadratic equations by factoring and using the Square Root Property. In this section, we will solve quadratic equations by a process called completing the square, which is important for our work on conics later.

## Complete the Square of a Binomial Expression

In the last section, we were able to use the Square Root Property to solve the equation  $(y - 7)^2 = 12$  because the left side was a perfect square.

$$\begin{aligned}(y - 7)^2 &= 12 \\ y - 7 &= \pm\sqrt{12} \\ y - 7 &= \pm 2\sqrt{3} \\ y &= 7 \pm 2\sqrt{3}\end{aligned}$$

We also solved an equation in which the left side was a perfect square trinomial, but we had to rewrite it the form  $(x - k)^2$  in order to use the Square Root Property.

$$\begin{aligned}x^2 - 10x + 25 &= 18 \\ (x - 5)^2 &= 18\end{aligned}$$

What happens if the variable is not part of a perfect square? Can we use algebra to make a perfect square?

Let's look at two examples to help us recognize the patterns.

---

$(x + 9)^2$	$(y - 7)^2$
$(x + 9)(x + 9)$	$(y - 7)(y - 7)$
$x^2 + 9x + 9x + 81$	$y^2 - 7y - 7y + 49$
$x^2 + 18x + 81$	$y^2 - 14y + 49$

---

We restate the patterns here for reference.

### Binomial Squares Pattern

If  $a$  and  $b$  are real numbers,

$$\begin{aligned}(a + b)^2 &= a^2 + 2ab + b^2 & \underbrace{(a + b)^2}_{(\text{binomial})^2} &= \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2 \times (\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2} \\ (a - b)^2 &= a^2 - 2ab + b^2 & \underbrace{(a - b)^2}_{(\text{binomial})^2} &= \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2 \times (\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}\end{aligned}$$

We can use this pattern to “make” a perfect square.

We will start with the expression  $x^2 + 6x$ . Since there is a plus sign between the two terms, we will use the  $(a + b)^2$  pattern,  $a^2 + 2ab + b^2 = (a + b)^2$ .

$$\begin{array}{r} a^2 + 2ab + b^2 \\ x^2 + 6x + \underline{\quad} \end{array}$$

We ultimately need to find the last term of this trinomial that will make it a perfect square trinomial. To do that we will need to find  $b$ . But first we start with determining  $a$ . Notice that the first term of  $x^2 + 6x$  is a square,  $x^2$ . This tells us that  $a = x$ .

$$\begin{array}{r} a^2 + 2ab + b^2 \\ x^2 + 2 \cdot x \cdot b + b^2 \end{array}$$

What number,  $b$ , when multiplied with  $2x$  gives  $6x$ ? It would have to be  $3$ , which is  $\frac{1}{2}(6)$ . So  $b = 3$ .

$$\begin{array}{r} a^2 + 2ab + b^2 \\ x^2 + 2 \cdot 3 \cdot x + \underline{\quad} \end{array}$$

Now to complete the perfect square trinomial, we will find the last term by squaring  $b$ , which is  $3^2 = 9$ .

$$\begin{array}{r} a^2 + 2ab + b^2 \\ x^2 + 6x + 9 \end{array}$$

We can now factor.

$$\begin{array}{r} (a+b)^2 \\ (x + 3)^2 \end{array}$$

So we found that adding  $9$  to  $x^2 + 6x$  'completes the square', and we write it as  $(x + 3)^2$ .

## HOW TO

### Complete a square of $x^2 + bx$ .

1. Identify  $b$ , the coefficient of  $x$ .
2. Find  $\left(\frac{1}{2}b\right)^2$ , the number to complete the square.
3. Add the  $\left(\frac{1}{2}b\right)^2$  to  $x^2 + bx$ .
4. Factor the perfect square trinomial, writing it as a binomial squared.

### Example 4.2.1

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

a.  $x^2 - 26x$

b.  $y^2 - 9y$

c.  $n^2 + \frac{1}{2}n$

#### Solution

a.

**Step 1: The coefficient of  $x$  is  $-26$ .**

$$\begin{array}{l} x^2 - bx \\ x^2 - 26x \end{array}$$

**Step 2: Find  $\left(\frac{1}{2}b\right)^2$ .**

$$\begin{aligned} &= \left(\frac{1}{2} \times (-26)\right)^2 \\ &= (13)^2 \\ &= 169 \end{aligned}$$

**Step 3: Add 169 to the binomial to complete the square.**

$$x^2 - 26x + 169$$

**Step 4: Factor the perfect square trinomial, writing it as a binomial squared.**

$$(x - 13)^2$$

b.

**Step 1: The coefficient of  $y$  is  $-9$ .**

$$\begin{array}{l} x^2 - bx \\ y^2 - 9y \end{array}$$

**Step 2: Find  $\left(\frac{1}{2}b\right)^2$ .**

$$\begin{aligned} &= \left(\frac{1}{2} \times (-9)\right)^2 \\ &= \left(-\frac{9}{2}\right)^2 \\ &= \frac{81}{4} \end{aligned}$$

**Step 3: Add  $\frac{81}{4}$  to the binomial to complete the square.**

$$y^2 - 9y + \frac{81}{4}$$

**Step 4: Factor the perfect square trinomial, writing it as a binomial squared.**

$$\left(y - \frac{9}{2}\right)^2$$


---

C.

**Step 1: The coefficient of  $n$  is  $\frac{1}{2}$ .**

$$\begin{array}{l} x^2 + bx \\ n^2 + \frac{1}{2}n \end{array}$$

**Step 2: Find  $\left(\frac{1}{2}b\right)^2$ .**

$$\begin{aligned}
 &= \left(\frac{1}{2} \times \frac{1}{2}\right)^2 \\
 &= \left(\frac{1}{4}\right)^2 \\
 &= \frac{1}{16}
 \end{aligned}$$

**Step 3:** Add  $\frac{1}{16}$  to the binomial to complete the square.

$$n^2 + \frac{1}{2}n + \frac{1}{16}$$

**Step 4:** Rewrite as a binomial square.

$$\left(n + \frac{1}{4}\right)^2$$

## Try It

4) Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

- a.  $a^2 - 20a$
- b.  $m^2 - 5m$
- c.  $p^2 + \frac{1}{4}p$

### Solution

- a.  $(a - 10)^2$

b.  $\left(b - \frac{5}{2}\right)^2$

c.  $\left(p + \frac{1}{8}\right)^2$

## Try It

5) Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

a.  $b^2 - 4b$

b.  $n^2 + 13n$

c.  $q^2 - \frac{2}{3}q$

**Solution**

a.  $(b - 2)^2$

b.  $\left(n + \frac{13}{2}\right)^2$

c.  $\left(q - \frac{1}{3}\right)^2$

## Solve Quadratic Equations of the Form $x^2 + bx + c = 0$ by Completing the Square

In solving equations, we must always do the same thing to both sides of the equation. This is true, of course, when we solve a quadratic equation by completing the square too. When we add a term to one side of the equation to make a perfect square trinomial, we must also add the same term to the other side of the equation.

For example, if we start with the equation  $x^2 + 6x = 40$ , and we want to complete the square on the left, we will add 9 to both sides of the equation.

$$x^2 + 6x = 40$$

$$x^2 + 6x + \underline{\quad} = 40 + \underline{\quad}$$

Add 9 to both sides to complete the square.

$$x^2 + 6x + 9 = 40 + 9$$

$$(x + 3)^2 = 49$$

Now the equation is in the form to solve using the Square Root Property! Completing the square is a way to transform an equation into the form we need to be able to use the Square Root Property.

### Example 4.2.2

How to Solve a Quadratic Equation of the Form  $x^2 + bx + c = 0$  by Completing the Square.

Solve by completing the square:  $x^2 + 8x = 48$ .

#### Solution

**Step 1: Isolate the variable terms on one side and the constant terms on the other.**

This equation has all the variables on the left.

$$x^2 + 8x = 48$$

**Step 2: Find  $\left(\frac{1}{2} \cdot b\right)^2$ , the number to complete the square. Add it to both sides of the equation.**

Take half of 8 and square it.  $4^2 = 16$

Add 16 to BOTH sides of the equation.

**Formula does not parse**

**Step 3: Factor the perfect square trinomial as a binomial square.**

$x^2 + 8x + 16 = (x + 4)^2$ , Add the terms on the right.

$$(x + 4)^2 = 64$$

**Step 4: Use the Square Root Property.**



$$x + 4 = \pm\sqrt{64}$$

**Step 5: Simplify the radical and then solve the two resulting equations.**

$$x + 4 = \pm 8$$

$$x + 4 = 8 \quad x + 4 = -8$$

$$x = 4 \quad x = -12$$

**Step 6: Check the solutions.**

Put each answer in the original equation to check.

$$\begin{array}{l} \begin{array}{l} x^2 + 8x + 16 = 48 \\ \text{Substitute } x=4: (4)^2 + 8(4) + 16 = 48 \\ 16 + 32 + 16 = 48 \\ 48 = 48 \\ \checkmark \end{array} \\ \begin{array}{l} x^2 + 8x + 16 = 48 \\ \text{Substitute } x=-12: (-12)^2 + 8(-12) + 16 = 48 \\ 144 - 96 + 16 = 48 \\ 48 = 48 \\ \checkmark \end{array} \end{array}$$

## Try It

6) Solve by completing the square:  $x^2 + 4x = 5$ .

**Solution**

$$x = -5, x = -1$$

## Try It

7) Solve by completing the square:  $y^2 - 10y = -9$ .

### Solution

$$y = 1, y = 9$$

The steps to solve a quadratic equation by completing the square are listed here.

## HOW TO

Solve a quadratic equation of the form  $x^2 + bx + c = 0$  by completing the square.

1. Isolate the variable terms on one side and the constant terms on the other.
2. Find  $\left(\frac{1}{2} \cdot b\right)^2$ , the number needed to complete the square. Add it to both sides of the equation.
3. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right
4. Use the Square Root Property.
5. Simplify the radical and then solve the two resulting equations.
6. Check the solutions.

When we solve an equation by completing the square, the answers will not always be integers.

### Example 4.2.3

Solve by completing the square:  $x^2 + 4x = -21$ .

#### Solution

**Step 1: The variable terms are on the left side.**

Take half of 4 and square it.

$$\left(\frac{1}{2}(4)\right)^2 = 4$$

$$x^2 + 4x + \frac{\quad}{\quad} = -21$$

rgb]1.0,0.0,0.0(1/2.4)^2

**Step 2: Add 4 to both sides.**

$$x^2 + 4x + 4 = -21 + 4$$

**Step 3: Factor the perfect square trinomial, writing it as a binomial squared.**

$$(x + 2)^2 = -17$$

**Step 4: Use the Square Root Property.**

$$x + 2 = \pm\sqrt{-17}$$

**Step 5: Simplify using complex numbers.**

$$x + 2 = \pm\sqrt{17}i$$

**Step 6: Subtract 2 from each side.**

$$x = -2 \pm \sqrt{17}i$$

**Step 7: Rewrite to show two solutions.**

$$x = -2 + \sqrt{17}i, \quad x = -2 - \sqrt{17}i$$

**Step 8: We leave the check to you.**

**Try It**

8) Solve by completing the square:  $y^2 - 10y = -35$ .

**Solution**

$$y = 5 + \sqrt{15}i, y = 5 - \sqrt{15}i$$

**Try It**

9) Solve by completing the square:  $z^2 + 8z = -19$ .

**Solution**

$$z = -4 + \sqrt{3}i, z = -4 - \sqrt{3}i$$

In the previous example, our solutions were complex numbers. In the next example, the solutions will be irrational numbers.

**Example 4.2.4**

Solve by completing the square:  $y^2 - 18y = -6$ .

**Solution**

**Step 1: The variable terms are on the left side.**

Take half of  $-18$  and square it.

**Formula does not parse**

**Step 2: Add 81 to both sides.**

$$y^2 - 18y + 81 = -6 + 81$$

**Step 3: Factor the perfect square trinomial, writing it as a binomial squared.**

$$(y - 9)^2 = 75$$

**Step 4: Use the Square Root Property.**

$$y - 9 = \pm\sqrt{75}$$

**Step 5: Simplify the radical.**

$$y - 9 = \pm 5\sqrt{3}$$

**Step 6: Solve for  $y$ .**

$$y = 9 \pm 5\sqrt{3}$$

**Step 7: Check.**

$$\begin{aligned} y^2 - 18y &= -6 \\ (rgb]1.0, 0.0, 0.09 + 5\sqrt{3})^2 - 18(rgb]1.0, 0.0, 0.09 + 5\sqrt{3}) &\stackrel{?}{=} -6 \\ 81 + 90\sqrt{3} + 75 - 162 + 90\sqrt{3} &\stackrel{?}{=} -6 \\ -6 &= -6 \checkmark \end{aligned}$$

$$\begin{aligned} y^2 - 18y &= -6 \\ (rgb]1.0, 0.0, 0.09 - 5\sqrt{3})^2 - 18(rgb]1.0, 0.0, 0.09 - 5\sqrt{3}) &\stackrel{?}{=} -6 \\ 81 + 90\sqrt{3} + 75 - 162 + 90\sqrt{3} &\stackrel{?}{=} -6 \\ -6 &= -6 \checkmark \end{aligned}$$

Another way to check this would be to use a calculator. Evaluate  $y^2 - 18y$  for both of the solutions. The answer should be  $-6$ .

**Try It**

10) Solve by completing the square:  $x^2 - 16x = -16$ .

**Solution**

$$x = 8 + 4\sqrt{3}, x = 8 - 4\sqrt{3}$$

**Try It**

11) Solve by completing the square:  $y^2 + 8y = 11$ .

**Solution**

$$y = -4 + 3\sqrt{3}, y = -4 - 3\sqrt{3}$$

We will start the next example by isolating the variable terms on the left side of the equation.

**Example 4.2.5**

Solve by completing the square:  $x^2 + 10x + 4 = 15$ .

**Solution**

**Step 1: Isolate the variable terms on the left side.**

Subtract 4 to get the constant terms on the right side.

$$x^2 + 10x = 11$$

**Step 2: Take half of 10 and square it.**

**Formula does not parse**

**Step 3: Add 25 to both sides.**

$$x^2 + 10x + 25 = 11 + 25$$

**Step 4: Factor the perfect square trinomial, writing it as a binomial squared.**

$$(x + 5)^2 = 36$$

**Step 5: Use the Square Root Property.**

$$x + 5 = \pm\sqrt{36}$$

**Step 6: Simplify the radical.**

$$x + 5 = \pm 6$$

**Step 7: Solve for  $x$ .**

$$x = -5 \pm 6$$

**Step 8: Rewrite to show two solutions.**

$$x = -5 + 6, \quad x = -5 - 6$$

**Step 9: Solve the equations.**

$$x = 1, \quad x = -11$$

**Step 10: Check.**

$$x^2 + 10x + 4 = 15$$

$$rgb]1.0, 0.0, 0.0(1)^2 + 10rgb]1.0, 0.0, 0.0(1) + 4 \stackrel{?}{=} 15$$

$$1 + 10 + 4 \stackrel{?}{=} 15$$

$$15 = 15 \checkmark$$

$$x^2 + 10x + 4 = 15$$

$$rgb]1.0, 0.0, 0.0(-11)^2 + 10rgb]1.0, 0.0, 0.0(-11) + 4 \stackrel{?}{=} 15$$

$$121 + 110 + 4 \stackrel{?}{=} 15$$

$$15 = 15 \checkmark$$

### Try It

12) Solve by completing the square:  $a^2 + 4a + 9 = 30$ .

**Solution**

$$a = -7, a = 3$$

### Try It

13) Solve by completing the square:  $b^2 + 8b - 4 = 16$ .

**Solution**

$$b = -10, b = 2$$



To solve the next equation, we must first collect all the variable terms on the left side of the equation. Then we proceed as we did in the previous examples.

### Example 4.2.6

Solve by completing the square:  $n^2 = 3n + 11$ .

#### Solution

**Step 1: Subtract  $3n$  to get the variable terms on the left side.**

$$n^2 - 3n = 11$$

**Step 2: Take half of  $-3$  and square it.**

Formula does not parse

**Step 3: Add  $\frac{9}{4}$  to both sides.**

$$n^2 - 3n + \frac{9}{4} = 11 + \frac{9}{4}$$

**Step 4: Factor the perfect square trinomial, writing it as a binomial squared.**

$$\left(n - \frac{3}{2}\right)^2 = \frac{44}{4} + \frac{9}{4}$$

**Step 5: Add the fractions on the right side.**

$$\left(n - \frac{3}{2}\right)^2 = \frac{53}{4}$$

**Step 6: Use the Square Root Property.**

$$n - \frac{3}{2} = \pm \sqrt{\frac{53}{4}}$$

**Step 7: Simplify the radical.**

$$n - \frac{3}{2} = \pm \frac{\sqrt{53}}{2}$$

**Step 8: Solve for  $n$ .**

$$n = 32 \pm \frac{\sqrt{53}}{2}$$

**Step 9: Rewrite to show two solutions.**

$$n = \frac{3}{2} + \frac{\sqrt{53}}{2}, \quad n = \frac{3}{2} - \frac{\sqrt{53}}{2}$$

**Step 10: Check.**

We leave the check for you!

## Try It

14) Solve by completing the square:  $p^2 = 5p + 9$ .

**Solution**

$$p = \frac{5}{2} + \frac{\sqrt{61}}{2}, \quad p = \frac{5}{2} - \frac{\sqrt{61}}{2}$$

## Try It

15) Solve by completing the square:  $q^2 = 7q - 3$ .

**Solution**

$$q = \frac{7}{2} + \frac{\sqrt{37}}{2}, q = \frac{7}{2} - \frac{\sqrt{37}}{2}$$

Notice that the left side of the next equation is in factored form. But the right side is not zero. So, we cannot use the Zero Product Property since it says “If  $a \cdot b = 0$ , then  $a = 0$  or  $b = 0$ .” Instead, we multiply the factors and then put the equation into standard form to solve by completing the square.

### Example 4.2.7

Solve by completing the square:  $(x - 3)(x + 5) = 9$ .

#### Solution

**Step 1: We multiply the binomials on the left.**

$$x^2 + 2x - 15 = 9$$

**Step 2: Add 15 to isolate the constant terms on the right.**

$$x^2 + 2x = 24$$

**Step 3: Take half of 2 and square it.**

Formula does not parse

**Step 4: Add 1 to both sides.**

$$x^2 + 2x + 1 = 24 + 1$$

**Step 5: Factor the perfect square trinomial, writing it as a binomial squared.**

$$(x + 1)^2 = 25$$

**Step 6: Use the Square Root Property.**

$$x + 1 = \pm\sqrt{25}$$

**Step 7: Solve for  $x$ .**

$$x = -1 \pm 5$$

**Step 8: Rewrite to show two solutions.**

$$x = -1 + 5, \quad x = -1 - 5$$

**Step 9: Simplify.**

$$x = 4, \quad x = -6$$

**Step 10: Check.**

We leave the check for you!

## Try It

16) Solve by completing the square:  $(c - 2)(c + 8) = 11$ .

**Solution**

$$c = -9, \quad c = 3$$

## Try It

17) Solve by completing the square:  $(d - 7)(d + 3) = 56$ .

**Solution**

$$d = 11, \quad d = -7$$

## Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

The process of completing the square works best when the coefficient of  $x^2$  is 1, so the left side of the equation is of the form  $x^2 + bx + c$ . If the  $x^2$  term has a coefficient other than 1, we take some preliminary steps to make the coefficient equal to 1.

Sometimes the coefficient can be factored from all three terms of the trinomial. This will be our strategy in the next example.

### Example 4.2.8

Solve by completing the square:  $3x^2 - 12x - 15 = 0$ .

#### Solution

To complete the square, we need the coefficient of  $x^2$  to be one. If we factor out the coefficient of  $x^2$  as a common factor, we can continue with solving the equation by completing the square.

**Step 1: Factor out the greatest common factor.**

$$3(x^2 - 4x - 5) = 0$$

**Step 2: Divide both sides by 3 to isolate the trinomial with coefficient 1.**

$$\frac{3(x^2 - 4x - 5)}{3} = \frac{0}{3}$$

**Step 3: Simplify.**

$$x^2 - 4x - 5 = 0$$

**Step 4: Add 5 to get the constant terms on the right side.**

$$x^2 - 4x = 5$$

**Step 5: Take half of 4 and square it.**

Formula does not parse

**Step 6: Add 4 to both sides.**

$$x^2 - 4x + 4 = 5 + 4$$

**Step 7: Factor the perfect square trinomial, writing it as a binomial squared.**

$$(x - 2)^2 = 9$$

**Step 8: Use the Square Root Property.**

$$x - 2 = \pm\sqrt{9}$$

**Step 9: Solve for  $x$ .**

$$x - 2 = \pm 3$$

**Step 10: Rewrite to show two solutions.**

$$x = 2 + 3, \quad x = 2 - 3$$

**Step 11: Simplify.**

$$x = 5, \quad x = -1$$

**Step 12: Check:**

$$x = 5$$

$$3x^2 - 12x - 15 = 0$$

$$3(5)^2 - 12(5) - 15 \stackrel{?}{=} 0$$

$$75 - 60 - 15 \stackrel{?}{=} 0$$

$$0 = 0 \checkmark$$

$$x = 1$$

$$3x^2 - 12x - 15 = 0$$

$$3(-1)^2 - 12(-1) - 15 \stackrel{?}{=} 0$$

$$3 + 12 - 15 \stackrel{?}{=} 0$$

$$0 = 0 \checkmark$$

**Try It**

18) Solve by completing the square:  $2m^2 + 16m + 14 = 0$ .

**Solution**

$$m = -7, m = -1$$

**Try It**

19) Solve by completing the square:  $4n^2 - 24n - 56 = 8$ .

**Solution**

$$n = -2, n = 8$$

To complete the square, the coefficient of the  $x^2$  must be 1. When the leading coefficient is not a factor of all the terms, we will divide both sides of the equation by the leading coefficient! This will give us a fraction for the second coefficient. We have already seen how to complete the square with fractions in this section.

**Example 4.2.9**

Solve by completing the square:  $2x^2 - 3x = 20$ .

**Solution**

To complete the square we need the coefficient of  $x^2$  to be one. We will divide both sides of the equation by the coefficient of  $x^2$ . Then we can continue with solving the equation by completing the square.

**Step 1: Divide both sides by 2 to get the coefficient of  $x^2$  to be 1.**

$$\frac{2x^2 - 3x}{2} = \frac{20}{2}$$

**Step 2: Simplify.**

$$x^2 - \frac{3}{2}x = 10$$

**Step 3: Take half of  $-\frac{3}{2}$  and square it.**

**Formula does not parse**

**Step 4: Add  $\frac{9}{16}$  to both sides.**

$$x^2 - \frac{3}{2}x + \frac{9}{16} = 10 + \frac{9}{16}$$

**Step 5: Factor the perfect square trinomial, writing it as a binomial squared.**

$$\left(x - \frac{3}{4}\right)^2 = \frac{160}{16} + \frac{9}{16}$$

**Step 6: Add the fractions on the right side.**

$$\left(x - \frac{3}{4}\right)^2 = \frac{169}{16}$$

**Step 7: Use the Square Root Property.**

$$x - \frac{3}{4} = \pm \sqrt{\frac{169}{16}}$$

**Step 8: Simplify the radical.**

$$x - \frac{3}{4} = \pm \frac{13}{4}$$

**Step 9: Solve for  $x$ .**



$$x = \frac{3}{4} \pm \frac{13}{4}$$

**Step 10: Rewrite to show two solutions.**

$$x = \frac{3}{4} + \frac{13}{4}, \quad x = \frac{3}{4} - \frac{13}{4}$$

**Step 11: Simplify.**

$$x = 4, \quad x = -\frac{5}{2}$$

**Step 12: Check.**

We leave the check for you!

## Try It

20) Solve by completing the square:  $3r^2 - 2r = 21$ .

**Solution**

$$r = -\frac{7}{3}, \quad r = 3$$

## Try It

21) Solve by completing the square:  $4t^2 + 2t = 20$ .

**Solution**

$$t = -\frac{5}{2}, t = 2$$

Now that we have seen that the coefficient of  $x^2$  must be 1 for us to complete the square, we update our procedure for solving a quadratic equation by completing the square to include equations of the form  $ax^2 + bx + c = 0$ .

## HOW TO

**Solve a quadratic equation of the form  $ax^2 + bx + c = 0$  by completing the square.**

1. Divide by  $a$  to make the coefficient of  $x^2$  term 1.
2. Isolate the variable terms on one side and the constant terms on the other.
3. Find  $\left(\frac{1}{2} \cdot b\right)^2$ , the number needed to complete the square. Add it to both sides of the equation.
4. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right
5. Use the Square Root Property.
6. Simplify the radical and then solve the two resulting equations.
7. Check the solutions.

### Example 4.2.10

Solve by completing the square:  $3x^2 + 2x = 4$ .

**Solution**

Again, our first step will be to make the coefficient of  $x^2$  one. By dividing both sides of the equation by the coefficient of  $x^2$ , we can then continue with solving the equation by completing the square.

**Step 1: Divide both sides by 3 to make the coefficient of  $x^2$  equal 1.**

$$\frac{3x^2 + 2x}{3} = \frac{4}{3}$$

**Step 2: Simplify.**

$$x^2 + \frac{2}{3}x = \frac{4}{3}$$

**Step 3: Take half of  $\frac{2}{3}$  and square it.**

Formula does not parse

**Step 4: Add  $\frac{1}{9}$  to both sides.**

$$x^2 + \frac{2}{3}x + \frac{1}{9} = \frac{4}{3} + \frac{1}{9}$$

**Step 5: Factor the perfect square trinomial, writing it as a binomial squared.**

$$\left(x + \frac{1}{3}\right)^2 = \frac{12}{9} + \frac{1}{9}$$

**Step 6: Use the Square Root Property.**

$$x + \frac{1}{3} = \pm \sqrt{\frac{13}{9}}$$

**Step 7: Simplify the radical.**

$$x + \frac{1}{3} = \pm \frac{\sqrt{13}}{3}$$

**Step 8: Solve for  $x$ .**

$$x = -\frac{1}{3} \pm \frac{\sqrt{13}}{3}$$

**Step 9: Rewrite to show two solutions.**

$$x = -\frac{1}{3} + \frac{\sqrt{13}}{3}, \quad x = -\frac{1}{3} - \frac{\sqrt{13}}{3}$$

**Step 10: Check.**

We leave the check for you!

**Try It**

22) Solve by completing the square:  $4x^2 + 3x = 2$ .

**Solution**

$$x = -\frac{3}{8} + \frac{\sqrt{41}}{8}, \quad x = -\frac{3}{8} - \frac{\sqrt{41}}{8}$$

**Try It**

23) Solve by completing the square:  $3y^2 - 10y = -5$ .

**Solution**

$$y = \frac{5}{3} + \frac{\sqrt{10}}{3}, \quad y = \frac{5}{3} - \frac{\sqrt{10}}{3}$$

Access these online resources for additional instruction and practice with completing the square.

- Completing Perfect Square Trinomials
- Completing the Square 1
- Completing the Square to Solve Quadratic Equations
- Completing the Square to Solve Quadratic Equations: More Examples
- Completing the Square 4

## Key Concepts

### ◦ Binomial Squares Pattern

If  $a$  and  $b$  are real numbers,

$$\begin{array}{l} \left(a+b\right)^2 = a^2 + 2ab + b^2 \\ \left(a-b\right)^2 = a^2 - 2ab + b^2 \end{array}$$

### ◦ How to Complete a Square

1. Identify  $b$ , the coefficient of  $x$ .
2. Find  $\left(\frac{1}{2}b\right)^2$ , the number to complete the square.
3. Add the  $\left(\frac{1}{2}b\right)^2$  to  $x^2 + bx$

4. Rewrite the trinomial as a binomial square
- **How to solve a quadratic equation of the form  $ax^2 + bx + c = 0$  by completing the square.**
  1. Divide by  $a$  to make the coefficient of  $x^2$  term **1**.
  2. Isolate the variable terms on one side and the constant terms on the other.
  3. Find  $\left(\frac{1}{2} \cdot b\right)^2$ , the number needed to complete the square. Add it to both sides of the equation.
  4. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right.
  5. Use the Square Root Property.
  6. Simplify the radical and then solve the two resulting equations.
  7. Check the solutions.

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=881#h5p-28>

b) After reviewing this checklist, what will you do to become confident for all objectives?

# 4.3 SOLVE QUADRATIC EQUATIONS USING THE QUADRATIC FORMULA

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## Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations using the Quadratic Formula
- Use the discriminant to predict the number and type of solutions of a quadratic equation
- Identify the most appropriate method to use to solve a quadratic equation

## Try It

Before you get started, take this readiness quiz:

- 1) Evaluate  $b^2 - 4ab$  when  $a = 3$  and  $b = -2$
- 2) Simplify:  $\sqrt{108}$
- 3) Simplify:  $\sqrt{50}$

## Solve Quadratic Equations Using the Quadratic Formula

When we solved quadratic equations in the last section by completing the square, we took the same steps every

time. By the end of the exercise set, you may have been wondering ‘isn’t there an easier way to do this?’ The answer is ‘yes’. Mathematicians look for patterns when they do things over and over in order to make their work easier. In this section we will derive and use a formula to find the solution of a quadratic equation.

We have already seen how to solve a formula for a specific variable ‘in general’, so that we would do the algebraic steps only once, and then use the new formula to find the value of the specific variable. Now we will go through the steps of completing the square using the general form of a quadratic equation to solve a quadratic equation for  $x$ .

We start with the standard form of a quadratic equation and solve it for  $x$  by completing the square.

$$ax^2 + bx + c = 0, \quad a \neq 0$$

**Step 1: Isolate the variable terms on one side.**

$$ax^2 + bx = -c$$

**Step 2: Make the coefficient of  $x^2$  equal to 1, by dividing by  $a$ .**

$$\frac{ax^2}{a} + \frac{b}{a}x = -\frac{c}{a}$$

**Step 3: Simplify.**

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

**Step 4: To complete the square, find  $\left(\frac{1}{2} \times \frac{b}{a}\right)^2$  and add it to both sides of the equation.**

$$\left(\frac{1}{2} \frac{b}{a}\right)^2 = \frac{b^2}{4a^2}$$

**Formula does not parse**

**Step 5: The left side is a perfect square, factor it.**

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

**Step 6: Find the common denominator of the right side and write equivalent fractions with the common denominator.**

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c \times 4a}{a \times 4a}$$

**Step 7: Simplify.**

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$

**Step 8: Combine to one fraction.**

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

**Step 9: Use the square root property.**



$$x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

**Step 10: Simplify the radical.**

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

**Step 11: Add  $-\frac{b}{2a}$  to both sides of the equation.**

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

**Step 12: Combine the terms on the right side.**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This equation is the Quadratic Formula.

## Quadratic Formula

The solutions to a quadratic equation of the form  $ax^2 + bx + c = 0$ , where  $a \neq 0$  are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To use the Quadratic Formula, we substitute the values of  $a$ ,  $b$ , and  $c$  from the standard form into the expression on the right side of the formula. Then we simplify the expression. The result is the pair of solutions to the quadratic equation. Notice the formula is an equation. Make sure you use both sides of the equation.

### Example 4.3.1

Solve by using the Quadratic Formula:  $2x^2 + 9x - 5 = 0$ .

**Solution****Step 1: Write the quadratic equation in standard form. Identify the  $a$ ,  $b$ ,  $c$  values.**

This equation is in standard form.

$$\begin{aligned} ax^2 + bx + c &= 0 \\ 2x^2 + 9x - 5 &= 0 \end{aligned}$$

State the  $a$ ,  $b$ ,  $c$  values.

$$a = 2, b = 9, c = -5$$

**Step 2: Write the quadratic formula. Then substitute in the values of  $a$ ,  $b$ ,  $c$ .**

Substitute in  $a = 2$ ,  $b = 9$ ,  $c = -5$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-9 \pm \sqrt{9^2 - 4 \times 2 \times (-5)}}{2 \times 2} \end{aligned}$$

**Step 3: Simplify the fraction.**

$$\begin{aligned} x &= \frac{-9 \pm \sqrt{81 - (-40)}}{4} \\ x &= \frac{-9 \pm \sqrt{121}}{4} \\ x &= \frac{-9 \pm 11}{4} \end{aligned}$$

**Step 4: Solve for  $x$ .**

$$\begin{aligned} x &= \frac{-9 + 11}{4} & x &= \frac{-9 - 11}{4} \\ x &= \frac{2}{4} & x &= \frac{-20}{4} \\ x &= \frac{1}{2} & x &= -5 \end{aligned}$$

**Step 4: Check the solutions.**

Put each answer in the original equation to check.

Substitute **Formula does not parse**

Formula does not parse

Substitute  $x = -5$

$$\begin{aligned}
 2x^2 + 9x - 5 &= 0 \\
 2(-5)^2 + 9 \times (-5) - 5 &\stackrel{?}{=} 0 \\
 2 \times 25 - 45 - 5 &\stackrel{?}{=} 0 \\
 50 - 45 - 5 &\stackrel{?}{=} 0 \\
 0 &= 0 \checkmark
 \end{aligned}$$

## Try It

4) Solve by using the Quadratic Formula:  $3y^2 - 5y + 2 = 0$ .

**Solution**

$$y = 1, y = \frac{2}{3}$$

## Try It

5) Solve by using the Quadratic Formula:  $4z^2 + 2z - 6 = 0$ .

**Solution**

$$z = 1, z = -\frac{3}{2}$$

## HOW TO

### Solve a quadratic equation using the quadratic formula.

1. Write the quadratic equation in standard form,  $ax^2 + bx + c = 0$ . Identify the values of  $a$ ,  $b$ , and  $c$ .
2. Write the Quadratic Formula. Then substitute in the values of  $a$ ,  $b$ , and  $c$ .
3. Simplify.
4. Check the solutions.

If you say the formula as you write it in each problem, you'll have it memorized in no time! And remember, the Quadratic Formula is an EQUATION. Be sure you start with " $x =$ ".

### Example 4.3.2

Solve by using the Quadratic Formula:  $x^2 - 6x = -5$ .

#### Solution

**Step 1: Write the equation in standard form by adding 5 to each side.**

$$x^2 - 6x + 5 = 0$$

This equation is now in standard form.

$$\begin{aligned} ax^2 + bx + c &= 0 \\ x^2 - 6x + 5 &= 0 \end{aligned}$$

**Step 2: Identify the values of  $a$ ,  $b$ ,  $c$ .**

$$a = 1, b = -6, c = 5$$

**Step 3: Write the Quadratic Formula.**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**Step 4:** Then substitute in the values of  $a$ ,  $b$ ,  $c$ .

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \times 0.0, 0.0, 1.01 \times (5)}}{2 \times 0.0, 0.0, 1.01}$$

**Step 5:** Simplify.

$$x = \frac{6 \pm \sqrt{36 - 20}}{2}$$

$$x = \frac{6 \pm \sqrt{16}}{2}$$

$$x = \frac{6 \pm 4}{2}$$

**Step 6:** Rewrite to show two solutions.

$$x = \frac{6 + 4}{2}, \quad x = \frac{6 - 4}{2}$$

**Step 7:** Simplify.

$$x = \frac{10}{2} \quad x = \frac{2}{2}$$

$$x = 5 \quad x = 1$$

**Step 8:** Check:

$$x^2 - 6x + 5 = 0$$

$$5^2 - 6 \times 5 + 5 \stackrel{?}{=} 0$$

$$25 - 30 + 5 \stackrel{?}{=} 0$$

$$0 = 0 \checkmark$$

$$x^2 - 6x + 5 = 0$$

$$1^2 - 6 \times 1 + 5 \stackrel{?}{=} 0$$

$$0 = 0 \checkmark$$

### Try It

6) Solve by using the Quadratic Formula:  $a^2 - 2a = 15$ .

**Solution**

$$a = -3, a = 5$$

### Try It

7) Solve by using the Quadratic Formula:  $b^2 + 24 = -10b$ .

**Solution**

$$b = -6, b = -4$$

When we solved quadratic equations by using the Square Root Property, we sometimes got answers that had radicals. That can happen, too, when using the Quadratic Formula. If we get a radical as a solution, the final answer must have the radical in its simplified form.

### Example 4.3.3

Solve by using the Quadratic Formula:  $2x^2 + 10x + 11 = 0$ .

**Solution**

This equation is in standard form.

$$\begin{aligned} ax^2 + bx + c &= 0 \\ 2x^2 + 10x + 11 &= 0 \end{aligned}$$

**Step 1: Identify the values of  $a$ ,  $b$ , and  $c$ .**

$$a = 2, b = 10, c = 11$$

**Step 2: Write the Quadratic Formula.**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**Step 3: Then substitute the values of  $a$ ,  $b$ , and  $c$ .**

$$x = \frac{-10 \pm \sqrt{10^2 - 4 \times 2 \times 11}}{2 \times 2}$$

**Step 4: Simplify.**

$$\begin{aligned} x &= \frac{-10 \pm \sqrt{100 - 88}}{4} \\ x &= \frac{-10 \pm \sqrt{12}}{4} \end{aligned}$$

**Step 5: Simplify the radical.**

$$x = \frac{-10 \pm 2\sqrt{3}}{4}$$

**Step 6: Factor out the common factor in the numerator.**

$$x = \frac{2(-5 \pm \sqrt{3})}{4}$$

**Step 7: Remove the common factors.**

$$x = \frac{-5 \pm \sqrt{3}}{2}$$

**Step 8: Rewrite to show two solutions.**

$$x = \frac{-5 + \sqrt{3}}{2}, \quad x = \frac{-5 - \sqrt{3}}{2}$$

**Step 9: Check:**

We leave the check for you!

**Try It**

8) Solve by using the Quadratic Formula:  $3m^2 + 12m + 7 = 0$ .

**Solution**

$$m = \frac{-6 + \sqrt{15}}{3}, m = \frac{-6 - \sqrt{15}}{3}$$

**Try It**

9) Solve by using the Quadratic Formula:  $5n^2 + 4n - 4 = 0$ .

**Solution**

$$n = \frac{-2 + 2\sqrt{6}}{5}, n = \frac{-2 - 2\sqrt{6}}{5}$$

When we substitute  $a$ ,  $b$ , and  $c$  into the Quadratic Formula and the radicand is negative, the quadratic equation will have imaginary or complex solutions. We will see this in the next example.



### Example 4.3.4

Solve by using the Quadratic Formula:  $3p^2 + 2p + 9 = 0$ .

#### Solution

This equation is in standard form

$$\begin{aligned} ax^2 + bx + c &= 0 \\ 3p^2 + 2p + 9 &= 0 \end{aligned}$$

**Step 1: Identify the values of  $a$ ,  $b$ , and  $c$ .**

$$a = 3, \quad b = 2, \quad c = 9$$

**Step 2: Write the Quadratic Formula.**

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**Step 3: Then substitute in the values of  $a$ ,  $b$ , and  $c$ .**

$$p = \frac{-(2) \pm \sqrt{(2)^2 - 4 \times 3 \times 9}}{2 \times 3}$$

**Step 4: Simplify.**

$$p = \frac{-2 \pm \sqrt{4 - 108}}{6}$$

$$p = \frac{-2 \pm \sqrt{-104}}{6}$$

**Step 5: Simplify the radical using complex numbers.**

$$p = \frac{-2 \pm \sqrt{104} i}{6}$$

**Step 6: Simplify the radical.**

$$p = \frac{-2 \pm 2\sqrt{26} i}{6}$$

**Step 7: Factor the common factor in the numerator.**

$$p = \frac{2(-1 \pm \sqrt{26}i)}{6}$$

**Step 8: Remove the common factors.**

$$p = \frac{-1 \pm \sqrt{26}i}{3}$$

**Step 9: Rewrite in standard  $a + bi$  form.**

$$p = -\frac{1}{3} \pm \frac{\sqrt{26}i}{3}$$

**Step 10: Write as two solutions.**

$$p = -\frac{1}{3} + \frac{\sqrt{26}i}{3}, \quad p = -\frac{1}{3} - \frac{\sqrt{26}i}{3}$$

## Try It

10) Solve by using the Quadratic Formula:  $4a^2 - 2a + 8 = 0$ .

**Solution**

$$a = \frac{1}{4} + \frac{\sqrt{31}i}{4}, \quad a = \frac{1}{4} - \frac{\sqrt{31}i}{4}$$

## Try It

11) Solve by using the Quadratic Formula:  $5b^2 + 2b + 4 = 0$ .

### Solution

$$b = -\frac{1}{5} + \frac{\sqrt{19}}{5}i, \quad b = -\frac{1}{5} - \frac{\sqrt{19}}{5}i$$

Remember, to use the Quadratic Formula, the equation must be written in standard form,  $ax^2 + bx + c = 0$ . Sometimes, we will need to do some algebra to get the equation into standard form before we can use the Quadratic Formula.

## Example 4.3.5

Solve by using the Quadratic Formula:  $x(x + 6) + 4 = 0$ .

### Solution

**Step 1: Distribute to get the equation in standard form.**

$$x^2 + 6x + 4 = 0$$

**Step 2: This equation is now in standard form**

$$\begin{aligned} ax^2 + bx + c &= 0 \\ x^2 + 6x + 4 &= 0 \end{aligned}$$

**Step 3: Identify the values of  $a$ ,  $b$ , and  $c$ .**

$$a = 1, \quad b = 6, \quad c = 4$$

**Step 4: Write the Quadratic Formula.**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**Step 5: Then substitute in the values of  $a$ ,  $b$ , and  $c$ .**

$$x = \frac{-(6) \pm \sqrt{(6)^2 - 4 \times rgb]0.0, 0.0, 1.01 \times (4)}}{2 \times rgb]0.0, 0.0, 1.01}$$

**Step 6: Simplify.**

$$x = \frac{-6 \pm \sqrt{36 - 16}}{2}$$

$$x = \frac{-6 \pm \sqrt{20}}{2}$$

**Step 7: Simplify the radical.**

$$x = \frac{-6 \pm 2\sqrt{5}}{2}$$

**Step 8: Factor the common factor in the numerator.**

$$x = \frac{2(-3 \pm 2\sqrt{5})}{2}$$

**Step 9: Remove the common factors.**

$$x = -3 \pm 2\sqrt{5}$$

**Step 10: Write as two solutions.**

$$x = -3 + 2\sqrt{5}, \quad x = -3 - 2\sqrt{5}$$

**Step 11: Check:**

We leave the check for you!

**Try It**

12) Solve by using the Quadratic Formula:  $x(x + 2) - 5 = 0$ .

**Solution**

$$x = -1 + \sqrt{6}, x = -1 - \sqrt{6}$$

**Try It**

13) Solve by using the Quadratic Formula:  $3y(y - 2) - 3 = 0$ .

**Solution**

$$y = 1 + \sqrt{2}, y = 1 - \sqrt{2}$$

When we solved linear equations, if an equation had too many fractions we cleared the fractions by multiplying both sides of the equation by the LCD. This gave us an equivalent equation—without fractions—to solve. We can use the same strategy with quadratic equations.

**Example 4.3.6**

Solve by using the Quadratic Formula:  $\frac{1}{2}u^2 + \frac{2}{3}u = \frac{1}{3}$ .

**Solution****Step 1: Multiply both sides by the LCD, 6, to clear the fractions.**

$$6 \left( \frac{1}{2}u^2 + 23u \right) = 6 \left( \frac{1}{3} \right)$$

**Step 2: Multiply.**

$$3u^2 + 4u = 2$$

**Step 3: Subtract 2 to get the equation in standard form.**

$$\begin{aligned} ax^2 + bx + c &= 0 \\ 3u^2 + 4u - 2 &= 0 \end{aligned}$$

**Step 4: Identify the values of  $a$ ,  $b$ , and  $c$ .**

$$a = 3, \quad b = 4, \quad c = -2$$

**Step 5: Write the Quadratic Formula.**

$$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**Step 6: Then substitute in the values of  $a$ ,  $b$ , and  $c$ .**

$$u = \frac{-(4) \pm \sqrt{(4)^2 - 4 \times 3 \times (-2)}}{2 \times 3}$$

**Step 7: Simplify.**

$$\begin{aligned} u &= \frac{-4 \pm \sqrt{16 + 24}}{6} \\ u &= \frac{-4 \pm \sqrt{40}}{6} \end{aligned}$$

**Step 8: Simplify the radical.**

$$u = \frac{-4 \pm 2\sqrt{10}}{6}$$

**Step 9: Factor the common factor in the numerator.**

$$u = \frac{2(-2 \pm \sqrt{10})}{6}$$

**Step 10: Remove the common factors.**

$$u = \frac{-2 \pm \sqrt{10}}{3}$$

**Step 11: Rewrite to show two solutions.**

$$u = \frac{-2 + \sqrt{10}}{3}, \quad u = \frac{-2 - \sqrt{10}}{3}$$

**Step 12: Check:**

We leave the check for you!

## Try It

14) Solve by using the Quadratic Formula:  $\frac{1}{4}c^2 - \frac{1}{3}c = \frac{1}{12}$ .

**Solution**

$$c = \frac{2 + \sqrt{7}}{3}, \quad c = \frac{2 - \sqrt{7}}{3}$$

## Try It

15) Solve by using the Quadratic Formula:  $\frac{1}{9}d^2 - \frac{1}{2}d = -\frac{1}{3}$ .

**Solution**

$$d = \frac{9 + \sqrt{33}}{4}, d = \frac{9 - \sqrt{33}}{4}$$

Think about the equation  $x - 3 = 0$ . We know from the Zero Product Property that this equation has only one solution,

$$x = 3$$

We will see in the next example how using the Quadratic Formula to solve an equation whose standard form is a perfect square trinomial equal to 0 gives just one solution. Notice that once the radicand is simplified it becomes 0, which leads to only one solution.

### Example 4.3.7

Solve by using the Quadratic Formula:  $4x^2 - 20x = -25$ .

#### Solution

**Step 1: Add 25 to get the equation in standard form.**

$$\begin{aligned} ax^2 + bx + c &= 0 \\ 4x^2 - 20x + 25 &= 0 \end{aligned}$$

**Step 2: Identify the values of  $a$ ,  $b$ , and  $c$ .**

$$rgb]0.0, 0.0, 1.0a = 4, b = -20, c = 25$$

**Step 3: Write the quadratic formula.**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**Step 4: Then substitute in the values of  $a$ ,  $b$ , and  $c$ .**

$$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4 \times rgb]0.0, 0.0, 1.04 \times (25)}}{2 \times rgb]0.0, 0.0, 1.04}$$

**Step 5: Simplify.**



$$x = \frac{20 \pm \sqrt{400 - 400}}{8}$$

$$x = \frac{20 \pm \sqrt{0}}{8}$$

**Step 6: Simplify the radical.**

$$x = \frac{20}{8}$$

**Step 7: Simplify the fraction.**

$$x = \frac{5}{2}$$

**Step 8: Check:**

We leave the check for you!

Did you recognize that  $4x^2 - 20x + 25$  is a perfect square trinomial. It is equivalent to  $(2x - 5)^2$ ? If you solve  $4x^2 - 20x + 25 = 0$  by factoring and then using the Square Root Property, do you get the same result?

## Try It

16) Solve by using the Quadratic Formula:  $r^2 + 10r + 25 = 0$ .

**Solution**

$$r = -5$$

## Try It

17) Solve by using the Quadratic Formula:  $25t^2 - 40t = -16$ .

**Solution**

$$t = \frac{4}{5}$$

## Use the Discriminant to Predict the Number and Type of Solutions of a Quadratic Equation

When we solved the quadratic equations in the previous examples, sometimes we got two real solutions, one real solution, and sometimes two complex solutions. Is there a way to predict the number and type of solutions to a quadratic equation without actually solving the equation?

Yes, the expression under the radical of the Quadratic Formula makes it easy for us to determine the number and type of solutions. This expression is called the **discriminant**.

### Discriminant

In the Quadratic Formula,  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ ,  
the quantity  $b^2 - 4ac$  is called the **discriminant**.

Let's look at the discriminant of the equations in some of the examples and the number and type of solutions to those quadratic equations.

Quadratic Equation (in standard form)	Discriminant $b^2 - 4ac$	Value of the Discriminant	Number and Type of solutions
$2x^2 + 9x - 5 = 0$	$9^2 - 4 \times 2(-5) = 121$	+	2 real
$4x^2 - 20x + 25 = 0$	$(-20)^2 - 4 \times 4 \times 25 = 0$	0	1 real
$3p^2 + 2p + 9 = 0$	$2^2 - 4 \times 3 \times 9 = -104$	-	2 complex

When the discriminant is **positive**, the quadratic equation has **2 real solutions**.

$$x = \frac{-b \pm \sqrt{+}}{2a}$$

When the discriminant is **zero**, the quadratic equation has **1 real solution**.

$$x = \frac{-b \pm \sqrt{0}}{2a}$$

When the discriminant is **negative**, the quadratic equation has **2 complex solutions**.

$$x = \frac{-b \pm \sqrt{-}}{2a}$$

### Using the Discriminant, $b^2 - 4ac$ , to Determine the Number and Type of Solutions of a Quadratic Equation

For a quadratic equation of the form  $ax^2 + bx + c = 0$ ,  $a \neq 0$ ,

- If  $b^2 - 4ac > 0$ , the equation has **2** real solutions.
- If  $b^2 - 4ac = 0$ , the equation has **1** real solution.
- If  $b^2 - 4ac < 0$ , the equation has **2** complex solutions.

### Example 4.3.8

Determine the number of solutions to each quadratic equation.

a.  $3x^2 + 7x - 9 = 0$

b.  $5n^2 + n + 4 = 0$

c.  $9y^2 - 6y + 1 = 0$

#### Solution

To determine the number of solutions of each quadratic equation, we will look at its discriminant.

a.

**Step 1: The equation is in standard form, identify  $a$ ,  $b$ , and  $c$ .**

$$a = 3, b = 7, c = -9$$

**Step 2: Write the discriminant.**

$$b^2 - 4ac$$

**Step 3: Substitute in the values of  $a$ ,  $b$ , and  $c$ .**

$$(7)^2 - 4 \times 3 \times (-9)$$

**Step 4: Simplify.**

$$49 + 108 = 157$$

Since the discriminant is positive, there are **2** real solutions to the equation.

b.

**Step 1: The equation is in standard form, identify  $a$ ,  $b$ , and  $c$ .**

$$a = 5, b = 1, c = 4$$

**Step 2: Write the discriminant.**

$$b^2 - 4ac$$

**Step 3: Substitute in the values of  $a$ ,  $b$ , and  $c$ .**

$$(1)^2 - 4 \times 5 \times 4$$

**Step 4: Simplify.**

$$1 - 80 = -79$$

Since the discriminant is negative, there are **2** complex solutions to the equation.

---

c.

**Step 1: The equation is in standard form, identify  $a$ ,  $b$ , and  $c$ .**

$$a = 9, b = -6, c = 1$$

**Step 2: Write the discriminant.**

$$b^2 - 4ac$$

**Step 3: Substitute in the values of  $a$ ,  $b$ , and  $c$ .**

$$(-6)^2 - 4 \times 9 \times 1$$

**Step 4: Simplify.**

$$36 - 36 = 0$$

Since the discriminant is **0**, there is **1** real solution to the equation.

## Try It

18) Determine the number and type of solutions to each quadratic equation.

a.  $8m^2 - 3m + 6 = 0$

b.  $5z^2 + 6z - 2 = 0$

c.  $9w^2 + 24w + 16 = 0$

**Solution**a. **2** complex solutionsb. **2** real solutionsc. **1** real solution**Try It**

19) Determine the number and type of solutions to each quadratic equation.

a.  $b^2 + 7b - 13 = 0$

b.  $5a^2 - 6a + 10 = 0$

c.  $4r^2 - 20r + 25 = 0$

**Solution**a. **2** real solutionsb. **2** complex solutionsc. **1** real solution

## Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

We summarize the four methods that we have used to solve quadratic equations below.

## Methods for Solving Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

Given that we have four methods to use to solve a quadratic equation, how do you decide which one to use? Factoring is often the quickest method and so we try it first. If the equation is  $ax^2 = k$  or  $a(x - h)^2 = k$  we use the Square Root Property. For any other equation, it is probably best to use the Quadratic Formula. Remember, you can solve any quadratic equation by using the Quadratic Formula, but that is not always the easiest method.

What about the method of Completing the Square? Most people find that method cumbersome and prefer not to use it. We needed to include it in the list of methods because we completed the square in general to derive the Quadratic Formula. You will also use the process of Completing the Square in other areas of algebra.

## HOW TO

Identify the most appropriate method to solve a quadratic equation.

1. Try **Factoring** first. If the quadratic factors easily, this method is very quick.
2. Try the **Square Root Property** next. If the equation fits the form  $ax^2 = k$  or  $a(x - h)^2 = k$ , it can easily be solved by using the Square Root Property.
3. Use the **Quadratic Formula**. Any other quadratic equation is best solved by using the Quadratic Formula.

The next example uses this strategy to decide how to solve each quadratic equation.

### Example 4.3.9

Identify the most appropriate method to use to solve each quadratic equation.

a.  $5z^2 = 17$

b.  $4x^2 - 12x + 9 = 0$

c.  $8u^2 + 6u = 11$

#### Solution

a.

Since the equation is in the  $ax^2 = k$  form, the most appropriate method is to use the Square Root Property.

---

b.

We recognize that the left side of the equation is a perfect square trinomial, and so factoring will be the most appropriate method.

---

c.

Put the equation in standard form.

$$8u^2 + 6u - 11 = 0$$

While our first thought may be to try factoring, thinking about all the possibilities for trial and error method leads us to choose the Quadratic Formula as the most appropriate method.



## Try It

20) Identify the most appropriate method to use to solve each quadratic equation.

a.  $x^2 + 6x + 8 = 0$

b.  $(n - 3)^2 = 16$

c.  $5p^2 - 6p = 9$

### Solution

- a. Factoring
- b. Square Root Property
- c. Quadratic Formula

## Try It

21) Identify the most appropriate method to use to solve each quadratic equation.

a.  $8a^2 + 3a - 9 = 0$

b.  $4b^2 + 4b + 1 = 0$

c.  $5c^2 = 125$

### Solution

- a. Quadratic Formula
- b. Factoring or Square Root Property
- c. Square Root Property

Access these online resources for additional instruction and practice with using the Quadratic Formula.

- Using the Quadratic Formula
- Solve a Quadratic Equation Using the Quadratic Formula with Complex Solutions
- Discriminant in Quadratic Formula

## Key Concepts

- Quadratic Formula
  - The solutions to a quadratic equation of the form  $ax^2 + bx + c = 0$ ,  $a \neq 0$  are given by the formula:
 
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
- How to solve a quadratic equation using the Quadratic Formula.
  1. Write the quadratic equation in standard form,  $ax^2 + bx + c = 0$ . Identify the values of  $a$ ,  $b$ , and  $c$ .
  2. Write the Quadratic Formula. Then substitute in the values of  $a$ ,  $b$ , and  $c$ .
  3. Simplify.
  4. Check the solutions.
- Using the Discriminant,  $b^2 - 4ac$ , to Determine the Number and Type of Solutions of a Quadratic Equation
  - For a quadratic equation of the form  $ax^2 + bx + c = 0$ ,  $a \neq 0$ ,
    - If  $b^2 - 4ac > 0$ , the equation has **2** real solutions.
    - If  $b^2 - 4ac = 0$ , the equation has **1** real solution.

- if  $b^2 - 4ac < 0$ , the equation has **2** complex solutions.
- Methods to Solve Quadratic Equations:
  - Factoring
  - Square Root Property
  - Completing the Square
  - Quadratic Formula
- How to identify the most appropriate method to solve a quadratic equation.
  1. Try Factoring first. If the quadratic factors easily, this method is very quick.
  2. Try the **Square Root Property** next. If the equation fits the form  $ax^2 = k$  or  $a(x - h)^2 = k$ , it can easily be solved by using the Square Root Property.
  3. Use the **Quadratic Formula**. Any other quadratic equation is best solved by using the Quadratic Formula.

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=973#h5p-29>

b) What does this checklist tell you about your mastery of this section? What steps will you take to improve?

## Glossary

### discriminant

In the Quadratic Formula,  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ , the quantity  $b^2 - 4ac$  is called the discriminant.

# 4.4 SOLVE QUADRATIC EQUATIONS IN QUADRATIC FORM

## Learning Objectives

By the end of this section, you will be able to:

- Solve equations in quadratic form

## Try It

Before you get started, take this readiness quiz:

1) Factor by substitution:  $y^4 - y^2 - 20$

2) Factor by substitution:  $(y - 4)^2 + 8(y - 4) + 15$

3) Simplify:

a.  $x^{\frac{1}{2}} \cdot x^{\frac{1}{4}}$

b.  $\left(x^{\frac{1}{3}}\right)^2$

c.  $(x^{-1})^2$

## Solve Equations in Quadratic Form

Sometimes when we factored trinomials, the trinomial did not appear to be in the  $ax^2 + bx + c$  form. So we factored by substitution allowing us to make it fit the  $ax^2 + bx + c$  form. We used the standard  $u$  for the substitution.

To factor the expression  $x^4 - 4x^2 - 5$ , we noticed the variable part of the middle term is  $x^2$  and its square,  $x^4$ , is the variable part of the first term. (We know  $(x^2)^2 = x^4$ .) So we let  $u = x^2$  and factored.

$$x^4 - 4x^2 - 5$$

$$(x^2)^2 - 4(x^2) - 5$$

Let  $u = x^2$  and substitute.

$$u^2 - 4u - 5$$

Factor the trinomial.

$$(u + 1)(u - 5)$$

Replace  $u$  with  $x^2$ .

$$(x^2 + 1)(x^2 - 5)$$

Similarly, sometimes an equation is not in the  $ax^2 + bx + c = 0$  form but looks much like a quadratic equation. Then, we can often make a thoughtful substitution that will allow us to make it fit the  $ax^2 + bx + c = 0$  form. If we can make it fit the form, we can then use all of our methods to solve quadratic equations. Notice that in the quadratic equation  $ax^2 + bx + c = 0$ , the middle term has a variable,  $x$ , and its square,  $x^2$ , is the variable part of the first term. Look for this relationship as you try to find a substitution.

Again, we will use the standard  $u$  to make a substitution that will put the equation in quadratic form. If the substitution gives us an equation of the form  $ax^2 + bx + c = 0$ , we say the original equation was of quadratic form.

The next example shows the steps for solving an equation in quadratic form.

### Example 4.4.1

How to Solve Equations in Quadratic Form. Solve:  $6x^4 - 7x^2 + 2 = 0$ .

#### Solution

**Step 1: Identify a substitution that will put the equation in quadratic form.**

Since  $(x^2)^2 = x^4$ , we let  $u = x^2$ .

$$6x^4 - 7x^2 + 2 = 0$$

**Step 2: Rewrite the equation with the substitution to put it in quadratic form.**

Rewrite to prepare for the substitution.

$$6(x^2)^2 - 7x^2 + 2 = 0$$

Substitute  $u = x^2$ .

$$6u^2 - 7u + 2 = 0$$

**Step 3: Solve the quadratic equation for  $u$ .**

We can solve by factoring.

$$(2u - 1)(3u - 2) = 0$$

Use the Zero Product Property

$$2u - 1 = 0$$

$$2u = 1$$

$$u = \frac{1}{2}$$

$$3u - 2 = 0$$

$$3u = 2$$

$$u = \frac{2}{3}$$

**Step 4: Substitute the original variable back into the results, using the substitution.**

Replace  $u$  with  $x^2$ .

$$x^2 = \frac{1}{2}, \quad x^2 = \frac{2}{3}$$

**Step 5: Solve for the original variable.**

Solve for  $x$ , using the Square Root Property.

$$x = \pm \sqrt{\frac{1}{2}}$$

$$x = \pm \frac{\sqrt{2}}{2}$$

$$x = \pm \sqrt{\frac{2}{3}}$$

$$x = \pm \frac{\sqrt{6}}{3}$$

There are four solutions.

$$x = \frac{\sqrt{2}}{2}$$

$$x = -\frac{\sqrt{2}}{2}$$

$$x = \frac{\sqrt{6}}{3}$$

$$x = -\frac{\sqrt{6}}{3}$$

**Step 6: Check the solutions.**

Check all four solutions.

$$x = \text{rgb}[1.0, 0.0, 0.0] \frac{\sqrt{2}}{2}$$

We will show one check here.



$$\begin{aligned}
 6x^4 - 7x^2 + 2 &= 0 \\
 6\left(\text{rgb}[1.0, 0.0, 0.0] \frac{\sqrt{2}}{2}\right)^4 - 7\left(\text{rgb}[1.0, 0.0, 0.0] \frac{\sqrt{2}}{2}\right)^2 + 2 &\stackrel{?}{=} 0 \\
 6\left(\frac{4}{16}\right) - 7\left(\frac{2}{4}\right)^2 + 2 &\stackrel{?}{=} 0 \\
 \frac{3}{2} - \frac{7}{2} + \frac{4}{2} &\stackrel{?}{=} 0 \\
 0 &= 0 \checkmark
 \end{aligned}$$

We leave the other checks to you!

## Try It

4) Solve:  $x^4 - 6x^2 + 8 = 0$ .

### Solution

$$x = \sqrt{2}, x = -\sqrt{2}, x = 2, x = -2$$

## Try It

5) Solve:  $x^4 - 11x^2 + 28 = 0$ .

### Solution

$$x = \sqrt{7}, x = -\sqrt{7}, x = 2, x = -2$$

We summarize the steps to solve an equation in quadratic form.

## HOW TO

### Solve equations in quadratic form.

1. Identify a substitution that will put the equation in quadratic form.
2. Rewrite the equation with the substitution to put it in quadratic form.
3. Solve the quadratic equation for  $u$ .
4. Substitute the original variable back into the results, using the substitution.
5. Solve for the original variable.
6. Check the solutions.

In the next example, the binomial in the middle term,  $(x - 2)$  is squared in the first term. If we let  $u = x - 2$  and substitute, our trinomial will be in  $ax^2 + bx + c$  form.

### Example 4.4.2

Solve:  $(x - 2)^2 + 7(x - 2) + 12 = 0$ .

#### Solution

**Step 1: Prepare for the substitution.**

$$(x - 2)^2 + 7(x - 2) + 12 = 0$$

**Step 2: Let  $u = x - 2$  and substitute.**

$$u^2 + 7u + 12 = 0$$

**Step 3: Solve by factoring.**

$$(u + 3)(u + 4) = 0$$

$$u + 3 = 0$$

$$u = -3$$

$$u + 4 = 0$$

$$u = -4$$

**Step 4: Replace  $u$  with  $x - 2$ .**

$$x - 2 = -3, \quad x - 2 = -4$$

**Step 5: Solve for  $x$ .**

$$x = -1, \quad x = -2$$

**Step 6: Check:**

$$x = -1$$

$$(x - 2)^2 + 7(x - 2) + 12 = 0$$

$$(-1 - 2)^2 + 7(-1 - 2) + 12 \stackrel{?}{=} 0$$

$$-3)^2 + 7(-3) + 12 \stackrel{?}{=} 0$$

$$9 - 21 + 12 \stackrel{?}{=} 0$$

$$0 = 0 \checkmark$$

$$x = -2$$

$$(x - 2)^2 + 7(x - 2) + 12 = 0$$

$$(-2 - 2)^2 + 7(-2 - 2) + 12 \stackrel{?}{=} 0$$

$$(-4)^2 + 7(-4) + 12 \stackrel{?}{=} 0$$

$$16 - 28 + 12 \stackrel{?}{=} 0$$

$$0 = 0 \checkmark$$

**Try It**

6) Solve:  $(x - 5)^2 + 6(x - 5) + 8 = 0$ .

**Solution**

$x = 3, x = 1$

**Try It**

7) Solve:  $(y - 4)^2 + 8(y - 4) + 15 = 0$ .

**Solution**

$y = -1, y = 1$

In the next example, we notice that  $(\sqrt{x})^2 = x$ . Also, remember that when we square both sides of an equation, we may introduce extraneous roots. Be sure to check your answers!

**Example 4.4.3**

Solve:  $x - 3\sqrt{x} + 2 = 0$ .

**Solution**

The  $\sqrt{x}$  in the middle term, is squared in the first term  $(\sqrt{x})^2 = x$ . If we let  $u = \sqrt{x}$  and substitute, our trinomial will be in  $ax^2 + bx + c = 0$  form.

**Step 1: Rewrite the trinomial to prepare for the substitution.**

**Formula does not parse**

**Step 2: Let  $u = \sqrt{x}$  and substitute.**

$$u^2 - 3u + 2 = 0$$

**Step 3: Solve by factoring.**

$$(u - 2)(u - 1) = 0$$

$$u - 2 = 0$$

$$u = 2$$

$$u - 1 = 0$$

$$u = 1$$

**Step 4: Replace  $u$  with  $\sqrt{x}$ .**

$$\sqrt{x} = 2, \quad \sqrt{x} = 1$$

**Step 5: Solve for  $x$ , by squaring both sides.**

$$x = 4, \quad x = 1$$

**Step 6: Check:**

$$x = rgb]1.0, 0.0, 0.04$$

$$x - 3\sqrt{x} + 2 = 0$$

$$4 - 3\sqrt{4} + 2 \stackrel{?}{=} 0$$

$$4 - 6 + 2 \stackrel{?}{=} 0$$

$$0 = 0 \checkmark$$

$$x = rgb]1.0, 0.0, 0.01$$

$$x - 3\sqrt{x} + 2 = 0$$

$$1 - 3\sqrt{1} + 2 \stackrel{?}{=} 0$$

$$1 - 3 + 2 \stackrel{?}{=} 0$$

$$0 = 0 \checkmark$$

### Try It

8) Solve:  $x - 7\sqrt{x} + 12 = 0$ .

**Solution**

$x = 9, x = 16$

### Try It

9) Solve:  $x - 6\sqrt{x} + 8 = 0$ .

**Solution**

$$x = 4, x = 16$$

Substitutions for rational exponents can also help us solve an equation in quadratic form. Think of the properties of exponents as you begin the next example.

**Example 4.4.4**

Solve:  $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 = 0$ .

**Solution**

The  $x^{\frac{1}{3}}$  in the middle term is squared in the first term  $\left(x^{\frac{1}{3}}\right)^2 = x^{\frac{2}{3}}$ . If we let  $u = x^{\frac{1}{3}}$  and substitute, our trinomial will be in  $ax^2 + bx + c = 0$  form.

**Step 1: Rewrite the trinomial to prepare for the substitution.**

$$\left(x^{\frac{1}{3}}\right)^2 - 2\left(x^{\frac{1}{3}}\right) - 24 = 0$$

**Step 2: Let  $u = x^{\frac{1}{3}}$  and substitute.**

$$u^2 - 2u - 24 = 0$$

**Step 3: Solve by factoring.**

$$(u - 6)(u + 4) = 0$$

$$u - 6 = 0$$

$$u = 6$$

$$u + 4 = 0$$

$$u = -4$$

**Step 4: Replace  $u$  with  $x^{\frac{1}{3}}$ .**

$$x^{\frac{1}{3}} = 6, \quad x^{\frac{1}{3}} = -4$$

**Step 5: Solve for  $x$  by cubing both sides.**

$$\begin{aligned} \left(x^{\frac{1}{3}}\right)^3 &= (6)^3 \\ x &= 216 \end{aligned}$$

$$\begin{aligned} \left(x^{\frac{1}{3}}\right)^3 &= (-4)^3 \\ x &= -64 \end{aligned}$$

**Step 6: Check:**

$$x = \text{rgb}[1.0, 0.0, 0.0]216$$

$$\begin{aligned} x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 &= 0 \\ (216)^{\frac{2}{3}} - 2(216)^{\frac{1}{3}} - 24 &\stackrel{?}{=} 0 \\ 36 - 12 - 24 &\stackrel{?}{=} 0 \\ 0 &= 0 \checkmark \end{aligned}$$

$$x = \text{rgb}[1.0, 0.0, 0.0] - 64$$

$$\begin{aligned} x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 &= 0 \\ (-64)^{\frac{2}{3}} - 2(-64)^{\frac{1}{3}} - 24 &\stackrel{?}{=} 0 \\ 16 + 8 - 24 &\stackrel{?}{=} 0 \\ 0 &= 0 \checkmark \end{aligned}$$



**Try It**

10) Solve:  $x^{\frac{2}{3}} - 5x^{\frac{1}{3}} - 14 = 0$ .

**Solution**

$x = -8, x = 343$

**Try It**

11) Solve:  $x^{\frac{1}{2}} + 8x^{\frac{1}{4}} + 15 = 0$ .

**Solution**

$x = 81, x = 625$

In the next example, we need to keep in mind the definition of a negative exponent as well as the properties of exponents.

**Example 4.4.5**

Solve:  $3x^{-2} - 7x^{-1} + 2 = 0$ .

**Solution**

The  $x^{-1}$  in the middle term is squared in the first term  $(x^{-1})^2 = x^{-2}$ . If we let  $u = x^{-1}$  and substitute, our trinomial will be in  $ax^2 + bx + c = 0$  form.

**Step 1: Rewrite the trinomial to prepare for the substitution.**

$$3(x^{-1})^2 - 7(x^{-1}) + 2 = 0$$

**Step 2: Let  $u = x^{-1}$  and substitute.**

$$3u^2 - 7u + 2 = 0$$

**Step 3: Solve by factoring.**

$$(3u - 1)(u - 2) = 0$$

$$3u - 1 = 0$$

$$u = \frac{1}{3}$$

$$u - 2 = 0$$

$$u = 2$$

**Step 4: Replace  $u$  with  $x^{-1}$ .**

$$x^{-1} = \frac{1}{3}, \quad x^{-1} = 2$$

**Step 5: Solve for  $x$  by taking the reciprocal since  $x^{-1} = \frac{1}{x}$ .**

$$x = 3, \quad x = \frac{1}{2}$$

**Step 6: Check:**

Formula does not parse

**Try It**

12) Solve:  $8x^{-2} - 10x^{-1} + 3 = 0$ .

**Solution**

$$x = \frac{4}{3}, x = 2$$

**Try It**

13) Solve:  $6x^{-2} - 23x^{-1} + 20 = 0$ .

**Solution**

$$x = \frac{2}{5}, x = \frac{3}{4}$$

Access this online resource for additional instruction and practice with solving quadratic equations.

- Solving Equations in Quadratic Form

## Key Concepts

- How to solve equations in quadratic form.
  1. Identify a substitution that will put the equation in quadratic form.
  2. Rewrite the equation with the substitution to put it in quadratic form.
  3. Solve the quadratic equation for  $u$ .
  4. Substitute the original variable back into the results, using the substitution.
  5. Solve for the original variable.
  6. Check the solutions.

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

*<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=1023#h5p-30>*

b) On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

# 4.5 SOLVE APPLICATIONS OF QUADRATIC EQUATIONS

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## Learning Objectives

By the end of this section, you will be able to:

- Solve applications modelled by quadratic equations

## Try It

Before you get started, take this readiness quiz:

- 1) The sum of two consecutive odd numbers is  $-100$ . Find the numbers.
- 2) Solve:  $\frac{2}{x+1} + \frac{1}{x-1} = \frac{1}{x^2-1}$ .
- 3) Find the length of the hypotenuse of a right triangle with legs  $5$  inches and  $12$  inches.

## Solve Applications Modelled by Quadratic Equations

We solved some applications that are modelled by quadratic equations earlier, when the only method we had to

solve them was factoring. Now that we have more methods to solve quadratic equations, we will take another look at applications.

Let's first summarize the methods we now have to solve quadratic equations.

## Methods to Solve Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

As you solve each equation, choose the method that is most convenient for you to work the problem. As a reminder, we will copy our usual Problem-Solving Strategy here so we can follow the steps.

## HOW TO

### Use a Problem-Solving Strategy.

1. Read the problem. Make sure all the words and ideas are understood.
2. Identify what we are looking for.
3. Name what we are looking for. Choose a variable to represent that quantity.
4. Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. Solve the equation using algebra techniques.
6. Check the answer in the problem and make sure it makes sense.
7. Answer the question with a complete sentence

We have solved number applications that involved consecutive even and odd integers, by modelling the situation with linear equations. Remember, we noticed each even integer is **2** more than the number preceding

it. If we call the first one  $n$ , then the next one is  $n + 2$ . The next one would be  $n + 2 + 2$  or  $n + 4$ . This is also true when we use odd integers. One set of even integers and one set of odd integers are shown below.

**Consecutive even integers**

64, 66, 68

$n$     1<sup>st</sup> even integer  
 $n + 2$     2<sup>nd</sup> consecutive even integer  
 $n + 4$     3<sup>rd</sup> consecutive even integer

**Consecutive odd integers**

77, 79, 81

$n$     1<sup>st</sup> odd integer  
 $n + 2$     2<sup>nd</sup> consecutive odd integer  
 $n + 4$     3<sup>rd</sup> consecutive odd integer

Some applications of odd or even consecutive integers are modelled by quadratic equations. The notation above will be helpful as you name the variables.

### Example 4.5.1

The product of two consecutive odd integers is **195**. Find the integers.

**Solution**

**Step 1: Read the problem.**

**Step 2: Identify what we are looking for.**

We are looking for two consecutive odd integers.

**Step 3: Name what we are looking for.**

Let  $n$  = the first odd integer.

Let  $n + 2$  = the next odd integer

**Step 4: Translate into an equation. State the problem in one sentence.**

“The product of two consecutive odd integers is **195**.”

The product of the first odd integer and the second odd integer is **195**.

Translate into the equation

$$n(n + 2) = 195$$

$$n^2 + 2n = 195$$

**Step 5: Solve the equation. Distribute. Write the equation in standard form. Factor.**

$$n^2 + 2n - 195 = 0$$

$$(n + 15)(n - 13) = 0$$

Use the Zero Product Property. Solve each equation.

$$n + 15 = 0$$

$$n = -15$$

$$n - 13 = 0$$

$$n = 13$$

There are two values of  $n$  that are solutions. This will give us two pairs of consecutive odd integers for our solutions.

First odd integer is 13.  
 Next odd integer is 15.  
 $13 \times 15 = 195$

First odd integer is -13.  
 Next odd integer is -15.  
 $-13 \times -15 = 195$

**Step 6: Check the answer. Do these pairs work?**

Are they consecutive odd integers?

$$13, 15 = \text{yes} \quad -13, -15 = \text{yes}$$

Is their product 195?

$$13 \times 15 = 195, \text{ yes.}$$

$$-13(-15) = 195, \text{ yes.}$$

**Step 7: Answer the question.**

Two consecutive odd integers whose product is 195 are 13, 15 and -13, -15.



## Try It

4) The product of two consecutive odd integers is **99**. Find the integers.

**Solution**

The two consecutive odd integers whose product is **99** are **9, 11**, and **−9, −11**

## Try It

5) The product of two consecutive even integers is **168**. Find the integers.

**Solution**

The two consecutive even integers whose product is **128** are **12, 14** and **−12, −14**.

We will use the formula for the area of a triangle to solve the next example.

## Area of a Triangle

For a triangle with base,  $b$ , and height,  $h$ , the area,  $A$ , is given by the formula  $A = \frac{1}{2}bh$

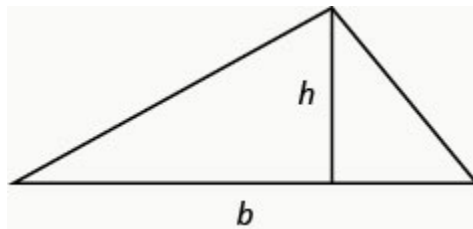


Figure 4.5.1

Recall that when we solve geometric applications, it is helpful to draw the figure.

### Example 4.5.2

An architect is designing the entryway of a restaurant. She wants to put a triangular window above the doorway. Due to energy restrictions, the window can only have an area of **120** square feet and the architect wants the base to be **4** feet more than twice the height. Find the base and height of the window.

#### Solution

##### Step 1: Read the problem.

Draw a picture.

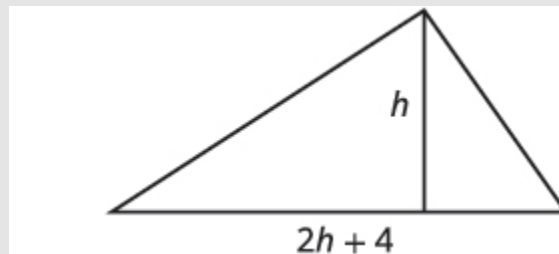


Figure 4.5.2

##### Step 2: Identify what we are looking for.

We are looking for the base and height.

##### Step 3: Name what we are looking for.

Let  $h$  = the height of the triangle.

Let  $2h + 4$  = the base of the triangle

##### Step 4: Translate into an equation.

We know the area. Write the formula for the area of a triangle.

$$A = \frac{1}{2}bh$$

**Step 5: Solve the equation.**

Substitute in the values.

$$120 = \frac{1}{2}(2h + 4)h$$

Distribute.

$$120 = h^2 + 2h$$

This is a quadratic equation, rewrite it in standard form.

$$h^2 + 2h - 120 = 0$$

Factor.

$$(h - 10)(h + 12) = 0$$

Use the Zero Product Property.

$$h - 10 = 0 \quad h + 12 = 0$$

Simplify.

$$h = 10, \quad \overline{)h = -12}$$

Since  $h$  is the height of a window, a value of  $h = -12$  does not make sense.

The height of the triangle  $h = 10$ .

The base of the triangle  $2h + 4$ .

$$2 \times 10 + 4 = 24$$

**Step 6: Check the answer.**

Does a triangle with height **10** and base **24** have area **120**? Yes.

**Step 7: Answer the question.**

The height of the triangular window is **10** feet and the base is **24** feet.

### Try It

6) Find the base and height of a triangle whose base is four inches more than six times its height and has an area of **456** square inches.

**Solution**

The height of the triangle is **12** inches and the base is **76** inches.

### Try It

7) If a triangle that has an area of **110** square feet has a base that is two feet less than twice the height, what is the length of its base and height?

**Solution**

The height of the triangle is **11** feet and the base is **20** feet.

In the two preceding examples, the number in the radical in the Quadratic Formula was a perfect square and so the solutions were rational numbers. If we get an irrational number as a solution to an application problem, we will use a calculator to get an approximate value.

We will use the formula for the area of a rectangle to solve the next example.

## Area of a Rectangle

For a rectangle with length,  $L$ , and width,  $W$ , the area,  $A$ , is given by the formula  $A = LW$ .

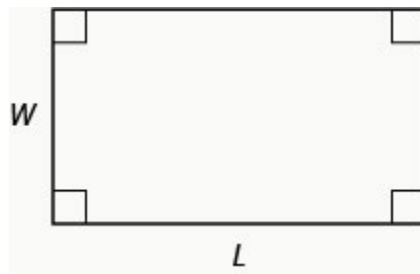


Figure 4.5.3

### Example 4.5.3

Mike wants to put **150** square feet of artificial turf in his front yard. This is the maximum area of artificial turf allowed by his homeowners association. He wants to have a rectangular area of turf with length one foot less than **3** times the width. Find the length and width. Round to the nearest tenth of a foot.

#### Solution

##### Step 1: Read the problem.

Draw a picture.



Figure 4.5.4

##### Step 2: Identify what we are looking for.

We are looking for the length and width.

##### Step 3: Name what we are looking for.

Let  $w$  = the width of the rectangle.

Let  $3w - 1$  = the length of the rectangle.

##### Step 4: Translate into an equation.

We know the area. Write the formula for the area of a rectangle.

$$A = L \times W$$

**Step 5: Solve the equation. Substitute in the values.**

$$150 = (3w - 1)w$$

Distribute.

$$150 = 3w^2 - w$$

This is a quadratic equation; rewrite it in standard form.

Solve the equation using the Quadratic Formula.

$$3w^2 - w - 150 = 0$$

Identify the  $a$ ,  $b$ ,  $c$  values.

$$a = 3, b = -1, c = -150$$

Write the Quadratic Formula.

$$w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Then substitute in the values of  $a$ ,  $b$ ,  $c$ .

$$w = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-150)}}{2(3)}$$

Simplify.

$$w = \frac{1 \pm \sqrt{1 + 1800}}{6}$$

$$w = \frac{1 \pm \sqrt{1801}}{6}$$

Rewrite to show two solutions.

$$w = \frac{1 + \sqrt{1801}}{6}, w = \frac{1 - \sqrt{1801}}{6}$$

Approximate the answers using a calculator.

We eliminate the negative solution for the width.

$$w \approx 7.2, \quad \cancel{w \approx -6.9}$$

$$\text{width } w \approx 7.2$$

$$\text{length} \approx 3w - 1$$

$$\approx 3(7.2) - 1$$

$$\approx 20.6$$

**Step 6: Check the answer.**

Make sure that the answers make sense. Since the answers are approximate, the area will not come out exactly to **150**

**Step 7: Answer the question.**

The width of the rectangle is approximately **7.2** feet and the length is approximately **20.6** feet.

**Try It**

8) The length of a **200** square foot rectangular vegetable garden is four feet less than twice the width. Find the length and width of the garden, to the nearest tenth of a foot.

**Solution**

The length of the garden is approximately **18** feet and the width **11** feet.

## Try It

9) A rectangular tablecloth has an area of **80** square feet. The width is **5** feet shorter than the length. What are the length and width of the tablecloth to the nearest tenth of a foot.?

### Solution

The length of the tablecloth is approximately **11.8** feet and the width **6.8** feet.

The Pythagorean Theorem gives the relation between the legs and hypotenuse of a right triangle. We will use the Pythagorean Theorem to solve the next example.

## Pythagorean Theorem

In any right triangle, where  $a$  and  $b$  are the lengths of the legs, and  $c$  is the length of the hypotenuse,  $a^2 + b^2 = c^2$ .

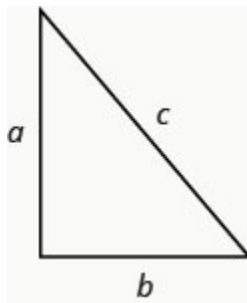


Figure 5.4.5



### Example 4.5.4

Rene is setting up a holiday light display. He wants to make a 'tree' in the shape of two right triangles, as shown below, and has two **10**-foot strings of lights to use for the sides. He will attach the lights to the top of a pole and to two stakes on the ground. He wants the height of the pole to be the same as the distance from the base of the pole to each stake. How tall should the pole be?

#### Solution

##### Step 1: Read the problem.

Draw a picture.

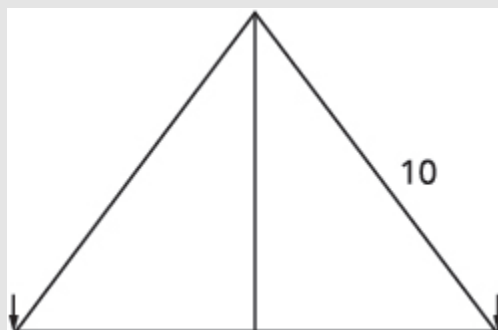


Figure 5.4.6

##### Step 2: Identify what we are looking for.

We are looking for the height of the pole.

##### Step 3: Name what we are looking for.

The distance from the base of the pole to either stake is the same as the height of the pole.

Let  $x$  = the height of the pole.

Let  $x$  = the distance from pole to stake.

Each side is a right triangle. We draw a picture of one of them.

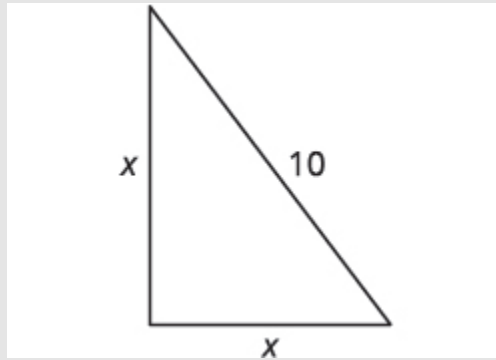


Figure 4.5.7

**Step 4: Translate into an equation.**

We can use the Pythagorean Theorem to solve for  $x$ .

Write the Pythagorean Theorem.

$$a^2 + b^2 = c^2$$

**Step 5: Solve the equation. Substitute.**

$$x^2 + x^2 = 10^2$$

Simplify.

$$2x^2 = 100$$

Divide by 2 to isolate the variable.

$$\frac{2x^2}{2} = \frac{100}{2}$$

Simplify.

$$x^2 = 50$$

Use the Square Root Property.

$$x = \pm\sqrt{50}$$

Simplify the radical.

$$x = \pm 5\sqrt{2}$$

Rewrite to show two solutions.

$$x = 5\sqrt{2}, \quad x = -5\sqrt{2}$$

If we approximate this number to the nearest tenth with a calculator, we find  $x \approx 7.1$ .

**Step 6: Check the answer.**

Check on your own in the Pythagorean Theorem.

**Step 7: Answer the question.**

The pole should be about **7.1** feet tall.

## Try It

10) The sun casts a shadow from a flag pole. The height of the flag pole is three times the length of its shadow. The distance between the end of the shadow and the top of the flag pole is **20** feet. Find the length of the shadow and the length of the flag pole. Round to the nearest tenth.

**Solution**

The length of the flag pole's shadow is approximately **6.3** feet and the height of the flag pole is **18.9** feet.

## Try It

11) The distance between opposite corners of a rectangular field is four more than the width of the field. The length of the field is twice its width. Find the distance between the opposite corners. Round to the nearest tenth.

**Solution**

The distance between the opposite corners is approximately **7.2** feet.

The height of a projectile shot upward from the ground is modelled by a quadratic equation. The initial velocity,  $v_0$ , propels the object up until gravity causes the object to fall back down.

## Projectile motion

The height in feet,  $h$ , of an object shot upwards into the air with initial velocity,  $v_0$ , after  $t$  seconds is given by the formula:

$$h = -16t^2 + v_0t$$

We can use this formula to find how many seconds it will take for a firework to reach a specific height.

### Example 4.5.5

A firework is shot upwards with initial velocity **130** feet per second. How many seconds will it take to reach a height of **260** feet? Round to the nearest tenth of a second.

#### Solution

**Step 1: Read the problem.**

**Step 2: Identify what we are looking for.**

We are looking for the number of seconds, which is time.

**Step 3: Name what we are looking for.**

Let  $t$  = the number of seconds.

**Step 4: Translate into an equation. Use the formula.**

$$h = -16t^2 + v_0t$$

**Step 5: Solve the equation.**

We know the velocity  $v_0$  is **130** feet per second.

The height is **260** feet. Substitute the values.

$$260 = -16t^2 + 130t$$

This is a quadratic equation, rewrite it in standard form.

Solve the equation using the Quadratic Formula.

$$16t^2 - 130t + 260 = 0$$

Identify the values of  $a$ ,  $b$ ,  $c$ .

Write the Quadratic Formula.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Then substitute in the values of  $a$ ,  $b$ ,  $c$ .

$$t = \frac{-(-130) \pm \sqrt{(-130)^2 - 4(16)(260)}}{2(16)}$$

Simplify.

$$t = \frac{130 \pm \sqrt{16,900 - 16,640}}{32}$$

$$t = \frac{130 \pm 260}{32}$$

Rewrite to show two solutions.

$$t = \frac{130 + 260}{32}, t = \frac{130 - 260}{32}$$

Approximate the answer with a calculator.

$$t \approx 4.6 \text{ seconds}, t \approx 3.6 \text{ seconds}$$

**Step 6: Check the answer.**

The check is left to you.

**Step 7: Answer the question.**

The firework will go up and then fall back down. As the firework goes up, it will reach **260** feet after approximately **3.6** seconds. It will also pass that height on the way down at **4.6** seconds.

## Try It

12) An arrow is shot from the ground into the air at an initial speed of **108** ft/s. Use the formula  $h = -16t^2 + v_0t$  to determine when the arrow will be **180** feet from the ground. Round the nearest tenth.

### Solution

The arrow will reach **180** feet on its way up after **3** seconds and again on its way down after approximately **3.8** seconds.

## Try It

13) A man throws a ball into the air with a velocity of **96** ft/s. Use the formula  $h = -16t^2 + v_0t$  to determine when the height of the ball will be **48** feet. Round to the nearest tenth.

### Solution

The ball will reach **48** feet on its way up after approximately **.6** second and again on its way down after approximately **5.4** seconds.

We have solved uniform motion problems using the formula  $D = rt$  in previous chapters. We used a table like the one below to organize the information and lead us to the equation.

	$\begin{array}{c} \text{Rate} \\ \text{Time} \end{array} = \text{Distance}$		

The formula  $D = rt$  assumes we know  $r$  and  $t$  and use them to find  $D$ . If we know  $D$  and  $r$  and need to find  $t$ , we would solve the equation for  $t$  and get the formula  $t = \frac{D}{r}$ .

Some uniform motion problems are also modelled by quadratic equations.

### Example 4.5.6

Professor Smith just returned from a conference that was 2,000 miles east of his home. His total time in the airplane for the round trip was 9 hours. If the plane was flying at a rate of 450 miles per hour, what was the speed of the jet stream?

#### Solution

This is a uniform motion situation. A diagram will help us visualize the situation.

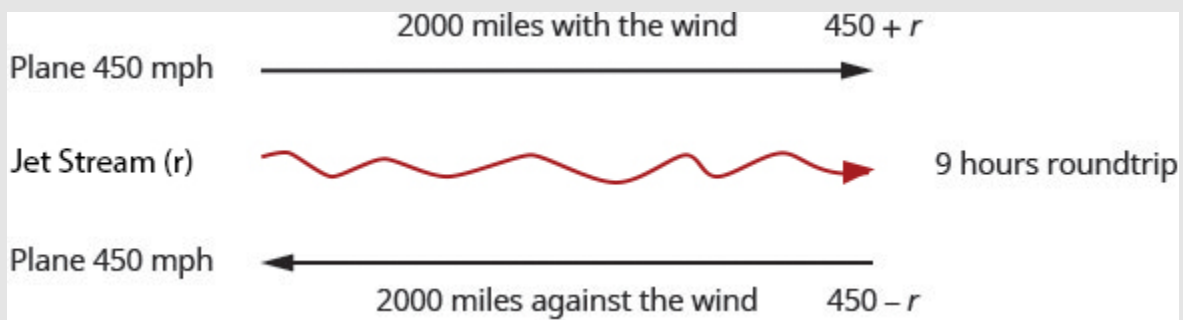


Figure 4.5.8

**Step 1: We fill in the chart to organize the information. We are looking for the speed of the jet stream.**

Let  $r$  = the speed of the jet stream.

When the plane flies with the wind, the wind increases its speed and so the rate is  $450 + r$ .

When the plane flies against the wind, the wind decreases its speed and the rate is  $450 - r$ .

**Step 2: Write in the rates. Write in the distances.**

Since  $D = r \times t$ , we solve for  $t$  and get  $t = \frac{D}{r}$ .

**Step 3: We divide the distance by the rate in each row, and place the expression in the time column.**

Type	\begin{array}{c}\text{Rate } \times \\ \text{Time } = \text{Distance}\end{array}		
Headwind	$450 - r$	$\frac{2000}{450-r}$	2000
Tailwind	$450 + r$	$\frac{2000}{450+r}$	2000
		9	

**Step 4: We know the times add to 9 and so we write our equation.**

$$\frac{2000}{450 - r} + \frac{2000}{450 + r} = 9$$

**Step 5: We multiply both sides by the LCD.**

$$(450 - r)(450 + r) \left( \frac{2000}{450 - r} + \frac{2000}{450 + r} \right) = 9(450 - r)(450 + r)$$

**Step 6: Simplify.**

$$2000(450 + r) + 2000(450 - r) = 9(450 - r)(450 + r)$$

**Step 7: Factor the 2,000.**

$$2000(450 + r + 450 - r) = 9(450^2 - r^2)$$

**Step 8: Solve.**

$$2000(900) = 9(450^2 - r^2)$$

**Step 9: Divide by 9.**

$$2000(100) = 450^2 - r^2$$

**Step 10: Simplify.**

$$200000 = 202500 - r^2$$

$$-2500 = -r^2$$

$$50 = r \text{ The speed of the jet stream.}$$



**Step 11: Check:**

Is 50 mph a reasonable speed for the jet stream? Yes.

If the plane is travelling 450 mph and the wind is 50 mph

Tailwind:

$$\begin{aligned} 450 + 50 &= 500 \text{ mph} \\ \frac{2000}{500} &= 4 \text{ hours} \end{aligned}$$

Headwind:

$$\begin{aligned} 450 - 50 &= 400 \text{ mph} \\ \frac{2000}{400} &= 5 \text{ hours} \end{aligned}$$

The times add to 9 hours, so it checks. The speed of the jet stream was 50 mph.

**Try It**

14) Mary Anne just returned from a visit with her grandchildren back east. The trip was 2400 miles from her home and her total time in the airplane for the round trip was 10 hours. If the plane was flying at a rate of 500 miles per hour, what was the speed of the jet stream?

**Solution**

The speed of the jet stream was 100 mph.

## Try It

15) Gerry just returned from a cross country trip. The trip was **3000** miles from his home and his total time in the airplane for the round trip was **11** hours. If the plane was flying at a rate of **550** miles per hour, what was the speed of the jet stream?

### Solution

The speed of the jet stream was **50** mph.

Work applications can also be modelled by quadratic equations. We will set them up using the same methods we used when we solved them with rational equations. We'll use a similar scenario now.

## Example 4.5.7

The weekly gossip magazine has a big story about the presidential election and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes **12** hours more than Press #2 to do the job and when both presses are running they can print the job in **8** hours. How long does it take for each press to print the job alone?

### Solution

This is a work problem. A chart will help us organize the information.

We are looking for how many hours it would take each press separately to complete the job.

#### **Step 1: State variables.**

Let  $x$  = the number of hours for Press #2 to complete the job.

#### **Step 2: Enter the hours per job for Press #1, Press #2, and when they work together.**

	Number of hours needed to complete the job.	Part of job completed/hour.
Press #1	$x + 12$	$\frac{1}{x+12}$
Press #2	$x$	$\frac{1}{x}$
Together	8	$\frac{1}{8}$

The part completed by Press #1 plus the part completed by Press #2 equals the amount completed together.

**Step 3: Translate to an equation.**

Work completed by Press #1 + Press #2 = Together

$$\frac{1}{x+12} + \frac{1}{x} = \frac{1}{8}$$

**Step 4: Solve.**

$$\frac{1}{x + 12} + \frac{1}{x} = \frac{1}{8}$$

**Step 5: Multiply by the LCD,  $8x(x + 12)$ .**

$$8x(x+12)\left(\frac{1}{x+12} + \frac{1}{x}\right) = 8x(x+12)\left(\frac{1}{8}\right)$$

**Step 6: Simplify.**

$$\begin{aligned} 8x + 8(x + 12) &= x(x + 12) \\ 8x + 8x + 96 &= x^2 + 12x \\ 0 &= x^2 - 4x - 96 \end{aligned}$$

**Step 7: Solve.**

$$0 = (x - 12)(x + 96)$$

$$\begin{aligned} x - 12 &= 0 \\ x &= 12 \text{ hours} \end{aligned}$$

$$\begin{aligned} x + 96 &= 0 \\ x &= -96 \text{ hours} \end{aligned}$$

Since the idea of negative hours does not make sense, we use the value  $x = 12$ .

$$rgb]1.0, 0.0, 0.012 + 12 = 24 \text{ hours}$$

$$rgb]1.0, 0.0, 0.012 = 12 \text{ hours}$$

**Step 8: Write our sentence answer.**

Press #1 would take **24** hours and Press #2 would take **12** hours to do the job alone.

## Try It

16) The weekly news magazine has a big story naming the Person of the Year and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes **6** hours more than Press #2 to do the job and when both presses are running they can print the job in **4** hours. How long does it take for each press to print the job alone?

**Solution**

Press #1 would take **12** hours, and Press #2 would take **6** hours to do the job alone.

## Try It

17) Erlinda is having a party and wants to fill her hot tub. If she only uses the red hose it takes **3** hours more than if she only uses the green hose. If she uses both hoses together, the hot tub fills in **2** hours. How long does it take for each hose to fill the hot tub?

**Solution**

The red hose take **6** hours and the green hose take **3** hours alone.

Access these online resources for additional instruction and practice with solving applications modelled by quadratic equations.

- Word Problems Involving Quadratic Equations
- Quadratic Equation Word Problems
- Applying the Quadratic Formula

## Key Concepts

- Methods to Solve Quadratic Equations
  - Factoring
  - Square Root Property
  - Completing the Square
  - Quadratic Formula
- How to use a Problem-Solving Strategy.
  1. Read the problem. Make sure all the words and ideas are understood.
  2. Identify what we are looking for.
  3. Name what we are looking for. Choose a variable to represent that quantity.
  4. Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
  5. Solve the equation using good algebra techniques.
  6. Check the answer in the problem and make sure it makes sense.

7. Answer the question with a complete sentence.

- Area of a Triangle

- For a triangle with base,  $b$ , and height,  $h$ , the area,  $A$ , is given by the formula

$$A = \frac{1}{2}bh.$$

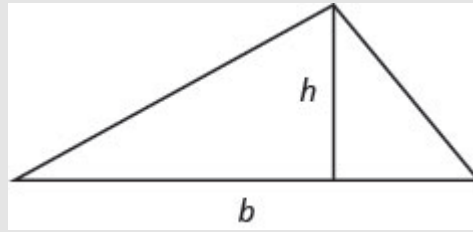


Figure 4.5.9

- Area of a Rectangle

- For a rectangle with length,  $L$ , and width,  $W$ , the area,  $A$ , is given by the formula

$$A = LW.$$

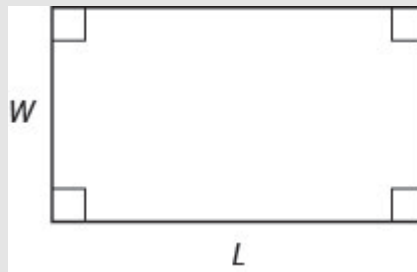


Figure 4.5.10

- Pythagorean Theorem

- In any right triangle, where  $a$  and  $b$  are the lengths of the legs, and  $c$  is the length of the hypotenuse,  $a^2 + b^2 = c^2$ .

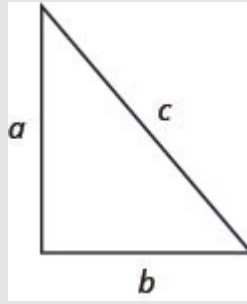


Figure 4.5.11

- Projectile motion
  - The height in feet,  $h$ , of an object shot upwards into the air with initial velocity,  $v_0$ , after  $t$  seconds is given by the formula  $h = -16t^2 + v_0t$ .

## Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

*<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=1073#h5p-31>*

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

# 4.6 GRAPH QUADRATIC FUNCTIONS USING PROPERTIES

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## Learning Objectives

By the end of this section, you will be able to:

- Recognize the graph of a quadratic function
- Find the axis of symmetry and vertex of a parabola
- Find the intercepts of a parabola
- Graph quadratic functions using properties
- Solve maximum and minimum applications

## Try It

Before you get started, take this readiness quiz:

- 1) Graph the function  $f(x) = x^2$  by plotting points.
- 2) Solve:  $2x^2 + 3x - 2 = 0$ .
- 3) Evaluate  $-\frac{b}{2a}$  when  $a = 3$  and  $b = -6$ .



## Recognize the Graph of a Quadratic Function

Previously we very briefly looked at the function  $f(x) = x^2$ , which we called the square function. It was one of the first non-linear functions we looked at. Now we will graph functions of the form  $f(x) = ax^2 + bx + c$  if  $a \neq 0$ . We call this kind of function a quadratic function.

### Quadratic Function

A **quadratic function**, where  $a$ ,  $b$ , and  $c$  are real numbers and  $a \neq 0$ , is a function of the form

$$f(x) = ax^2 + bx + c$$

We graphed the quadratic function  $f(x) = x^2$  by plotting points.

$x$	$f(x) = x^2$	$(x, f(x))$
-3	9	$(-3, 9)$
-2	4	$(-2, 4)$
-1	1	$(-1, 1)$
0	0	$(0, 0)$
1	1	$(1, 1)$
2	4	$(2, 4)$
3	9	$(3, 9)$

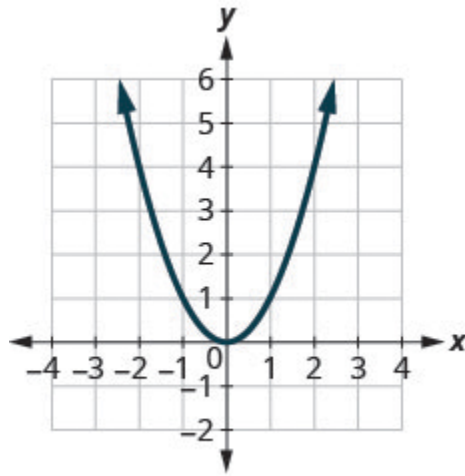


Figure 4.6.1

Every quadratic function has a graph that looks like this. We call this figure a parabola.

Let's practice graphing a parabola by plotting a few points.

### Example 4.6.1

Graph  $f(x) = x^2 - 1$ .

#### Solution

We will graph the function by plotting points.

**Step 1: Choose integer values for  $x$ , substitute them into the equation and simplify to find  $f(x)$ .**

Record the values of the ordered pairs in the chart.

$f(x) = x^2 - 1$	
$x$	$f(x)$
0	-1
1	0
-1	0
2	3
-2	3

**Step 2:** Plot the points, and then connect them with a smooth curve. The result will be the graph of the function  $f(x) = x^2 - 1$ .

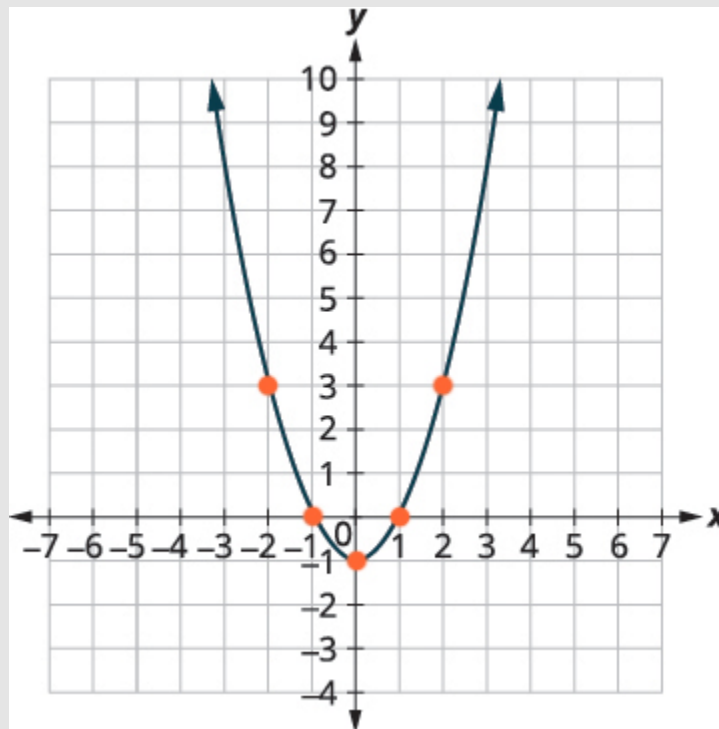


Figure 4.6.2

## Try It

4) Graph  $f(x) = -x^2$ .

### Solution

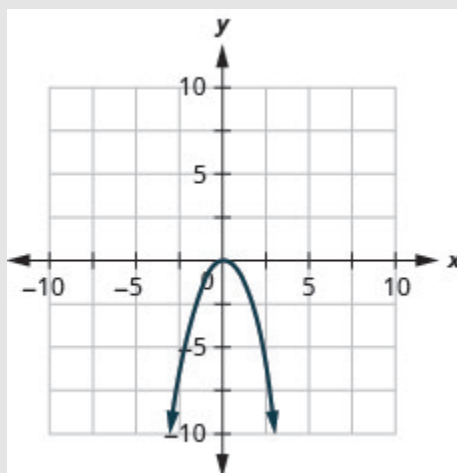


Figure 4.6.3

## Try It

5) Graph  $f(x) = x^2 + 1$ .

### Solution

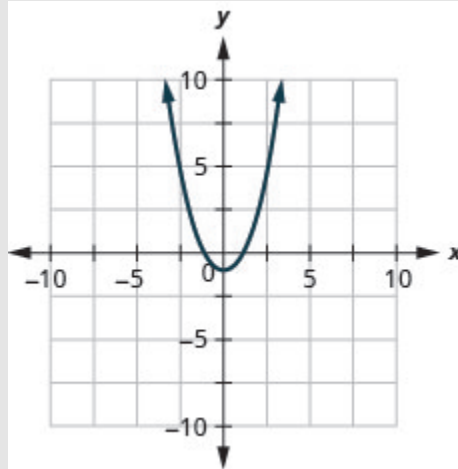
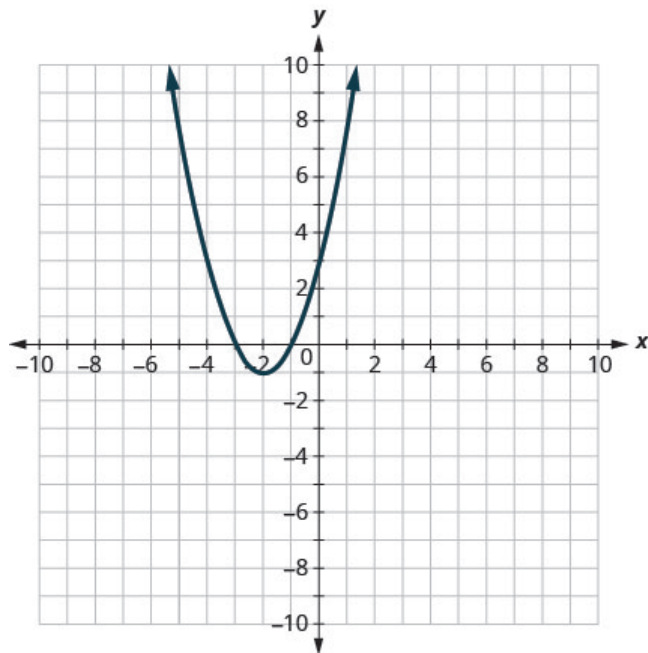


Figure 4.6.4

All graphs of quadratic functions of the form  $f(x) = ax^2 + bx + c$  are parabolas that open upward or downward. See Figure 4.6.5 below.

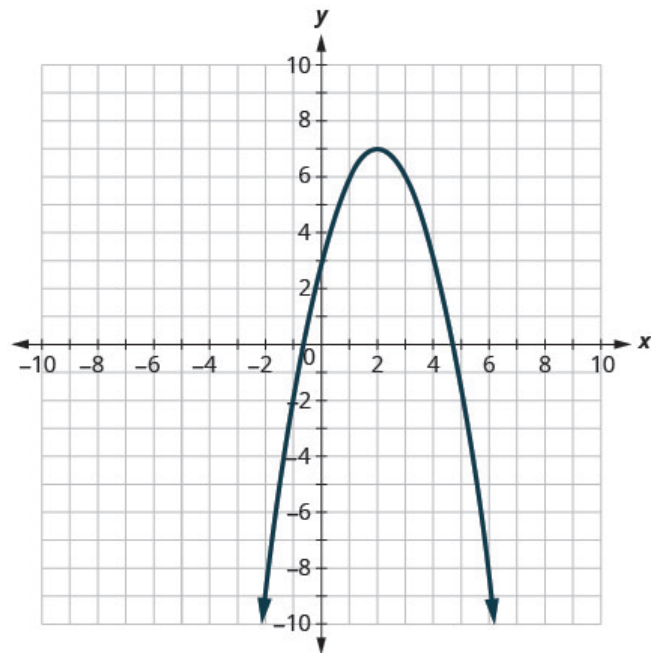


$$f(x) = a^2 + bx + c$$

$$f(x) = x^2 + 4x + 3$$

$$a > 0$$

opens upward



$$f(x) = a^2 + bx + c$$

$$f(x) = -x^2 + 4x + 3$$

$$a < 0$$

opens downward

Figure 4.6.5

Notice that the only difference in the two functions is the negative sign before the quadratic term ( $x^2$ ) in the equation of the graph above. When the quadratic term is positive, the parabola opens upward, and when the quadratic term is negative, the parabola opens downward.

## Parabola Orientation

For the graph of the quadratic function  $f(x) = ax^2 + bx + c$ , if:

- $a > 0$ , the parabola opens upward
- $a < 0$ , the parabola opens downward

### Example 4.6.2

Determine whether each parabola opens upward or downward:

a.  $f(x) = -3x^2 + 2x - 4$

b.  $f(x) = 6x^2 + 7x - 9$

#### Solution

a.

**Step 1: Find the value of  $a$ .**

$$f(x) = ax^2 + bx + c$$

$$f(x) = -3x^2 + 2x - 4$$

$$a = -3$$

**Step 2: Determine which direction the parabola will open.**

Since the  $a$  is negative, the parabola will open downward.

b.

**Step 1: Find the value of  $a$ .**

$$f(x) = a(x - h)^2 + k$$

$$f(x) = a(x - 1)^2 + 0$$

$$f(0) = a(0 - 1)^2 + 0 = 6$$

**Step 2: Determine which direction the parabola will open.**

Since the  $a$  is positive, the parabola will open upward.

## Try It

6) Determine whether the graph of each function is a parabola that opens upward or downward:

a.  $f(x) = 2x^2 + 5x - 2$

b.  $f(x) = -3x^2 - 4x + 7$

**Solution**

a. up

b. down

## Try It

7) Determine whether the graph of each function is a parabola that opens upward or downward:

- a.  $f(x) = -2x^2 - 2x - 3$   
 b.  $f(x) = 5x^2 - 2x - 1$

**Solution**

- a. down  
 b. up

## Find the Axis of Symmetry and Vertex of a Parabola

Look again at Figure 4.6.5. Do you see that we could fold each parabola in half and then one side would lie on top of the other? The ‘fold line’ is a line of symmetry. We call it the axis of symmetry of the parabola.

We show the same two graphs again with the axis of symmetry. See Figure 4.6.6 below.

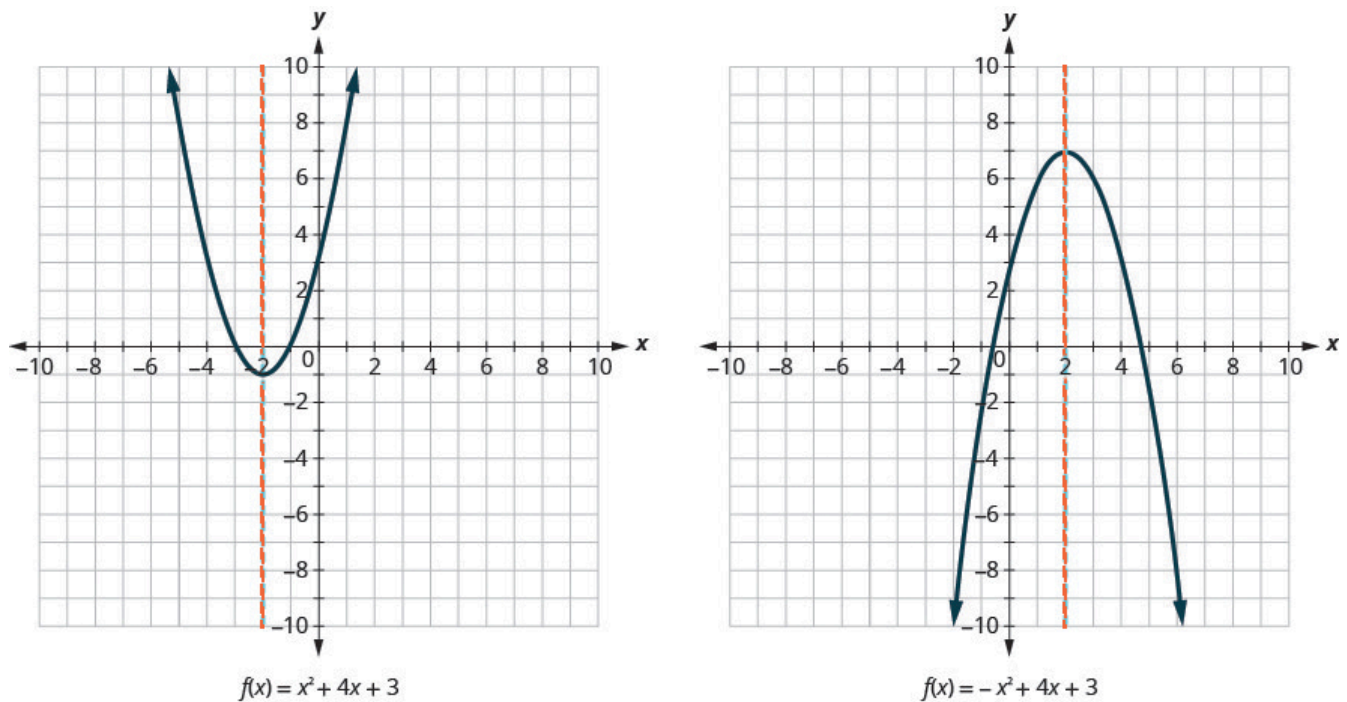


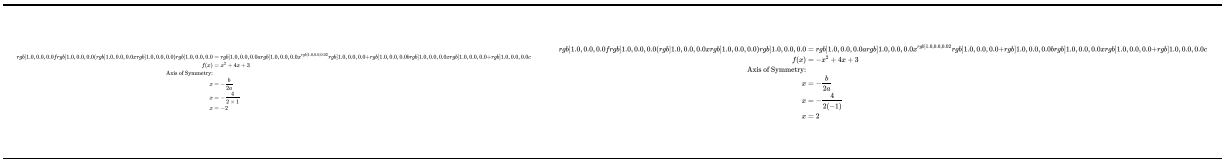
Figure 4.6.6

The equation of the axis of symmetry can be derived by using the Quadratic Formula. We will omit the derivation here and proceed directly to using the result. The equation of the axis of symmetry of the graph of

$$f(x) = ax^2 + bx + c \text{ is } x = -\frac{b}{2a}$$



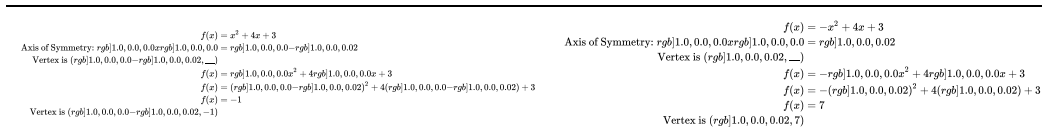
So to find the equation of symmetry of each of the parabolas we graphed above, we will substitute into the formula  $x = -\frac{b}{2a}$ .



Notice that these are the equations of the dashed blue lines on the graphs.

The point on the parabola that is the lowest (parabola opens up), or the highest (parabola opens down), lies on the axis of symmetry. This point is called the vertex of the parabola.

We can easily find the coordinates of the vertex, because we know it is on the axis of symmetry. This means its  $x$ -coordinate is  $-\frac{b}{2a}$ . To find the  $y$ -coordinate of the vertex we substitute the value of the  $x$ -coordinate into the quadratic function.



### Axis of Symmetry and Vertex of a Parabola

The graph of the function  $f(x) = ax^2 + bx + c$  is a parabola where:

- the axis of symmetry is the vertical line  $x = -\frac{b}{2a}$ .
- the vertex is a point on the axis of symmetry, so its  $x$ -coordinate is  $-\frac{b}{2a}$ .
- the  $y$ -coordinate of the vertex is found by substituting  $x = -\frac{b}{2a}$  into the quadratic equation.

### Example 4.6.3

For the graph of  $f(x) = 3x^2 - 6x + 2$  find:

- the axis of symmetry
- the vertex

#### Solution

a.

$$f(x) = 3x^2 - 6x + 2$$

#### Step 1: Find the axis of symmetry.

The axis of symmetry is the vertical line  $x = -\frac{b}{2a}$ .

#### Step 2: Substitute the values of $a$ , $b$ into the equation.

$$x = -\frac{-6}{2 \times 3}$$

#### Step 3: Simplify.

$$x = 1$$

#### Step 4: Write in a statement.

The axis of symmetry is the line  $x = 1$ .

b.

#### Step 1: Find the vertex.

The vertex is a point on the line of symmetry, so its  $x$ -coordinate will be  $x = 1$ .

#### Step 2: Find $f(1)$ .

$$f(1) = 3(1)^2 - 6(1) + 2$$

#### Step 3: Simplify.

$$f(1) = 3 \times [1.0, 0.0, 0.01] - 6 + 2$$

**Step 4: The result is the  $y$ -coordinate.**

$$f(1) = -1$$

**Step 5: Write as a statement.**

The vertex is  $(1, -1)$ .

## Try It

8) For the graph of  $f(x) = 2x^2 - 8x + 1$  find:

- the axis of symmetry
- the vertex

### Solution

- $x = 2$
- $(2, -7)$

## Try It

9) For the graph of  $f(x) = 2x^2 - 4x - 3$  find:

- the axis of symmetry
- the vertex

### Solution

- a.  $x = 1$   
 b.  $(1, -5)$

## Find the Intercepts of a Parabola

When we graphed linear equations, we often used the  $x$ - and  $y$ -intercepts to help us graph the lines. Finding the coordinates of the intercepts will help us to graph parabolas, too.

Remember, at the  $y$ -intercept the value of  $x$  is zero. So to find the  $y$ -intercept, we substitute  $x = 0$  into the function.

Let's find the  $y$ -intercepts of the two parabolas shown in Figure 4.6.7.

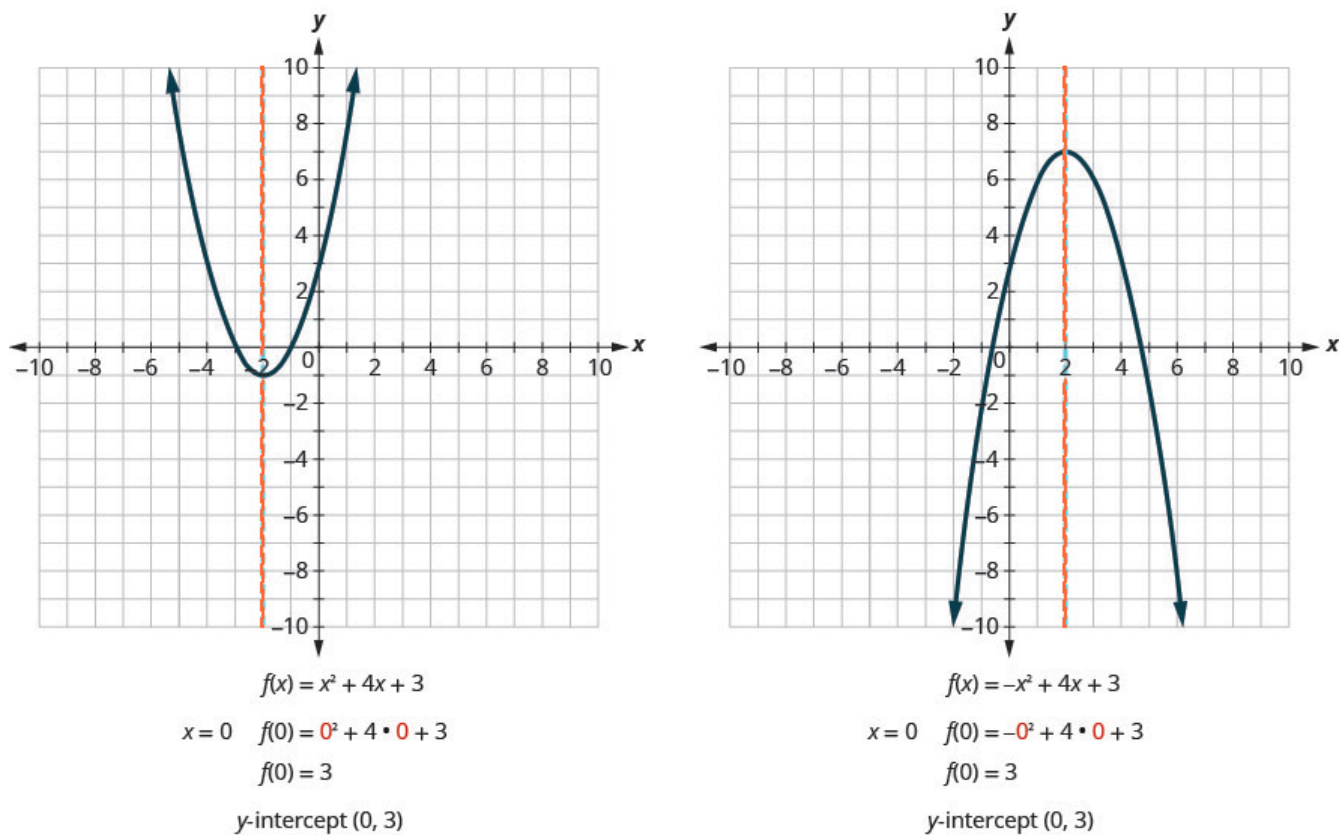


Figure 4.6.7

An  $x$ -intercept results when the value of  $f(x)$  is zero. To find an  $x$ -intercept, we let  $f(x) = 0$ . In other words, we will need to solve the equation  $0 = ax^2 + bx + c$  for  $x$ .

$$f(x) = ax^2 + bx + c$$

$$0 = ax^2 + bx + c$$

Solving quadratic equations like this is exactly what we have done earlier in this chapter!

We can now find the  $x$ -intercepts of the two parabolas we looked at. First we will find the  $x$ -intercepts of the parabola whose function is:

$$f(x) = x^2 + 4x + 3.$$

**Step 1: Let  $f(x) = 0$ .**

$$0 = x^2 + 4x + 3$$

**Step 2: Factor.**

$$0 = (x + 1)(x + 3)$$

**Step 3: Solve using the Zero Product Property.**

$$\begin{aligned} x + 1 &= 0 \\ x &= -1 \end{aligned}$$

$$\begin{aligned} x + 3 &= 0 \\ x &= -3 \end{aligned}$$

**Step 4: Write as a statement.**

The  $x$ -intercepts are  $(-1, 0)$  and  $(-3, 0)$ .

Now we will find the  $x$ -intercepts of the parabola whose function is:

$$f(x) = -x^2 + 4x + 3$$

**Step 1: Let  $f(x) = 0$ .**

This quadratic does not factor, so we use the Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**Step 2: Substitute in known values.**

$$a = rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.01, \quad b = rgb]0.0, 0.0, 1.04, \quad c = rgb]0.0, 0.5, 0.03$$

$$x = \frac{-rgb]0.0, 0.0, 1.04 \pm \sqrt{rgb]0.0, 0.0, 1.04^2 - 4(rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.01)(rgb]0.0, 0.5, 0.03)}}{2(rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.01)}$$

**Step 3: Simplify.**

$$x = \frac{-4 \pm \sqrt{16 + 12}}{-2}$$

$$x = \frac{-4 \pm \sqrt{28}}{-2}$$

$$x = \frac{-4 \pm 2\sqrt{7}}{-2}$$

$$x = \frac{-2(\pm 2\sqrt{7})}{-2}$$

$$x = 2 \pm \sqrt{7}$$

**Step 4: Write as a statement.**

The  $x$ -intercepts are  $(2 + \sqrt{7}, 0)$  and  $(2 - \sqrt{7}, 0)$ .

We will use the decimal approximations of the  $x$ -intercepts, so that we can locate these points on the graph:

$$(2 + \sqrt{7}, 0) \approx (4.6, 0) \qquad (2 - \sqrt{7}, 0) \approx (-0.6, 0)$$

Do these results agree with our graphs? See Figure 4.6.8.

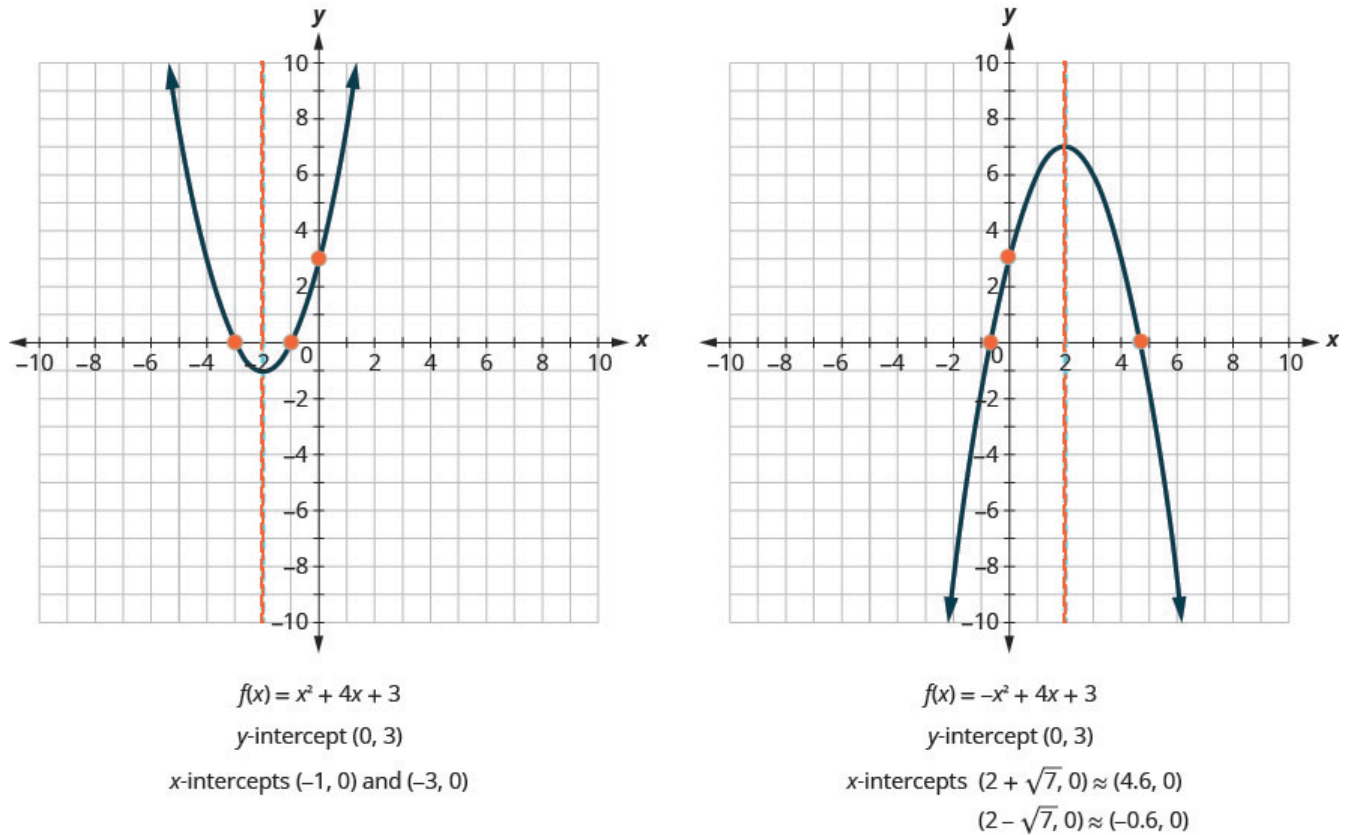


Figure 4.6.8

## Find the Intercepts of a Parabola

To find the intercepts of a parabola whose function is  $f(x) = ax^2 + bx + c$ :

$\begin{array}{l} \text{y-intercept} \\ \text{intercepts} \end{array}$ 
 $\begin{array}{l} \text{Let } x=0 \text{ and solve for } f(x) \\ \text{Let } f(x)=0 \text{ and solve for } x \end{array}$

### Example 4.6.3

Find the intercepts of the parabola whose function is  $f(x) = x^2 - 2x - 8$ .

#### Solution

**Step 1: To find the  $y$ -intercept**

Let  $x = 0$  and solve for  $f(x)$ .

$$f(x) = x^2 - 2x - 8$$

$$f(0) = (0)^2 - 2 \times (0) - 8$$

$$f(0) = -8$$

When  $x = 0$ , then  $f(0) = -8$ .

The  $y$ -intercept is the point  $(0, -8)$ .

**Step 2: To find the  $x$ -intercept**

Let  $f(x) = 0$  and solve for  $x$ .

$$f(x) = x^2 - 2x - 8$$

$$0 = x^2 - 2x - 8$$

**Step 3: Solve by factoring.**

$$(x - 4)(x + 2) = 0$$

$$x - 4 = 0$$

$$x = 4$$

$$x + 2 = 0$$

$$x = -2$$

When  $f(x) = 0$ , then  $x = 4$  or  $x = -2$ .

The  $x$ -intercepts are the points  $(4, 0)$  and  $(-2, 0)$ .

**Try It**

10) Find the intercepts of the parabola whose function is  $f(x) = x^2 + 2x - 8$ .



**Solution** $y$ -intercept:  $(0, -8)$  $x$ -intercepts:  $(-4, 0), (2, 0)$ **Try It**11) Find the intercepts of the parabola whose function is  $f(x) = x^2 - 4x - 12$ .**Solution** $y$ -intercept:  $(0, -12)$  $x$ -intercepts:  $(-2, 0), (6, 0)$ 

In this chapter, we have been solving quadratic equations of the form  $ax^2 + bx + c = 0$ . We solved for  $x$  and the results were the solutions to the equation.

We are now looking at quadratic functions of the form  $f(x) = ax^2 + bx + c$ . The graphs of these functions are parabolas. The  $x$ -intercepts of the parabolas occur where  $f(x) = 0$ .

For example:

**Quadratic equation**

$$\begin{aligned}x^2 - 2x - 15 &= 0 \\(x - 5)(x + 3) &= 0 \\x - 5 = 0 \quad x + 3 &= 0 \\x = 5 \quad x &= -3\end{aligned}$$

Let  $f(x) = 0$ .

**Quadratic function**

$$\begin{aligned}f(x) &= x^2 - 2x - 15 \\0 &= x^2 - 2x - 15 \\0 &= (x - 5)(x + 3) \\x - 5 &= 0 \quad x + 3 = 0 \\x &= 5 \quad x = -3 \\(5, 0) \text{ and } (-3, 0) & \\x\text{-intercepts}\end{aligned}$$

The solutions of the quadratic function are the  $x$  values of the  $x$ -intercepts.

Earlier, we saw that quadratic equations have 2, 1, or 0 solutions. The graphs below show examples of parabolas for these three cases. Since the solutions of the functions give the  $x$ -intercepts of the graphs, the number of  $x$ -intercepts is the same as the number of solutions.

Previously, we used the discriminant to determine the number of solutions of a quadratic function of the

form  $ax^2 + bx + c = 0$ . Now we can use the discriminant to tell us how many  $x$ -intercepts there are on the graph.

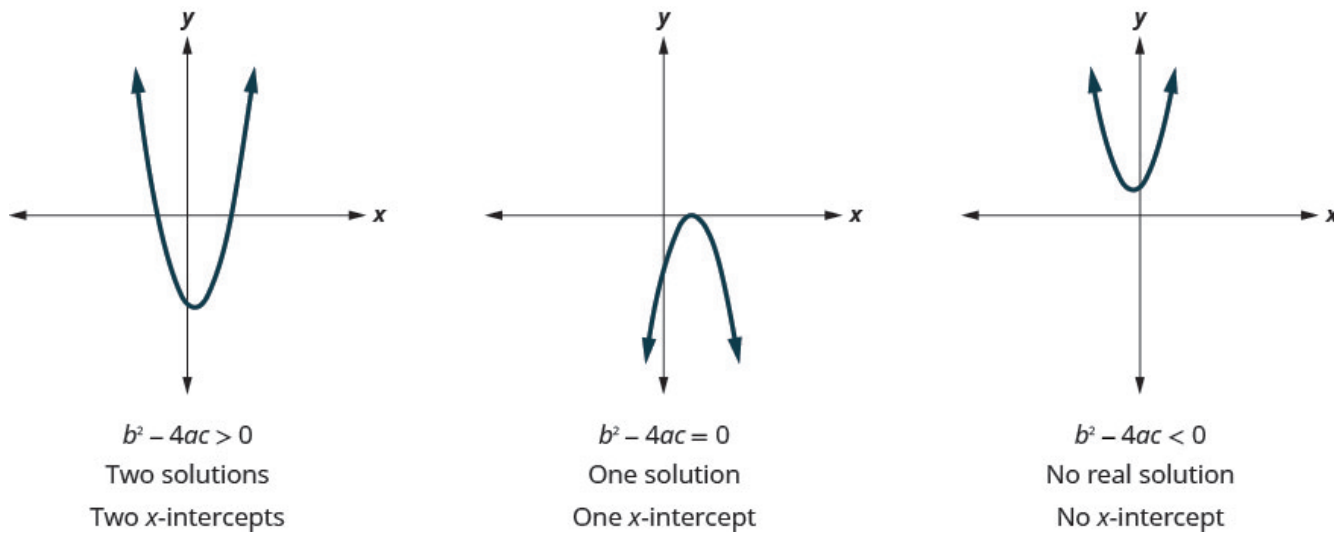


Figure 4.6.9

Before you to find the values of the  $x$ -intercepts, you may want to evaluate the discriminant so you know how many solutions to expect.

### Example 4.6.4

Find the intercepts of the parabola for the function  $f(x) = 5x^2 + x + 4$ .

#### Solution

##### Step 1: To find the $y$ -intercept

Let  $x = 0$  and solve for  $f(x)$ .

$$f(0) = 5 \times [1.0, 0.0, 0.00]^2 + [1.0, 0.0, 0.00] + 4$$

$$f(0) = 4$$

When  $x = 0$ , then  $f(0) = 4$ .

The  $y$ -intercept is the point  $(0, 4)$ .

**Step 2: To find the  $x$ -intercept**

Let  $f(x) = 0$  and solve for  $x$ .

$$f(x) = 5x^2 + x + 4$$

$$0 = 5x^2 + x + 4$$

**Step 3: Find the value of the discriminant to predict the number of solutions which is also the number of  $x$ -intercepts.**

$$\begin{aligned} &= b^2 - 4ac \\ &= 1^2 - 4 \times 5 \times 4 \\ &= 1 - 80 \\ &= -79 \end{aligned}$$

Since the value of the discriminant is negative, there is no real solution to the equation. There are no  $x$ -intercepts.

**Try It**

12) Find the intercepts of the parabola whose function is  $f(x) = 3x^2 + 4x + 4$ .

**Solution**

$y$ -intercept:  $(0, 4)$  no  $x$ -intercept

## Try It

13) Find the intercepts of the parabola whose function is  $f(x) = x^2 - 4x - 5$ .

### Solution

$y$ -intercept:  $(0, -5)$

$x$ -intercepts:  $(-1, 0)$ ,  $(5, 0)$

## Graph Quadratic Functions Using Properties

Now we have all the pieces we need in order to graph a quadratic function. We just need to put them together. In the next example we will see how to do this.

### Example 4.6.5

Graph  $f(x) = x^2 - 6x + 8$  by using its properties.

### Solution

**Step 1: Determine whether the parabola opens upward or downward.**

Look at  $a$  in the equation.

$rgb]1.0, 0.0, 0.0argb]1.0, 0.0, 0.0 =rgb]1.0, 0.0, 0.01rgb]1.0, 0.0, 0.0, rgb]1.0, 0.0, 0.0 =rgb]1.0, 0.0, 0.0 -rgb]1.0, 0.0, 0.06rgb]1.0, 0.0, 0.0, rgb]1.0, 0.0, 0.0crgb]1.0, 0.0, 0.0 =rgb]1.0, 0.0, 0.08$

Since  $a$  is positive, the parabola opens upward.  $rgb]1.0, 0.0, 0.0\langle$

**Step 2: Find the axis of symmetry**

$$f(x) = x^2 - 6x + 8$$

The axis of symmetry is the line  $x = -\frac{b}{2a}$ .

**Axis of Symmetry**

$$x = -\frac{b}{2a}$$

$$x = -\frac{(-6)}{2 \times 1}$$

$$x = 3$$

The axis of symmetry is the line  $x = 3$ .

**Step 3: Find the vertex.**

The vertex is on the axis of symmetry. Substitute  $x = 3$  into the function.

**Vertex**

$$f(x) = x^2 - 6x + 8$$

$$f(3) = (rgb[1.0, 0.0, 0.03])^2 - 6(rgb[1.0, 0.0, 0.03]) + 8$$

$$f(3) = -1$$

The vertex is  $(3, -1)$ .

**Step 4: Find the  $y$ -intercept.**

Find the point symmetric to the  $y$ -intercept across the axis of symmetry.

We find  $f(0)$ .

**$y$ -intercept**

$$f(x) = x^2 - 6x + 8$$

$$f(0) = (rgb[1.0, 0.0, 0.00])^2 - 6(rgb[1.0, 0.0, 0.00]) + 8$$

$$f(0) = 8$$

The  $y$ -intercept is  $(0, 8)$ .

We use the axis of symmetry to find a point symmetric to the  $y$ -intercept. The  $y$ -intercept is **3** units left of the axis of symmetry,  $x = 3$ .

A point **3** units to the right of the axis of symmetry has  $x = 6$ .

Point symmetric to  $y$ -intercept:  $(6, 8)$ .

**Step 5: Find the  $x$ -intercepts.**

Find additional points if needed.

We solve  $f(x) = 0$ .

We can solve this quadratic equation by factoring.

***x*-intercepts**

$$f(x) = x^2 - 6x + 8$$

$$rgb]1.0, 0.0, 0.00 = x^2 - 6x + 8$$

$$rgb]1.0, 0.0, 0.00 = (x - 2)(x - 4)$$

$$x = 2 \text{ or } x = 4$$

The  $x$ -intercepts are  $(2, 0)$  and  $(4, 0)$ .

**Step 6: Graph the parabola.**

We graph the vertex, intercepts, and the point symmetric to the  $y$ -intercept.

We connect these 5 points to sketch the parabola.

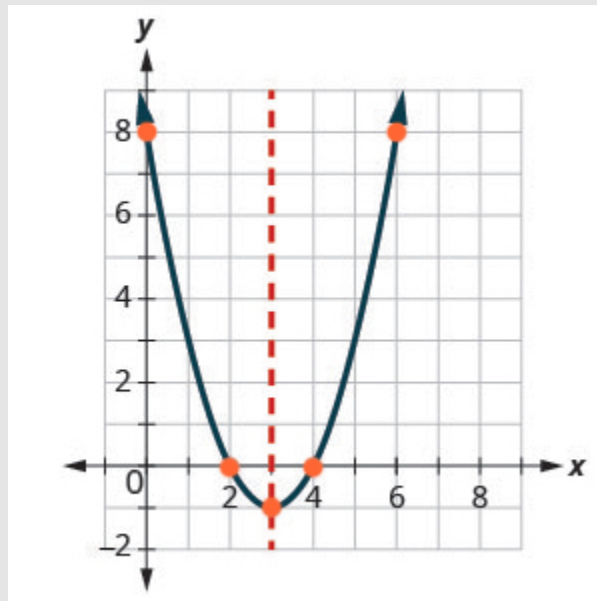


Figure 4.6.10

## Try It

14) Graph  $f(x) = x^2 + 2x - 8$  by using its properties.

### Solution

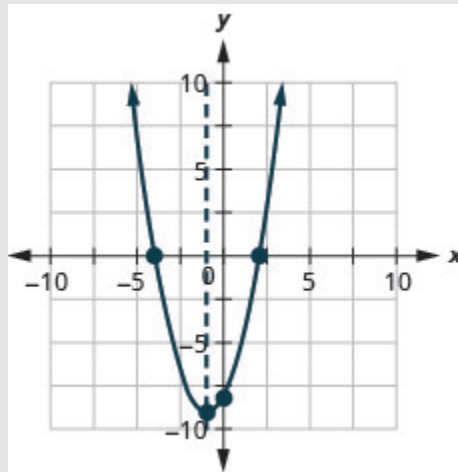


Figure 4.6.11

## Try It

15) Graph  $f(x) = x^2 - 8x + 12$  by using its properties.

### Solution

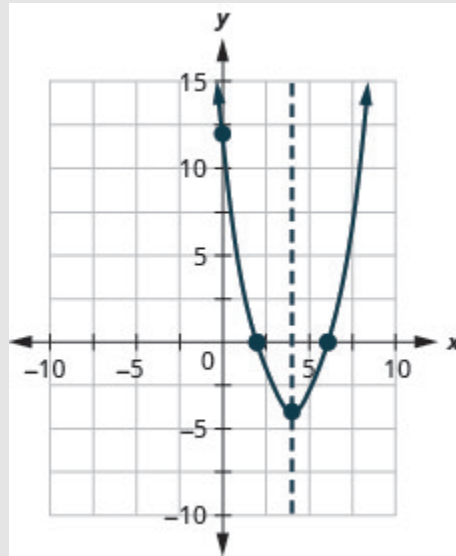


Figure 4.6.12

We list the steps to take in order to graph a quadratic function here.

## HOW TO

### To graph a quadratic function using properties.

1. Determine whether the parabola opens upward or downward.
2. Find the equation of the axis of symmetry.
3. Find the vertex.
4. Find the  $y$ -intercept. Find the point symmetric to the  $y$ -intercept across the axis of symmetry.
5. Find the  $x$ -intercepts. Find additional points if needed.
6. Graph the parabola.

We were able to find the  $x$ -intercepts in the last example by factoring. We find the  $x$ -intercepts in the next example by factoring, too.



## Example 4.6.7

Graph  $f(x) = x^2 + 6x - 9$  by using its properties.

### Solution

$$f(x) = -x^2 + 6x - 9$$

#### Step 1: Find the direction of the parabola.

Since  $a$  is  $-1$ , the parabola opens downward.

#### Step 2: Find the axis of symmetry and vertex.

To find the equation of the axis of symmetry, use  $x = -\frac{b}{2a}$ .

$$\begin{aligned} x &= -\frac{b}{2a} \\ x &= -\frac{6}{2(-1)} \\ x &= 3 \end{aligned}$$

The axis of symmetry is  $x = 3$ .

The vertex is on the line  $x = 3$ .

#### Step 3: Graph the axis of symmetry and vertex.

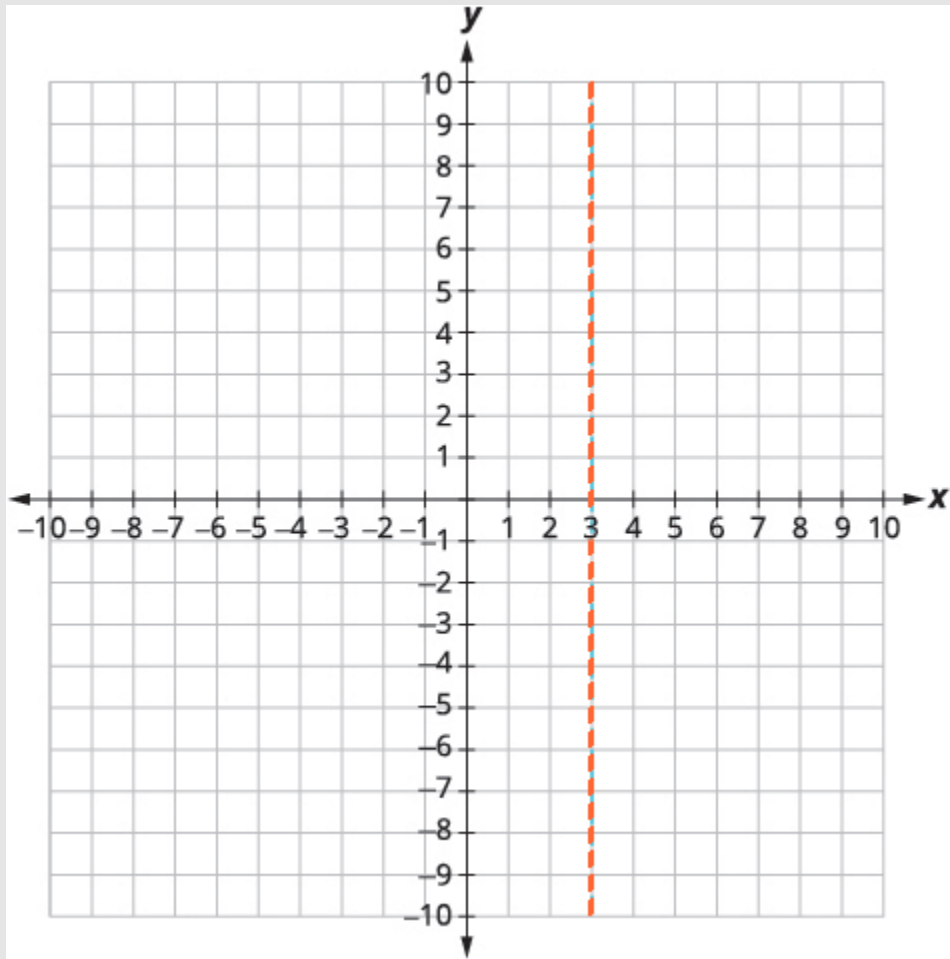


Figure 4.6.13

**Step 4: Find  $f(3)$ .**

$$f(x) = -x^2 + 6x - 9$$

$$f(3) = -1.0, 0.0, 0.03^2 + 6 \times 1.0, 0.0, 0.03 - 9$$

$$f(3) = -9 + 18 - 9$$

$$f(3) = 0$$

**Step 5: Graph  $f(3)$ .**

The vertex is  $(3, 0)$ .

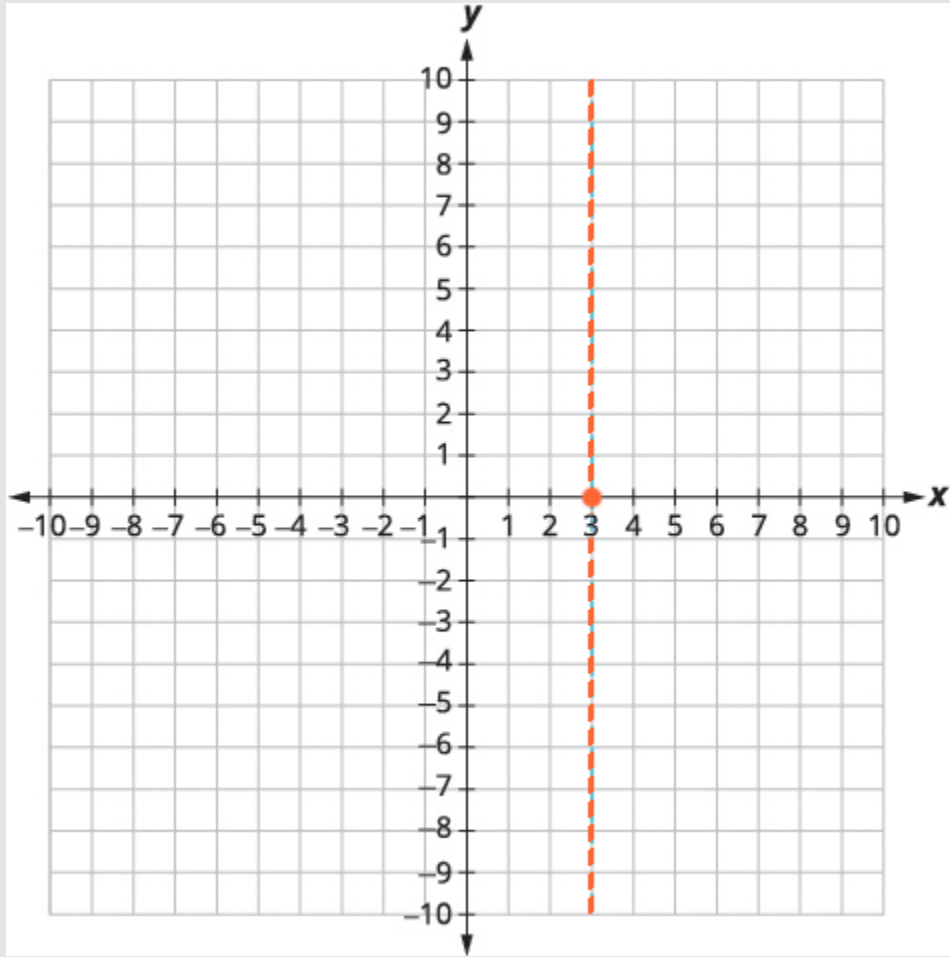


Figure 4.6.14

**Step 6: Find  $f(0)$ .** The  $y$ -intercept occurs when  $x = 0$ .

$$f(x) = -x^2 + 6x - 9$$

**Step 7: Substitute  $x = 0$ .**

$$f(0) = -[1.0, 0.0, 0.00]^2 + 6 \times [1.0, 0.0, 0.00] - 9$$

**Step 8: Simplify.**

$$f(0) = -9$$

The  $y$ -intercept is  $(0, -9)$ .

**Step 9: Graph the points of symmetry.**

The point  $(0, -9)$  is three units to the left of the line of symmetry. The point three units to the right of the line of symmetry is  $(6, -9)$ .

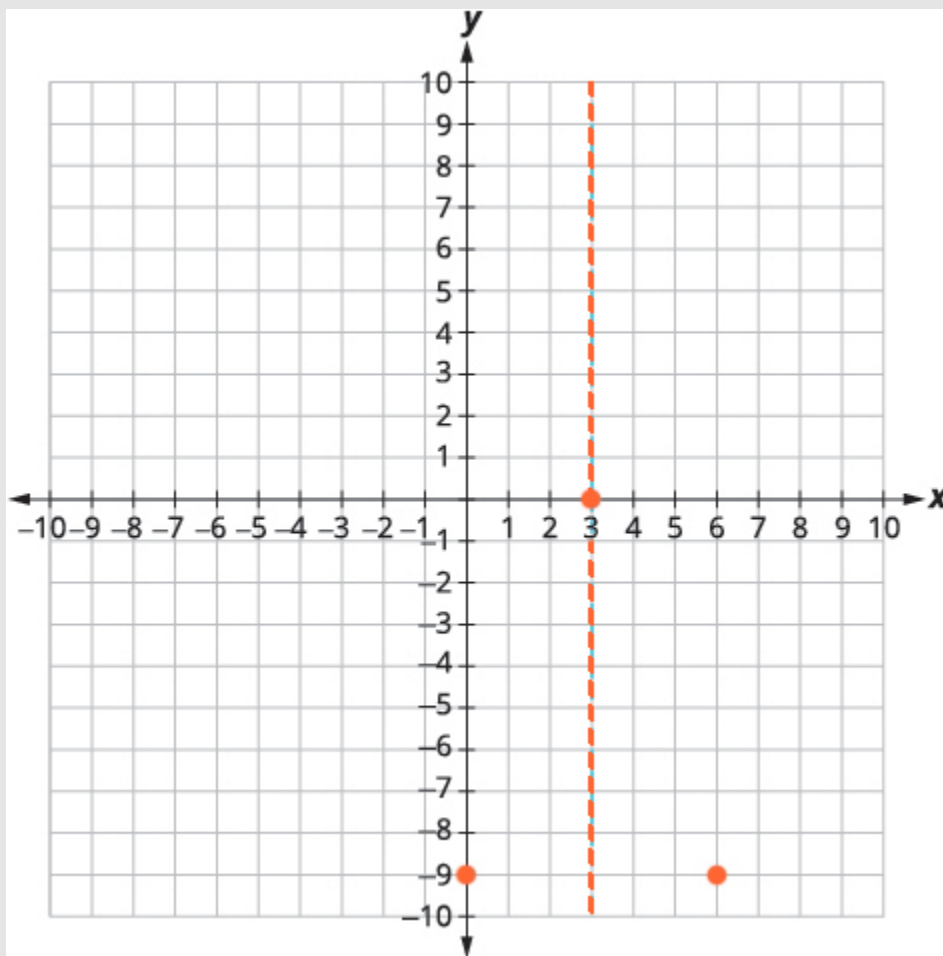


Figure 4.6.15

Point symmetric to the  $y$ -intercept is  $(6, -9)$

The  $x$ -intercept occurs when  $f(x) = 0$ .

$$f(x) = -x^2 + 6x - 9$$

**Step 10: Find  $f(x) = 0$ .**

$$0 = -x^2 + 6x - 9$$

**Step 11: Factor the GCF.**

$$0 = -(x^2 - 6x + 9)$$

**Step 12: Factor the trinomial.**

$$0 = -(x - 3)^2$$

**Step 13: Solve for  $x$ .**

$$0 = 3$$

**Step 14: Connect the points to graph the parabola.**

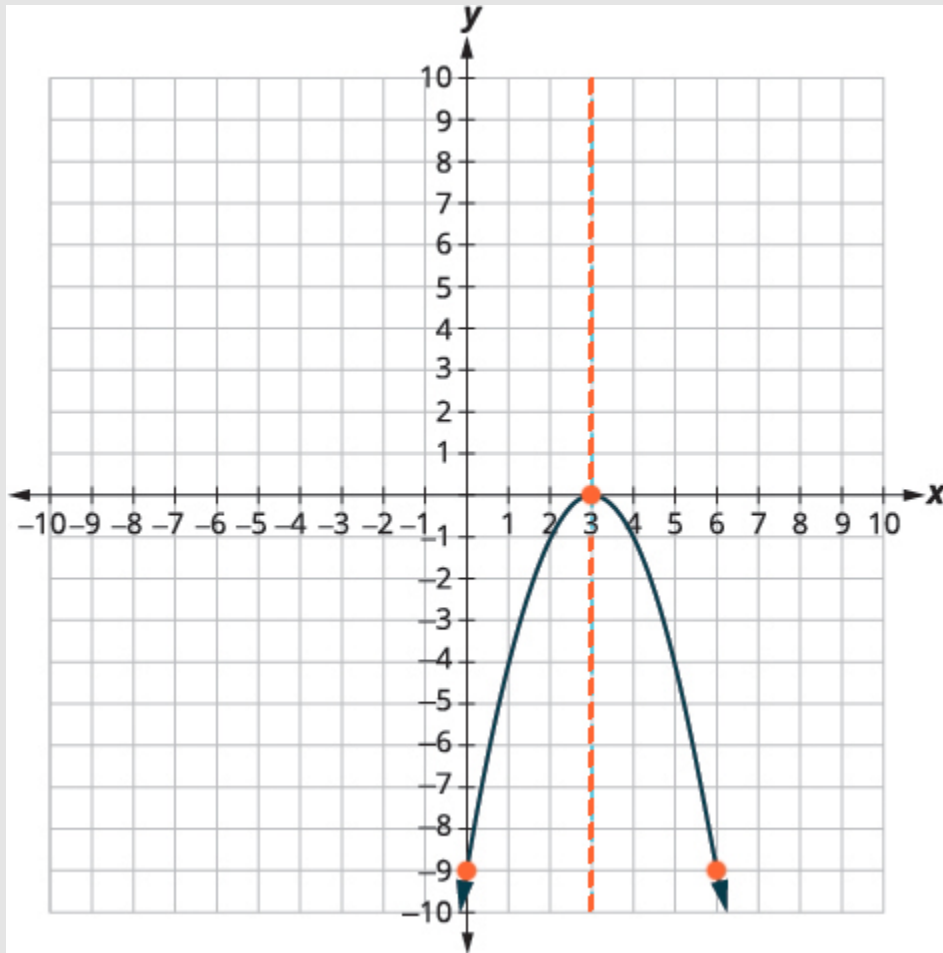


Figure 4.6.16

## Try It

16) Graph  $f(x) = 3x^2 + 12x - 12$  by using its properties.

### Solution

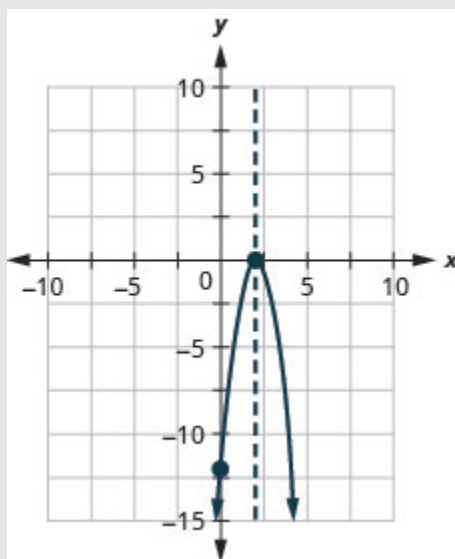


Figure 4.6.17

## Try It

17) Graph  $f(x) = 4x^2 + 24x + 36$  by using its properties.

### Solution

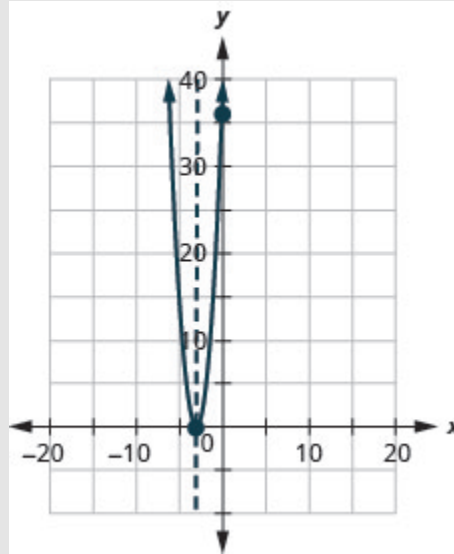


Figure 4.6.18

For the graph of  $f(x) = -x^2 + 6x - 9$ , the vertex and the  $x$ -intercept were the same point. Remember how the discriminant determines the number of solutions of a quadratic equation? The discriminant of the equation  $0 = -x^2 + 6x - 9$  is  $0$ , so there is only one solution. That means there is only one  $x$ -intercept, and it is the vertex of the parabola.

How many  $x$ -intercepts would you expect to see on the graph of  $f(x) = x^2 + 4x + 5$ ?

### Example 4.6.8

Graph  $f(x) = x^2 + 4x + 5$  by using its properties.

#### Solution

rgb]1.0,0.0,0.0frgb]1.0,0.0,0.0(rgb]1.0,0.0,0.0xrgb]1.0,0.0,0.0)rgb]1.0,0.0,0.0=rgb]1.0,0.0,0.0argb]1.0,0.0,0.0x<sup>rgb]1.0,0.0,0.02</sup>rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.0brgb]1.0,0.0,0.0xrgb]1.0,0.0,0.0+rgb]1.0,0.0,0.0x  
 $f(x) = x^2 + 4x + 5$

#### Step 1: Find the direction of the parabola.

Since  $a$  is  $1$ , the parabola opens upward. rgb]1.0,0.0,0.0

**Step 2: Find the axis of symmetry.**

To find the axis of symmetry, find  $x = -\frac{b}{2a}$ .

$$x = -\frac{b}{2a}$$

$$x = -\frac{4}{(2)1}$$

$$x = -2$$

**Step 3: Graph the axis of symmetry.**

The equation of the axis of symmetry is  $x = -2$ .

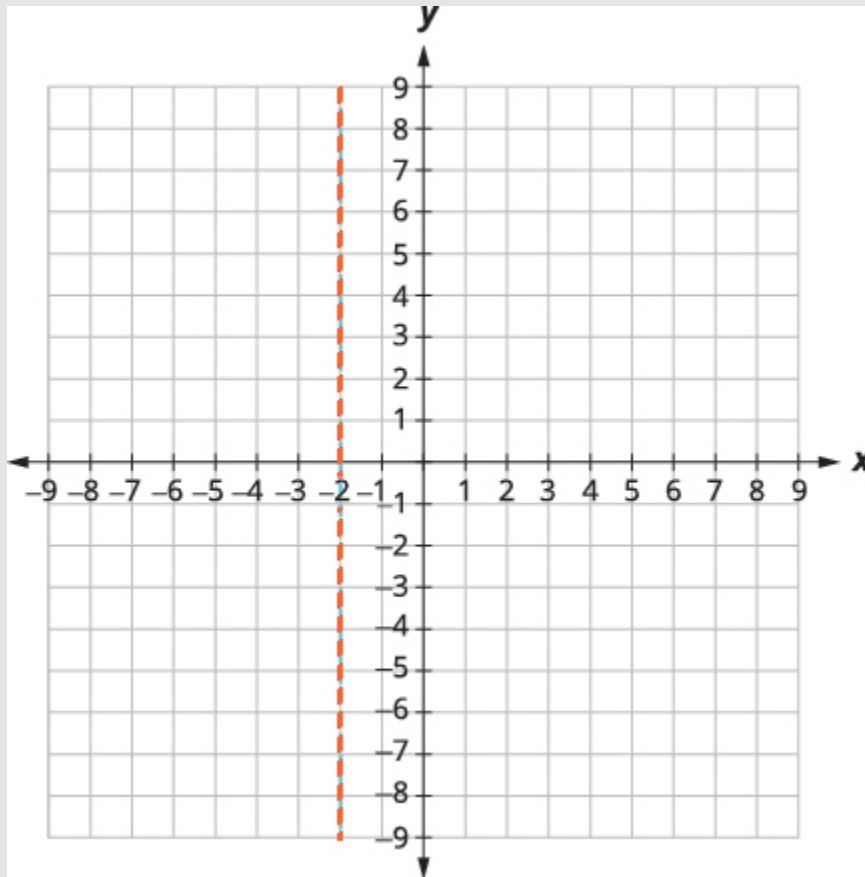


Figure 4.6.19

The vertex is on the line  $x = -2$ .

**Step 4: Find  $f(x)$  when  $x = -2$ .**



$$f(x) = x^2 + 4x + 5$$

$$f(-2) = (-2)^2 + 4(-2) + 5$$

$$f(-2) = 4 - 8 + 5$$

$$f(-2) = 1$$

**Step 5: Graph the vertex.**

The vertex is  $(-2, 1)$ .

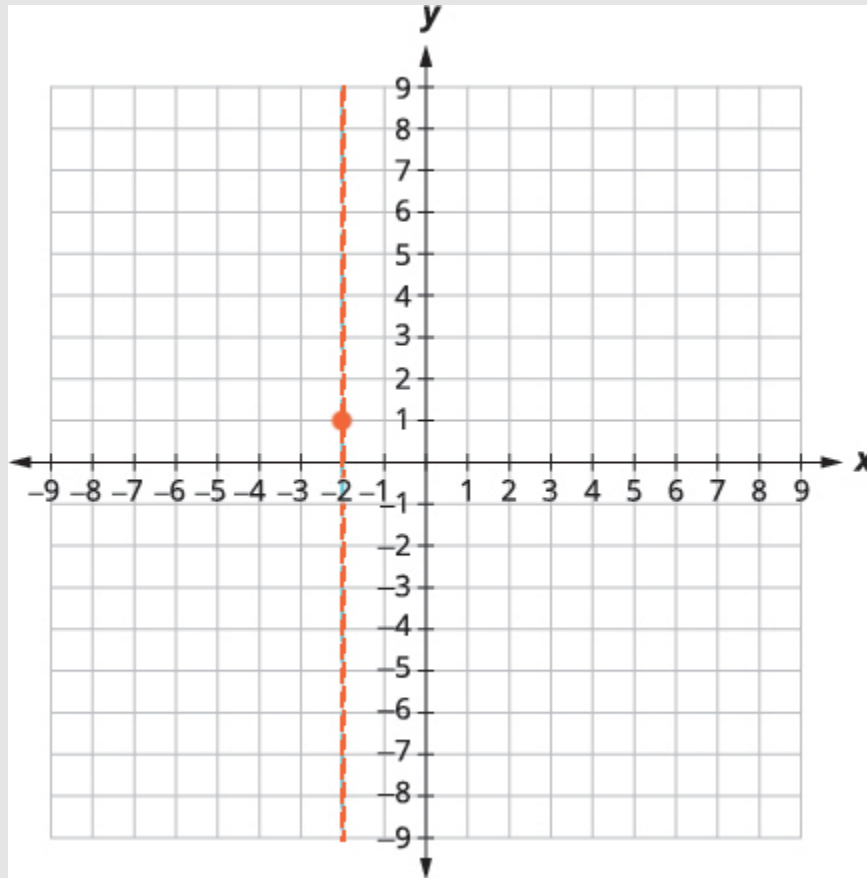


Figure 4.6.20

The  $y$ -intercept occurs when  $x = 0$ .

$$f(x) = x^2 + 4x - 5$$

**Step 6: Find  $f(0)$ .**

$$f(0) = 5$$

**Step 7: Simplify.**

$$f(0) = 5$$

The  $y$ -intercept is  $(0, 5)$ .

**Step 8: Graph the points.**

The point  $(-4, 5)$  is two units to the left of the line of symmetry.

The point two units to the right of the line of symmetry is  $(0, 5)$ .

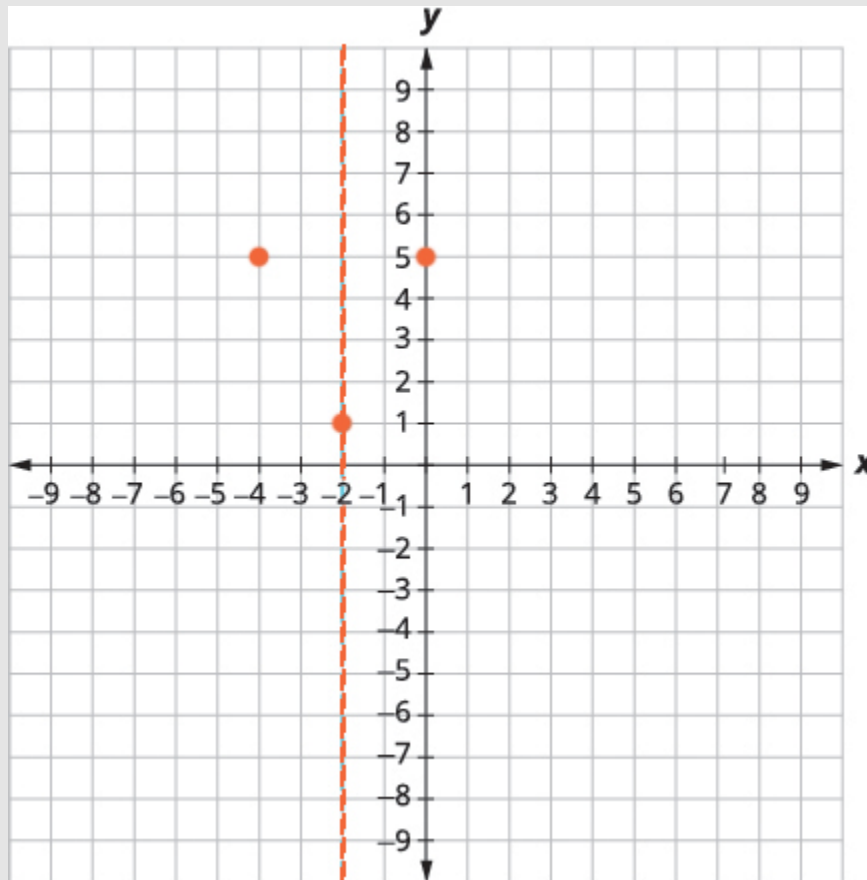


Figure 4.6.21

**Step 9: State the point that is symmetric to the  $y$ -intercept.**

Point symmetric to the  $y$ -intercept is  $(-4, 5)$ .

The  $x$ -intercept occurs when  $f(x) = 0$ .

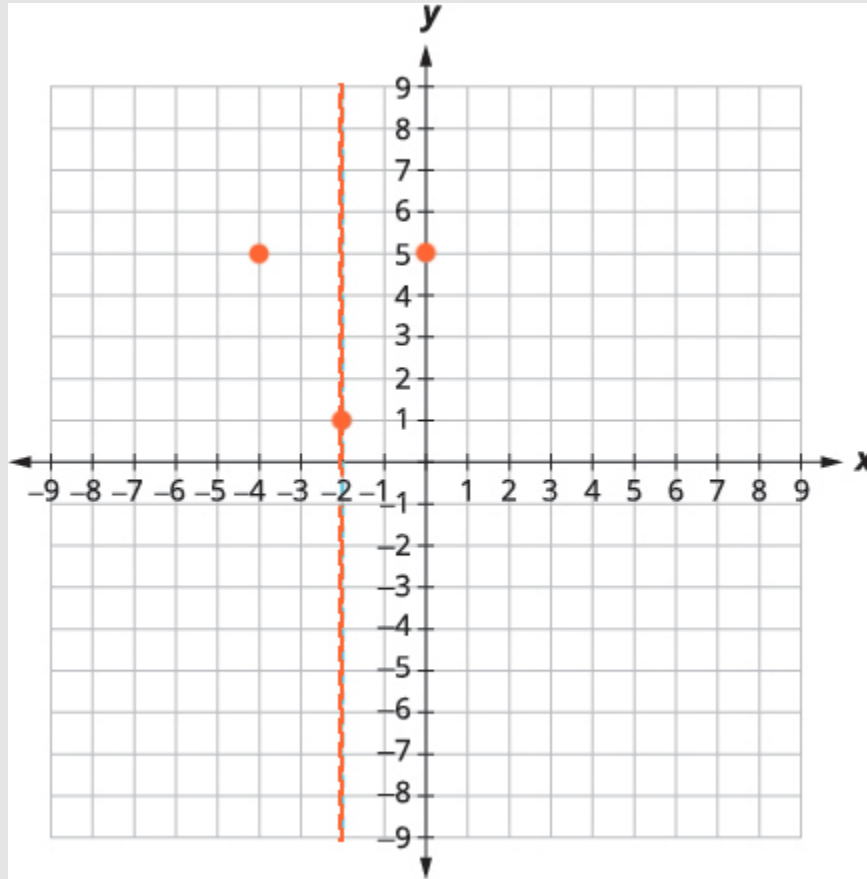


Figure 4.6.22

**Step 10: Find  $f(x) = 0$ .**

$$x^2 + 4x + 5 = 0$$

**Step 11: Test the discriminant.**

$$\begin{aligned} &= b^2 - 4ac \\ &= 4^2 - 4 \times 1 \times 5 \\ &= 16 - 20 \\ &= -4 \end{aligned}$$

Since the value of the discriminant is negative, there is no real solution and so no  $x$ -intercept.

Connect the points to graph the parabola. You may want to choose two more points for greater accuracy.

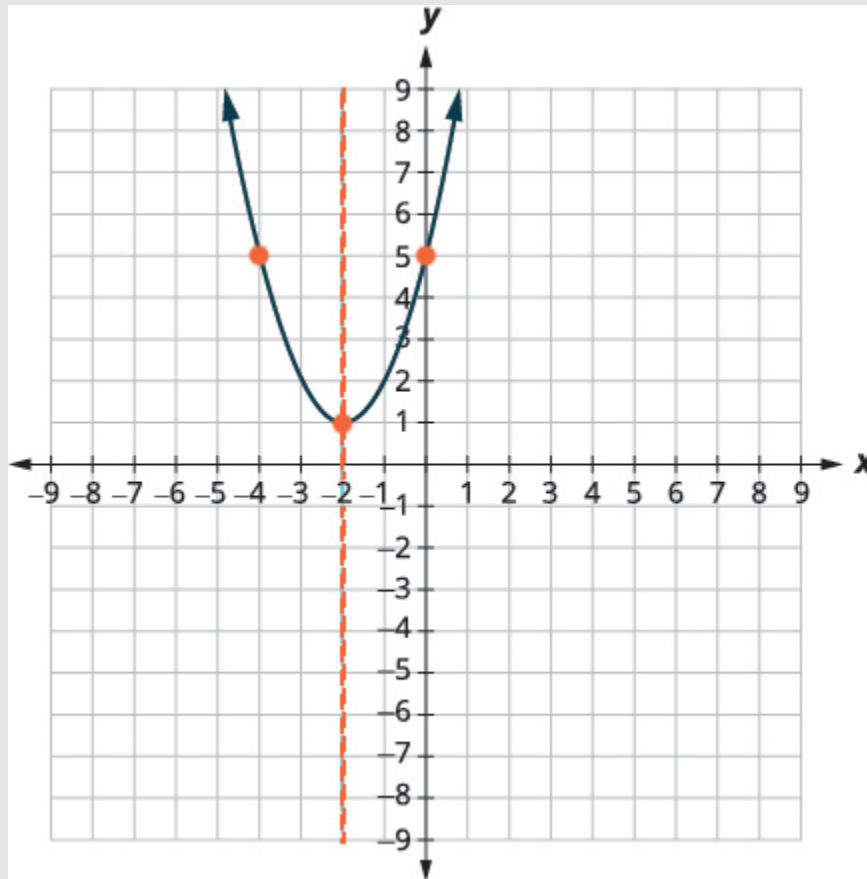


Figure 4.6.23

## Try It

18) Graph  $f(x) = x^2 - 2x + 3$  by using its properties.

### Solution

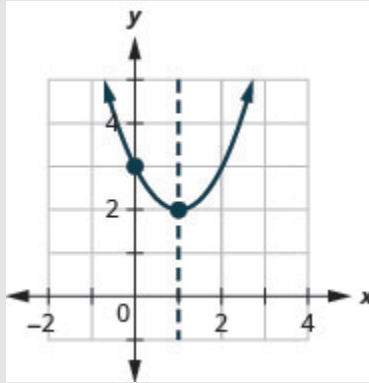


Figure 4.6.24

## Try It

19) Graph  $f(x) = -3x^2 - 6x - 4$  by using its properties.

### Solution

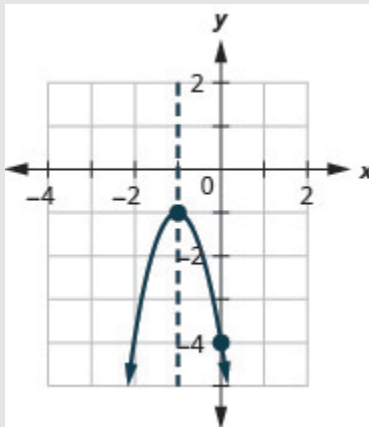


Figure 4.6.25

Finding the  $y$ -intercept by finding  $f(0)$  is easy, isn't it? Sometimes we need to use the Quadratic Formula to find the  $x$ -intercepts.

### Example 4.6.9

Graph  $f(x) = 2x^2 - 4x - 3$  by using its properties.

#### Solution

$$f(x) = 2x^2 - 4x - 3$$

#### Step 1: Find the direction of the parabola.

Since  $a$  is  $2$ , the parabola opens upward.

#### Step 2: Find the axis of symmetry.

To find the equation of the axis of symmetry, use  $x = -\frac{b}{2a}$ .

$$\begin{aligned} x &= -\frac{b}{2a} \\ x &= -\frac{-4}{2 \times 2} \\ x &= 1 \end{aligned}$$

The equation of the axis of symmetry is  $x = 1$ .

The vertex is on the line  $x = 1$ .

$$f(x) = 2x^2 - 4x - 3$$

#### Step 3: Find $f(1)$ .

$$f(x) = 2(1)^2 - 4(1) - 3$$

$$f(1) = 2 - 4 - 3$$

$$f(1) = -5$$

The vertex is  $(1, -5)$ .

The  $y$ -intercept occurs when  $x = 0$ .

$$f(x) = 2x^2 - 4x - 3$$

**Step 4: Find  $f(0)$ .**

$$f(x) = 2(0.5, 0.0, 0.00)^2 - 4(0.5, 0.0, 0.00) - 3$$

**Step 5: Simplify.**

$$f(0) = -3$$

The  $y$ -intercept is  $(0, -3)$ .

**Step 6: State the point that is symmetric to the  $y$ -intercept.**

The point  $(0, -3)$  is one unit to the left of the line of symmetry.

Point symmetric to the  $y$ -intercept is  $(2, -3)$

The point one unit to the right of the line of symmetry is  $(2, -3)$ .

The  $x$ -intercept occurs when  $y = 0$ .

$$f(x) = 2x^2 - 4x - 3$$

**Step 7: Find  $f(x) = 0$ .**

$$0 = 2x^2 - 4x - 3$$

**Step 8: Use the Quadratic Formula.**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

**Step 9: Substitute in the values of  $a$ ,  $b$ , and  $c$ .**

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)} = \frac{4 \pm \sqrt{16 + 24}}{4}$$

**Step 10: Simplify.**

$$x = \frac{4 \pm \sqrt{16 + 24}}{4}$$

**Step 11: Simplify inside the radical.**

$$x = \frac{4 \pm \sqrt{40}}{4}$$

**Step 12: Simplify the radical.**

$$x = \frac{4 \pm 2\sqrt{10}}{4}$$

**Step 13: Factor the GCF.**

$$x = \frac{2(2 \pm \sqrt{10})}{4}$$

**Step 14: Remove common factors.**

$$x = \frac{2 \pm \sqrt{10}}{2}$$

**Step 15: Write as two equations.**

$$x = \frac{2 + \sqrt{10}}{2}, \quad x = \frac{2 - \sqrt{10}}{2}$$

**Step 16: Approximate the values.**

$$x \approx 2.5, \quad x \approx -0.6$$

The approximate values of the  $x$ -intercepts are  $(2.5, 0)$  and  $(-0.6, 0)$ .

**Step 17: Graph the parabola using the points found.**



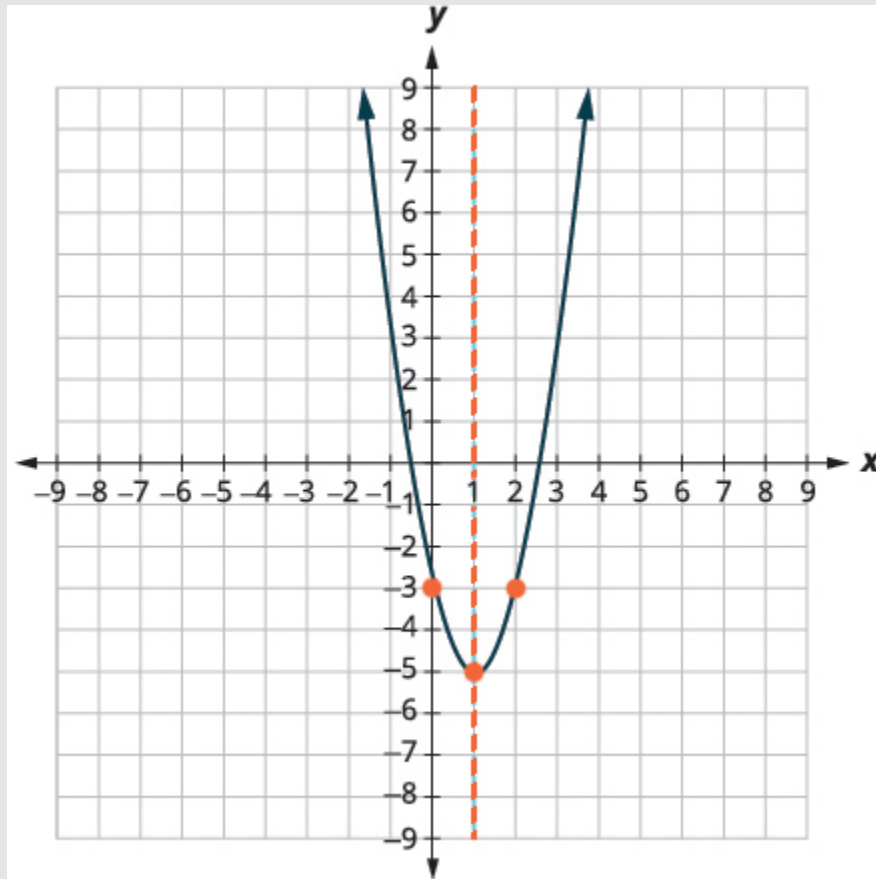


Figure 4.6.26

## Try It

20) Graph  $f(x) = 5x^2 + 10x + 3$  by using its properties.

### Solution

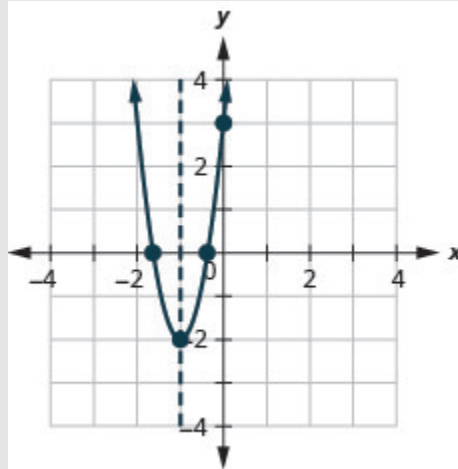


Figure 4.6.27

## Try It

21) Graph  $f(x) = -3x^2 - 6x + 5$  by using its properties.

### Solution

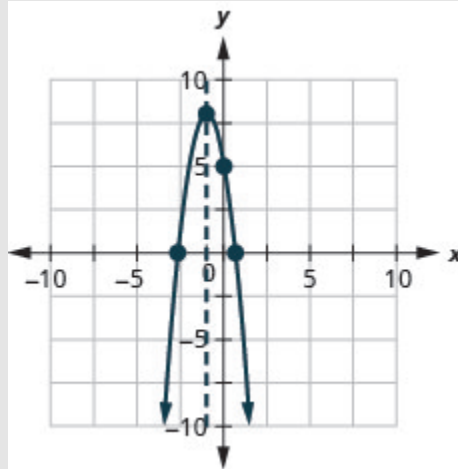


Figure 4.6.28

## Solve Maximum and Minimum Applications

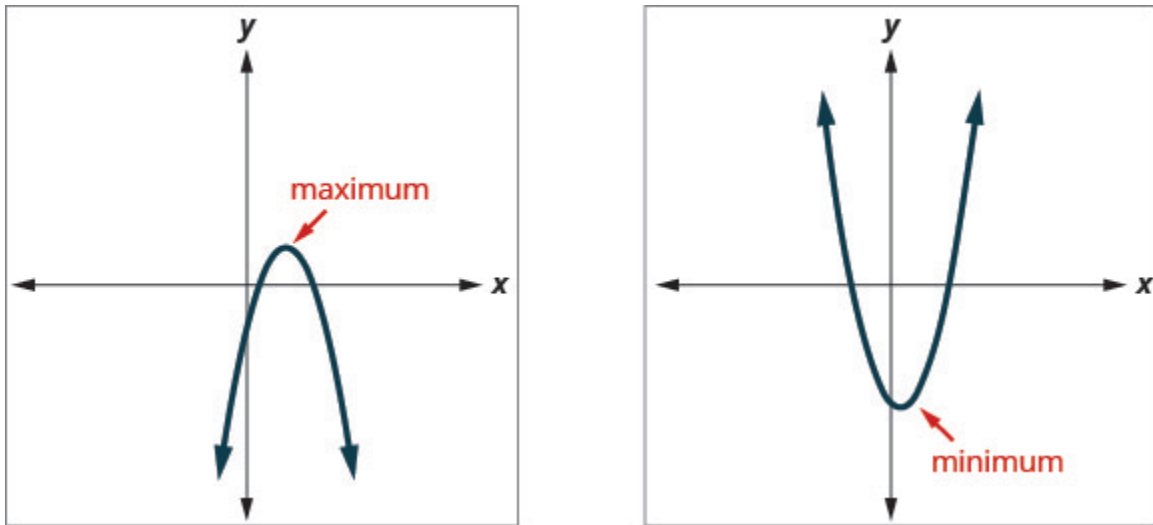


Figure 4.6.29

Knowing that the vertex of a parabola is the lowest or highest point of the parabola gives us an easy way to determine the minimum or maximum value of a quadratic function. The  $y$ -coordinate of the vertex is the minimum value of a parabola that opens upward. It is the maximum value of a parabola that opens downward. See 4.6.29.

## Minimum or Maximum Values of a Quadratic Function

The ***y*-coordinate of the vertex** of the graph of a quadratic function is the:

- *minimum* value of the quadratic equation if the parabola opens *upward*.
- *maximum* value of the quadratic equation if the parabola opens *downward*.

### Example 4.6.10

Find the minimum or maximum value of the quadratic function  $f(x) = x^2 + 2x - 8$ .

#### Solution

**Step 1: Find the direction of the parabola.**

Since  $a$  is positive, the parabola opens upward.

The quadratic equation has a minimum.

**Step 2: Find the equation of the axis of symmetry.**

$$x = -\frac{b}{2a}$$

$$x = -\frac{2}{2 \times 1}$$

$$x = -1$$

The equation of the axis of symmetry is  $x = -1$ .

The vertex is on the line  $x = -1$ .

$$f(x) = x^2 + 2x - 8$$

**Step 3: Find  $f(-1)$ .**

$$f(-1) = (rgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.01)^2 + 2(rgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.01) - 8$$

$$f(-1) = 1 - 2 - 8$$

$$f(-1) = -9$$

The vertex is  $(-1, -9)$ .

Since the parabola has a minimum, the  $y$ -coordinate of the vertex is the minimum  $y$ -value of the quadratic equation.

The minimum value of the quadratic is  $-9$  and it occurs when  $x = -1$ .

**Step 4: Show the graph to verify the result.**

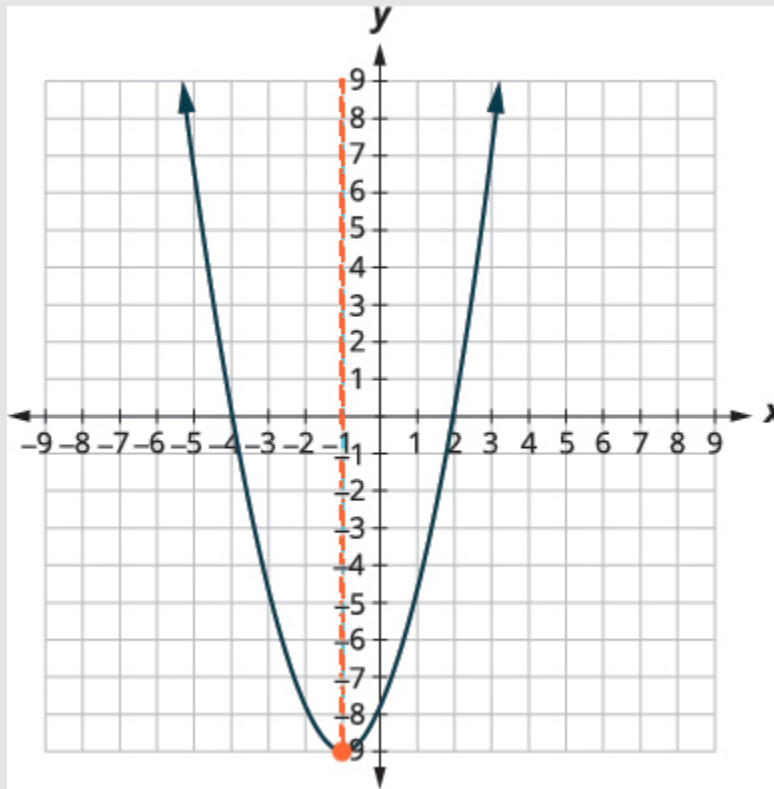


Figure 4.6.30

**Try It**

22) Find the maximum or minimum value of the quadratic function  $f(x) = x^2 - 8x + 12$ .

**Solution**

The minimum value of the quadratic function is  $-4$  and it occurs when  $x = 4$ .

**Try It**

23) Find the maximum or minimum value of the quadratic function  $f(x) = -4x^2 + 16x - 11$ .

**Solution**

The maximum value of the quadratic function is  $5$  and it occurs when  $x = 2$ .

We have used the formula

$$h(t) = -16t^2 + v_0t + h_0$$

to calculate the height in feet,  $h$ , of an object shot upwards into the air with initial velocity,  $v_0$ , after  $t$  seconds.

This formula is a quadratic function, so its graph is a parabola. By solving for the coordinates of the vertex  $(t, h)$ , we can find how long it will take the object to reach its maximum height. Then we can calculate the maximum height.

### Example 4.6.11

The quadratic equation  $h(t) = -16t^2 + 176t + 4$  models the height of a volleyball hit straight upwards with velocity **176** feet per second from a height of 4 feet.

- How many seconds will it take the volleyball to reach its maximum height?
- Find the maximum height of the volleyball.

#### Solution

Since  $a$  is negative, the parabola opens downward.

The quadratic function has a maximum.

a.

#### Step 1: Find the equation of the axis of symmetry.

$$t = -\frac{b}{2a}$$

$$t = -\frac{176}{2(-16)}$$

$$t = 5.5$$

#### Step 2: Find the axis of symmetry.

The equation of the axis of symmetry is  $t = 5.5$ .

#### Step 3: Find the vertex.

The vertex is on the line  $t = 5.5$ .

#### Step 4: Where is the maximum?

The maximum occurs when  $t = 5.5$  seconds.

b.

#### Step 1: Find $h(5.5)$ .

$$h(t) = -16t^2 + 176t + 4$$

$$h(t) = -16(\text{rgb}[1.0, 0.0, 0.05\text{rgb}[1.0, 0.0, 0.0.\text{rgb}[1.0, 0.0, 0.05]^2 + 176(\text{rgb}[1.0, 0.0, 0.05\text{rgb}[1.0, 0.0, 0.0.\text{rgb}[1.0, 0.0, 0.05] + 4$$

**Step 2: Use a calculator to simplify.**

$$h(t) = 488$$

**Step 3: State the vertex.**

The vertex is  $(5.5, 488)$ .

**Step 4: Where is the maximum?**

Since the parabola has a maximum, the  $h$ -coordinate of the vertex is the maximum value of the quadratic function.

The maximum value of the quadratic is **488** feet and it occurs when  $t = 5.5$  seconds.

**Step 5: State the findings.**

After **5.5** seconds, the volleyball will reach its maximum height of **488** feet.

## Try It

24) Solve, rounding answers to the nearest tenth.

The quadratic function  $h(t) = -16t^2 + 128t + 32$  is used to find the height of a stone thrown upward from a height of **32** feet at a rate of **128** ft/sec.

How long will it take for the stone to reach its maximum height? What is the maximum height?

### Solution

It will take **4** seconds for the stone to reach its maximum height of **288** feet.



## Try It

25) A path of a toy rocket thrown upward from the ground at a rate of **208** ft/sec is modelled by the quadratic function of  $h(t) = -16t^2 + 208t$ .

When will the rocket reach its maximum height? What will be the maximum height?

### Solution

It will **6.5** seconds for the rocket to reach its maximum height of **676** feet.

Access these online resources for additional instruction and practice with graphing quadratic functions using properties.

- Quadratic Functions: Axis of Symmetry and Vertex
- Finding x- and y-intercepts of a Quadratic Function
- Graphing Quadratic Functions
- Solve Maximum or Minimum Applications
- Quadratic Applications: Minimum and Maximum

## Key Concepts

- Parabola Orientation
  - For the graph of the quadratic function  $f(x) = ax^2 + bx + c$ , if

- $a > 0$ , the parabola opens upward.
  - $a < 0$ , the parabola opens downward.
- Axis of Symmetry and Vertex of a Parabola The graph of the function  $f(x) = ax^2 + bx + c$  is a parabola where:
    - the axis of symmetry is the vertical line  $x = -\frac{b}{2a}$ .
    - the vertex is a point on the axis of symmetry, so its  $x$ -coordinate is  $-\frac{b}{2a}$ .
    - the  $y$ -coordinate of the vertex is found by substituting  $x = -\frac{b}{2a}$  into the quadratic equation.
  - Find the Intercepts of a Parabola
    - To find the intercepts of a parabola whose function is  $f(x) = ax^2 + bx + c$  :
 

$\begin{array}{l} \text{intercept} \\ \text{intercepts} \end{array}$	\textbf{-}	\textbf{intercept}	\textbf{and}	\textbf{solve}	\textbf{for}	\textbf{x}	\textbf{and}	\textbf{solve}	\textbf{for}	\textbf{x}
$\begin{array}{l} \text{intercept} \\ \text{intercepts} \end{array}$	\textbf{=}	\textbf{0}	\textbf{and}	\textbf{solve}	\textbf{for}	\textbf{x}	\textbf{and}	\textbf{solve}	\textbf{for}	\textbf{x}
  - How to graph a quadratic function using properties.
    1. Determine whether the parabola opens upward or downward.
    2. Find the equation of the axis of symmetry.
    3. Find the vertex.
    4. Find the  $y$ -intercept. Find the point symmetric to the  $y$ -intercept across the axis of symmetry.
    5. Find the  $x$ -intercepts. Find additional points if needed.
    6. Graph the parabola.
  - Minimum or Maximum Values of a Quadratic Equation
    - The  $y$ -coordinate of the vertex of the graph of a quadratic equation is the
    - *minimum* value of the quadratic equation if the parabola opens *upward*.
    - *maximum* value of the quadratic equation if the parabola opens *downward*.

## Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=1224#h5p-32>

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

## Glossary

### Quadratic Function

A quadratic function, where  $a$ ,  $b$ , and  $c$  are real numbers and  $a \neq 0$ , is a function of the form  $f(x) = ax^2 + bx + c$ .

# 4.7 GRAPH QUADRATIC FUNCTIONS USING TRANSFORMATIONS

---

## Learning Objectives

By the end of this section, you will be able to:

- Graph quadratic functions of the form  $f(x) = x^2 + k$
- Graph quadratic functions of the form  $f(x) = (x - h)^2$
- Graph quadratic functions of the form  $f(x) = ax^2$
- Graph quadratic functions using transformations
- Find a quadratic function from its graph

## Try It

Before you get started, take this readiness quiz:

- 1) Graph the function  $f(x) = x^2$  by plotting points.
- 2) Factor completely:  $y^2 - 14y + 49$ .
- 3) Factor completely:  $2x^2 - 16x + 32$ .

## Graph Quadratic Functions of the form $f(x) = x^2 + k$

In the last section, we learned how to graph quadratic functions using their properties. Another method involves starting with the basic graph of  $f(x) = x^2$  and ‘moving’ it according to information given in the function equation. We call this graphing quadratic functions using transformations.

In the first example, we will graph the quadratic function  $f(x) = x^2$  by plotting points. Then we will see what effect adding a constant,  $k$ , to the equation will have on the graph of the new function  $f(x) = x^2 + k$ .

### Example 4.7.1

Graph  $f(x) = x^2$ ,  $g(x) = x^2 + 2$ , and  $h(x) = x^2 - 2$  on the same rectangular coordinate system. Describe what effect adding a constant to the function has on the basic parabola.

#### Solution

**Step 1:** Plotting points will help us see the effect of the constants on the basic

$f(x) = x^2$  graph.

We fill in the chart for all three functions.

$x$	$f(x) = x^2$	$g(x) = x^2 + 2$	$h(x) = x^2 - 2$
-3	9	11	7
-2	4	6	2
-1	1	3	-1
0	0	2	-2
1	1	3	-1
2	4	6	2
3	9	11	7

**Step 2:** The  $g(x)$  values are two more than the  $f(x)$  values. Also, the  $h(x)$  values are two less than the  $f(x)$  values. Now we will graph all three functions on the same rectangular coordinate system.

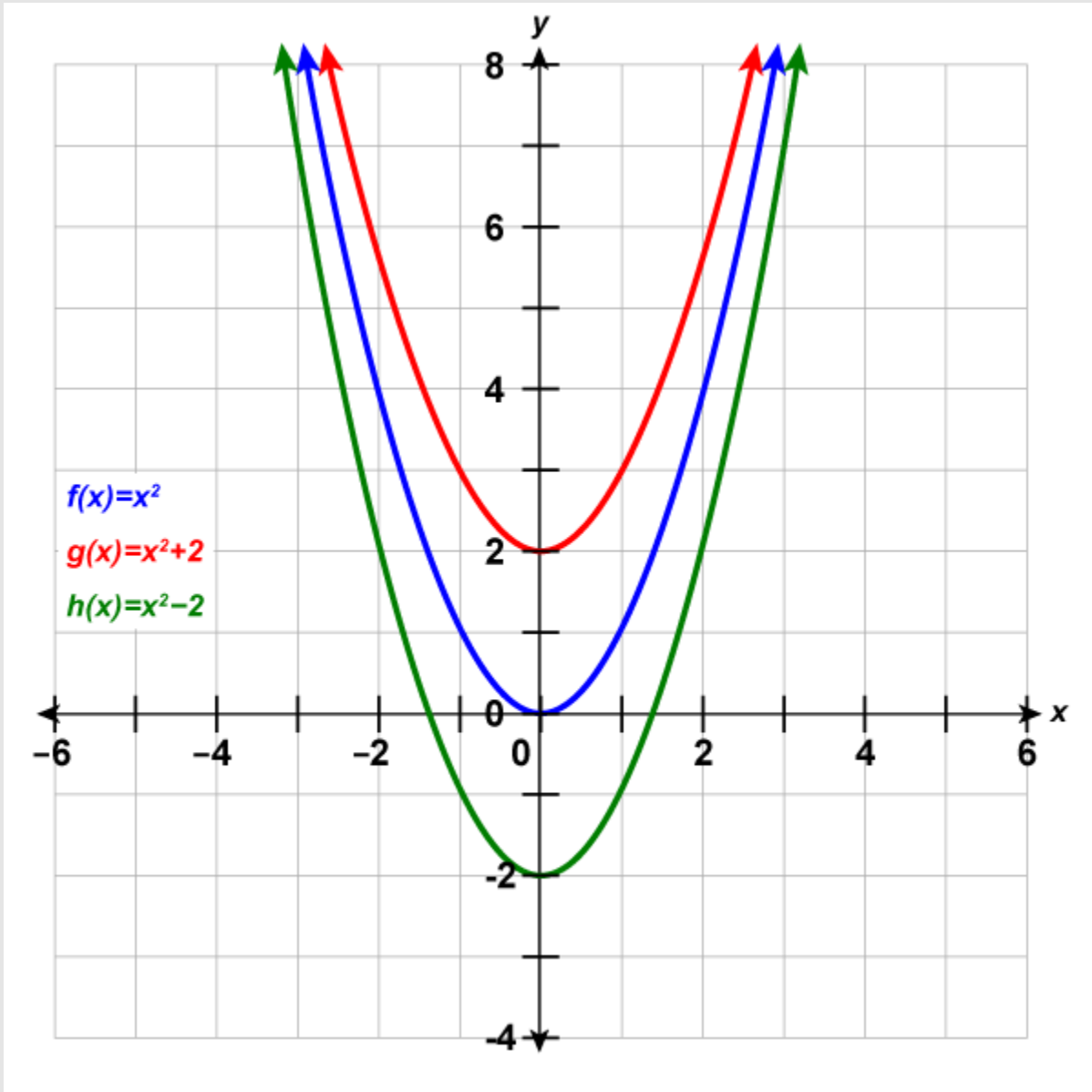


Figure 4.7.1

The graph of

$y = x^2 + 2$  is the same as the graph of

$y = x^2$  but shifted up 2 units.

The graph of

$y = x^2 - 2$  is the same as the graph of

is the same as the graph of

$y = x^2 + 1$  but shifted down 2 units.

## Try It

4) Complete the following:

- Graph  $f(x) = x^2$ ,  $g(x) = x^2 + 1$ , and  $h(x) = x^2 - 1$  on the same rectangular coordinate system.
- Describe what effect adding a constant to the function has on the basic parabola.

### Solution

a.



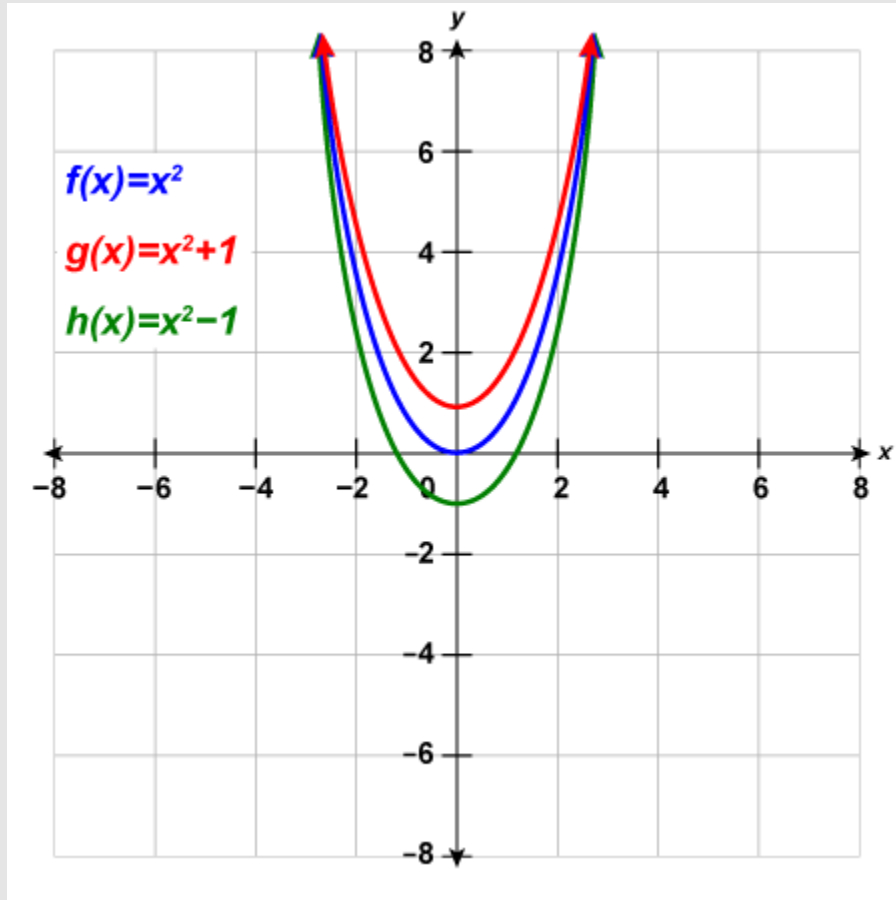


Figure 4.7.2

b. The graph of  $g(x) = x^2 + 1$  is the same as the graph of  $f(x) = x^2$  but shifted up 1 unit. The graph of  $h(x) = x^2 - 1$  is the same as the graph of  $f(x) = x^2$  but shifted down 1 unit.

## Try It

5) Complete the following:

a. Graph  $f(x) = x^2$ ,  $g(x) = x^2 + 6$ , and  $h(x) = x^2 - 6$  on the same rectangular

coordinate system.

b. Describe what effect adding a constant to the function has on the basic parabola.

**Solution**

a.

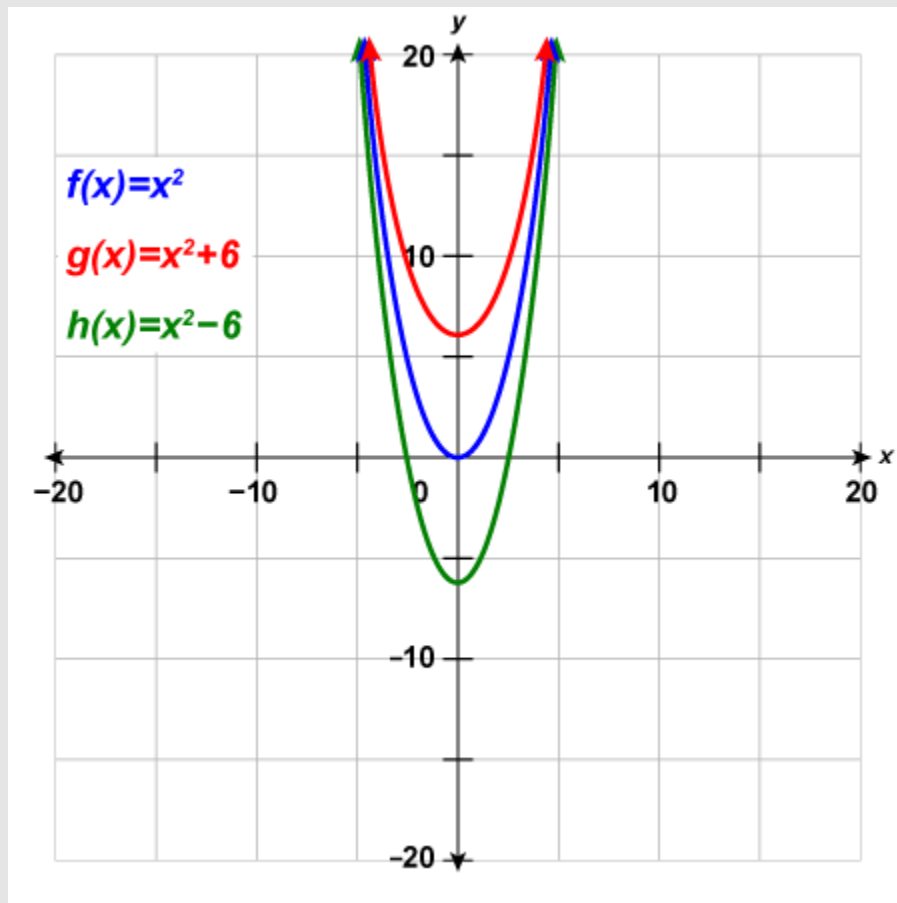


Figure 4.7.3

b. The graph of  $h(x) = x^2 + 6$  is the same as the graph of  $f(x) = x^2$  but shifted up 6 units.

The graph of  $h(x) = x^2 - 6$  is the same as the graph of  $f(x) = x^2$  but shifted down 6 units.

The last example shows us that to graph a quadratic function of the form  $f(x) = x^2 + k$ , we take the basic parabola graph of  $f(x) = x^2$  and vertically shift it up ( $k > 0$ ) or shift it down ( $k < 0$ ).

*This transformation is called a vertical shift.*

## HOW TO

### Graph a Quadratic Function of the form $f(x) = x^2 + k$ Using a Vertical Shift

The graph of  $f(x) = x^2 + k$  shifts the graph of  $f(x) = x^2$  vertically  $k$  units.

- If  $k > 0$ , shift the parabola vertically up  $k$  units.
- If  $k < 0$ , shift the parabola vertically down  $|k|$  units.

Now that we have seen the effect of the constant,  $k$ , it is easy to graph functions of the form  $f(x) = x^2 + k$ . We just start with the basic parabola of  $f(x) = x^2$  and then shift it up or down.

It may be helpful to practice sketching  $f(x) = x^2$  quickly. We know the values and can sketch the graph from there.

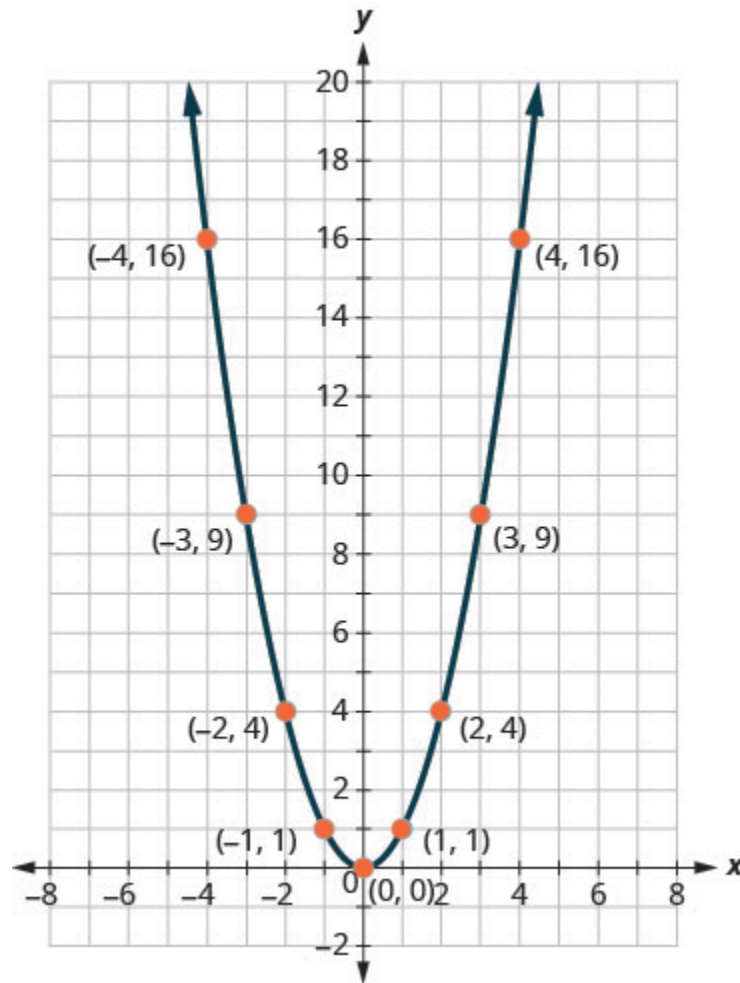


Figure 4.7.4

Once we know this parabola, it will be easy to apply the transformations. The next example will require a vertical shift.

### Example 4.7.2

Graph  $f(x) = x^2 - 3$  using a vertical shift.

#### Solution

**Step 1:** We first draw the graph of  $f(x) = x^2$  on the grid.

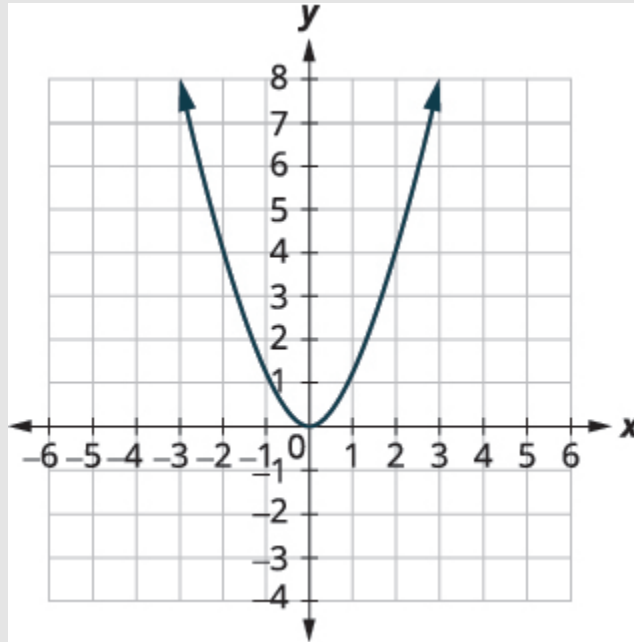


Figure 4.7.5

**Step 2: Determine  $k$ .**

$$f(x) = x^2 + k$$

$rgb[1.0, 0.0, 0.0]f[rgb[1.0, 0.0, 0.0](rgb[1.0, 0.0, 0.0]xrgb[1.0, 0.0, 0.0])rgb[1.0, 0.0, 0.0] = rgb[1.0, 0.0, 0.0]x^{rgb[1.0, 0.0, 0.0]2}rgb[1.0, 0.0, 0.0] - rgb[1.0, 0.0, 0.0]3$

State  $k$ .

$$k = -3$$

**Step 4: Shift the graph  $f(x) = x^2$  down 3.**

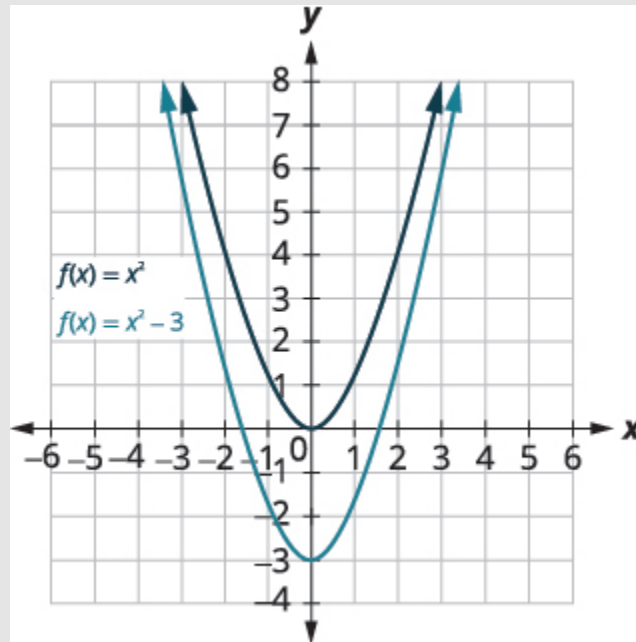


Figure 4.7.6

## Try It

6) Graph  $f(x) = x^2 - 5$  using a vertical shift.

**Solution**

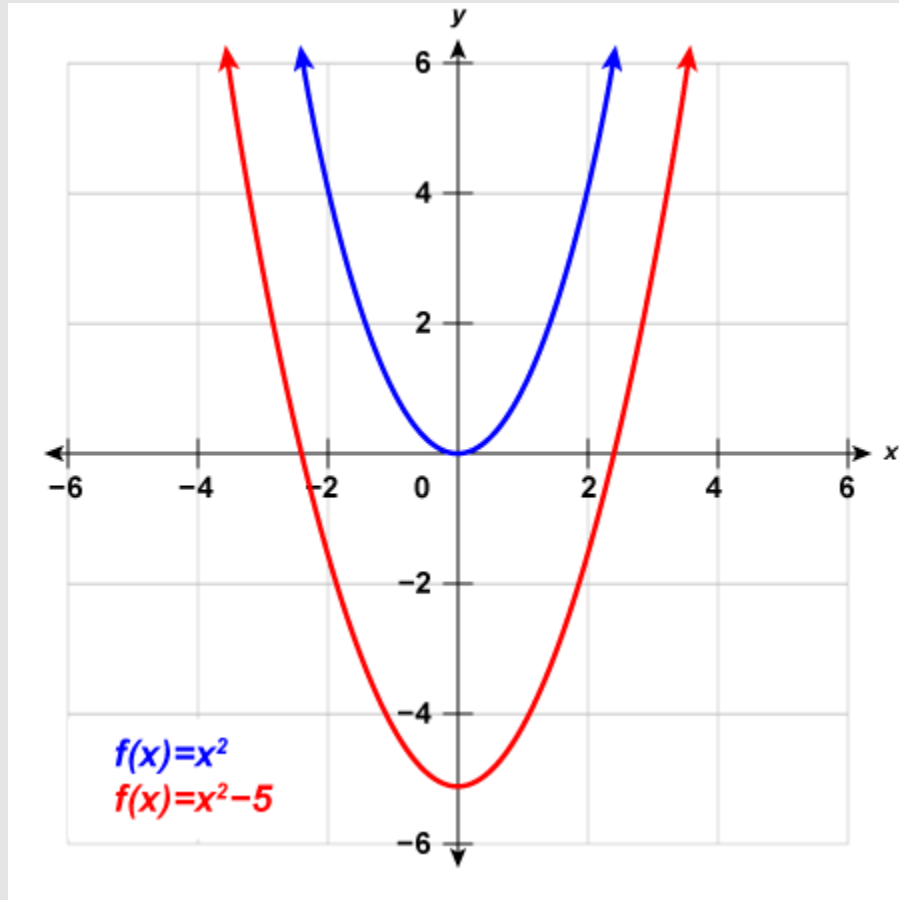


Figure 4.7.7

## Try It

7) Graph  $f(x) = x^2 + 7$  using a vertical shift.

### Solution

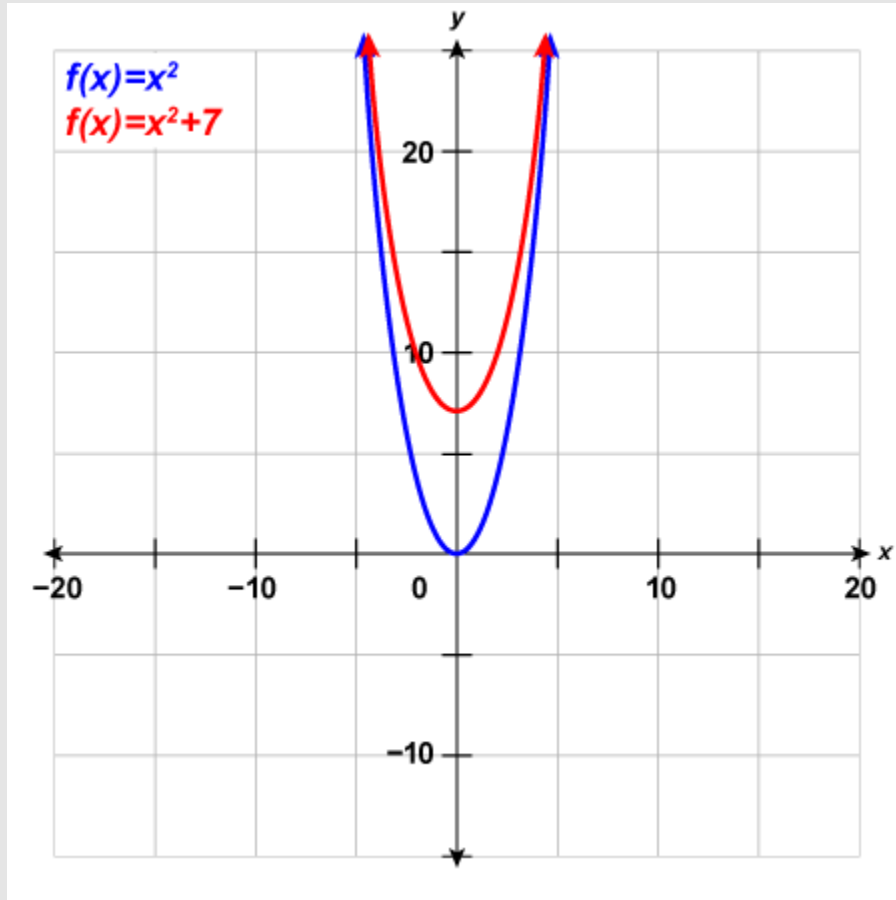


Figure 4.7.8

## Graph Quadratic Functions of the form $f(x) = (x - h)^2$

In the first example, we graphed the quadratic function  $f(x) = x^2$  by plotting points and then saw the effect of adding a constant  $k$  to the function had on the resulting graph of the new function  $f(x) = x^2 + k$ .

We will now explore the effect of subtracting a constant,  $h$ , from  $x$  has on the resulting graph of the new function  $f(x) = (x - h)^2$ .



### Example 4.7.3

Graph  $f(x) = x^2$ ,  $g(x) = (x - 1)^2$ , and  $h(x) = (x + 1)^2$  on the same rectangular coordinate system. Describe what effect adding a constant to the function has on the basic parabola.

#### Solution

**Step 1:** Plotting points will help us see the effect of the constants on the basic  $f(x) = x^2$  graph. We fill in the chart for all three functions.

$x$	$f(x) = x^2$	$g(x) = (x - 1)^2$	$h(x) = (x + 1)^2$
-3	9	4	16
-2	4	1	9
-1	1	0	4
0	0	1	1
1	1	4	0
2	4	9	1
3	9	16	4

**Step 2:** The  $g(x)$  values and the  $h(x)$  values share the common numbers 0, 1, 4, 9, and 16, but are shifted.

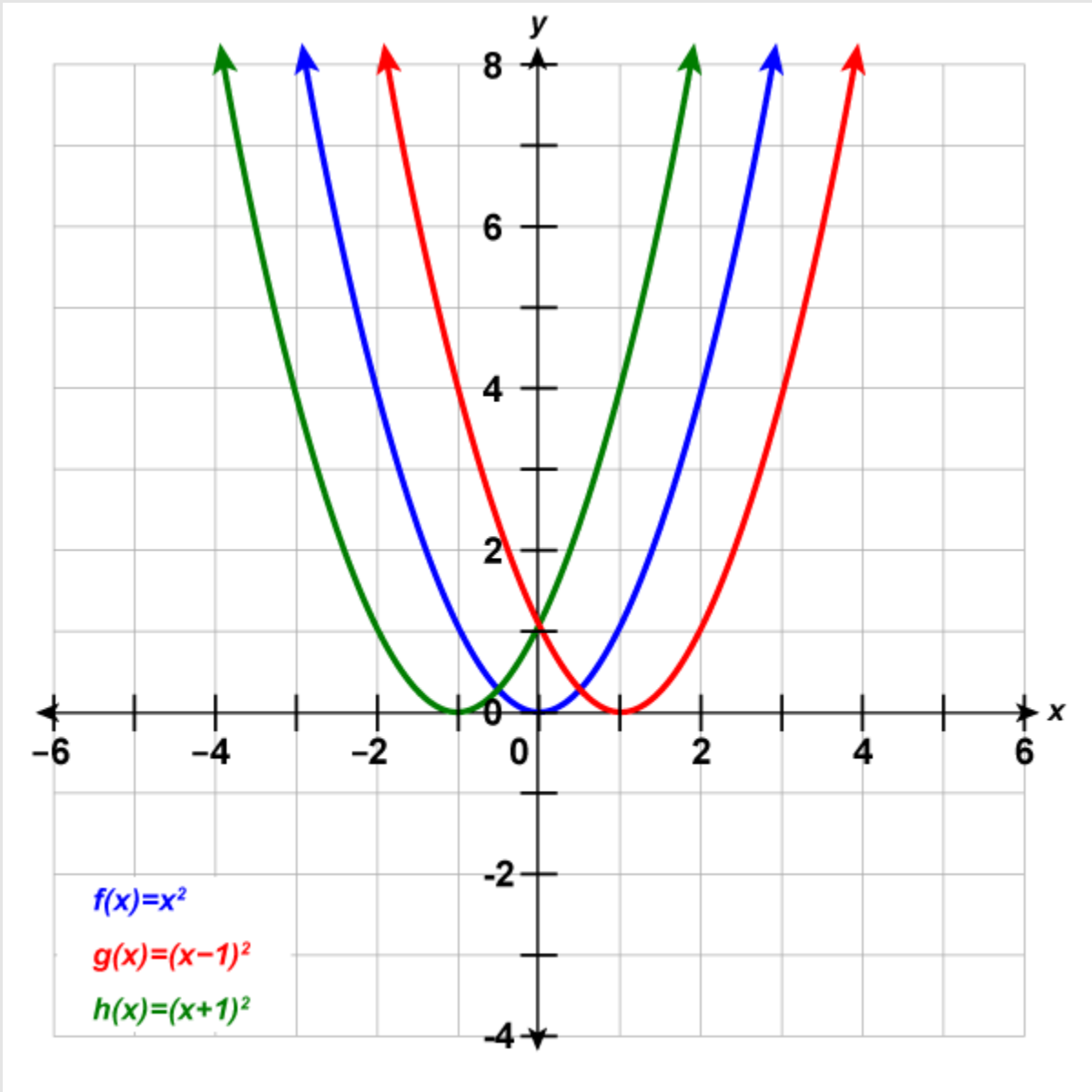


Figure 4.7.9

The graph of

$$f(x) = x^2$$

is the same as the graph of

$$g(x) = (x-1)^2$$

but shifted right 1 unit.

The graph of

$$h(x) = (x+1)^2$$

is the same as the graph of

$y = x^2$  but shifted left **1** unit.

## Try It

8) Complete the following:

- Graph  $f(x) = x^2$ ,  $g(x) = (x + 2)^2$ , and  $h(x) = (x - 2)^2$  on the same rectangular coordinate system.
- Describe what effect adding a constant to the function has on the basic parabola.

### Solution

a.

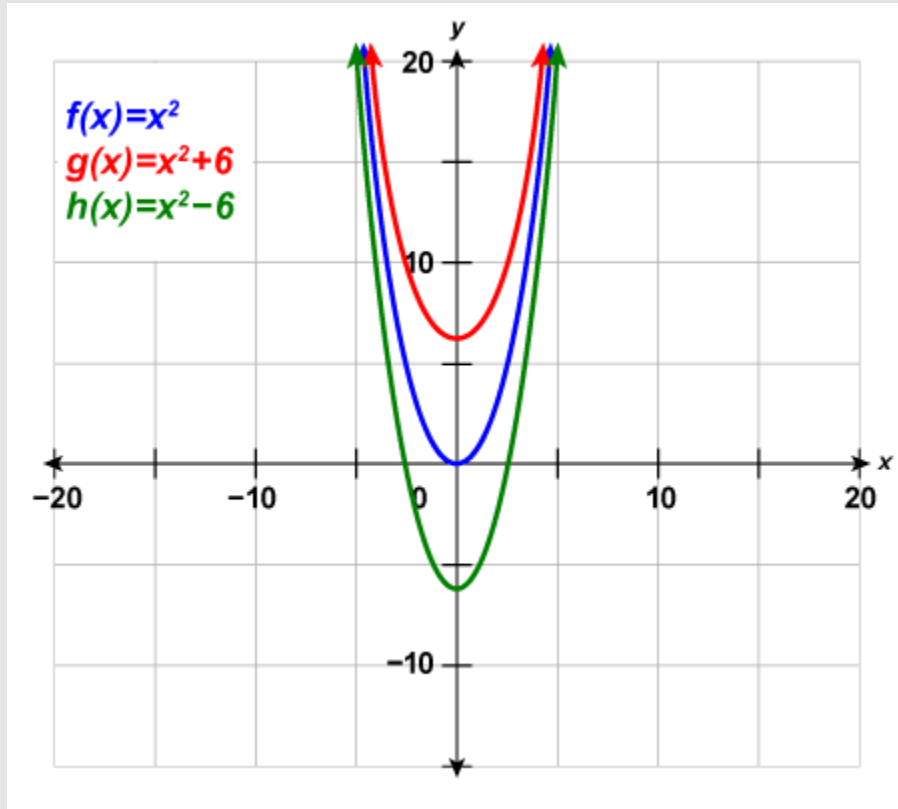


Figure 4.7.10

b.

The graph of  $g(x) = (x + 2)^2$  is the same as the graph of  $f(x) = x^2$  but shifted left 2 units.  
 The graph of  $h(x) = (x - 2)^2$  is the same as the graph of  $f(x) = x^2$  but shift right 2 units.

## Try It

9) Complete the following:

a. Graph  $f(x) = x^2$ ,  $g(x) = x^2 + 5$ , and  $h(x) = x^2 - 5$  on the same rectangular

coordinate system.

b. Describe what effect adding a constant to the function has on the basic parabola.

### Solution

a.

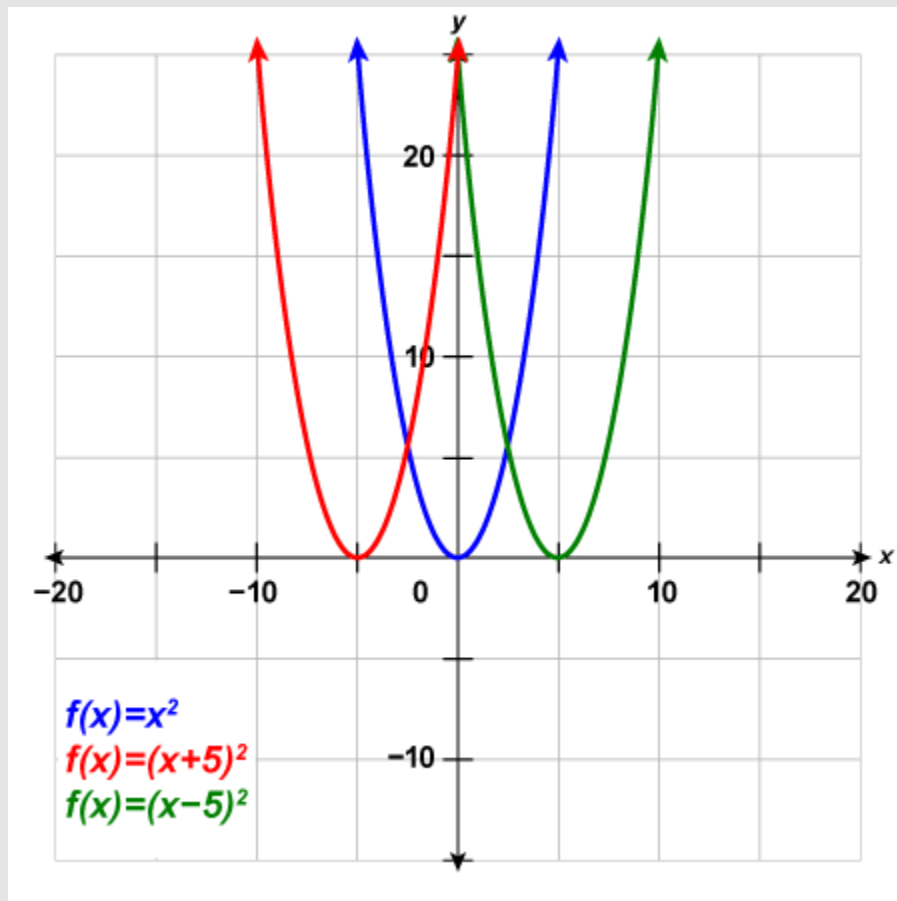


Figure 4.7.11

b.

The graph of  $g(x) = (x + 5)^2$  is the same as the graph of  $f(x) = x^2$  but shifted left 5 units.

The graph of  $h(x) = (x - 5)^2$  is the same as the graph of  $f(x) = x^2$  but shifted right 5 units.

The last example shows us that to graph a quadratic function of the form  $f(x) = (x - h)^2$ , we take the basic parabola graph of  $f(x) = x^2$  and shift it left ( $h > 0$ ) or shift it right ( $h < 0$ ).

*This transformation is called a horizontal shift.*

## HOW TO

### Graph a Quadratic Function of the form $f(x) = (x - h)^2$ Using a Horizontal Shift

The graph of  $f(x) = (x - h)^2$  shifts the graph of  $f(x) = x^2$  horizontally  $h$  units.

- If  $h > 0$ , shift the parabola horizontally left  $h$  units.
- If  $h < 0$ , shift the parabola horizontally right  $|h|$  units.

Now that we have seen the effect of the constant,  $h$ , it is easy to graph functions of the form  $f(x) = (x - h)^2$ . We just start with the basic parabola of  $f(x) = x^2$  and then shift it left or right.

The next example will require a horizontal shift.

### Example 4.7.4

Graph  $f(x) = (x - 6)^2$  using a horizontal shift.

#### **Solution**

**Step 1:** We first draw the graph of  $f(x) = x^2$  on the grid.

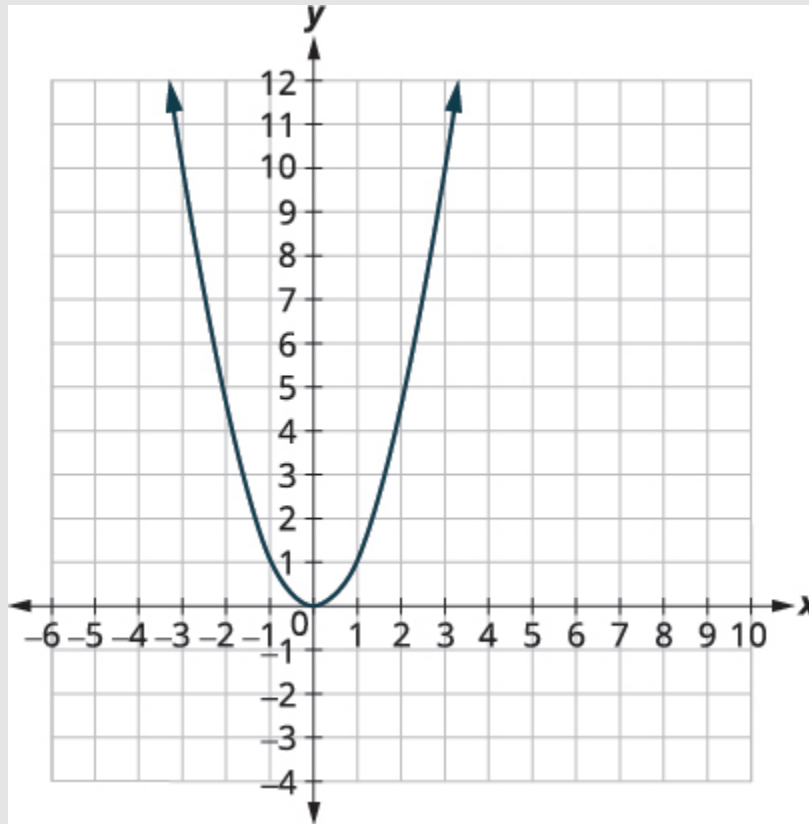


Figure 4.7.12

**Step 2: Determine  $h$ .**

$$f(x) = (x - h)^2$$

$$f(x) = (x - 6)^2$$

State  $h$ .

$$h = 6$$

**Step 3: Shift the graph  $f(x) = x^2$  to the right 6 units.**

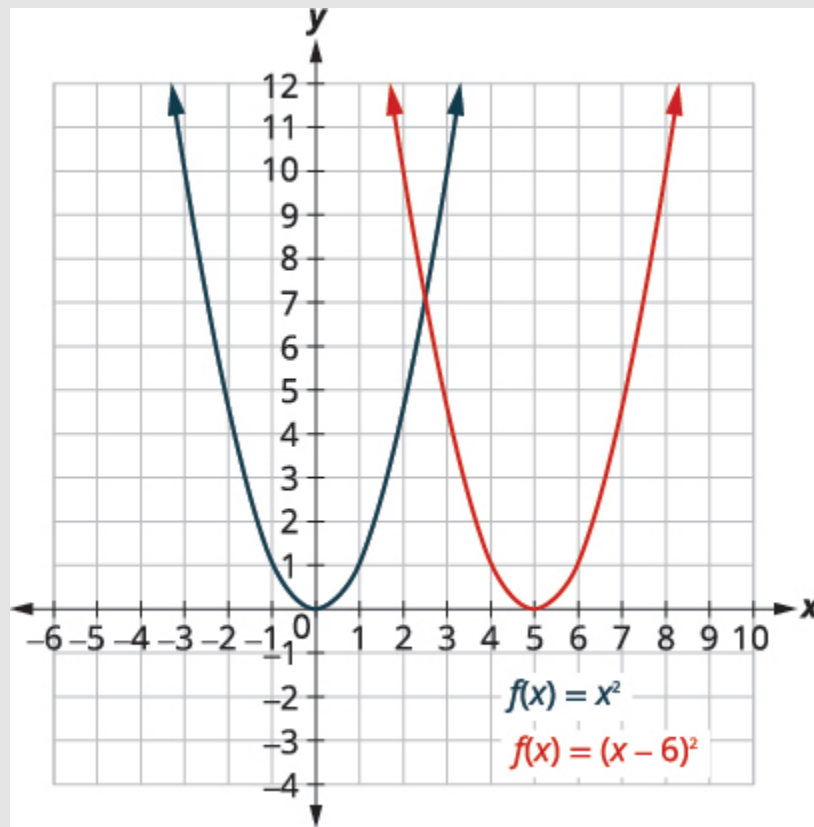


Figure 4.7.13

## Try It

10) Graph  $f(x) = (x - 4)^2$  using a horizontal shift.

### Solution



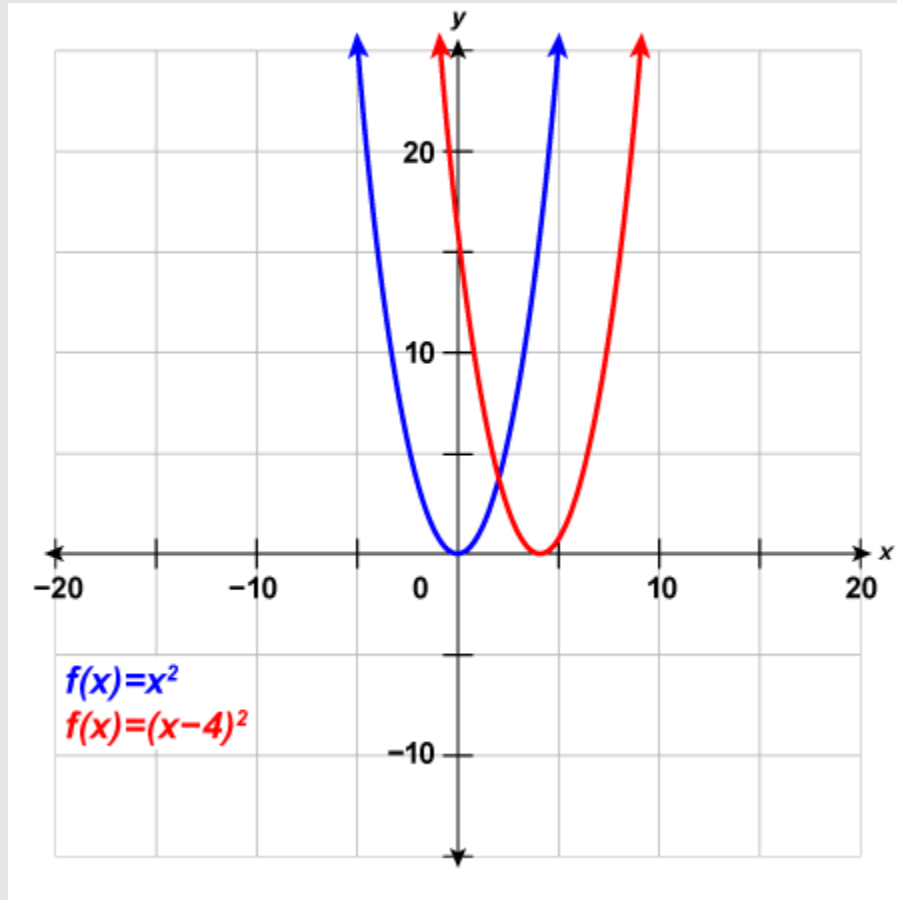


Figure 4.7.14

## Try It

11) Graph  $f(x) = (x + 6)^2$  using a horizontal shift.

### Solution

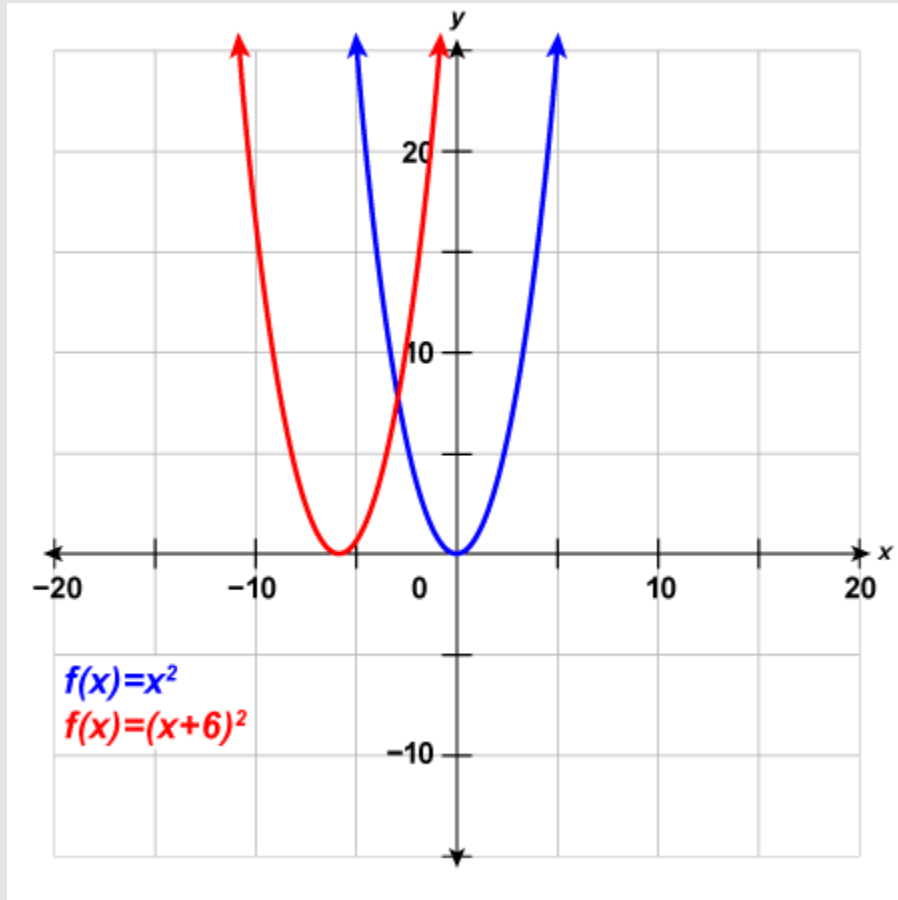


Figure 4.7.15

Now that we know the effect of the constants  $h$  and  $k$ , we will graph a quadratic function of the form  $f(x) = (x - h)^2 + k$  by first drawing the basic parabola and then making a horizontal shift followed by a vertical shift. We could do the vertical shift followed by the horizontal shift, but most students prefer the horizontal shift followed by the vertical.

## Example 4.7.5

Graph  $f(x) = (x + 1)^2 - 2$  using transformations.

### Solution

**Step 1: This function will involve two transformations and we need a plan.**

Let's first identify the constants  $h$ ,  $k$ .

$$f(x) = (x + 1)^2 - 2$$

$$f(x) = (x - h)^2 + k$$

$$f(x) = (x - (-1))^2 + (-2)$$

$$h = -1, k = -2$$

**Step 2: The  $h$  constant gives us a horizontal shift and the  $k$  gives us a vertical shift.**

$$f(x) = x^2 \xrightarrow{h = -1} f(x) = (x + 1)^2 \xrightarrow{k = -2} f(x) = (x + 1)^2 - 2$$

Shift left 1 unit

Shift down 2 units

**Step 3: We first draw the graph of  $f(x) = x^2$  on the grid.**

To graph

$$f(x) = (x + 1)^2$$

, shift the graph

$$f(x) = (x + 1)^2$$

to the left 1 unit.

To graph

$$f(x) = (x + 1)^2 - 2$$

, shift the graph

$$f(x) = (x + 1)^2 - 2$$

down 2 units.

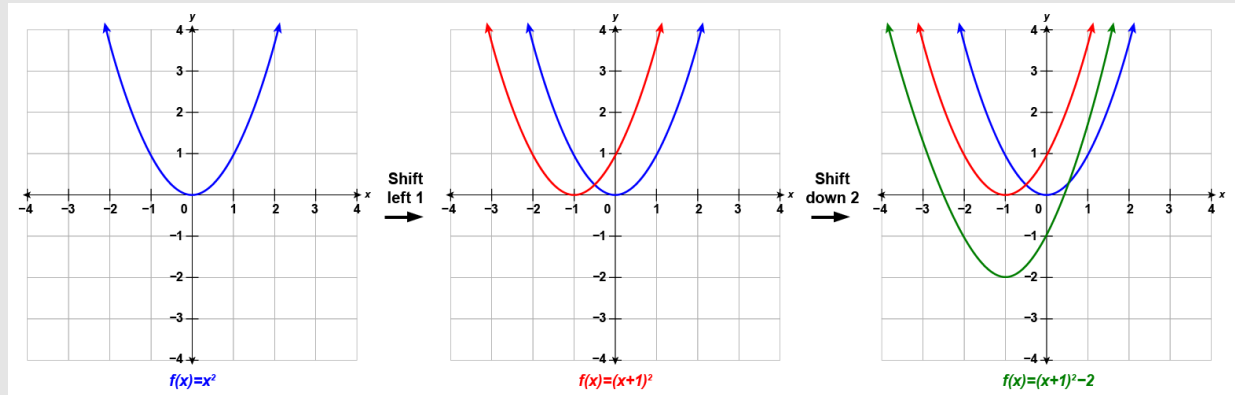


Figure 4.7.16

## Try It

12) Graph  $f(x) = (x + 2)^2 - 3$  using transformations.

### Solution

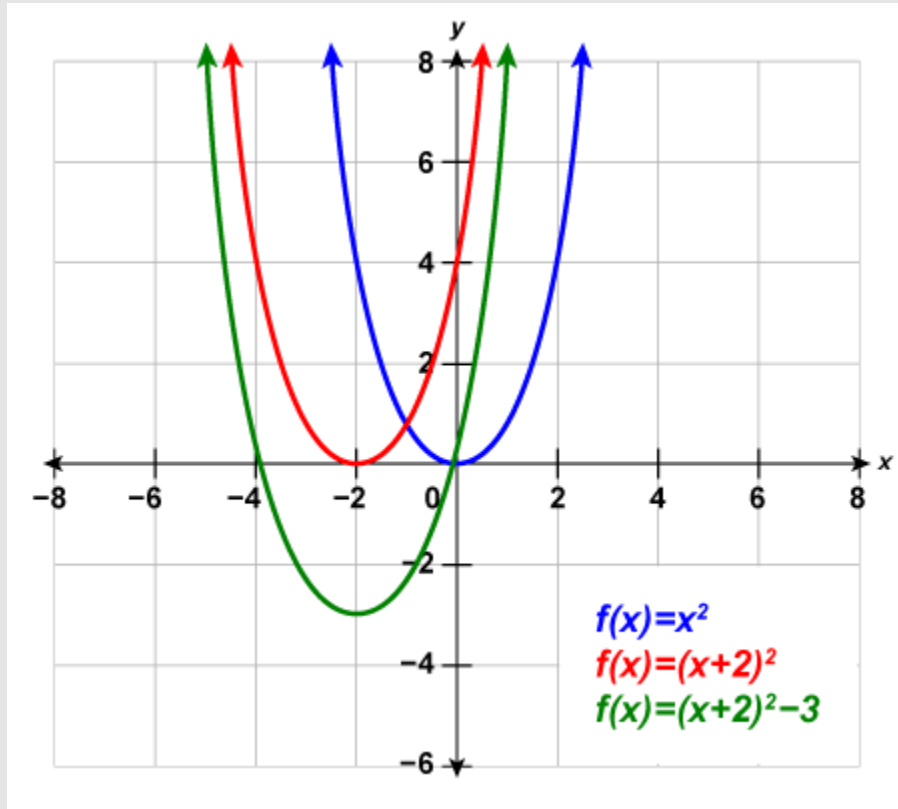


Figure 4.7.17

## Try It

13) Graph  $f(x) = (x - 3)^2 + 1$  using transformations.

**Solution**

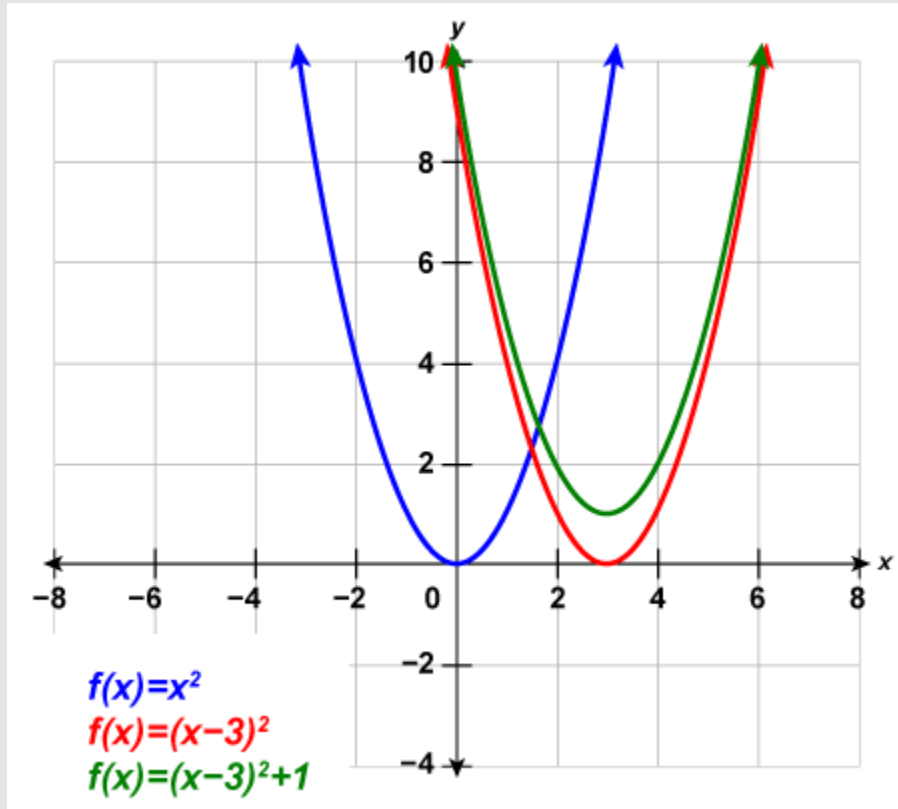


Figure 4.7.18

## Graph Quadratic Functions of the Form $f(x) = ax^2$

So far we graphed the quadratic function  $f(x) = x^2$  and then saw the effect of including a constant  $h$  or  $k$  in the equation had on the resulting graph of the new function. We will now explore the effect of the coefficient  $a$  on the resulting graph of the new function  $f(x) = ax^2$ .

Lets look at the quadratic functions

$f(x) = x^2$ ,  $f(x) = 2x^2$ ,  $f(x) = \frac{1}{2}x^2$ ,  $f(x) = -x^2$ ,  $f(x) = -2x^2$ ,  $f(x) = -\frac{1}{2}x^2$

and

$f(x) = x^2$ ,  $f(x) = 2x^2$ ,  $f(x) = \frac{1}{2}x^2$ ,  $f(x) = -x^2$ ,  $f(x) = -2x^2$ ,  $f(x) = -\frac{1}{2}x^2$

$x$	$f(x) = x^2$	$g(x) = 2x^2$	$h(x) = \frac{1}{2}x^2$
-2	4	8	2
-1	1	2	0.5
0	0	0	0
1	1	2	0.5
2	4	8	2

If we graph these functions, we can see the effect of the constant  $a$ , assuming  $a > 0$ .

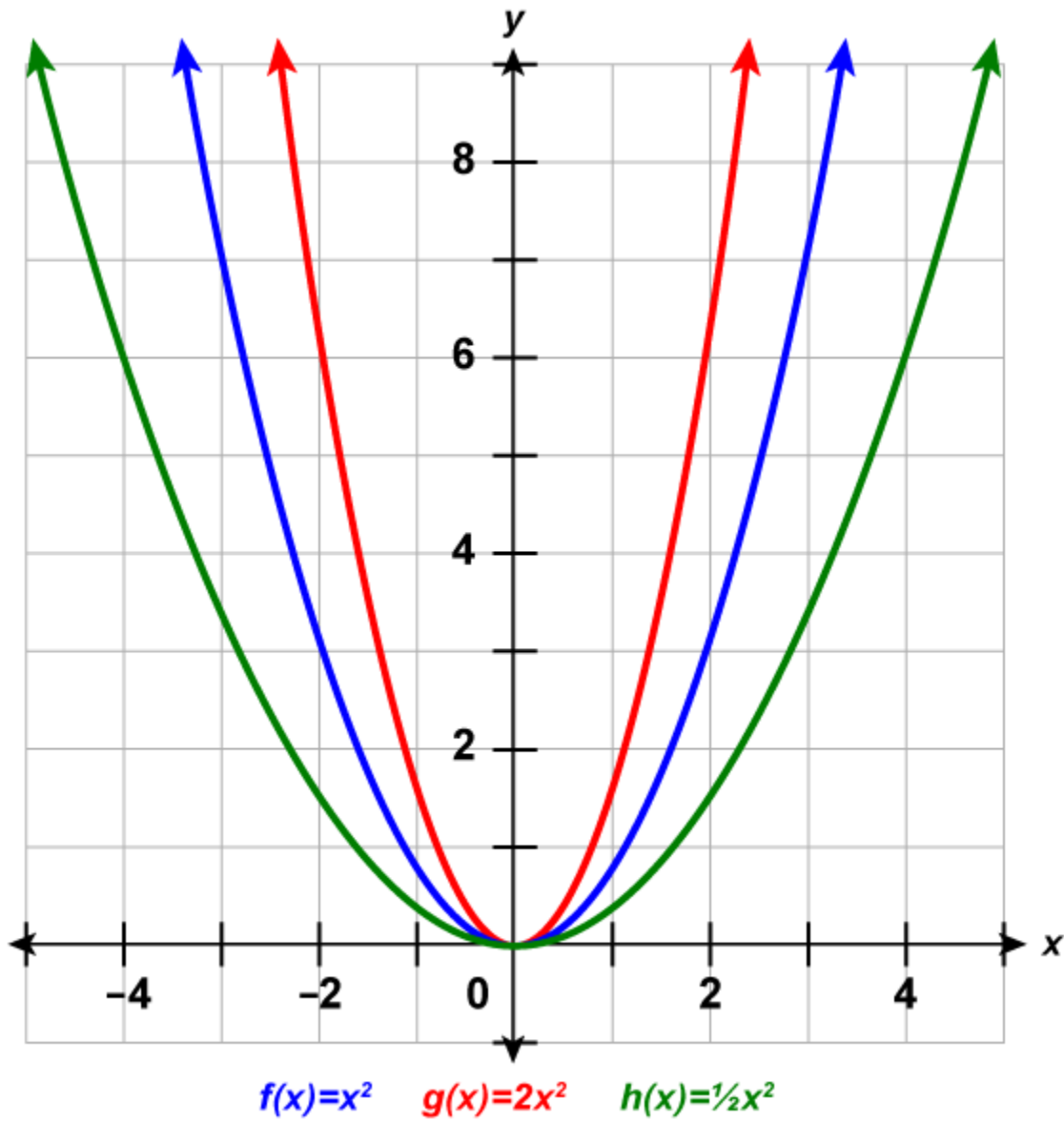


Figure 4.7.19

The graph of the function  $f(x) = x^2$  is “skinnier” than the graph of  $g(x) = 2x^2$ .

The graph of the function  $h(x) = \frac{1}{2}x^2$  is “wider” than the graph of  $f(x) = x^2$ .



$rgb]0.0, 0.0, 1.0frgb]0.0, 0.0, 1.0(rgb]0.0, 0.0, 1.0xrgb]0.0, 0.0, 1.0)rgb]0.0, 0.0, 1.0 =rgb]0.0, 0.0, 1.0x^{rgb]0.0, 0.0, 1.02}$

To graph a function with constant  $a$  it is easiest to choose a few points on  $f(x) = x^2$  and multiply the  $y$ -values by  $a$ .

### Graph of a Quadratic Function of the form $f(x) = ax^2$

The coefficient  $a$  in the function  $f(x) = ax^2$  affects the graph of  $f(x) = x^2$  by stretching or compressing it.

- If  $0 < |a| < 1$ , the graph of  $f(x) = ax^2$  will be “wider” than the graph of  $f(x) = x^2$ .
- If  $|a| > 1$ , the graph of  $f(x) = ax^2$  will be “skinnier” than the graph of  $f(x) = x^2$ .

### Example 4.7.6

Graph  $f(x) = 3x^2$ .

#### Solution

**Step 1:** We will graph the functions  $f(x) = x^2$  and  $g(x) = 3x^2$  on the same grid.

We will choose a few points on  $f(x) = x^2$  and then multiply the  $y$ -values by  $3$  to get the points for  $g(x) = 3x^2$ .

**Table 4.7.4**

$x$	$0.0, 0.0, 1.0(x, f(x))$	$1.0, 0.0, 0.0(x, g(x))$	
-2	$(-2, 4)$	$(-2, 12)$	$3 \times 4 = 12$
-1	$(-1, 1)$	$(-1, 3)$	$3 \times 1 = 3$
0	$(0, 0)$	$(0, 0)$	$3 \times 0 = 0$
1	$(1, 1)$	$(1, 3)$	$3 \times 1 = 3$
2	$(2, 4)$	$(2, 12)$	$3 \times 4 = 12$

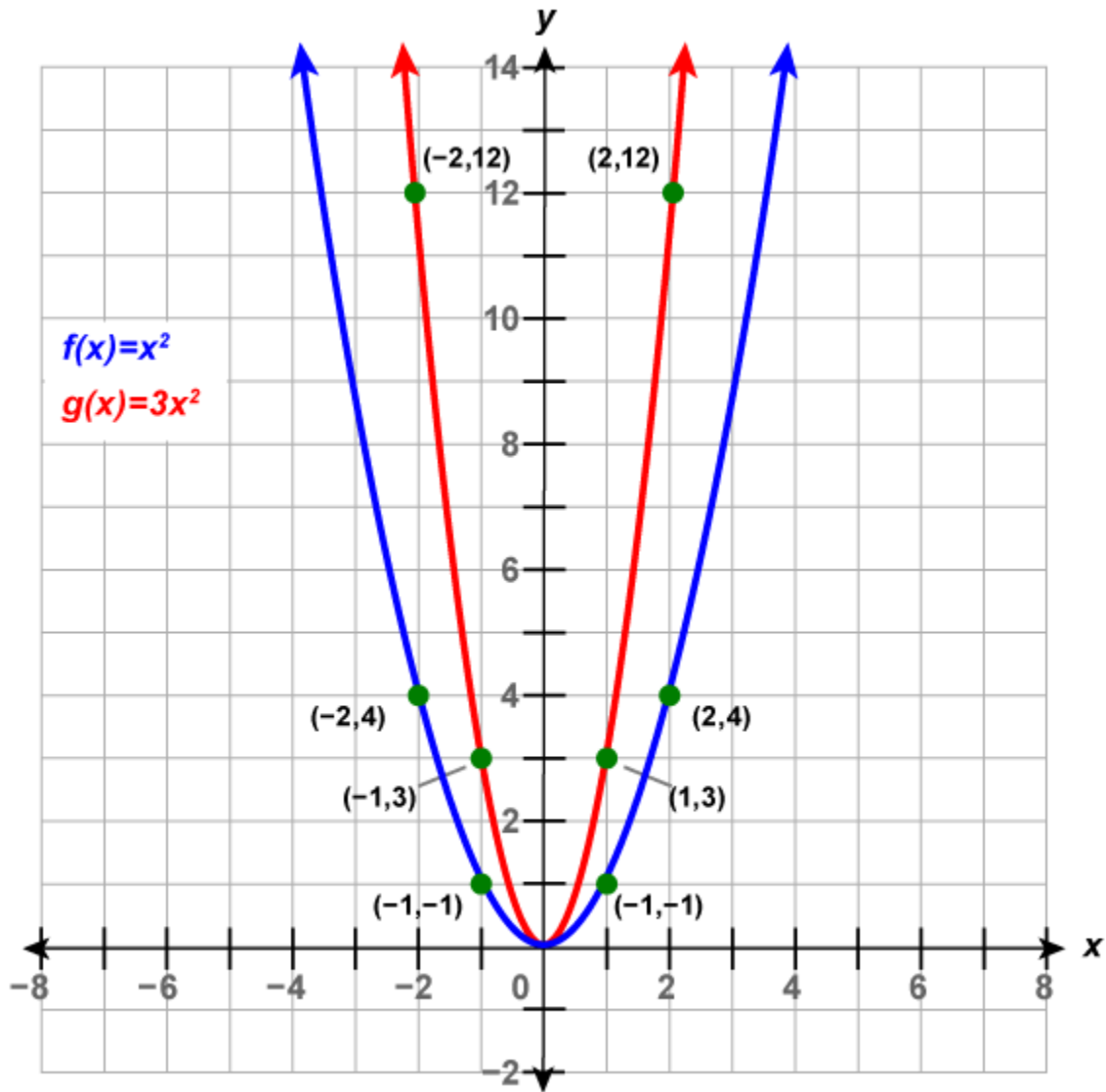


Figure 4.7.20

## Try It

14) Graph  $f(x) = -3x^2$ .

### Solution

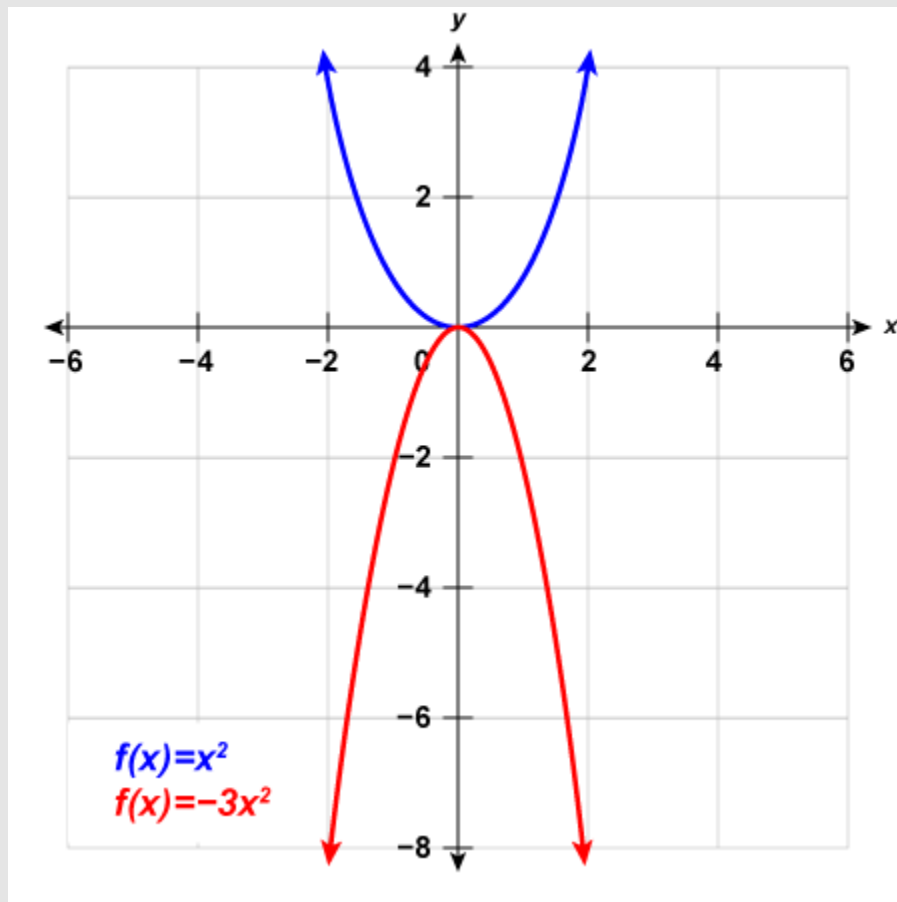


Figure 4.7.21

## Try It

15) Graph  $f(x) = 2x^2$ .

### Solution

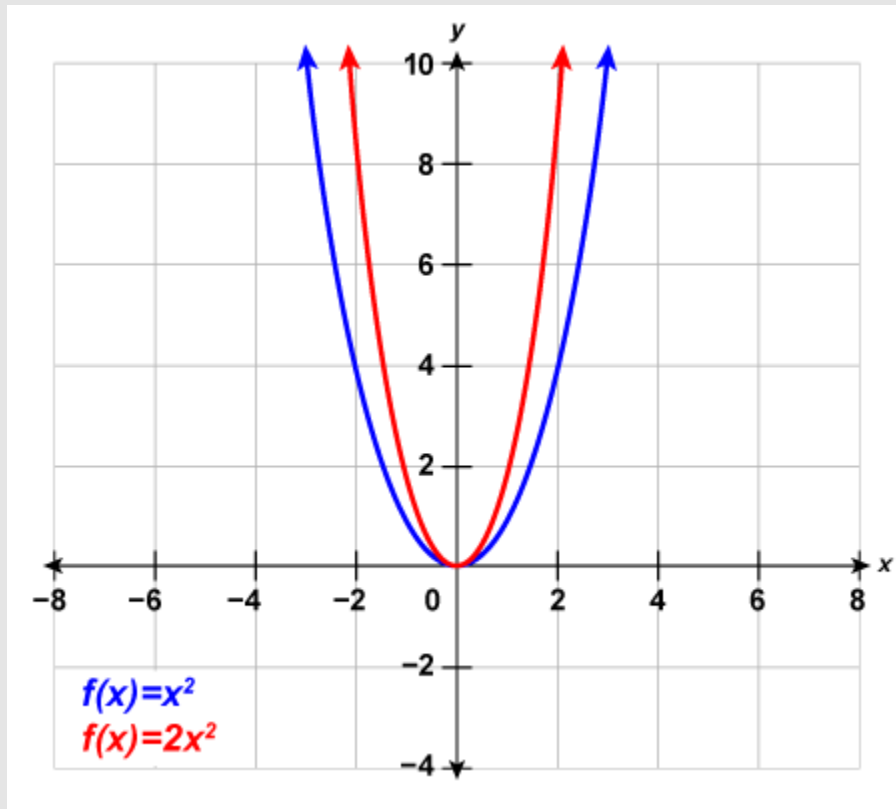


Figure 4.7.22

## Graph Quadratic Functions Using Transformations

We have learned how the constants  $a$ ,  $h$ , and  $k$  in the functions,  $f(x) = x^2 + k$ ,  $f(x) = (x - h)^2$ , and  $f(x) = ax^2$  affect their graphs. We can now put this together and graph quadratic functions  $f(x) = ax^2 + bx + c$  by first putting them into the form  $f(x) = a(x - h)^2 + k$  by completing the square. This form is sometimes known as the vertex form or standard form.

We must be careful to both add and subtract the number to the SAME side of the function to complete the square. We cannot add the number to both sides as we did when we completed the square with quadratic equations.

Quadratic Equation	Quadratic Function
$x^2 + 8x + 6 = 0$	$f(x) = x^2 + 8x + 6$
$x^2 + 8x = -6$	$f(x) = x^2 + 8x + 6$
$x^2 + 8x + 16 = -6 + 16$	$f(x) = x^2 + 8x + 16 + 6 - 16$
$(x + 4)^2 = 10$	$f(x) = (x + 4)^2 - 10$
<b>Add 16 to both sides</b>	<b>Add and subtract 16 from the same side</b>

Figure 4.7.23

When we complete the square in a function with a coefficient of  $x^2$  that is not one, we have to factor that coefficient from just the  $x$ -terms. We do not factor it from the constant term. It is often helpful to move the constant term a bit to the right to make it easier to focus only on the  $x$ -terms.

Once we get the constant we want to complete the square, we must remember to multiply it by that coefficient before we then subtract it.

### Example 4.7.7

Rewrite  $f(x) = -3x^2 - 6x - 1$  in the  $f(x) = a(x - h)^2 + k$  form by completing the square.

#### Solution

**Step 1: Separate the  $x$ -terms from the constant.**

$$f(x) = -3x^2 - 6x - 1$$

**Step 2: Factor the coefficient of  $x^2$ ,  $-3$ .**

$$f(x) = -3(x^2 + 2x) - 1$$

**Step 3: Prepare to complete the square.**

$$f(x) = -3(x^2 + 2x \quad ) - 1$$

**Step 4: Take half of 2 and then square it to complete the square.**

$$\left(\frac{1}{2} \cdot 2\right)^2 = 1$$

**Step 5: The constant 1 completes the square in the parentheses, but the parentheses is multiplied by  $-3$ .**

So we are really adding  $-3$ . We must then add  $3$  to not change the value of the function.

$$f(x) = -3(x^2 + 2x + 1) - 1 + 3$$

$-3 \cdot 1 = -3$  so add 3

Figure 4.7.24

**Step 6: Rewrite the trinomial as a square and subtract the constants.**

$$f(x) = -3(x + 1) + 2$$

**Step 7: The function is now in the  $f(x) = a(x - h)^2 + k$  form.**

$rgb[1.0, 0.0, 0.0]f[1.0, 0.0, 0.0]rgb[1.0, 0.0, 0.0]x[1.0, 0.0, 0.0]rgb[1.0, 0.0, 0.0] = rgb[1.0, 0.0, 0.0]a[1.0, 0.0, 0.0](x - h)^{2} + k$

$$f(x) = -3(x + 1)^2 + 2$$

## Try It

16) Rewrite  $f(x) = -4x^2 - 8x + 1$  in the  $f(x) = a(x - h)^2 + k$  form by completing the square.

**Solution**

$$f(x) = -4(x + 1)^2 + 5$$

## Try It

17) Rewrite  $f(x) = 2x^2 - 8x + 3$  in the  $f(x) = a(x - h)^2 + k$  form by completing the square.

### Solution

$$f(x) = 2(x - 2)^2 - 5$$

Once we put the function into the  $f(x) = a(x - h)^2 + k$  form, we can then use the transformations as we did in the last few problems. The next example will show us how to do this.

## Example 4.7.8

Graph  $f(x) = x^2 + 6x + 5$  by using transformations.

### Solution

**Step 1: Rewrite the function in  $f(x) = a(x - h)^2 + k$  vertex form by completing the square.**

Separate the  $x$  terms from the constant.

$$x^2 + 6x + 5$$

Take half of **6** and then square it to complete the square.

$$\left(\frac{1}{2} \times 6\right)^2 = 9$$

We both add **9** and subtract **9** to not change the value of the function.

$$f(x) = x^2 + 6x + 9 + 5 - 9$$



Rewrite the trinomial as a square and subtract the constants.

$$f(x) = (x + 3)^2 - 4$$

The function is now in the  $f(x) = (x - h)^2 + k$  form.

$$f(x) = (x + 3)^2 - 4$$

**Step 2: Graph the function using transformations.**

Looking at the  $h$ ,  $k$  values, we see the graph will take the graph of  $f(x) = x^2$  and shift it to the left 3 units and down 4 units.

$$f(x) = x^2 \xrightarrow{h = -3} f(x) = (x + 3)^2 \xrightarrow{k = -4} f(x) = (x + 3)^2 - 4$$

Shift left 3 units
Shift down 4 units

We first draw the graph of  $f(x) = x^2$  on the grid.

To graph  $f(x) = (x + 3)^2$ , shift the graph  $f(x) = x^2$  to the left 3 units.

To graph  $f(x) = (x + 3)^2 - 4$ , shift the graph  $f(x) = (x + 3)^2$  down 4 units.

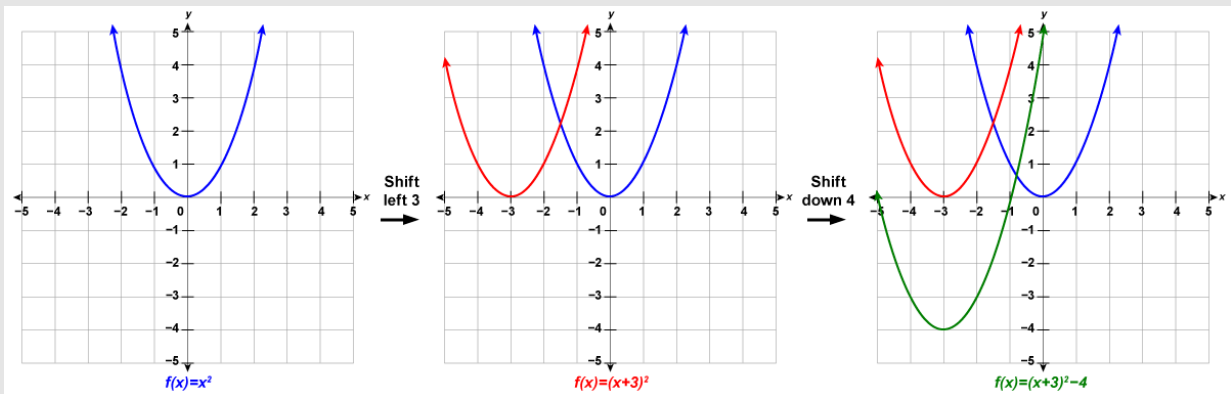


Figure 4.7.25

## Try It

18) Graph  $f(x) = x^2 + 2x - 3$  by using transformations.

### Solution

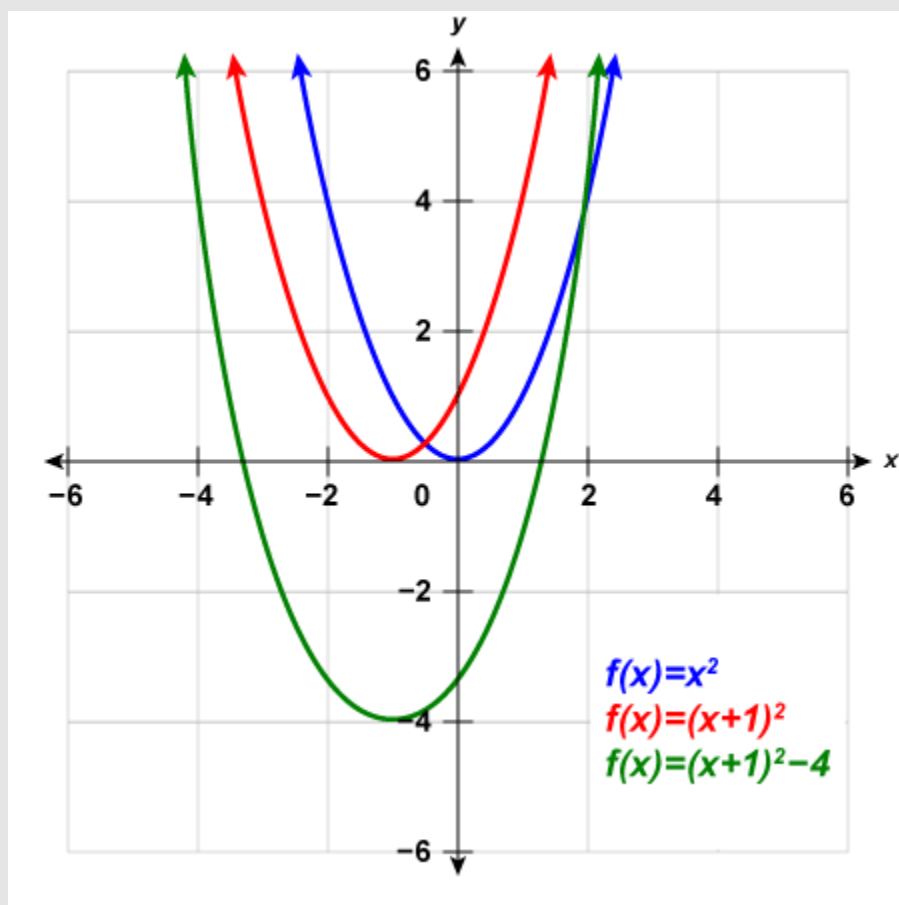


Figure 4.7.26

## Try It

19) Graph  $f(x) = x^2 - 8x + 12$  by using transformations.

### Solution

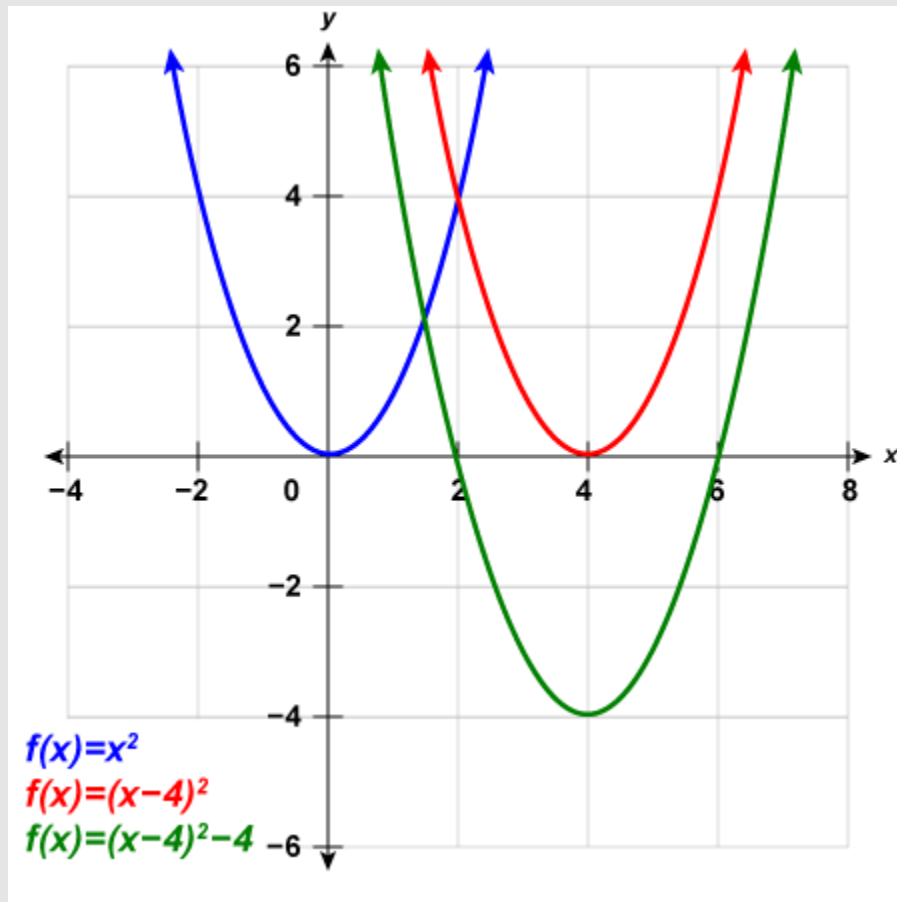


Figure 4.7.27

We list the steps to take to graph a quadratic function using transformations here.

## HOW TO

### Graph a quadratic function using transformations.

1. Rewrite the function in  $f(x) = a(x - h)^2 + k$  form by completing the square.
2. Graph the function using transformations.

### Example 4.7.9

Graph  $f(x) = -2x^2 - 4x + 2$  by using transformations.

#### Solution

**Step 1: Rewrite the function in  $f(x) = a(x - h)^2 + k$  vertex form by completing the square.**

Separate the  $x$  terms from the constant.

$$f(x) = -2x^2 - 4x + 2$$

We need the coefficient of  $x^2$  to be one. We factor  $-2$  from the  $x$ -terms.

$$f(x) = -2(x^2 + 2x) + 2$$

Take half of  $2$  and then square it to complete the square.

$$\left(\frac{1}{2} \times 2\right)^2 = 1$$

We add  $1$  to complete the square in the parentheses, but the parentheses is multiplied by  $-2$ . So we are really adding  $-2$ . To not change the value of the function we add  $2$ .

$$f(x) = -2(x^2 + 2x + 1) + 2 + 2$$

Figure 4.7.28

The function is now in the  $f(x) = a(x - h)^2 + k$  form.

rgb(1.0, 0.0, 0.0) f(rgb)1.0, 0.0, 0.0 (rgb)1.0, 0.0, 0.0 z(rgb)1.0, 0.0, 0.0 = (rgb)1.0, 0.0, 0.0 a(rgb)1.0, 0.0, 0.0 (rgb)1.0, 0.0, 0.0 z(rgb)1.0, 0.0, 0.0 - (rgb)1.0, 0.0, 0.0 h(rgb)1.0, 0.0, 0.0 (rgb)1.0, 0.0, 0.0 + (rgb)1.0, 0.0, 0.0 k

$$f(x) = -2(x + 1)^2 + 4$$

### Step 2: Graph the function using transformations.

$f(x) = x^2$  &  $f(x) = -2x^2$  &  $f(x) = -2(x+1)^2$  &  $f(x) = -2(x+1)^2 + 4$   
 $a = -2$  &  $h = -1$  &  $k = 4$   
 Multiply values by -2  
 Shift left 1 unit  
 Shift up 4 units

We first draw the graph of  $f(x) = x^2$  on the grid.

To graph  $f(x) = -2x^2$ , multiply the  $y$ -values in parabola of  $f(x) = x^2$  by  $-2$ .

To graph  $f(x) = -2(x + 1)^2$ , shift the graph  $f(x) = -2x^2$  to the left 1 unit.

To graph  $f(x) = -2(x + 1)^2 + 4$ , shift the graph of  $f(x) = -2(x + 1)^2$  up 4 units.

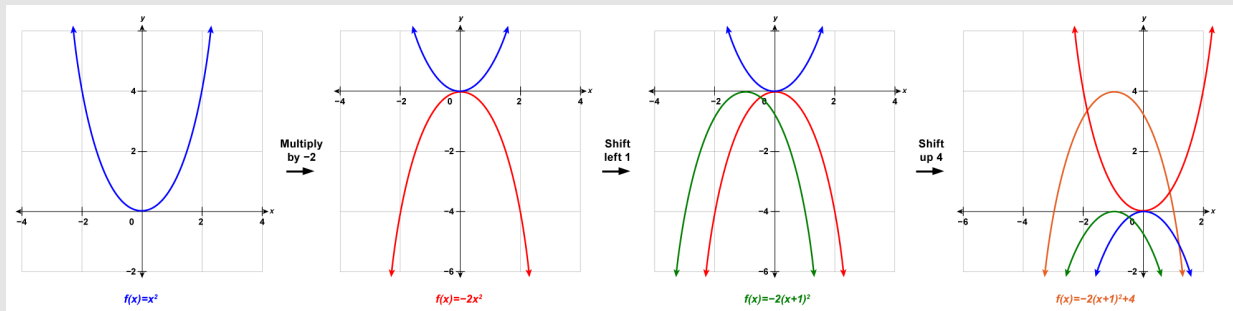


Figure 4.7.29

## Try It

20) Graph  $f(x) = -3x^2 + 12x - 4$  by using transformations.

### Solution

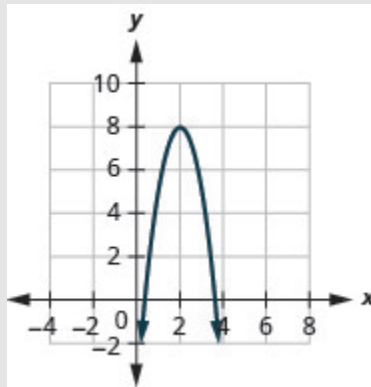


Figure 4.7.30

## Try It

21) Graph  $f(x) = -2x^2 + 12x - 9$  by using transformations.

### Solution

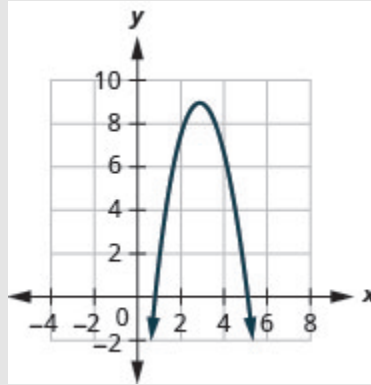


Figure 4.7.31

Now that we have completed the square to put a quadratic function into  $f(x) = a(x - h)^2 + k$  form, we can also use this technique to graph the function using its properties as in the previous section.

If we look back at the last few examples, we see that the vertex is related to the constants  $h$  and  $k$ .

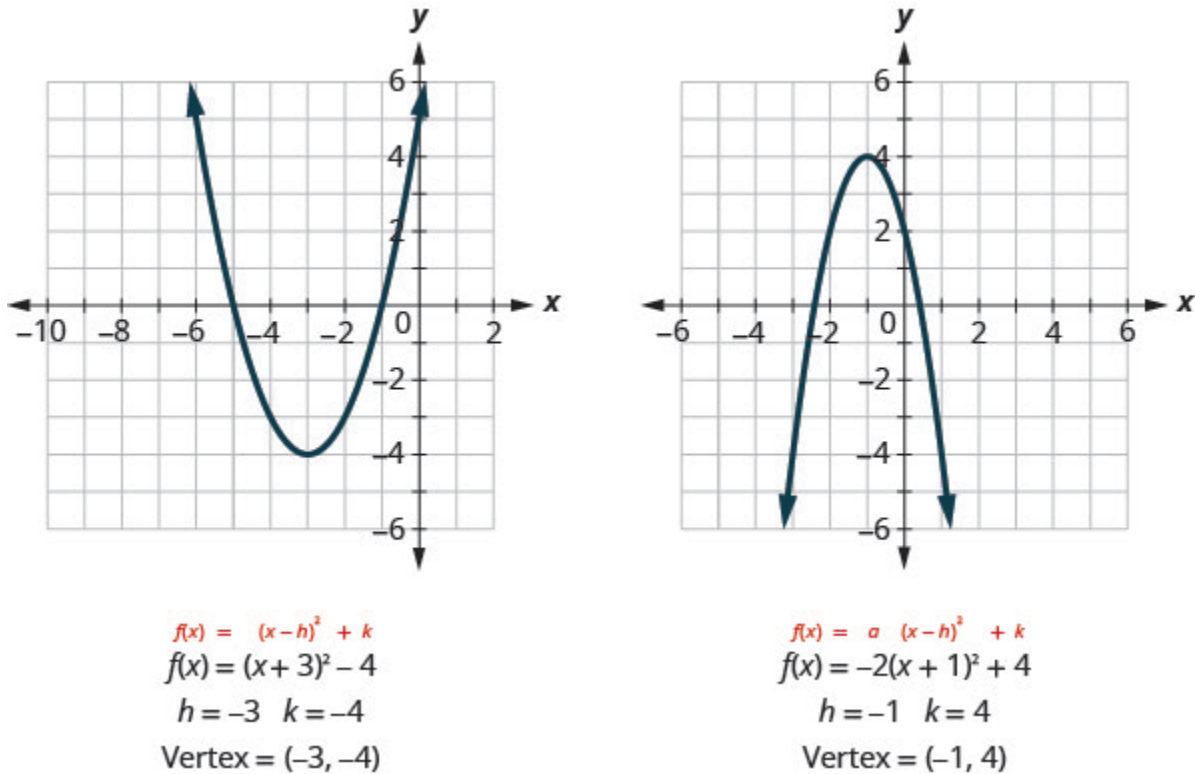


Figure 4.7.32

In each case, the vertex is  $(h, k)$ . Also the axis of symmetry is the line  $x = h$ .

We rewrite our steps for graphing a quadratic function using properties for when the function is in  $f(x) = a(x - h)^2 + k$  form.

## HOW TO

Graph a quadratic function in the form  $f(x) = a(x - h)^2 + k$  using properties.

1. Rewrite the function in  $f(x) = a(x - h)^2 + k$  form.
2. Determine whether the parabola opens upward,  $a > 0$ , or downward,  $a < 0$ .
3. Find the axis of symmetry,  $x = h$ .
4. Find the vertex,  $(h, k)$ .
5. Find the  $y$ -intercept. Find the point symmetric to the  $y$ -intercept across the axis of symmetry.
6. Find the  $x$ -intercepts.
7. Graph the parabola.

### Example 4.7.10

- a. Rewrite  $f(x) = 2x^2 + 4x + 5$  in  $f(x) = a(x - h)^2 + k$  form and
- b. graph the function using properties.

#### Solution

a.

**Step 1:** Rewrite the function in  $f(x) = a(x - h)^2 + k$  form by completing the square.

$$f(x) = 2x^2 + 4x + 5$$

**Step 2:** Identify the constants  $a, h, k$ .



$$f(x) = 2(x^2 + 2x) + 5$$

$$f(x) = 2(x^2 + 2x + 1) + 5 - 2$$

$$f(x) = 2(x + 1)^2 + 3$$

$$a = 2 \quad h = -1 \quad k = 3$$

**Step 3:** Since  $a = 2$ , the parabola opens upward.



Figure  
4.7.33

**Step 4:** The axis of symmetry is  $x = h$ .

The axis of symmetry is  $x = -1$ .

**Step 5:** The vertex is  $(h, k)$ .

The vertex is  $(-1, 3)$ .

**Step 6:** Find the  $y$ -intercept by finding  $f(0)$ .

$$f(0) = 2 \cdot 0^2 + 4 \cdot 0 + 5$$

**Step 7:** Find the point symmetric to  $(0, 5)$  across the axis of symmetry.

$$(-2, 5)$$

**Step 8:** Find the  $x$ -intercepts.

The discriminant is negative, so there are no  $x$ -intercepts.

b.

**Step 1:** Graph the parabola.

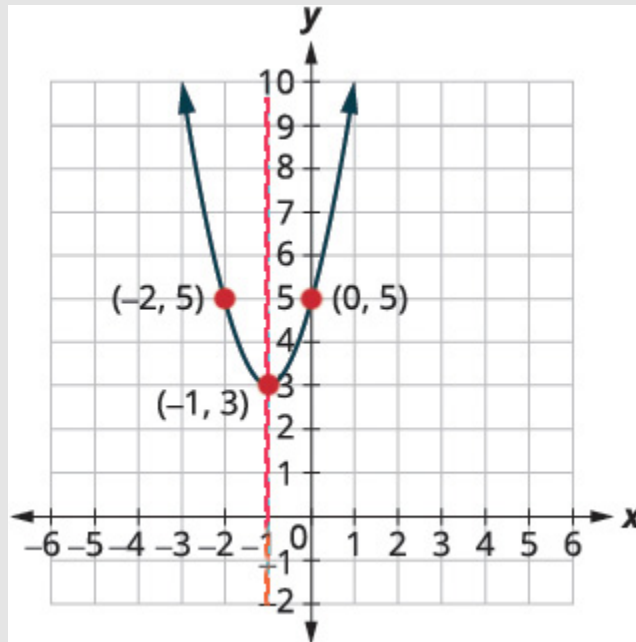


Figure 4.7.34

## Try It

22) Complete the following:

- Rewrite  $f(x) = 3x^2 - 6x + 5$  in  $f(x) = a(x - h)^2 + k$  form and
- Graph the function using properties.

### Solution

- $f(x) = 3(x - 1)^2 + 2$
-

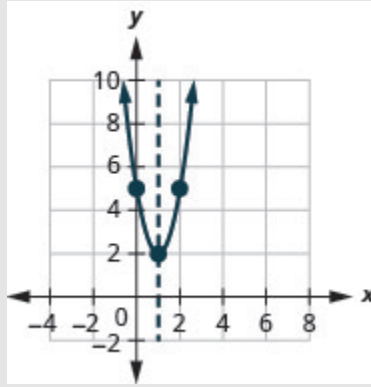


Figure 4.7.35

## Try It

23) Complete the following:

- Rewrite  $f(x) = -2x^2 + 8x - 7$  in  $f(x) = a(x - h)^2 + k$  form and
- graph the function using properties.

### Solution

- $f(x) = -2(x - 2)^2 + 1$
-

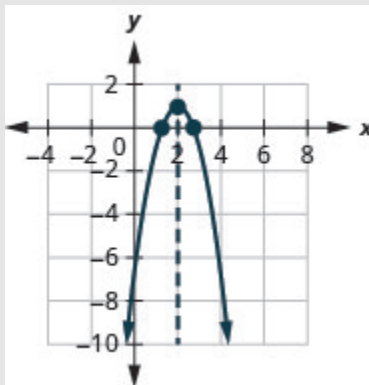


Figure 4.7.36

## Find a Quadratic Function from its Graph

So far we have started with a function and then found its graph.

Now we are going to reverse the process. Starting with the graph, we will find the function.

### Example 4.7.11

Determine the quadratic function whose graph is shown.

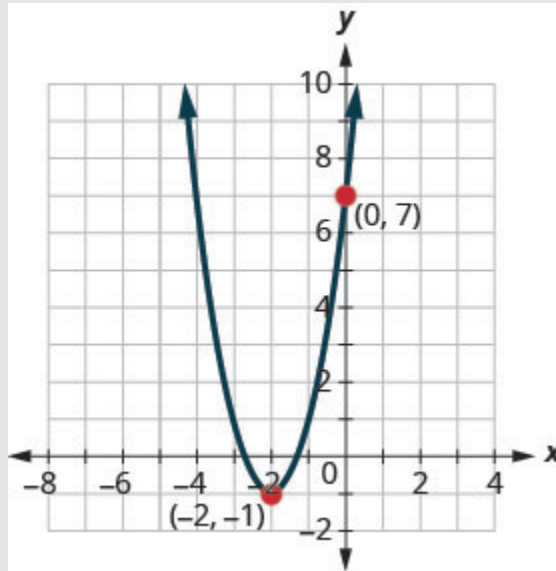


Figure 4.7.37

**Solution**

**Step 1: Since it is quadratic, we start with the  $f(x) = a(x - h)^2 + k$  form.**

The vertex,  $(h, k)$  is  $(-2, -1)$ , so  $h = -2$  and  $k = -1$ .

$$f(x) = a(x - (-2))^2 - 1$$

**Step 2: To find  $a$ , we use the  $y$ -intercept,  $(0, 7)$ .**

So  $f(0) = 7$ .

$$7 = a(x - (-2))^2 - 1$$

**Step 3: Solve for  $a$ .**

$$7 = a(0 + 2)^2 - 1$$

$$7 = 4a - 1$$

$$8 = 4a$$

$$2 = a$$

**Step 4: Write the function.**

$$f(x) = a(x - h)^2 + k$$

**Step 5: Substitute in  $h = -2$ ,  $k = -1$ , and  $a = 2$ .**

$$f(x) = 2(x + 2)^2 - 1$$

## Try It

24) Write the quadratic function in  $f(x) = a(x - h)^2 + k$  form whose graph is shown.

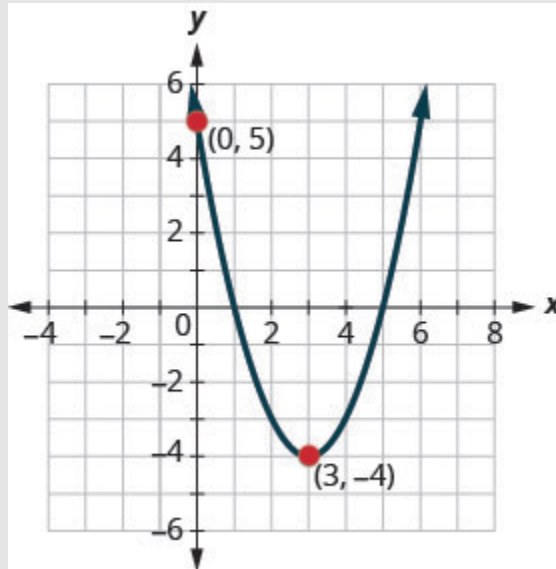


Figure 4.7.38

### Solution

$$f(x) = (x - 3)^2 - 4$$

## Try It

25) Determine the quadratic function whose graph is shown.

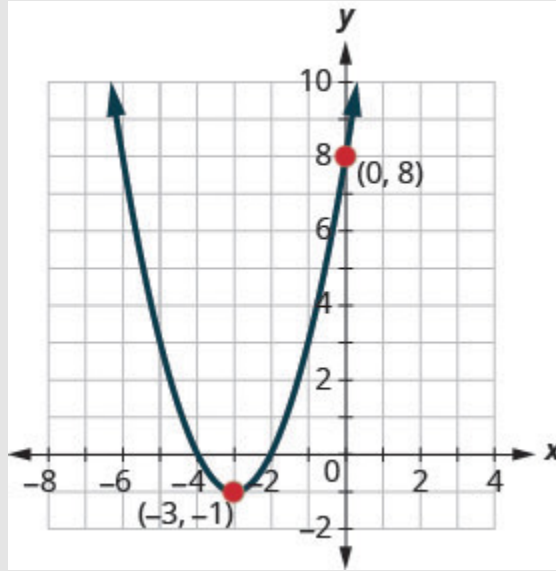


Figure 4.7.39

**Solution**

$$f(x) = (x + 3)^2 - 1$$

Access these online resources for additional instruction and practice with graphing quadratic functions using transformations.

- [Function Shift Rules Applied to Quadratic Functions](#)
- [Changing a Quadratic from Standard Form to Vertex Form](#)
- [Using Transformations to Graph Quadratic Functions](#)
- [Finding Quadratic Equation in Vertex Form from Graph](#)

## Key Concepts

- **Graph a Quadratic Function of the form  $f(x) = x^2 + k$  Using a Vertical Shift**
  - The graph of  $f(x) = x^2 + k$  shifts the graph of  $f(x) = x^2$  vertically  $k$  units.
    - If  $k > 0$ , shift the parabola vertically up  $k$  units.
    - If  $k < 0$ , shift the parabola vertically down  $|k|$  units.
- **Graph a Quadratic Function of the form  $f(x) = (x - h)^2$  Using a Horizontal Shift**
  - The graph of  $f(x) = (x - h)^2$  shifts the graph of  $f(x) = x^2$  horizontally  $h$  units.
    - If  $h > 0$ , shift the parabola horizontally left  $h$  units.
    - If  $h < 0$ , shift the parabola horizontally right  $|h|$  units.
- **Graph of a Quadratic Function of the form  $f(x) = ax^2$** 
  - The coefficient  $a$  in the function  $f(x) = ax^2$  affects the graph of  $f(x) = x^2$  by stretching or compressing it.
    - If  $0 < |a| < 1$ , then the graph of  $f(x) = ax^2$  will be “wider” than the graph of  $f(x) = x^2$ .
    - If  $|a| > 1$ , then the graph of  $f(x) = ax^2$  will be “skinnier” than the graph of  $f(x) = x^2$ .
- **How to graph a quadratic function using transformations**
  1. Rewrite the function in  $f(x) = a(x - h)^2 + k$  form by completing the square.
  2. Graph the function using transformations.
- **Graph a quadratic function in the vertex form  $f(x) = a(x - h)^2 + k$  using properties**



1. Rewrite the function in  $f(x) = a(x - h)^2 + k$  form.
2. Determine whether the parabola opens upward,  $a > 0$ , or downward,  $a < 0$ .
3. Find the axis of symmetry,  $x = h$ .
4. Find the vertex,  $(h, k)$ .
5. Find the  $y$ -intercept. Find the point symmetric to the  $y$ -intercept across the axis of symmetry.
6. Find the  $x$ -intercepts, if possible.
7. Graph the parabola.

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

*<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=1334#h5p-33>*

b) After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

## 4.8 UNIT 4 SOURCES

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**4.0** “9.0 Quadratic Equations and Functions Introduction” from Intermediate Algebra by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted.

**4.1** “9.1 Solve Quadratic Equations Using the Square Root Property” from Intermediate Algebra by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted.

**4.2** “9.2 Solve Quadratic Equations by Completing the Square” from Intermediate Algebra by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted.

**4.3** “9.3 Solve Quadratic Equations Using the Quadratic Formula” from Intermediate Algebra by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted.

**4.4** “9.4 Solve Equations in Quadratic Form” from Intermediate Algebra by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted.

**4.5** “9.5 Solve Applications of Quadratic Equations” from Intermediate Algebra by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted.

**4.6** “9.6 Graph Quadratic Functions Using Properties” from Intermediate Algebra by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted.

**4.7** “9.7 Graph Quadratic Functions Using Transformations” from Intermediate Algebra by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted.

# UNIT 5: EXPONENTIAL AND LOGARITHMIC FUNCTIONS

## Chapter Outline

- 5.0 Introduction
- 5.1 Evaluate and Graph Exponential Functions
- 5.2 Evaluate and Graph Logarithmic Functions
- 5.3 Use the Properties of Logarithms
- 5.4 Solve Exponential and Logarithmic Equations
- 5.5 Unit 5 Sources



## 5.0 INTRODUCTION

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“Hydroponics- Salad Dengan Sistem” by Izhamwong CC BY 3.0

Hydroponic systems allow botanists to grow crops without land. (credit: “Izhamwong”/Wikimedia Commons)

As the world population continues to grow, food supplies are becoming less able to meet the increasing demand. At the same time, available resources of fertile soil for growing plants is dwindling. One possible solution—grow plants without soil. Botanists around the world are expanding the potential of hydroponics, which is the process of growing plants without soil. To provide the plants with the nutrients they need, the botanists keep careful growth records. Some growth is described by the types of functions you will explore in this chapter—exponential and logarithmic. You will evaluate and graph these functions, and solve equations using them.

# 5.1 EVALUATE AND GRAPH EXPONENTIAL FUNCTIONS

---

## Learning Objectives

By the end of this section, you will be able to:

- Graph exponential functions
- Solve Exponential equations
- Use exponential models in applications

## Try It

Before you get started, take this readiness quiz:

1) Simplify:  $\left(\frac{x^3}{x^2}\right)$

2) Evaluate: a.  $2^0$  b.  $\left(\frac{1}{3}\right)^0$

3) Evaluate: a.  $2^{-1}$  b.  $\left(\frac{1}{3}\right)^{-1}$

## Graph Exponential Functions

The functions we have studied so far do not give us a model for many naturally occurring phenomena. From the growth of populations and the spread of viruses to radioactive decay and compounding interest, the models are very different from what we have studied so far. These models involve **exponential functions**.

An exponential function is a function of the form  $f(x) = a^x$  where  $a > 0$  and  $a \neq 1$ .

### Exponential Function

An exponential function, where  $a > 0$  and  $a \neq 1$ , is a function of the form

$$f(x) = a^x$$

Notice that in this function, the variable is the exponent. In our functions so far, the variables were the base.

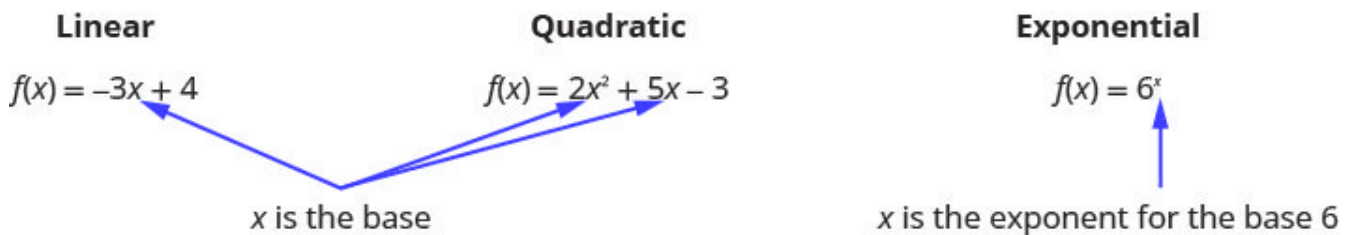


Figure 5.11

Our definition says  $a \neq 1$ . If we let  $a = 1$ , then  $f(x) = a^x$  becomes  $f(x) = 1^x$ . Since  $1^x = 1$  for all real numbers,  $f(x) = 1$ . This is the constant function.

Our definition also says  $a > 0$ . If we let a base be negative, say  $-4$ , then  $f(x) = (-4)^x$  is not a real number when  $x = \frac{1}{2}$ .

$$\begin{aligned} f(x) &= (-4)^x \\ f\left(\frac{1}{2}\right) &= (-4)^{\frac{1}{2}} \\ f\left(\frac{1}{2}\right) &= \sqrt{-4} \text{ Not a real number.} \end{aligned}$$

In fact,  $f(x) = (-4)^x$  would not be a real number any time  $x$  is a fraction with an even denominator. So our definition requires  $a > 0$ .

By graphing a few exponential functions, we will be able to see their unique properties.

### Example 5.1.1

On the same coordinate system graph  $f(x) = 2^x$  and  $g(x) = 3^x$ .

#### Solution

**Step 1:** We will use point plotting to graph the functions.

$x$	$f(x) = 2^x$	$(x, f(x))$	$g(x) = 3^x$	$(x, g(x))$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$	$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$	$\left(-2, \frac{1}{9}\right)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$	$3^{-1} = \frac{1}{3^1} = \frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$
0	$2^0 = 1$	$(0, 1)$	$3^0 = 1$	$(0, 1)$
1	$2^1 = 2$	$(1, 2)$	$3^1 = 3$	$(1, 3)$
2	$2^2 = 4$	$(2, 4)$	$3^2 = 9$	$(2, 9)$
3	$2^3 = 8$	$(3, 8)$	$3^3 = 27$	$(3, 27)$



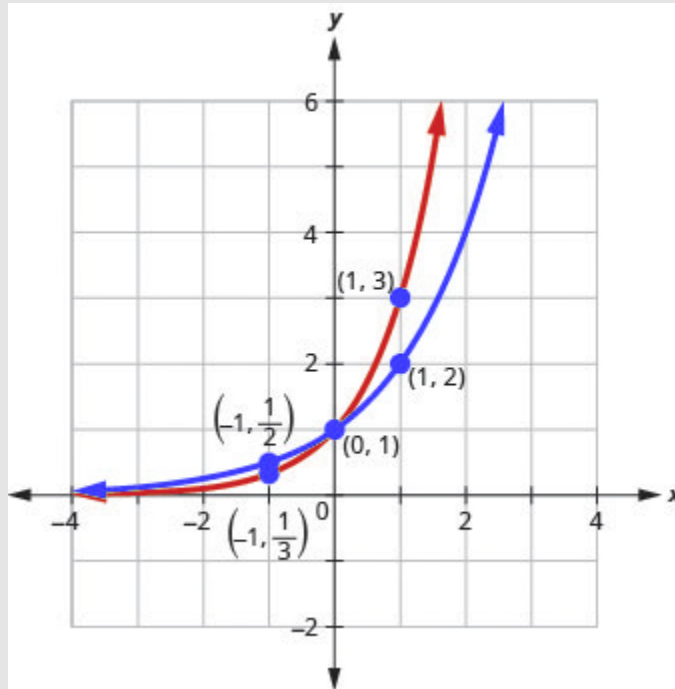


Figure 5.1.2

## Try It

4) Graph:  $f(x) = 4^x$

**Solution**

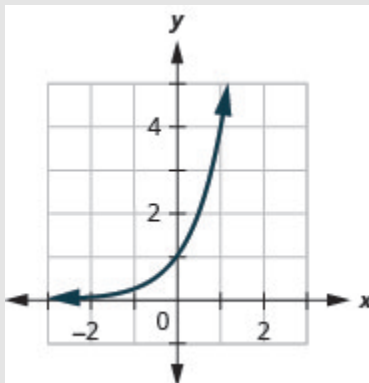


Figure 5.13

## Try It

5) Graph:  $g(x) = 5^x$

### Solution

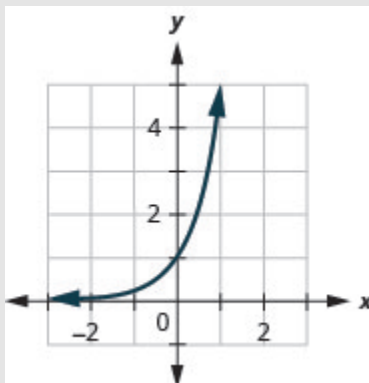


Figure 5.14

If we look at the graphs from the previous Example and Try Its, we can identify some of the properties of exponential functions. The graphs of  $f(x) = 2^x$  and  $g(x) = 3^x$ , as well as the graphs of  $f(x) = 4^x$

and  $g(x) = 5^x$ , all have the same basic shape. This is the shape we expect from an exponential function where  $a > 1$ .

We notice, that for each function, the graph contains the point  $(0, 1)$ . This makes sense because  $a^0 = 1$  for any  $a$ .

The graph of each function,  $f(x) = a^x$  also contains the point  $(1, a)$ . The graph of  $f(x) = 2^x$  contained  $(1, 2)$  and the graph of  $g(x) = 3^x$  contained  $(1, 3)$ . This makes sense as  $a^1 = a$ .

Notice too, the graph of each function  $f(x) = a^x$  also contains the point  $\left(-1, \frac{1}{a}\right)$ . The graph of  $f(x) = 2^x$  contained  $\left(-1, \frac{1}{2}\right)$  and the graph of  $g(x) = 3^x$  contained  $\left(-1, \frac{1}{3}\right)$ . This makes sense as  $a^{-1} = \frac{1}{a}$ .

What is the domain for each function? From the graphs we can see that the domain is the set of all real numbers. There is no restriction on the domain. We write the domain in interval notation as  $(-\infty, \infty)$ .

Look at each graph. What is the range of the function? The graph never hits the  $x$ -axis. The range is all positive numbers. We write the range in interval notation as  $(0, \infty)$ .

Whenever a graph of a function approaches a line but never touches it, we call that line an **asymptote**. For the exponential functions we are looking at, the graph approaches the  $x$ -axis very closely but will never cross it, we call the line  $y = 0$ , the  $x$ -axis, a horizontal asymptote.

Properties of the Graph of  $f(x) = a^x$  when  $a > 1$

<b>Domain</b>	$(-\infty, \infty)$
<b>Range</b>	$(0, \infty)$
<b><math>x</math>-intercept</b>	None
<b><math>y</math>-intercept</b>	$(0, 1)$
<b>Contains</b>	$(1, a), \left(-1, \frac{1}{a}\right)$
<b>Asymptote</b>	$x$ -axis, the line $y = 0$

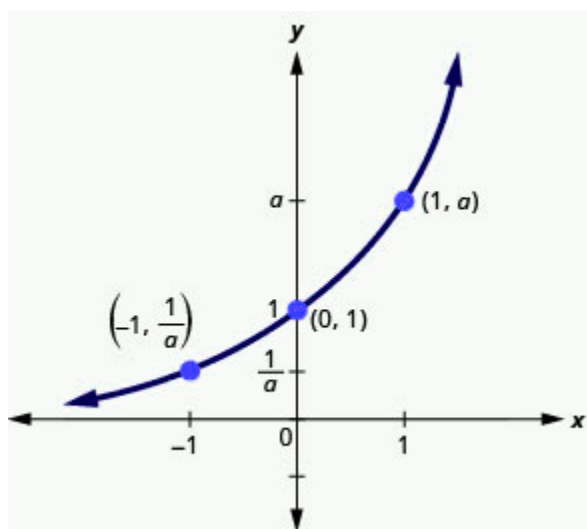


Figure 5.1.5

Our definition of an exponential function  $f(x) = a^x$  says  $a > 0$ , but the examples and discussion so far has been about functions where  $a > 1$ . What happens when  $0 < a < 1$ ? The next example will explore this possibility.

### Example 5.1.2

On the same coordinate system, graph  $f(x) = \left(\frac{1}{2}\right)^x$  and  $g(x) = \left(\frac{1}{3}\right)^x$

#### Solution

**Step 1:** We will use point plotting to graph the functions.

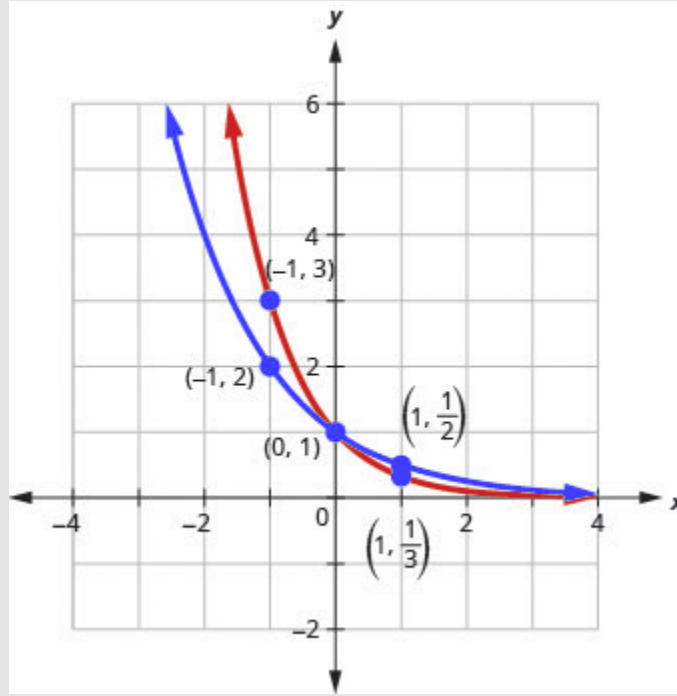


Figure 5.1.6

$x$	$f(x) = \left(\frac{1}{2}\right)^x$	$(x, f(x))$	$g(x) = \left(\frac{1}{3}\right)^x$	$(x, g(x))$
-2	$\left(\frac{1}{2}\right)^{-2} = 2^2 = 4$	$(-2, 4)$	$\left(\frac{1}{3}\right)^{-2} = 3^2 = 9$	$(-2, 9)$
-1	$\left(\frac{1}{2}\right)^{-1} = 2^1 = 2$	$(-1, 2)$	$\left(\frac{1}{3}\right)^{-1} = 3^1 = 3$	$(-1, 3)$
0	$\left(\frac{1}{2}\right)^0 = 1$	$(0, 1)$	$\left(\frac{1}{3}\right)^0 = 1$	$(0, 1)$
1	$\left(\frac{1}{2}\right)^1 = \frac{1}{2}$	$\left(1, \frac{1}{2}\right)$	$\left(\frac{1}{3}\right)^1 = \frac{1}{3}$	$\left(1, \frac{1}{3}\right)$
2	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$	$\left(2, \frac{1}{4}\right)$	$\left(\frac{1}{3}\right)^2 = \frac{1}{9}$	$\left(2, \frac{1}{9}\right)$
3	$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$	$\left(3, \frac{1}{8}\right)$	$\left(\frac{1}{3}\right)^3 = \frac{1}{27}$	$\left(3, \frac{1}{27}\right)$

### Try It

6) Graph:  $f(x) = \left(\frac{1}{4}\right)^x$

### Solution

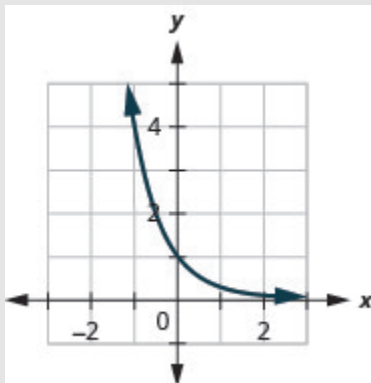


Figure 5.1.7

## Try It

7) Graph:  $g(x) = \left(\frac{1}{5}\right)^x$

**Solution**

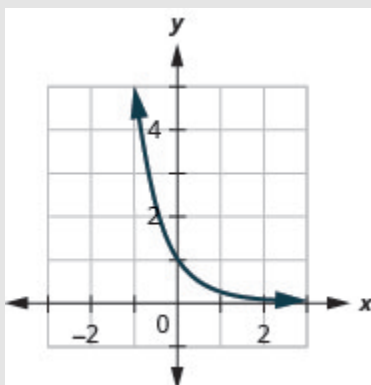


Figure 5.1.8

Now let's look at the graphs from the previous Example and Try Its so we can now identify some of the properties of exponential functions where  $0 < a < 1$ .

The graphs of  $f(x) = \left(\frac{1}{2}\right)^x$  and  $g(x) = \left(\frac{1}{3}\right)^x$  as well as the graphs of  $f(x) = \left(\frac{1}{4}\right)^x$  and  $g(x) = \left(\frac{1}{5}\right)^x$  all have the same basic shape. While this is the shape we expect from an exponential function where  $0 < a < 1$ , the graphs go down from left to right while the previous graphs, when  $a > 1$ , went from up from left to right.

We notice that for each function, the graph still contains the point  $(0, 1)$ . This makes sense because  $a^0 = 1$  for any  $a$ .

As before, the graph of each function,  $f(x) = a^x$ , also contains the point  $(1, a)$ . The graph of  $f(x) = \left(\frac{1}{2}\right)^x$  contained  $\left(1, \frac{1}{2}\right)$  and the graph of  $g(x) = \left(\frac{1}{3}\right)^x$  contained  $\left(1, \frac{1}{3}\right)$ . This makes sense as  $a^1 = a$ .

Notice too that the graph of each function,  $f(x) = a^x$ , also contains the point  $\left(-1, \frac{1}{a}\right)$ . The graph of  $f(x) = \left(\frac{1}{2}\right)^x$  contained  $(-1, 2)$  and the graph of  $g(x) = \left(\frac{1}{3}\right)^x$  contained  $(-1, 3)$ . This makes sense as  $a^{-1} = \frac{1}{a}$ .

What is the domain and range for each function? From the graphs we can see that the domain is the set of all real numbers and we write the domain in interval notation as  $(-\infty, \infty)$ . Again, the graph never hits the  $x$ -axis. The range is all positive numbers. We write the range in interval notation as  $(0, \infty)$ .

We will summarize these properties in the chart below. Which also include when  $a > 1$ .

Properties of the Graph of  $f(x) = a^x$



when $a > 1$		when $0 < a < 1$	
Domain	$(-\infty, \infty)$	Domain	$(-\infty, \infty)$
Range	$(0, \infty)$	Range	$(0, \infty)$
$x$ -intercept	none	$x$ -intercept	none
$y$ -intercept	$(0, 1)$	$y$ -intercept	$(0, 1)$
Contains	$(1, a), \left(-1, \frac{1}{a}\right)$	Contains	$(1, a), \left(-1, \frac{1}{a}\right)$
Asymptote	$x$ -axis, the line $y = 0$	Asymptote	$x$ -axis, the line $y = 0$
Basic shape	increasing	Basic shape	decreasing

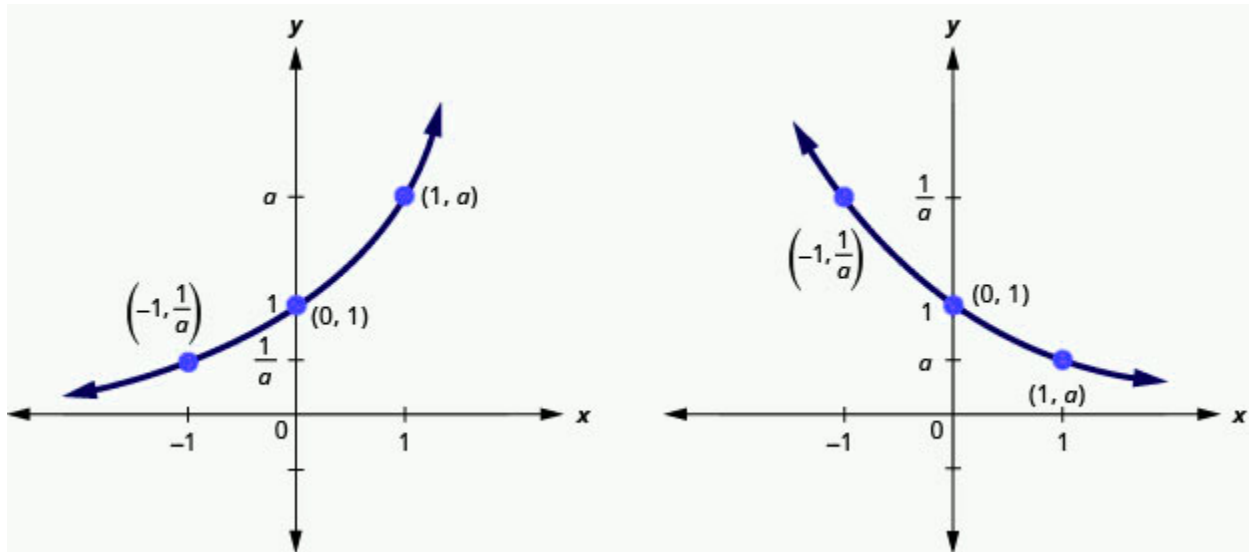


Figure 5.1.9

It is important for us to notice that both of these graphs are one-to-one, as they both pass the horizontal line test. This means the exponential function will have an inverse. We will look at this later.

When we graphed quadratic functions, we were able to graph using translation rather than just plotting points. Will that work in graphing exponential functions?

### Example 5.1.3

On the same coordinate system graph  $f(x) = 2^x$  and  $g(x) = 2^{x+1}$

#### Solution

We will use point plotting to graph the functions.

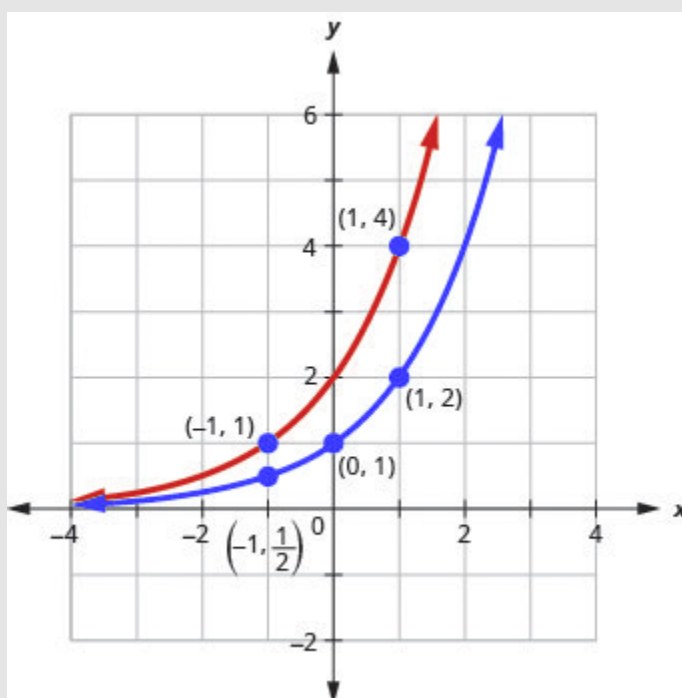


Figure 5.1.10

$x$	$f(x) = 2^x$	$(x, f(x))$	$g(x) = 2^{x+1}$	$(x, g(x))$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$	$2^{-2+1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-2, \frac{1}{2}\right)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$	$2^{-1+1} = 2^0 = 1$	$(-1, 1)$
0	$2^0 = 1$	$(0, 1)$	$2^{0+1} = 2^1 = 2$	$(0, 2)$
1	$2^1 = 2$	$(1, 2)$	$2^{1+1} = 2^2 = 4$	$(1, 4)$
2	$2^2 = 4$	$(2, 4)$	$2^{2+1} = 2^3 = 8$	$(2, 8)$
3	$2^3 = 8$	$(3, 8)$	$2^{3+1} = 2^4 = 16$	$(3, 16)$

### Try It

8) On the same coordinate system, graph:  $f(x) = 2^x$  and  $g(x) = 2^{x-1}$

### Solution

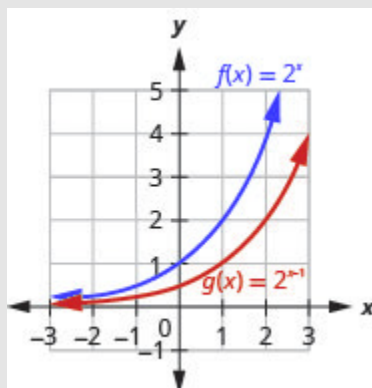


Figure 5.1.11

## Try It

9) On the same coordinate system, graph:  $f(x) = 3^x$  and  $g(x) = 3^{x+1}$

### Solution

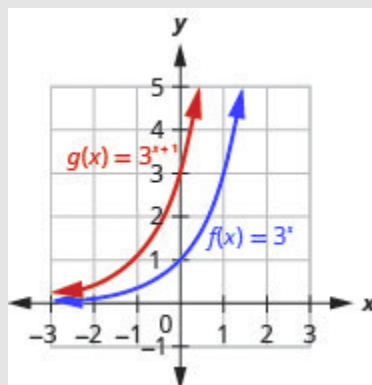


Figure 5.1.12

Looking at the graphs of the functions  $f(x) = 2^x$  and  $g(x) = 2^{x+1}$  in the last example, we see that

adding one in the exponent caused a horizontal shift of one unit to the left. Recognizing this pattern allows us to graph other functions with the same pattern by translation.

Let's now consider another situation that might be graphed more easily by translation, once we recognize the pattern.

### Example 5.1.4

On the same coordinate system graph  $f(x) = 3^x$  and  $g(x) = 3^x - 2$ .

#### Solution

We will use point plotting to graph the functions.

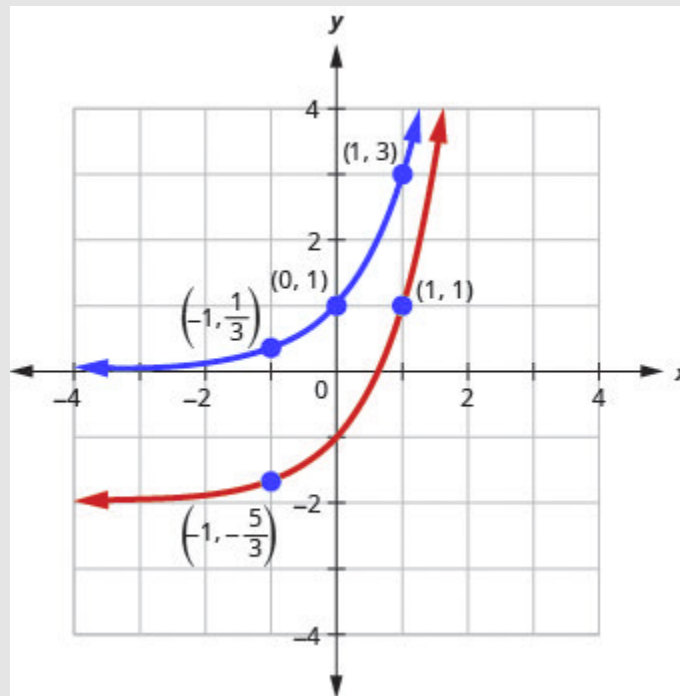


Figure 5.1.13

$x$	$f(x) = 3^x$	$(x, f(x))$	$g(x) = 3^x - 2$	$(x, g(x))$
-2	$3^{-2} = \frac{1}{9}$	$\left(-2, \frac{1}{9}\right)$	$3^{-2} - 2 = \frac{1}{9} - 2 = -\frac{17}{9}$	$\left(-2, -\frac{17}{9}\right)$
-1	$3^{-1} = \frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$	$3^{-1} - 2 = \frac{1}{3} - 2 = -\frac{5}{3}$	$\left(-1, -\frac{5}{3}\right)$
0	$3^0 = 1$	$(0, 1)$	$3^0 - 2 = 1 - 2 = -1$	$(0, -1)$
1	$3^1 = 3$	$(1, 3)$	$3^1 - 2 = 3 - 2 = 1$	$(1, 1)$
2	$3^2 = 9$	$(2, 9)$	$3^2 - 2 = 9 - 2 = 7$	$(2, 8)$

## Try It

10) On the same coordinate system, graph:  $f(x) = 3^x$  and  $g(x) = 3^x + 2$

### Solution

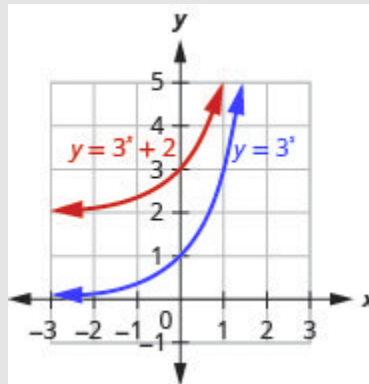


Figure 5.114

## Try It

11) On the same coordinate system, graph:  $f(x) = 4^x$  and  $g(x) = 4^x - 2$

### Solution

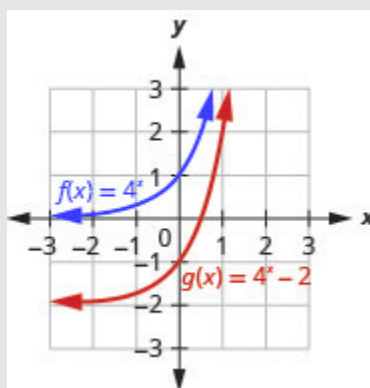


Figure 5.115

Looking at the graphs of the functions  $f(x) = 3^x$  and  $g(x) = 3^x - 2$  in the last example, we see that subtracting 2 caused a vertical shift of down two units. Notice that the horizontal asymptote also shifted down 2 units. Recognizing this pattern allows us to graph other functions with the same pattern by translation.

All of our exponential functions have had either an integer or a rational number as the base. We will now look at an exponential function with an irrational number as the base.

Before we can look at this exponential function, we need to define the irrational number,  $e$ . This number is used as a base in many applications in the sciences and business that are modelled by exponential functions.

The number is defined as the value of  $\left(1 + \frac{1}{n}\right)^n$  as  $n$  gets larger and larger. We say, as  $n$  approaches infinity, or increases without bound. The table shows the value of  $\left(1 + \frac{1}{n}\right)^n$  for several values of  $n$ .

$n$	$\left(1 + \frac{1}{n}\right)^n$
1	2
2	2.25
5	2.48832
10	2.59374246
100	2.704813829...
1,000	2.716923932...
10,000	2.718145927...
100,000	2.718268237...
1,000,000	2.718280469...
1,000,000,000	2.718281827...

$$e \approx 2.718281827$$

The number  $e$  is like the number  $\pi$  in that we use a symbol to represent it because its decimal representation never stops or repeats. The irrational number  $e$  is called the **natural base**.

### Natural Base $e$

The number  $e$  is defined as the value of  $\left(1 + \frac{1}{n}\right)^n$ , as  $n$  increases without bound. We say, as  $n$  approaches infinity,

$$e \approx 2.718281827\dots$$

The exponential function whose base is  $e$ ,  $f(x) = e^x$  is called the **natural exponential function**.

## Natural Exponential Function

The natural exponential function is an exponential function whose base is  $e$



$$f(x) = e^x$$

The domain is  $(-\infty, \infty)$  and the range is  $(0, \infty)$ .

Let's graph the function  $f(x) = e^x$  on the same coordinate system as  $g(x) = 2^x$  and  $h(x) = 3^x$ .

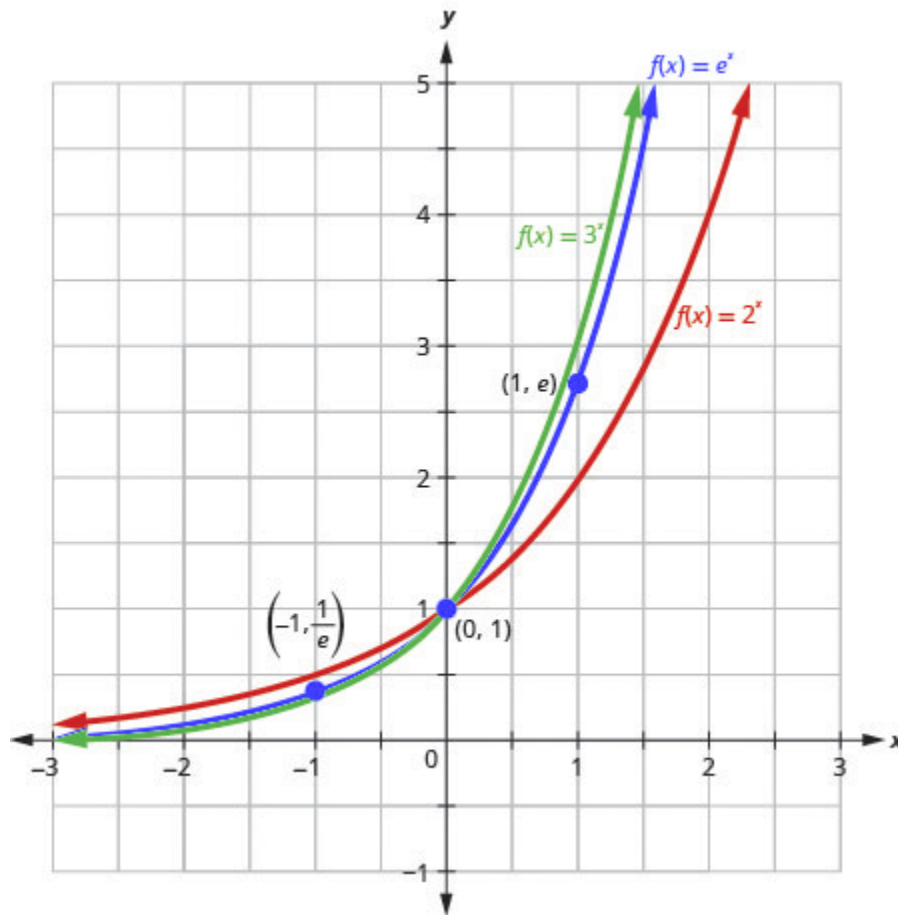


Figure 5.116

Notice that the graph of  $f(x) = e^x$  is “between” the graphs of  $g(x) = 2^x$  and  $h(x) = 3^x$ . Does this make sense as  $2 < e < 3$ ?

## Solve Exponential Equations

Equations that include an exponential expression  $a^x$  are called exponential equations. To solve them we use a property that says as long as  $a > 0$  and  $a \neq 1$ , if  $a^x = a^y$  then it is true that  $x = y$ . In other words, in an exponential equation, if the bases are equal then the exponents are equal.

## One-to-One Property of Exponential Equations

For  $a > 0$  and  $a \neq 1$ ,  
If  $a^x = a^y$ , then  $x = y$ .

To use this property, we must be certain that both sides of the equation are written with the same base.

### Example 5.1.5

Solve:  $3^{2x-5} = 27$

#### Solution

**Step 1: Write both sides of the equation with the same base.**

Since the left side has base 3, we write the right side with base 3.  $27 = 3^3$

$$3^{2x-5} = 27$$

$$3^{2x-5} = 3^3$$

**Step 2: Write a new equation by setting the exponents equal.**

Since the bases are the same, the exponents must be equal.

$$2x - 5 = 3$$

**Step 3: Solve the equation.**

Add 5 to each side.

Divide by 2.

$$2x = 8$$

$$x = 4$$

**Step 4: Check the solution.**

Substitute  $x = 4$  into the original equation.

$$3^{2x-5} = 27$$

$$3^{2 \cdot 4 - 5} \stackrel{?}{=} 27$$

$$3^3 \stackrel{?}{=} 27$$

$$27 = 27 \checkmark$$

## Try It

12) Solve:  $3^{3x-2} = 81$

**Solution**

$$x = 2$$

## Try It

13) Solve:  $7^{x-3} = 7$

**Solution**

$$x = 4$$

The steps are summarized below.

## HOW TO

### How to Solve an Exponential Equation

1. Write both sides of the equation with the same base, if possible.
2. Write a new equation by setting the exponents equal.
3. Solve the equation.
4. Check the solution.

In the next example, we will use our properties on exponents.

### Example 5.1.6

Solve  $\frac{e^{x^2}}{e^3} = e^{2x}$

#### Solution

**Step 1: Use the Property of Exponents:**  $\frac{a^m}{a^n} = a^{m-n}$

$$e^{x^2-3} = e^{2x}$$

**Step 2: Write a new equation by setting the exponents equal.**

$$x^2 - 3 = 2x$$

**Step 3: Solve the equation.**

$$x^2 - 2x - 3 = 0$$

$$(x - 3)(x + 1) = 0$$

$$x = 3, x = -1$$

**Step 4: Check the solutions.**

$$x = 3$$

$$\frac{e^{rgb[1.0,0.0,0.0]x^2}}{e^3} \stackrel{?}{=} e^{2 \cdot rgb[1.0,0.0,0.0]x}$$

$$\frac{e^{rgb[1.0,0.0,0.0]3^2}}{e^3} \stackrel{?}{=} e^{2 \cdot rgb[1.0,0.0,0.0]3}$$

$$\frac{e^9}{e^3} \stackrel{?}{=} e^6$$

$$e^6 = e^6 \checkmark$$

$$x = -1$$

$$\frac{e^{x^2}}{e^3} \stackrel{?}{=} e^{2x}$$

$$\frac{e^{rgb[1.0,0.0,0.0](-1)^2}}{e^3} \stackrel{?}{=} e^{2 \cdot rgb[1.0,0.0,0.0](-1)}$$

$$\frac{e^1}{e^3} \stackrel{?}{=} e^{-2}$$

$$e^{-2} = e^{-2} \checkmark$$

**Try It**

14) Solve:  $\frac{e^{x^2}}{e^x} = e^2$

**Solution**

$x = -1, x = 2$

**Try It**

15) Solve:  $\frac{e^{x^2}}{e^x} = e^6$

**Solution**

$x = -2, x = 3$

## Use Exponential Models in Applications

Exponential functions model many situations. If you own a bank account, you have experienced the use of an exponential function. There are two formulas that are used to determine the balance in the account when interest is earned. If a principal,  $P$ , is invested at an interest rate,  $r$ , for  $t$  years, the new balance,  $A$ , will depend on how often the interest is compounded. If the interest is compounded  $n$  times a year we use the formula  $A = P\left(1 + \frac{r}{n}\right)^{nt}$ . If the interest is compounded continuously, we use the formula  $A = Pe^{rt}$ . These are the formulas for compound interest.

## Compound Interest

For a principal,  $P$ , invested at an interest rate,  $r$ , for  $t$  years, the new balance,  $A$ , is:

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{when compounded } n \text{ times a year.}$$

$$A = Pe^{rt} \quad \text{when compounded continuously.}$$

As you work with the Interest formulas, it is often helpful to identify the values of the variables first and then substitute them into the formula.

### Example 5.1.7

A total of \$10,000 was invested in a college fund for a new grandchild. If the interest rate is 5%, how much will be in the account in 18 years by each method of compounding?

- compound quarterly
- compound monthly
- compound continuously

#### Solution

Identify the values of each variable in the formulas. Remember to express the percent as a decimal.

$$A = ?$$

$$P = \$10,000$$

$$r = 0.05$$

$$t = 18 \text{ years}$$

a.

For quarterly compounding,  $n = 4$ . There are 4 quarters in a year.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

**Step 1: Substitute the values in the formula.**

$$A = 10,000\left(1 + \frac{0.05}{4}\right)^{4 \cdot 18}$$

**Step 2: Compute the amount.**

Be careful to consider the order of operations as you enter the expression into your calculator.

$$A = \$24,459.20$$


---

b.

For monthly compounding,  $n = 12$ . There are **12** months in a year.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

**Step 1: Substitute the values in the formula.**

$$A = 10,000\left(1 + \frac{0.05}{12}\right)^{12 \cdot 18}$$

**Step 2: Compute the amount.**

$$A = \$24,550.08$$


---

c.

For compounding continuously,  $A = Pe^{rt}$ .

**Step 1: Substitute the values in the formula.**

$$A = 10,000e^{0.05 \cdot 18}$$

**Step 2: Compute the amount.**

$$A = \$24,596.03$$

## Try It

16) Angela invested **\$15,000** in a savings account. If the interest rate is **4%**, how much will be in the account in **10** years by each method of compounding?

- a. compound quarterly
- b. compound monthly
- c. compound continuously

### Solution

- a. **\$22,332.96**
- b. **\$22,362.49**
- c. **\$22,377.37**

## Try It

17) Allan invested **\$10,000** in a mutual fund. If the interest rate is **5%**, how much will be in the account in **15** years by each method of compounding?

- a. compound quarterly
- b. compound monthly
- c. compound continuously

### Solution

- a. **\$21,071.81**
- b. **\$21,137.04**
- c. **\$21,170.00**

Other topics that are modelled by exponential functions involve growth and decay. Both also use the formula



$A = Pe^{rt}$  we used for the growth of money. For growth and decay, generally we use  $A_0$ , as the original amount instead of calling it  $P$ , the principal. We see that exponential growth has a positive rate of growth and exponential decay has a negative rate of growth.

## Exponential Growth and Decay

For an original amount,  $A_0$ , that grows or decays at a rate,  $r$ , for a certain time,  $t$ , the final amount,  $A$ , is:

$$A = A_0e^{rt}$$

Exponential growth is typically seen in the growth of populations of humans or animals or bacteria. Our next example looks at the growth of a virus.

### Example 5.1.8

Chris is a researcher at the Center for Disease Control and Prevention and he is trying to understand the behaviour of a new and dangerous virus. He starts his experiment with **100** of the virus that grows at a rate of **25%** per hour. He will check on the virus in **24** hours. How many viruses will he find?

#### Solution

**Step 1: Identify the values of each variable in the formulas.**

Be sure to put the percent in decimal form.  
Be sure the units match - the rate is per hour  
and the time is in hours.

$$\begin{aligned} A &=? \\ A_0 &= 100 \\ r &= 0.25/\text{hour} \\ t &= 24 \text{ hours} \end{aligned}$$

**Step 2: Substitute the values in the formula:  $A = A_0e^{rt}$**

$$A = 100e^{0.25 \cdot 24}$$

**Step 3: Compute the amount.**

$$A = 40,342.88$$

**Step 4: Round to the nearest whole virus.**

$$A = 40,343$$

The researcher will find **40,343** viruses.

### Try It

18) Another researcher at the Center for Disease Control and Prevention, Lisa, is studying the growth of a bacteria. She starts his experiment with **50** of the bacteria that grows at a rate of **15%** per hour. She will check on the bacteria every **8** hours. How many bacteria will he find in **8** hours?

**Solution**

She will find **166** bacteria.

### Try It

19) Maria, a biologist is observing the growth pattern of a virus. She starts with **100** of the virus that grows at a rate of **10%** per hour. She will check on the virus in **24** hours. How many viruses will she find?

**Solution**

She will find **1,102** viruses.

Access these online resources for additional instruction and practice with evaluating and graphing exponential functions.

- Graphing Exponential Functions
- Solving Exponential Equations
- Applications of Exponential Functions
- Continuously Compound Interest
- Radioactive Decay and Exponential Growth

## Key Concepts

- **Properties of the Graph of  $f(x) = a^x$ :**

when $a > 1$		when $0 < a < 1$	
Domain	$(-\infty, \infty)$	Domain	$(-\infty, \infty)$
Range	$(0, \infty)$	Range	$(0, \infty)$
$x$ -intercept	none	$x$ -intercept	none
$y$ -intercept	$(0, 1)$	$y$ -intercept	$(0, 1)$
Contains	$(1, a), \left(-1, \frac{1}{a}\right)$	Contains	$(1, a), \left(-1, \frac{1}{a}\right)$
Asymptote	$x$ -axis, the line $y = 0$	Asymptote	$x$ -axis, the line $y = 0$
Basic shape	increasing	Basic shape	decreasing

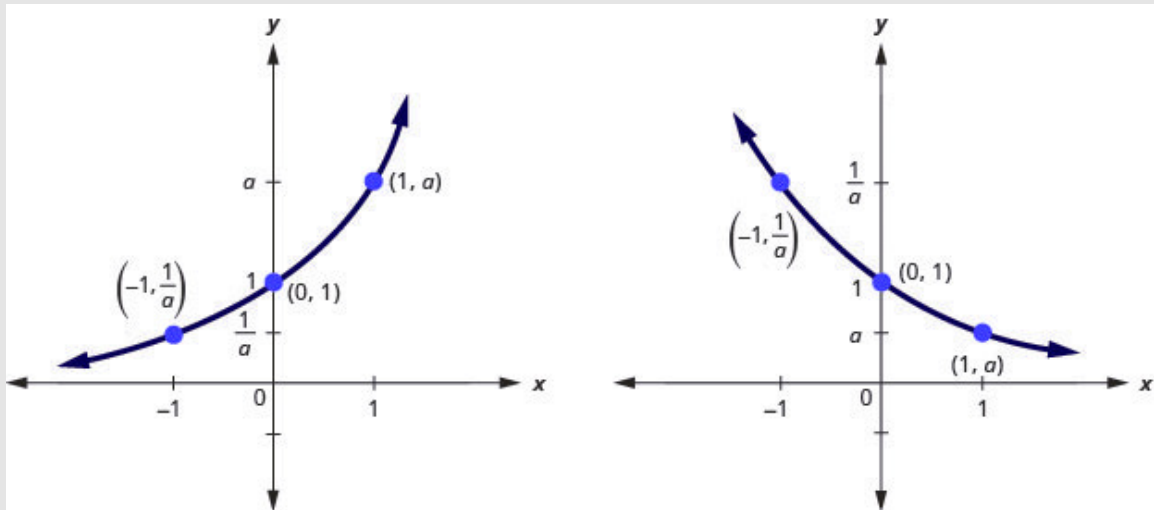


Figure 5.117

- **One-to-One Property of Exponential Equations:** For  $a > 0$  and  $a \neq 1$ ,  

$$A = A_0 e^{rt}$$
- **How to Solve an Exponential Equation**
  1. Write both sides of the equation with the same base, if possible.
  2. Write a new equation by setting the exponents equal.
  3. Solve the equation.
  4. Check the solution.
- **Compound Interest:** For a principal,  $P$ , invested at an interest rate,  $r$ , for  $t$  years, the new balance,  $A$ , is

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{when compounded } n \text{ times a year.}$$

$$A = Pe^{rt} \quad \text{when compounded continuously.}$$

- **Exponential Growth and Decay:** For an original amount,  $A_0$  that grows or decays at a rate,  $r$ , for a certain time  $t$ , the final amount,  $A$ , is  $A = A_0 e^{rt}$ .

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=1517#h5p-34>

b) After reviewing this checklist, what will you do to become confident for all objectives?

## Glossary

### asymptote

A line which a graph of a function approaches closely but never touches.

### exponential function

An exponential function, where  $a > 0$  and  $a \neq 1$ , is a function of the form  $f(x) = a^x$ .

### natural base

The number  $e$  is defined as the value of  $\left(1 + \frac{1}{n}\right)^n$ , as  $n$  gets larger and larger. We say, as

$n$  increases without bound,  $e \approx 2.718281827\dots$

### natural exponential function

The natural exponential function is an exponential function whose base is  $e$ :  $f(x) = e^x$ . The domain is  $(-\infty, \infty)$  and the range is  $(0, \infty)$ .

# 5.2 EVALUATE AND GRAPH LOGARITHMIC FUNCTIONS

---

## Learning Objectives

By the end of this section, you will be able to:

- Convert between exponential and logarithmic form
- Evaluate logarithmic functions
- Graph logarithmic functions
- Solve logarithmic equations
- Use logarithmic models in applications

## Try It

Before you get started, take this readiness quiz:

1) Solve:  $x^2 = 81$

2) Evaluate:  $3^{-2}$

3) Solve:  $2^4 = 3x - 5$

We have spent some time finding the inverse of many functions. It works well to ‘undo’ an operation with

another operation. Subtracting ‘undoes’ addition, multiplication ‘undoes’ division, taking the square root ‘undoes’ squaring.

As we studied the exponential function, we saw that it is one-to-one as its graphs pass the horizontal line test. This means an exponential function does have an inverse. If we try our algebraic method for finding an inverse, we run into a problem.

$$f(x) = a^x$$

$$y = a^x$$

$$x = a^y$$

Rewrite with  $y = f(x)$

Interchange the variables  $x$  and  $y$ .

Solve for  $y$ .

Oops! We have now way to solve for  $y$ !

To deal with this we define the logarithm function with base  $a$  to be the inverse of the exponential function  $f(x) = a^x$ . We use the notation  $f^{-1}(x) = \log_a x$  and say the inverse function of the exponential function is the **logarithmic function**.

## Logarithmic Function

The function  $f(x) = \log_a x$  is the logarithmic function with base  $a$ , where  $a > 0$ ,  $x > 0$ , and  $a \neq 1$ .

$$y = \log_a x \text{ is equivalent to } x = a^y$$

## Convert Between Exponential and Logarithmic Form

Since the equations  $y = \log_a x$  and  $x = a^y$  are equivalent, we can go back and forth between them. This will often be the method to solve some exponential and logarithmic equations. To help with converting back and forth let’s take a close look at the equations. See Figure 5.2.1. Notice the positions of the exponent and base.

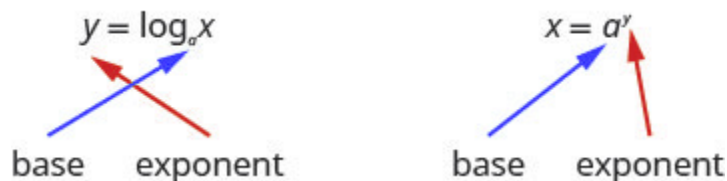


Figure 5.2.1

If we realize the logarithm is the exponent it makes the conversion easier. You may want to repeat, “base to the exponent give us the number.”

### Example 5.2.1

Convert to logarithmic form:

a.  $2^3 = 8$ ,

b.  $5^{\frac{1}{2}} = \sqrt{5}$

c.  $\left(\frac{1}{2}\right)^x = \frac{1}{16}$

#### Solution

**Step 1: Identify the base and the exponent.**

$$\begin{array}{l} (a) \\ \text{rgb}[1.0, 0.0, 0.02^{\text{rgb}[0.0, 0.0, 1.03}} = 8 \end{array}$$

$$\text{rgb}[0.0, 0.0, 1.0y = \log_{\text{rgb}[1.0, 0.0, 0.0a} x$$

$$3 = \log_2 8$$

$$\text{If } 2^3 = 8, \text{ then } 3 = \log_2 8$$

$$\begin{array}{l} (b) \\ \text{rgb}[1.0, 0.0, 0.05^{\text{rgb}[0.0, 0.0, 1.0\frac{1}{2}}} = \sqrt{5} \end{array}$$

$$\text{rgb}[0.0, 0.0, 1.0y = \log_{\text{rgb}[1.0, 0.0, 0.0a} x$$

$$\frac{1}{2} = \log_5 \sqrt{5}$$

$$\text{If } 5^{\frac{1}{2}} = \sqrt{5}, \text{ then } \frac{1}{2} = \log_5 \sqrt{5}.$$

$$\begin{array}{l} (c) \\ \text{rgb}[1.0, 0.0, 0.0\left(\frac{1}{2}\right)^{\text{rgb}[0.0, 0.0, 1.04}} = \frac{1}{16} \end{array}$$

$$\text{rgb}[0.0, 0.0, 1.0y = \log_{\text{rgb}[1.0, 0.0, 0.0a} x$$

$$4 = \log_{\frac{1}{2}} \frac{1}{16}$$

$$\text{If } \left(\frac{1}{2}\right)^4 = \frac{1}{16}, \text{ then } 4 = \log_{\frac{1}{2}} \frac{1}{16}.$$

### Try It

4) Convert to logarithmic form:

a.  $3^2 = 9$

b.  $7^{\frac{1}{2}} = \sqrt{7}$

c.  $\left(\frac{1}{3}\right)^x = \frac{1}{27}$

#### Solution



$$\text{a. } \log_3 9 = 2$$

$$\text{b. } \log_7 \sqrt{7} = \frac{1}{2}$$

$$\text{c. } \log_{\frac{1}{3}} \frac{1}{27} = x$$

## Try It

5) Convert to logarithmic form:

$$\text{a. } 4^3 = 64$$

$$\text{b. } 4^{\frac{1}{3}} = \sqrt[3]{4}$$

$$\text{c. } \left(\frac{1}{2}\right)^x = \frac{1}{32}$$

### Solution

$$\text{a. } \log_4 64 = 3$$

$$\text{b. } \log_4 \sqrt[3]{4} = \frac{1}{3}$$

$$\text{c. } \log_{\frac{1}{2}} \frac{1}{32} = x$$

In the next example we do the reverse—convert logarithmic form to exponential form.

## Example 5.2.2

Convert to exponential form:

a.  $2 = \log_8 64$

b.  $0 = \log_4 1$

c.  $-3 = \log_{10} \frac{1}{1000}$

### Solution

**Step 1: Identify the base and the exponent.**

(a)

$$\log_{8} 64 = 2$$

$$x = \log_{8} 64$$

$$64 = 8^2$$

If  $2 = \log_8 64$ , then  $64 = 8^2$

(b)

$$\log_{4} 1 = 0$$

$$x = \log_{4} 1$$

$$1 = 4^0$$

If  $0 = \log_4 1$ , then  $1 = 4^0$

(c)

$$\log_{10} \frac{1}{1000} = -3$$

$$x = \log_{10} \frac{1}{1000}$$

$$\frac{1}{1000} = 10^{-3}$$

If  $-3 = \log_{10} \frac{1}{1000}$ , then

$$\frac{1}{1000} = \frac{1}{1000} = 10^{-3}$$

## Try It

6) Convert to exponential form:

a.  $3 = \log_4 64$

b.  $0 = \log_x 1$

c.  $-2 = \log_{10} \frac{1}{100}$

### Solution

a.  $64 = 4^3$

$$\text{b. } 1 = x^0$$

$$\text{c. } \frac{1}{100} = 10^{-2}$$

## Try It

7) Convert to exponential form:

$$\text{a. } 3 = \log_3 27$$

$$\text{b. } 0 = \log_x 1$$

$$\text{c. } -1 = \log_{10} \frac{1}{10}$$

### Solution

$$\text{a. } 27 = 3^3$$

$$\text{b. } 1 = x^0$$

$$\text{c. } \frac{1}{10} = 10^{-1}$$

## Evaluate Logarithmic Functions

We can solve and evaluate logarithmic equations by using the technique of converting the equation to its equivalent exponential equation.

### Example 5.2.3

Find the value of  $x$ :

a.  $\log_x 36 = 2$

b.  $\log_4 x = 3$

c.  $\log_{\frac{1}{2}} \frac{1}{8} = x$

#### Solution

a.

**Step 1: Convert to exponential form.**

$$x^2 = 36$$

**Step 2: Solve the quadratic.**

$$x = 6, \overline{)x = -6}$$

**Step 3: The base of a logarithmic function must be positive, so we eliminate  $x = -6$**

Therefore,  $\log_6 36 = 2$

---

b.

**Step 1: Convert to exponential form.**

$$4^3 = x$$

Simplify.  $x = 64$

Therefore,  $\log_4 64 = 3$

---

c.

**Step 1: Convert to exponential form.**

$$\left(\frac{1}{2}\right)^x = \frac{1}{8}$$

**Step 2:** Rewrite  $\frac{1}{8}$  as  $\left(\frac{1}{2}\right)^3$

$$\left(\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^3$$

**Step 3:** With the same base, the exponents must be equal.

$$x = 3 \text{ Therefore, } \log_{\frac{1}{2}} \frac{1}{8} = 3$$

## Try It

8) Find the value of  $x$ :

a.  $\log_x 64 = 2$

b.  $\log_5 x = 3$

c.  $\log_{\frac{1}{2}} \frac{1}{4} = x$

### Solution

a.  $x = 8$

b.  $x = 125$

c.  $x = 2$

## Try It

9) Find the value of  $x$ :

a.  $\log_x 81 = 2$

b.  $\log_3 x = 5$

c.  $\log_{\frac{1}{3}} \frac{1}{27} = x$

### Solution

a.  $x = 9$

b.  $x = 243$

c.  $x = 3$

When see an expression such as  $\log_3 27$ , we can find its exact value two ways. By inspection we realize it means “3 to what power will be 27”? Since  $3^3 = 27$ , we know  $\log_3 27 = 3$ . An alternate way is to set the expression equal to  $x$  and then convert it into an exponential equation.

## Example 5.2.4

Find the exact value of each logarithm without using a calculator:

a.  $\log_5 25$

b.  $\log_9 3$

c.  $\log_2 \frac{1}{16}$

### Solution

a.

**Step 1: 5 to what power will be 25?**

$$\log_5 25 = 2$$

OR

**Step 1: Set the expression equal to  $x$ .**

$$\log_5 25 = x$$

**Step 2: Change to exponential form.**

$$5^x = 25$$

**Step 3: Rewrite 25 as  $5^2$**

$$5^x = 5^2$$

**Step 4: With the same base the exponents must be equal.**

$$x = 2 \text{ Therefore, } \log_5 25 = 2$$


---

b.

**Step 1: Set the expression equal to  $x$ .**

$$\log_9 3 = x$$

**Step 2: Change to exponential form.**

$$9^x = 3$$

**Step 3: Rewrite 9 as  $3^2$**

$$(3^2)^x = 3^1$$

**Step 4: Simplify the exponents.**

$$3^{2x} = 3^1$$

**Step 5: With the same base the exponents must be equal.**

$$2x = 1$$

**Step 6: Solve the equation.**

$$x = \frac{1}{2} \text{ Therefore, } \log_9 3 = \frac{1}{2}$$


---

c.

**Step 1: Set the expression equal to  $x$ .**

$$\log_2 \frac{1}{16} = x$$

**Step 2: Change to exponential form.**

$$2^x = \frac{1}{16}$$

**Step 3: Rewrite 16 as  $2^4$ .**

$$2^x = \frac{1}{16}$$

$$2^x = 2^{-4}$$

**Step 4: With the same base the exponents must be equal.**

$$x = -4 \text{ Therefore, } \log_2 \frac{1}{16} = -4$$

## Try It

10) Find the exact value of each logarithm without using a calculator:

a.  $\log_{12} 144$

b.  $\log_4 2$

c.  $\log_2 \frac{1}{32}$

**Solution**

a. 2

b.  $\frac{1}{2}$

c.  $-5$



## Try It

11) Find the exact value of each logarithm without using a calculator:

a.  $\log_9 81$

b.  $\log_8 2$

c.  $\log_3 \frac{1}{9}$

### Solution

a. 2

b.  $\frac{1}{3}$

c.  $-2$

## Graph Logarithmic Functions

To graph a logarithmic function  $y = \log_a x$ , it is easiest to convert the equation to its exponential form,  $x = a^y$ . Generally, when we look for ordered pairs for the graph of a function, we usually choose an  $x$ -value and then determine its corresponding  $y$ -value. In this case you may find it easier to choose  $y$ -values and then determine its corresponding  $x$ -value.

### Example 5.2.5

Graph  $y = \log_2 x$ .

### Solution

To graph the function, we will first rewrite the logarithmic equation,  $y = \log_2 x$ , in exponential

form,  $2^y = x$ .

We will use point plotting to graph the function. It will be easier to start with values of  $y$  and then get  $x$ .

$y$	$2^y = x$	$(x, y)$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(\frac{1}{4}, -2\right)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(\frac{1}{2}, -1\right)$
0	$2^0 = 1$	$(1, 0)$
1	$2^1 = 2$	$(2, 1)$
2	$2^2 = 4$	$(4, 2)$
3	$2^3 = 8$	$(8, 3)$

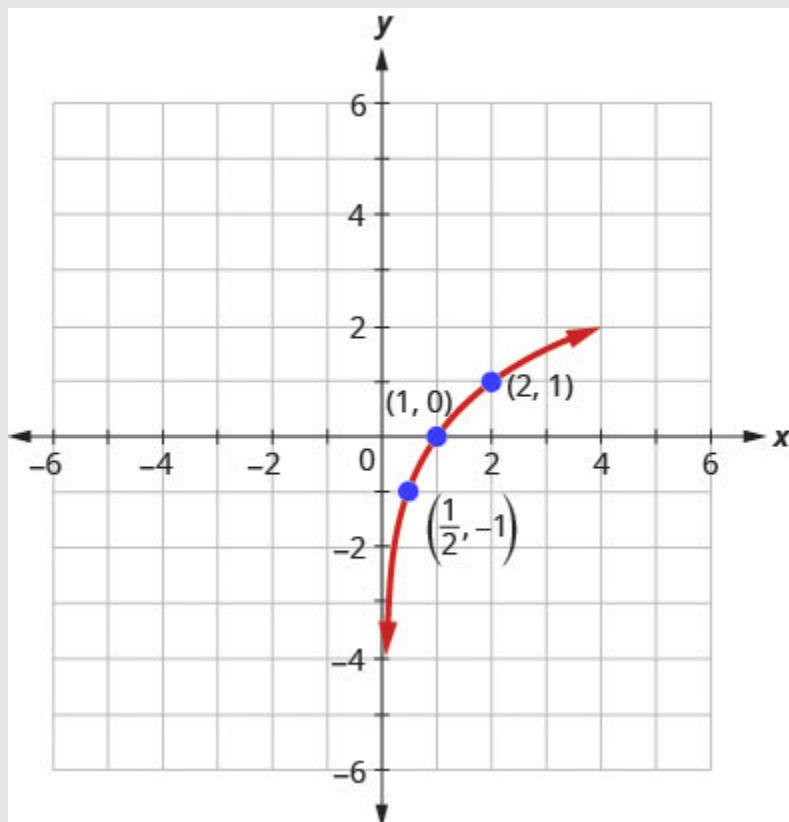


Figure 5.2.2

## Try It

12) Graph:  $y = \log_3 x$ .

**Solution**

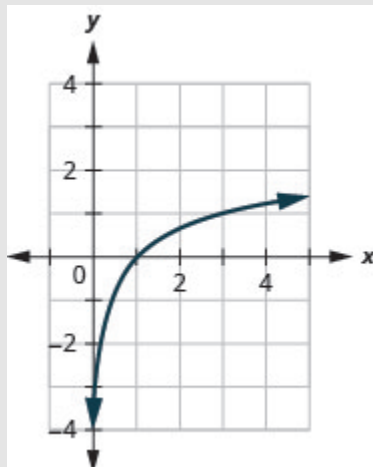


Figure 5.2.3

## Try It

13) Graph:  $y = \log_5 x$ .

**Solution**

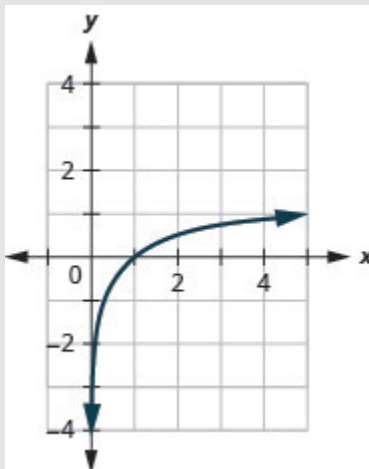


Figure 5.2.4

The graphs of  $y = \log_2 x$ ,  $y = \log_3 x$ , and  $y = \log_5 x$  are the shape we expect from a logarithmic function where  $a > 1$ . We notice that for each function the graph contains the point  $(1, 0)$ . This makes sense because  $0 = \log_a 1$  means  $a^0 = 1$  which is true for any  $a$ .

The graph of each function, also contains the point  $(a, 1)$ . This makes sense as  $1 = \log_a a$  means  $a^1 = a$ . Which is true for any  $a$ .

Notice too, the graph of each function  $y = \log_a x$  also contains the point  $\left(\frac{1}{a}, -1\right)$ . This makes sense as  $-1 = \log_a \frac{1}{a}$  means  $a^{-1} = \frac{1}{a}$ , which is true for any  $a$ .

Look at each graph again. Now we will see that many characteristics of the logarithm function are simply 'mirror images' of the characteristics of the corresponding exponential function.

What is the domain of the function? The graph never hits the  $y$ -axis. The domain is all positive numbers. We write the domain in interval notation as  $(0, \infty)$ .

What is the range for each function? From the graphs we can see that the range is the set of all real numbers. There is no restriction on the range. We write the range in interval notation as  $(-\infty, \infty)$ .

When the graph approaches the  $y$ -axis so very closely but will never cross it, we call the line  $x = 0$ , the  $y$ -axis, a vertical asymptote.

Properties of the Graph of  $y = \log_a x$  when  $a > 1$

<b>Domain</b>	$(0, \infty)$
<b>Range</b>	$(-\infty, \infty)$
<b><math>x</math>-intercept</b>	$(1, 0)$
<b><math>y</math>-intercept</b>	None
<b>Contains</b>	$(a, 1), \left(\frac{1}{a}, -1\right)$
<b>Asymptote</b>	$y$ -axis

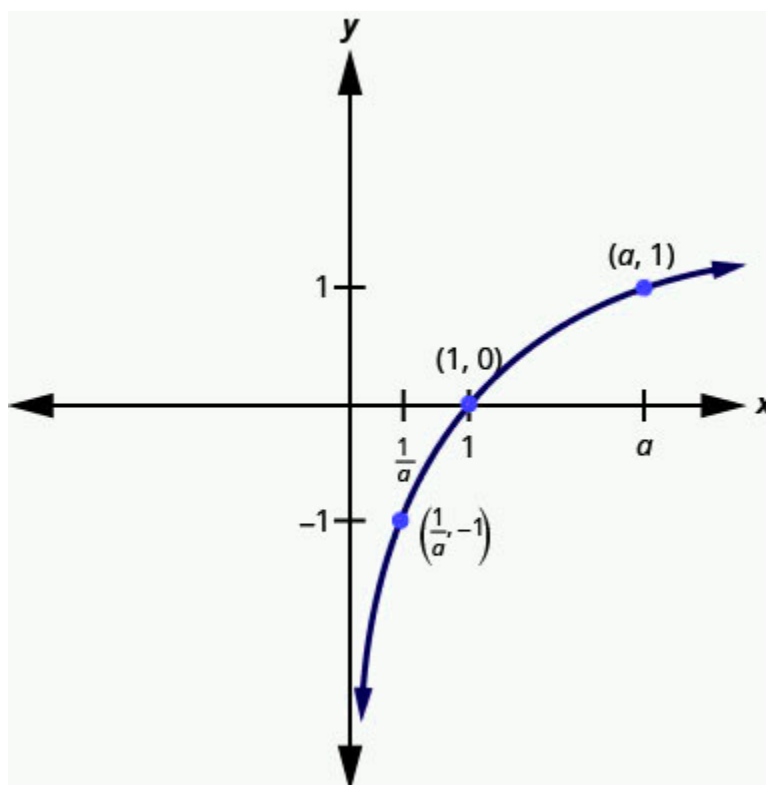


Figure 5.2.5

Our next example looks at the graph of  $y = \log_a x$  when  $0 < a < 1$ .

### Example 5.2.6

Graph  $y = \log_{\frac{1}{3}} x$ .

#### Solution

To graph the function, we will first rewrite the logarithmic equation,  $y = \log_{\frac{1}{3}} x$ , in exponential form,  $\left(\frac{1}{3}\right)^y = x$ .

We will use point plotting to graph the function. It will be easier to start with values of  $y$  and then get  $x$ .

$y$	$\left(\frac{1}{3}\right)^y = x$	$(x, y)$
-2	$\left(\frac{1}{3}\right)^{-2} = 3^2 = 9$	$(9, -2)$
-1	$\left(\frac{1}{3}\right)^{-1} = 3^1 = 3$	$(3, -1)$
0	$\left(\frac{1}{3}\right)^0 = 1$	$(1, 0)$
1	$\left(\frac{1}{3}\right)^1 = \frac{1}{3}$	$\left(\frac{1}{3}, 1\right)$
2	$\left(\frac{1}{3}\right)^2 = \frac{1}{9}$	$\left(\frac{1}{9}, 2\right)$
3	$\left(\frac{1}{3}\right)^3 = \frac{1}{27}$	$\left(\frac{1}{27}, 3\right)$

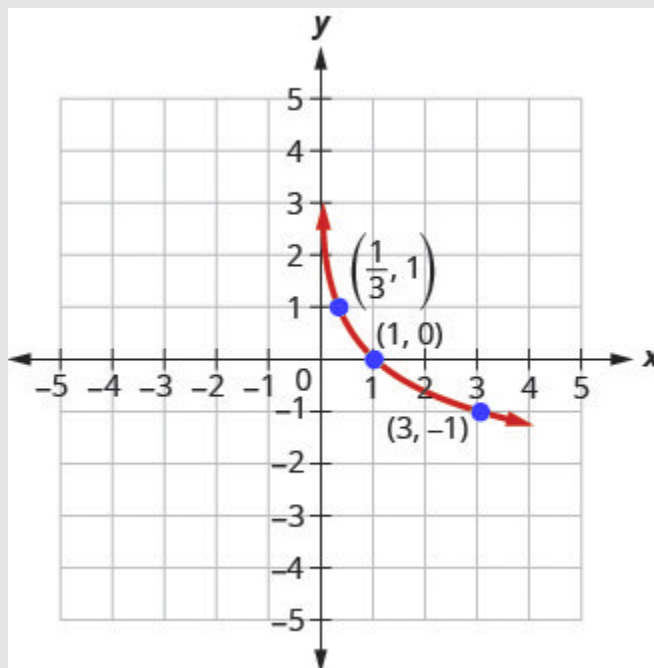


Figure 5.2.6

## Try It

14) Graph:  $y = \log_{\frac{1}{2}} x$ .

**Solution**



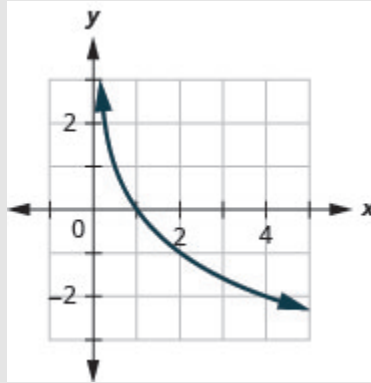


Figure 5.2.7

## Try It

15) Graph:  $y = \log_{\frac{1}{4}} x$ .

### Solution

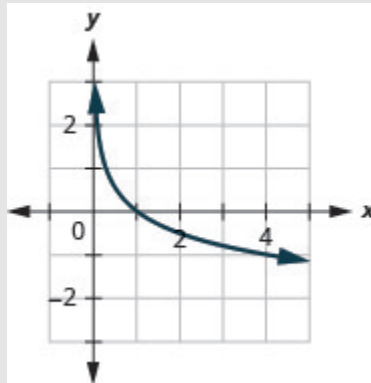


Figure 5.2.8

Now, let's look at the graphs  $y = \log_{\frac{1}{2}} x$ ,  $y = \log_{\frac{1}{3}} x$  and  $y = \log_{\frac{1}{4}} x$ , so we can identify some of the

properties of logarithmic functions where  $0 < a < 1$ . The graphs of all have the same basic shape. While this is the shape we expect from a logarithmic function where  $0 < a < 1$ .

We notice, that for each function again, the graph contains the points,  $(1, 0)$ ,  $(a, 1)$ ,  $\left(\frac{1}{a}, -1\right)$ .

This make sense for the same reasons we argued above. We notice the domain and range are also the same—the domain is  $(0, \infty)$  and the range is  $(-\infty, \infty)$ .

The  $y$ -axis is again the vertical asymptote. We will summarize these properties in the chart below. Which also include when  $a > 1$ .

### Properties of the Graph of $y = \log_a x$

when $a > 1$		when $0 < a < 1$	
Domain	$(0, \infty)$	Domain	$(0, \infty)$
Range	$(-\infty, \infty)$	Range	$(-\infty, \infty)$
$x$ -intercept	$(1, 0)$	$x$ -intercept	$(1, 0)$
$y$ -intercept	none	$y$ -intercept	None
Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$	Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$
Asymptote	$y$ -axis	Asymptote	$y$ -axis
Basic shape	increasing	Basic shape	Decreasing

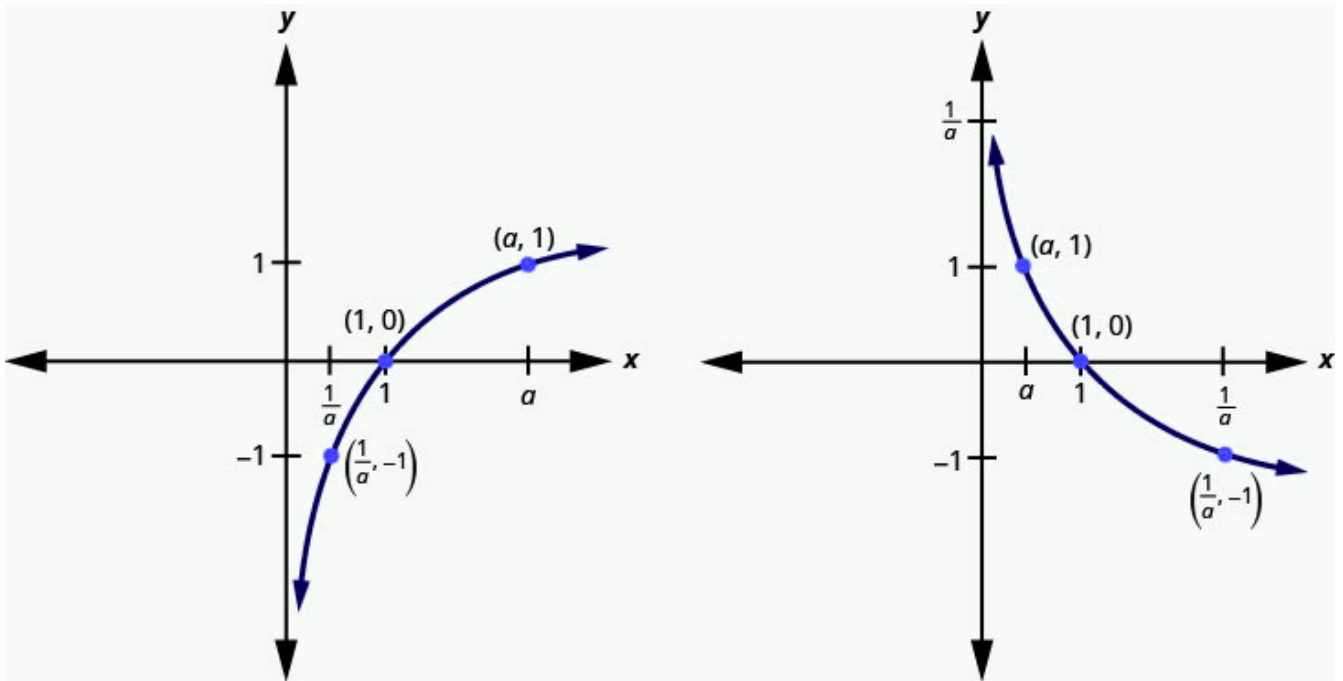


Figure 5.2.9

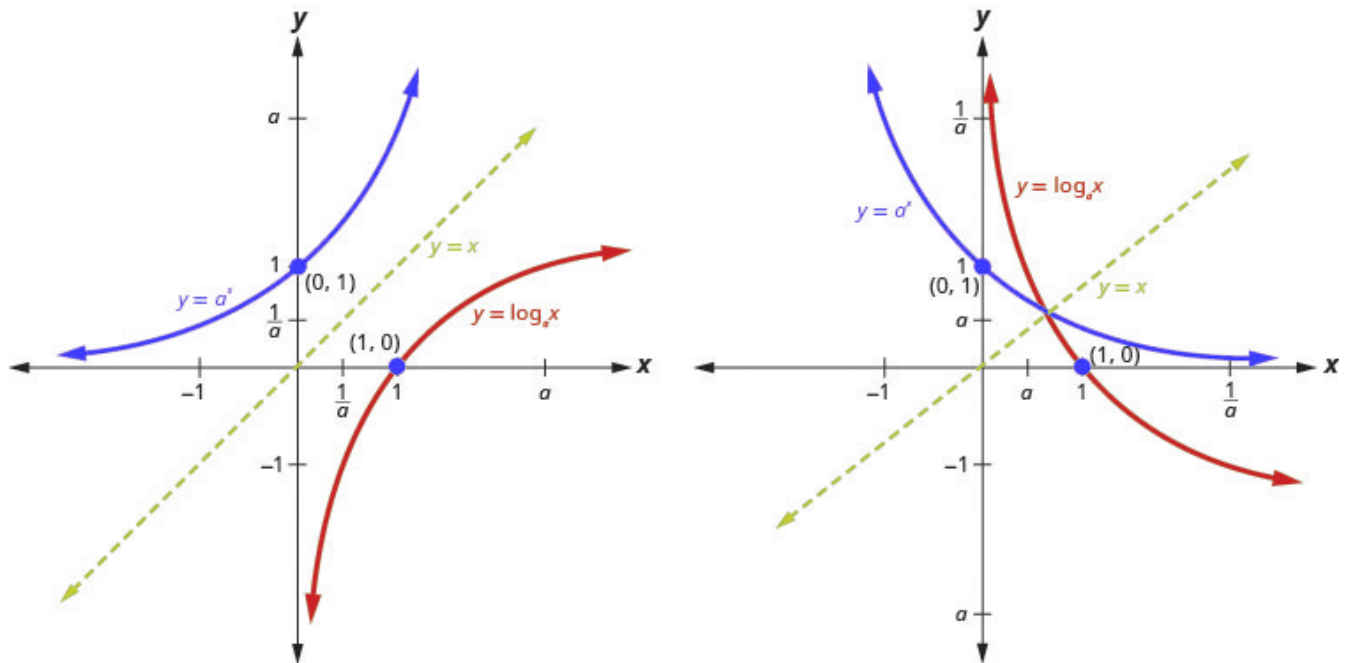


Figure 5.2.10

We talked earlier about how the logarithmic function  $f^{-1}(x) = \log_a x$  is the inverse of the exponential function  $f(x) = a^x$ . The graphs in Figure 5.2.10 show both the exponential (blue) and logarithmic (red) functions on the same graph for both  $a > 1$  and  $0 < a < 1$ .

Notice how the graphs are reflections of each other through the line  $y = x$ . We know this is true of inverse functions. Keeping a visual in your mind of these graphs will help you remember the domain and range of each function. Notice the  $x$ -axis is the horizontal asymptote for the exponential functions and the  $y$ -axis is the vertical asymptote for the logarithmic functions.

## Solve Logarithmic Equations

When we talked about exponential functions, we introduced the number  $e$ . Just as  $e$  was a base for an exponential function, it can be used a base for logarithmic functions too. The logarithmic function with base  $e$  is called the **natural logarithmic function**. The function  $f(x) = \log_e x$  is generally written  $f(x) = \ln x$  and we read it as “el en of  $x$ .”

### Natural Logarithmic Function

The function  $f(x) = \ln x$  is the natural logarithmic function with base  $e$ , where  $x > 0$ .

$$y = \ln x \text{ is equivalent to } x = e^y$$

When the base of the logarithm function is  $10$ , we call it the **common logarithmic function** and the base is not shown. If the base  $a$  of a logarithm is not shown, we assume it is  $10$ .

### Common Logarithmic Function

The function  $f(x) = \log x$  is the common logarithmic function with base  $10$ , where  $x > 0$ .

$$y = \log x \text{ is equivalent to } x = 10^y$$

It will be important for you to use your calculator to evaluate both common and natural logarithms. Look for the **log** and **ln** keys on your calculator.

To solve logarithmic equations, one strategy is to change the equation to exponential form and then solve the exponential equation as we did before. As we solve logarithmic equations,  $y = \log_a x$ , we need to remember that for the base  $a$ ,  $a > 0$  and  $a \neq 1$ . Also, the domain is  $x > 0$ . Just as with radical equations, we must check our solutions to eliminate any extraneous solutions.

## Example 5.2.7

Solve:

a.  $\log_a 49 = 2$

b.  $\ln x = 3$

### Solution

a.

**Step 1: Rewrite in exponential form.**

$$a^2 = 49$$

**Step 2: Solve the equation using the square root property.**

$$a = \pm 7$$

**Step 3: The base cannot be negative, so we eliminate  $a = -7$**

$$a = 7, \quad \overline{)a = -7}$$

**Step 4: Check**

$$\begin{aligned} a &= 7 \\ \log_{[1.0,0.0,0.0]a} 49 &= 2 \\ \log_{[1.0,0.0,0.0]7} 49 &\stackrel{?}{=} 2 \\ 7^2 &\stackrel{?}{=} 49 \\ 49 &= 49 \checkmark \end{aligned}$$

b.

**Step 1: Rewrite in exponential form.**

$$e^3 = x$$

**Step 2: Check**

$$\begin{aligned}
 x &= e^3 \\
 \ln rgb]1.0, 0.0, 0.0x &= 3 \\
 \ln rgb]1.0, 0.0, 0.0e^3 &\stackrel{?}{=} 3 \\
 e^3 &= e^3 \checkmark
 \end{aligned}$$

## Try It

16) Solve:

a.  $\log_a 121 = 2$

b.  $\ln x = 7$

### Solution

a.  $a = 11$

b.  $x = e^7$

## Try It

17) Solve:

a.  $\log_a 64 = 3$

b.  $\ln x = 9$

### Solution

a.  $a = 4$

b.  $x = e^9$

## Example 5.2.8

Solve:

a.  $\log_2(3x - 5) = 4$

b.  $\ln e^{2x} = 4$

### Solution

a.

**Step 1: Rewrite in exponential form.**

$$\begin{aligned} 2^4 &= 3x - 5 \\ \text{Simplify.} \quad 16 &= 3x - 5 \end{aligned}$$

**Step 2: Solve the equation.**

$$\begin{aligned} 21 &= 3x \\ 7 &= x \end{aligned}$$

**Step 3: Check**

$$\begin{aligned} x &= 7 \\ \log_2(3 \cdot 7 - 5) &= 4 \\ \log_2(3 \cdot 7 - 5) &\stackrel{?}{=} 4 \\ \log_2(16) &\stackrel{?}{=} 4 \\ 2^4 &\stackrel{?}{=} 16 \\ 16 &= 16 \checkmark \end{aligned}$$

b.

**Step 1: Rewrite in exponential form.**

$$e^4 = e^{2x}$$

**Step 2: Since the bases are the same the exponents are equal.**

$$4 = 2x$$

**Step 3: Solve the equation.**

$$2 = x$$

**Step 4: Check**

$$\begin{aligned} x &= 2 \\ \ln e^{2 \cdot \log_2 1.0, 0.0, 0.0, 0.0 x} &= 4 \\ \ln e^{2 \cdot \log_2 1.0, 0.0, 0.0, 0.02} &\stackrel{?}{=} 4 \\ \ln e^4 &\stackrel{?}{=} 4 \\ e^4 &= e^4 \checkmark \end{aligned}$$

## Try It

18) Solve:

a.  $\log_2 (5x - 1) = 6$

b.  $\ln e^{3x} = 6$

**Solution**

a.  $x = 13$

b.  $x = 2$

## Try It

19) Solve:



a.  $\log_3 (4x + 3) = 3$

b.  $\ln e^{4x} = 4$

**Solution**

a.  $x = 6$

b.  $x = 1$

## Use Logarithmic Models in Applications

There are many applications that are modelled by logarithmic equations. We will first look at the logarithmic equation that gives the decibel (dB) level of sound. Decibels range from 0, which is barely audible to 160, which can rupture an eardrum. The  $10^{-12}$  in the formula represents the intensity of sound that is barely audible.

### Decibel Level of Sound

The loudness level,  $D$ , measured in decibels, of a sound of intensity,  $I$ , measured in watts per square inch is

$$D = 10 \log \left( \frac{I}{10^{-12}} \right)$$

#### Example 5.2.9

Extended exposure to noise that measures 85 dB can cause permanent damage to the inner ear which will result in hearing loss. What is the decibel level of music coming through ear phones with intensity  $10^{-2}$  watts per square inch?

**Solution**

$$D = 10 \log \left( \frac{I}{10^{-12}} \right)$$

**Step 1: Substitute in the intensity level,  $I$ .**

$$D = 10 \log \left( \frac{[1.0, 0.0, 0.010]^{1.0, 0.0, 0.0-2}}{10^{-12}} \right)$$

**Step 2: Simplify.**

$$D = 10 \log(10^{10})$$

Since  $\log 10^{10} = 10$ .

$$D = 10 \times 10$$

**Step 3: Multiply.**

$$D = 100$$

The decibel level of music coming through earphones is **100** dB.

## Try It

20) What is the decibel level of one of the new quiet dishwashers with intensity  $10^{-7}$  watts per square inch?

### Solution

The quiet dishwashers have a decibel level of **50** dB.

## Try It

21) What is the decibel level heavy city traffic with intensity  $10^{-3}$  watts per square inch?

**Solution**

The decibel level of heavy traffic is **90** dB.

## Earthquake Intensity

The magnitude  $R$  of an earthquake is measured by a logarithmic scale called the Richter scale. The model is  $R = \log I$ , where  $I$  is the intensity of the shock wave. This model provides a way to measure earthquake intensity.

The magnitude  $R$  of an earthquake is measured by  $R = \log I$ , where  $I$  is the intensity of its shock wave.

### Example 5.2.10

In 1906, San Francisco experienced an intense earthquake with a magnitude of **7.8** on the Richter scale. Over **80%** of the city was destroyed by the resulting fires. In 2014, Los Angeles experienced a moderate earthquake that measured **5.1** on the Richter scale and caused **\$108** million dollars of damage. Compare the intensities of the two earthquakes.

**Solution**

**Step 1: Convert the magnitudes to intensities.**

	$R = \log I$
1906 earthquake	$7.8 = \log I$
Convert to exponential form.	$I = 10^{7.8}$
2014 earthquake	$5.1 = \log I$
Convert to exponential form.	$I = 10^{5.1}$

**Step 2: Form a ratio of the intensities.**

$$\frac{\text{Intensity for 1906}}{\text{Intensity for 2014}}$$

**Step 3: Substitute in the values**

$$\frac{10^{7.8}}{10^{5.1}}$$

**Step 4: Divide by subtracting the exponents.**

$$10^{2.7}$$

**Step 5: Evaluate**

$$501$$

The intensity of the 1906 earthquake was about **501** times the intensity of the 2014 earthquake.

**Try It**

22) In 1906, San Francisco experienced an intense earthquake with a magnitude of **7.8** on the Richter scale. In 1989, the Loma Prieta earthquake also affected the San Francisco area, and measured **6.9** on the Richter scale. Compare the intensities of the two earthquakes.

**Solution**

The intensity of the 1906 earthquake was about **8** times the intensity of the 1989 earthquake.

**Try It**

23) In 2014, Chile experienced an intense earthquake with a magnitude of **8.2** on the Richter scale.

In 2014, Los Angeles also experienced an earthquake which measured **5.1** on the Richter scale. Compare the intensities of the two earthquakes.

**Solution**

The intensity of the earthquake in Chile was about **1,259** times the intensity of the earthquake in Los Angeles.

Access these online resources for additional instruction and practice with evaluating and graphing logarithmic functions.

- Re-writing logarithmic equations in exponential form
- Simplifying Logarithmic Expressions
- Graphing logarithmic functions
- Using logarithms to calculate decibel levels

## Key Concepts

- **Properties of the Graph of  $y = \log_a x$ :**

when $a > 1$		when $0 < a < 1$	
Domain	$(0, \infty)$	Domain	$(0, \infty)$
Range	$(-\infty, \infty)$	Range	$(-\infty, \infty)$
$x$ -intercept	$(1, 0)$	$x$ -intercept	$(1, 0)$
$y$ -intercept	none	$y$ -intercept	none
Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$	Contains	$(a, 1), \left(\frac{1}{a}, -1\right)$
Asymptote	$y$ -axis	Asymptote	$y$ -axis
Basic shape	increasing	Basic shape	decreasing

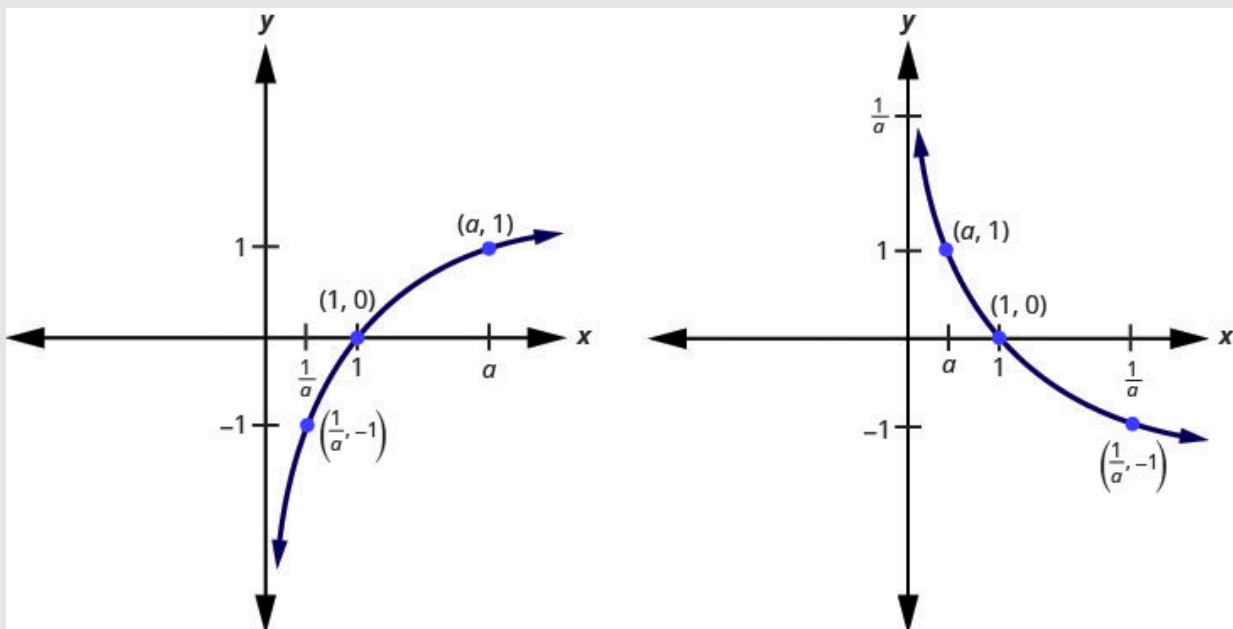


Figure 5.2.11

- **Decibel Level of Sound:** The loudness level,  $D$ , measured in decibels, of a sound of intensity,  $I$ , measured in watts per square inch is  $D = 10 \log \left( \frac{I}{10^{-12}} \right)$ .
- **Earthquake Intensity:** The magnitude  $R$  of an earthquake is measured by  $R = \log I$ , where  $I$  is the intensity of its shock wave.

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



An interactive H5P element has been excluded from this version of the text. You can view it online here:

<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=1543#h5p-35>

b) After reviewing this checklist, what will you do to become confident for all objectives?

## Glossary

### common logarithmic function

The function  $f(x) = \log x$  is the common logarithmic function with base 10, where  $x > 0$ .

$y = \log x$  is equivalent to  $x = 10^y$

### logarithmic function

The function  $f(x) = \log_a x$  is the logarithmic function with base  $a$ , where  $a > 0$ ,  $x > 0$ , and  $a \neq 1$ .

$y = \log_a x$  is equivalent to  $x = a^y$

### natural logarithmic function

The function  $f(x) = \ln x$  is the natural logarithmic function with base  $e$ , where  $x > 0$ .

$y = \ln x$  is equivalent to  $x = e^y$

## 5.3 USE THE PROPERTIES OF LOGARITHMS

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### Learning Objectives

By the end of this section, you will be able to:

- Use the properties of logarithms
- Use the Change of Base Formula

### Try It

Before you get started, take this readiness quiz:

1) Evaluate:

a)  $a^0$

b)  $a^1$

2) Write with a rational exponent:  $\sqrt[3]{x^2y}$

3) Round to three decimal places: **2.5646415**

### Use the Properties of Logarithms

Now that we have learned about exponential and logarithmic functions, we can introduce some of the



properties of logarithms. These will be very helpful as we continue to solve both exponential and logarithmic equations.

The first two properties derive from the definition of logarithms. Since  $a^0 = 1$ , we can convert this to logarithmic form and get  $\log_a 1 = 0$ . Also, since  $a^1 = a$ , we get  $\log_a a = 1$ .

## Properties of Logarithms

$$\log_a 1 = 0$$

$$\log_a a = 1$$

In the next example we could evaluate the logarithm by converting to exponential form, as we have done previously, but recognizing and then applying the properties saves time.

### Example 5.3.1

Evaluate using the properties of logarithms:

- a.  $\log_8 1$
- b.  $\log_6 6$

#### Solution

a.

**Step 1: Use the property  $\log_a 1 = 0$ .**

$$\log_8 1 = 0$$


---

b.

**Step 1: Use the property  $\log_a a = 1$ .**

$$\log_6 6 = 1$$

## Try It

4) Evaluate using the properties of logarithms:

a.  $\log_{13} 1$

b.  $\log_9 9$

### Solution

a. 0

b. 1

## Try It

5) Evaluate using the properties of logarithms:

a.  $\log_5 1$

b.  $\log_7 7$

### Solution

a. 0

b. 1

The next two properties can also be verified by converting them from exponential form to logarithmic form, or the reverse.

The exponential equation  $a^{\log_a x} = x$  converts to the logarithmic equation  $\log_a x = \log_a x$ , which is a true statement for positive values for  $x$  only.

The logarithmic equation  $\log_a a^x = x$  converts to the exponential equation  $a^x = a^x$ , which is also a true statement.

These two properties are called inverse properties because, when we have the same base, raising to a power

“undoes” the log and taking the log “undoes” raising to a power. These two properties show the composition of functions. Both ended up with the identity function which shows again that the exponential and logarithmic functions are inverse functions.

## Inverse Properties of Logarithms

$$\text{For } a > 0, x > 0 \text{ and } a \neq 1, \\ a^{\log_a x} = x \qquad \log_a a^x = x$$

In the next example, apply the inverse properties of logarithms.

### Example 5.3.2

Evaluate using the properties of logarithms:

- a.  $4^{\log_4 9}$   
 b.  $\log_3 3^5$

#### Solution

a.

**Step 1: Use the property  $\log_a x = x$ .**

$$4^{\log_4 9} = 9$$


---

b.

**Step 1: Use the property  $\log_a x = x$ .**

$$\log_3 3^5 = 5$$

## Try It

6) Evaluate using the properties of logarithms:

a.  $5^{\log_5 15}$

b.  $\log_7 7^4$

### Solution

a. 15

b. 4

## Try It

7) Evaluate using the properties of logarithms:

a.  $2^{\log_2 8}$

b.  $\log_2 2^{15}$

### Solution

a. 8

b. 15

There are three more properties of logarithms that will be useful in our work. We know exponential functions and logarithmic function are very interrelated. Our definition of logarithm shows us that a logarithm is the exponent of the equivalent exponential. The properties of exponents have related properties for exponents.

In the Product Property of Exponents,  $a^m \cdot a^n = a^{m+n}$ , we see that to multiply the same base, we add the exponents. The Product Property of Logarithms,  $\log_a M \cdot N = \log_a M + \log_a N$  tells us to take the log of a product, we add the log of the factors.

## Product Property of Logarithms

$$\text{If } M > 0, N > 0, a > 0 \text{ and } a \neq 1, \text{ then,}$$

$$\log_a (M \cdot N) = \log_a M + \log_a N$$

The logarithm of a product is the sum of the logarithms.

We use this property to write the log of a product as a sum of the logs of each factor.

### Example 5.3.3

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible:

a.  $\log_3 7x$

b.  $\log_4 64xy$

#### Solution

a.

**Step 1: Use the Product Property**  $\log_a (M \cdot N) = \log_a M + \log_a N$ .

$$= \log_3 7 + \log_3 x$$

$$\log_3 7x = \log_3 7 + \log_3 x$$

b.

**Step 1: Use the Product Property**  $\log_a (M \cdot N) = \log_a M + \log_a N$ .

$$\log_4 64 + \log_4 x + \log_4 y$$

**Step 2: Simplify by evaluating**  $\log_4 64$ .

$$= 3 + \log_4 x + \log_4 y$$

$$\log_4 64xy = 3 + \log_4 x + \log_4 y$$

## Try It

8) Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible.

a.  $\log_3 3x$

b.  $\log_2 8xy$

### Solution

a.  $1 + \log_3 x$

b.  $3 + \log_2 x + \log_2 y$

## Try It

9) Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible.

a.  $\log_9 9x$

b.  $\log_3 27xy$

### Solution

a.  $1 + \log_9 x$

b.  $3 + \log_3 x + \log_3 y$

Similarly, in the Quotient Property of Exponents,  $\frac{a^m}{a^n} = a^{m-n}$ , we see that to divide the same base, we subtract the exponents. The Quotient Property of Logarithms,  $\log_a \frac{M}{N} = \log_a M - \log_a N$  tells us to take the log of a quotient, we subtract the log of the numerator and denominator.

## Quotient Property of Logarithms

If  $M > 0$ ,  $N > 0$ ,  $a > 0$  and  $a \neq 1$ , then,

$$\log_a \frac{M}{N} = \log_a M - \log_a N$$

The logarithm of a quotient is the difference of the logarithms.

Note that  $\log_a M - \log_a N \neq \log_a (M - N)$ .

We use this property to write the log of a quotient as a difference of the logs of each factor.

### Example 5.3.4

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a.  $\log_5 \frac{5}{7}$

b.  $\log \frac{x}{100}$

#### Solution

a.

**Step 1: Use the Quotient Property**  $\log_a \frac{M}{N} = \log_a M - \log_a N$ .

$$= \log_5 5 - \log_5 7$$

$$= 1 - \log_5 7$$

**Step 2: Simplify.**

$$\log_5 \frac{5}{7} = 1 - \log_5 7$$

b.

**Step 1: Use the Quotient Property**  $\log_a \frac{M}{N} = \log_a M - \log_a N$ .

$$= x - \log 100$$

$$= \log x - 2$$

**Step 2: Simplify.**

$$\log \frac{x}{100} = \log x - 2$$

## Try It

10) Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a.  $\log_4 \frac{3}{4}$

b.  $\log \frac{x}{1000}$

**Solution**

a.  $\log_4 3 - 1$

b.  $\log x - 3$



## Try It

11) Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a.  $\log_2 \frac{5}{4}$

b.  $\log \frac{10}{y}$

### Solution

a.  $\log_2 5 - 2$

b.  $1 - \log y$

The third property of logarithms is related to the Power Property of Exponents,  $(a^m)^n = a^{m \cdot n}$ , we see that to raise a power to a power, we multiply the exponents. The Power Property of Logarithms,  $\log_a M^p = p \log_a M$  tells us to take the log of a number raised to a power, we multiply the power times the log of the number.

## Power Property of Logarithms

$$\text{If } M > 0, a > 0, a \neq 1 \text{ and } p \text{ is any real number then,}$$

$$\log_a M^p = p \log_a M$$

The log of a number raised to a power is the product of the power times the log of the number.

We use this property to write the log of a number raised to a power as the product of the power times the log of the number. We essentially take the exponent and throw it in front of the logarithm.

### Example 5.3.5

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

a.  $\log_5 4^3$

b.  $\log x^{10}$

#### Solution

a.

**Step 1: Use the Power Property**  $\log_a M^p = p \log_a M$ .

$$= 3 \log_5 4$$

$$\log_5 4^3 = 3 \log_5 4$$


---

b.

**Step 1: Use the Power Property**  $\log_a M^p = p \log_a M$ .

$$= 10 \log x$$

$$\log x^{10} = 10 \log x$$

### Try It

12) Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

- a.  $\log_7 5^4$   
 b.  $\log x^{100}$

**Solution**

- a.  $4\log_7 5$   
 b.  $100 \cdot \log x$

**Try It**

13) Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

- a.  $\log_2 3^7$   
 b.  $\log x^{20}$

**Solution**

- a.  $7\log_2 3$   
 b.  $20 \cdot \log x$

We summarize the Properties of Logarithms here for easy reference. While the natural logarithms are a special case of these properties, it is often helpful to also show the natural logarithm version of each property.

## Properties of Logarithms

If  $M > 0$ ,  $a > 0$ ,  $a \neq 1$  and  $p$  is any real number then,

Property	Base $a$	Base $e$
	$\log_a 1 = 0$	$\ln 1 = 0$
	$\log_a a = 1$	$\ln e = 1$
<b>Inverse Properties</b>	$a^{\log_a x} = x$ $\log_a a^x = x$	$e^{\ln x} = x$ $\ln e^x = x$
<b>Product Property of Logarithms</b>	$\log_a (M \cdot N) = \log_a M + \log_a N$	$\ln (M \cdot N) = \ln M + \ln N$
<b>Quotient Property of Logarithms</b>	$\log_a \frac{M}{N} = \log_a M - \log_a N$	$\ln \frac{M}{N} = \ln M - \ln N$
<b>Power Property of Logarithms</b>	$\log_a M^p = p \log_a M$	$\ln M^p = p \ln M$

Now that we have the properties we can use them to “expand” a logarithmic expression. This means to write the logarithm as a sum or difference and without any powers.

We generally apply the Product and Quotient Properties before we apply the Power Property.

### Example 5.3.6

Use the Properties of Logarithms to expand the logarithm  $\log_4 (2x^3y^2)$ . Simplify, if possible.

#### Solution

**Step 1: Use the Product Property**  $\log_a (M \cdot N) = \log_a M + \log_a N$ .

$$\log_4 2 + \log_4 x^3 + \log_4 y^2$$

**Step 2: Use the Power Property**,  $\log_a M^p = p \log_a M$ , on the last two terms.

$$\log_4 2 + 3\log_4 x + 2\log_4 y$$

**Step 3: Simplify.**

$$= \frac{1}{2} + 3\log_4 x + 2\log_4 y$$

$$\log_4 (2x^3 y^2) = \frac{1}{2} + 3\log_4 x + 2\log_4 y$$

### Try It

14) Use the Properties of Logarithms to expand the logarithm  $\log_2 (5x^4 y^2)$ . Simplify, if possible.

**Solution**

$$\log_2 5 + 4\log_2 x + 2\log_2 y$$

### Try It

15) Use the Properties of Logarithms to expand the logarithm  $\log_3 (7x^5 y^3)$ . Simplify, if possible.

**Solution**

$$\log_3 7 + 5\log_3 x + 3\log_3 y$$

When we have a radical in the logarithmic expression, it is helpful to first write its radicand as a rational exponent.

### Example 5.3.7

Use the Properties of Logarithms to expand the logarithm  $\log_2 \sqrt[4]{\frac{x^3}{3y^2z}}$ . Simplify, if possible.

#### Solution

**Step 1: Rewrite the radical with a rational exponent.**

$$\begin{aligned} &= \log_2 \sqrt[4]{\frac{x^3}{3y^2z}} \\ &= \log_2 \left( \frac{x^3}{3y^2z} \right)^{\frac{1}{4}} \end{aligned}$$

**Step 2: Use the Power Property,  $\log_a M^p = p \log_a M$ .**

$$\frac{1}{4} \log_2 \left( \frac{x^3}{3y^2z} \right)$$

**Step 3: Use the Quotient Property,  $\log_a M \times N = \log_a M - \log_a N$ .**

$$\frac{1}{4} (\log_2(x^3) - \log_2(3y^2z))$$

**Step 4: Use the Product Property,  $\log_a M \times N = \log_a M + \log_a N$ , in the second term.**

$$\frac{1}{4} (\log_2(x^3) - (\log_2 3 + \log_2 y^2 + \log_2 z))$$

**Step 5: Use the Power Property inside the parentheses.**

$$\frac{1}{4} (3 \log_2 x - (\log_2 3 + 2 \log_2 y + \log_2 z))$$

**Step 6: Simplify by distributing.**

$$\frac{1}{4} (3 \log_2 x - \log_2 3 - 2 \log_2 y - \log_2 z)$$

## Try It

16) Use the Properties of Logarithms to expand the logarithm  $\log_4 \sqrt[5]{\frac{x^4}{2y^3 z^2}}$ . Simplify, if possible.

### Solution

$$\frac{1}{5} \left( 4\log_4 x - \frac{1}{2} - 3\log_4 y - 2\log_4 z \right)$$

## Try It

17) Use the Properties of Logarithms to expand the logarithm  $\log_3 \sqrt[3]{\frac{x^2}{5yz}}$ . Simplify, if possible.

### Solution

$$\frac{1}{3} (2\log_3 x - \log_3 5 - \log_3 y - \log_3 z)$$

The opposite of expanding a logarithm is to condense a sum or difference of logarithms that have the same base into a single logarithm. We again use the properties of logarithms to help us, but in reverse.

To condense logarithmic expressions with the same base into one logarithm, we start by using the Power Property to get the coefficients of the log terms to be one and then the Product and Quotient Properties as needed.

### Example 5.3.8

Use the Properties of Logarithms to condense the logarithm  $\log_4 3 + \log_4 x - \log_4 y$ . Simplify, if possible.

#### Solution

**Step 1: Check the log base. The log expressions all have the same base, 4.**

$$\log_4 3 + \log_4 x - \log_4 y$$

**Step 2: The first two terms are added, so we use the Product Property,**

$$\log_a M \times N = \log_a M + \log_a N.$$

$$\log_4 3x - \log_4 y$$

**Step 3: Since the logs are subtracted, we use the Quotient Property,**

$$\log_a M \times N = \log_a M - \log_a N.$$

$$= \log_4 \frac{3x}{y}$$

$$\log_4 3 + \log_4 x - \log_4 y = \log_4 \frac{3x}{y}$$

### Try It

18) Use the Properties of Logarithms to condense the logarithm  $\log_2 5 + \log_2 x - \log_2 y$ . Simplify, if possible.

#### Solution

$$\log_2 \frac{5x}{y}$$



## Try It

19) Use the Properties of Logarithms to condense the logarithm  $\log_3 6 - \log_3 x - \log_3 y$ . Simplify, if possible.

### Solution

$$\log_3 \frac{6}{xy}$$

## Example 5.3.9

Use the Properties of Logarithms to condense the logarithm  $2\log_3 x + 4\log_3 (x + 1)$ . Simplify, if possible.

### Solution

**Step 1: Check the log base. The log expressions have the same base, 3.**

$$2\log_3 x + 4\log_3 (x + 1)$$

**Step 2: Use the Power Property,  $\log_a M^p = p\log_a M$ .**

$$\log_3 x^2 + \log_3 (x + 1)^4$$

**Step 3: The terms are added, so we use the Product Property,**

$$\log_a M \times N = \log_a M + \log_a N.$$

$$= \log_3 x^2 (x + 1)^4$$

$$\log_3 x + 4\log_3 (x + 1) = \log_3 x^2 (x + 1)^4$$

### Try It

20) Use the Properties of Logarithms to condense the logarithm  $3\log_2 x + 2\log_2 (x - 1)$ . Simplify, if possible.

**Solution**

$$\log_2 x^3 (x - 1)^2$$

### Try It

21) Use the Properties of Logarithms to condense the logarithm  $2\log x + 2\log (x + 1)$ . Simplify, if possible.

**Solution**

$$\log x^2 (x + 1)^2$$

## Use the Change-of-Base Formula

To evaluate a logarithm with any other base, we can use the Change-of-Base Formula. We will show how this is derived.

Suppose we want to evaluate  $\log_a M$ .

Let  $y = \log_a M$ .

Rewrite the expression in exponential form.

Take the  $\log_b$  of each side.

Use the Power Property.

Solve for  $y$ .

Substitute  $y = \log_a M$ .

$$\log_a M$$

$$y = \log_a M$$

$$a^y = M$$

$$\log_b a^y = \log_b M$$

$$y \log_b a = \log_b M$$

$$y = \frac{\log_b M}{\log_b a}$$

$$\log_a M = \frac{\log_b M}{\log_b a}$$

The Change-of-Base Formula introduces a new base  $b$ . This can be any base  $b$  we want where  $b > 0$ ,  $b \neq 1$ . Because our calculators have keys for logarithms base 10 and base  $e$ , we will rewrite the Change-of-Base Formula with the new base as 10 or  $e$ .

## Change-of-Base Formula

For any logarithmic bases  $a$ ,  $b$  and  $M > 0$ ,

$$\log_a M = \frac{\log_b M}{\log_b a}$$

new base  $b$

$$\log_a M = \frac{\log M}{\log a}$$

new base 10

$$\log_a M = \frac{\ln M}{\ln a}$$

new base  $e$

When we use a calculator to find the logarithm value, we usually round to three decimal places. This gives us an approximate value and so we use the approximately equal symbol ( $\approx$ ).

### Example 5.3.10

Rounding to three decimal places, approximate  $\log_4 35$ .

#### Solution

$$\log_4 35$$

**Step 1: Use the Change-of-Base Formula.**

$$\log_{rgb} 1.0, 0.0, 0.0 M = \frac{\log_b rgb 1.0, 0.0, 0.0 M}{\log_b rgb 0.0, 0.0, 1.0 a}$$

**Step 2: Identify  $a$  and  $M$ . Choose 10 for  $b$ .**

$$\log_{rgb]0.0,0.0,1.04} rgb]1.0,0.0,0.035 = \frac{\log rgb]1.0,0.0,0.035}{\log rgb]0.0,0.0,1.04}$$

**Step 3: Enter the expression  $\frac{\log 35}{\log 4}$  in the calculator using the log button for base 10.**

**Round to three decimal places.**

$$\log_4 35 \approx 2.565$$

## Try It

22) Rounding to three decimal places, approximate  $\log_3 42$ .

**Solution**

3.402

## Try It

23) Rounding to three decimal places, approximate  $\log_5 46$ .

**Solution**

2.379

Access these online resources for additional instruction and practice with using the properties of logarithms.

- Using Properties of Logarithms to Expand Logs
- Using Properties of Logarithms to Condense Logs
- Change of Base

## Key Concepts

- **Properties of Logarithms**

$$\log_a 1 = 0$$

$$\log_a a = 1$$

- **Inverse Properties of Logarithms**

- For  $a > 0$ ,  $x > 0$  and  $a \neq 1$

$$a^{\log_a x} = x$$

$$\log_a a^x = x$$

- **Product Property of Logarithms**

- If  $M > 0$ ,  $N > 0$ ,  $a > 0$  and  $a \neq 1$ , then,

$$\log_a M \cdot N = \log_a M + \log_a N$$

The logarithm of a product is the sum of the logarithms.

- **Quotient Property of Logarithms**

- If  $M > 0$ ,  $N > 0$ ,  $a > 0$  and  $a \neq 1$ , then,

$$\log_a \frac{M}{N} = \log_a M - \log_a N$$

The logarithm of a quotient is the difference of the logarithms.

- **Power Property of Logarithms**

- If  $M > 0$ ,  $a > 0$ ,  $a \neq 1$  and  $p$  is any real number then,

$$\log_a M^p = p \log_a M$$

The log of a number raised to a power is the product of the power times the log of the number.

### • Properties of Logarithms Summary

- If  $M > 0$ ,  $a > 0$ ,  $a \neq 1$  and  $p$  is any real number then,

Property	Base $a$	Base $e$
	$\log_a 1 = 0$	$\ln 1 = 0$
	$\log_a a = 1$	$\ln e = 1$
<b>Inverse Properties</b>	$a^{\log_a x} = x$ $\log_a a^x = x$	$e^{\ln x} = x$ $\ln e^x = x$
<b>Product Property of Logarithms</b>	$\log_a (M \cdot N) = \log_a M + \log_a N$	$\ln (M \cdot N) = \ln M + \ln N$
<b>Quotient Property of Logarithms</b>	$\log_a \frac{M}{N} = \log_a M - \log_a N$	$\ln \frac{M}{N} = \ln M - \ln N$
<b>Power Property of Logarithms</b>	$\log_a M^p = p \log_a M$	$\ln M^p = p \ln M$

### • Change-of-Base Formula

- For any logarithmic bases  $a$  and  $b$ , and  $M > 0$ ,

$$\log_a M = \frac{\log_b M}{\log_b a}$$

new base  $b$

$$\log_a M = \frac{\log M}{\log a}$$

new base 10

$$\log_a M = \frac{\ln M}{\ln a}$$

new base  $e$

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

*<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=1549#h5p-36>*

b) On a scale of **1** — **10**, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

# 5.4 SOLVE EXPONENTIAL AND LOGARITHMIC EQUATIONS

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## Learning Objectives

By the end of this section, you will be able to:

- Solve logarithmic equations using the properties of logarithms
- Solve exponential equations using logarithms
- Use exponential models in applications

## Try It

Before you get started, take this readiness quiz:

- 1) Solve:  $x^2 = 16$ .
- 2) Solve:  $x^2 - 5x + 6 = 0$ .
- 3) Solve:  $x(x + 6) = 2x + 5$ .

## Solve Logarithmic Equations Using the Properties of Logarithms

In the section on logarithmic functions, we solved some equations by rewriting the equation in exponential



form. Now that we have the properties of logarithms, we have additional methods we can use to solve logarithmic equations.

If our equation has two logarithms we can use a property that says that if  $\log_a M = \log_a N$  then it is true that  $M = N$ . This is the One-to-One Property of Logarithmic Equations.

## One-to-One Property of Logarithmic Equations

$\begin{array}{c} \text{For } M > 0, N > 0, a > 0, \text{ and } a \neq 1 \\ \text{is any real number:} \\ \text{If } \log_a M = \log_a N, \text{ then } M = N \end{array}$

To use this property, we must be certain that both sides of the equation are written with the same base.

Remember that logarithms are defined only for positive real numbers. Check your results in the original equation. You may have obtained a result that gives a logarithm of zero or a negative number.

### Example 5.4.1

Solve:  $2\log_5 x = \log_5 81$ .

#### Solution

**Step 1: Use the Power Property.**

$$\log_5 x^2 = \log_5 81$$

**Step 2: Use the One-to-One Property, if  $\log_a M = \log_a N$  then  $M = N$ .**

$$x^2 = 81$$

**Step 3: Solve using the Square Root Property.**

$$x = \pm 9$$

**Step 4: We eliminate  $x = -9$  as we cannot take the logarithm of a negative number.**

$$x = 9, \quad \overline{x = -9}$$

**Step 5: Check.**

$$\begin{aligned}x &= 9 \\2 \log_5 x &= \log_5 81 \\2 \log_5 9 &\stackrel{?}{=} \log_5 81 \\ \log_5 9^2 &\stackrel{?}{=} \log_5 81 \\ \log_5 81 &= \log_5 81 \checkmark\end{aligned}$$

### Try It

4) Solve:  $2\log_3 x = \log_3 36$

**Solution**

$$x = 6$$

### Try It

5) Solve:  $3\log x = \log 64$

**Solution**

$$x = 4$$

Another strategy to use to solve logarithmic equations is to condense sums or differences into a single logarithm.

## Example 5.4.2

Solve:  $\log_3 x + \log_3 (x - 8) = 2$ .

### Solution

**Step 1: Use the Product Property,**  $\log_a M + \log_a N = \log_a M \cdot N$ .

$$\log_3 x(x - 8) = 2$$

**Step 2: Rewrite in exponential form.**

$$3^2 = x(x - 8)$$

**Step 3: Simplify.**

$$9 = x^2 - 8x$$

**Step 4: Subtract 9 from each side.**

$$0 = x^2 - 8x - 9$$

**Step 5: Factor.**

$$0 = (x - 9)(x + 1)$$

**Step 6: Use the Zero-Product Property.**

$$x - 9 = 0, \quad x + 1 = 0$$

**Step 7: Solve each equation.**

$$x = 9, \quad x = -1$$

**Step 8: Check.**

$\log_3 (x) + \log_3 (x - 8) = 2$   
 $\log_3 (9) + \log_3 (9 - 8) = 2$   
 $\log_3 (9) + \log_3 (1) = 2$   
 $2 + 0 = 2$   
 $2 = 2 \checkmark$

$\log_3 (x) + \log_3 (x - 8) = 2$   
 $\log_3 (-1) + \log_3 (-1 - 8) = 2$   
 $\log_3 (-1) + \log_3 (-9) = 2$   
 $-0.301 + (-0.954) = 2$   
 $-1.255 = 2$   
 $-1.255 \neq 2$   
 We cannot take the log of a negative number.

$$\begin{aligned}
 & \log_3 (x) + \log_3 (x - 8) = 2 \\
 & \log_3 (9) + \log_3 (9 - 8) = 2 \\
 & \log_3 (9) + \log_3 (1) = 2 \\
 & 2 + 0 = 2 \\
 & 2 = 2 \checkmark
 \end{aligned}$$

**Try It**

6) Solve:  $\log_2 x + \log_2 (x - 2) = 3$

**Solution**

$x = 4$

**Try It**

7) Solve:  $\log_2 x + \log_2 (x - 6) = 4$

**Solution**

$x = 8$

When there are logarithms on both sides, we condense each side into a single logarithm. Remember to use the Power Property as needed.

**Example 5.4.3**

Solve:  $\log_4 (x + 6) - \log_4 (2x + 5) = -\log_4 x$ .

**Solution**

**Step 1: Use the Quotient Property on the left side and the Power Property on the right.**

$$\log_4 \left( \frac{x+6}{2x+5} \right) = \log_4 x^{-1}$$

**Step 2: Rewrite**  $x^{-1} = \frac{1}{x}$ .

$$\log_4 \left( \frac{x+6}{2x+5} \right) = \log_4 x^{\frac{1}{4}}$$

**Step 3: Use the One-to-One Property, if  $\log_a M = \log_a N$ , then  $M = N$ .**

$$\frac{x+6}{2x+5} = \frac{1}{x}$$

**Step 4: Solve the rational equation.**

$$x(x+6) = 2x+5$$

**Step 5: Distribute.**

$$x^2 + 6x = 2x + 5$$

**Step 6: Write in standard form.**

$$x^2 + 4x - 5 = 0$$

**Step 7: Factor.**

$$(x+5)(x-1) = 0$$

**Step 8: Use the Zero-Product Property.**

$$x+5=0, \quad x-1=0$$

**Step 9: Solve each equation.**

$$\cancel{x = -5}, \quad x = 1$$

**Step 10: Check.**

We leave the check for you.

**Try It**

8) Solve:  $\log(x + 2) - \log(4x + 3) = -\log x$ .

**Solution**

$x = 3$

**Try It**

9) Solve:  $\log(x - 2) - \log(4x + 16) = \log \frac{1}{x}$ .

**Solution**

$x = 8$

## Solve Exponential Equations Using Logarithms

In the section on exponential functions, we solved some equations by writing both sides of the equation with the same base. Next we wrote a new equation by setting the exponents equal.

It is not always possible or convenient to write the expressions with the same base. In that case we often take the common logarithm or natural logarithm of both sides once the exponential is isolated.

### Example 5.4.4

Solve  $5^x = 11$ . Find the exact answer and then approximate it to three decimal places.

**Solution**

**Step 1: Since the exponential is isolated, take the logarithm of both sides.**

$$\log 5^x = \log 11$$

**Step 2: Use the Power Property to get the  $x$  as a factor, not an exponent.**

$$x \log 5 = \log 11$$

**Step 3: Solve for  $x$ . Find the exact answer.**

$$x = \frac{\log 11}{\log 5}$$

**Step 4: Approximate the answer.**

$$x \approx 1.490$$

**Step 5: Check — Does it make sense?**

Since  $5^1 = 5$ ,  $5^2 = 25$  does it make sense that  $5^{1.490} \approx 11$ ?

### Try It

10) Solve  $7^x = 43$ . Find the exact answer and then approximate it to three decimal places.

**Solution**

$$x = \frac{\log 43}{\log 7} \approx 1.933$$

## Try It

11) Solve  $8^x = 98$ . Find the exact answer and then approximate it to three decimal places.

### Solution

$$x = \frac{\log 98}{\log 8} \approx 2.205$$

When we take the logarithm of both sides we will get the same result whether we use the common or the natural logarithm (try using the natural log in the last example. Did you get the same result?) When the exponential has base  $e$ , we use the natural logarithm.

## Example 5.4.5

Solve  $3e^{x+2} = 24$ . Find the exact answer and then approximate it to three decimal places.

### Solution

**Step 1: Isolate the exponential by dividing both sides by 3.**

$$e^{x+2} = 8$$

**Step 2: Take the natural logarithm of both sides.**

$$\ln e^{x+2} = \ln 8$$

**Step 3: Use the Power Property to get the  $x$  as a factor, not an exponent.**

$$(x + 2)\ln e = \ln 8$$

**Step 4: Use the property  $\ln e = 1$  to simplify.**

$$x + 2 = \ln 8$$



**Step 5: Solve the equation. Find the exact answer.**

$$x = \ln 8 - 2$$

**Step 6: Approximate the answer.**

$$x \approx 0.079$$

## Try It

12) Solve  $2e^{x-2} = 18$ . Find the exact answer and then approximate it to three decimal places.

**Solution**

$$x = \ln 9 + 2 \approx 4.197$$

## Try It

13) Solve  $5e^{2x} = 25$ . Find the exact answer and then approximate it to three decimal places.

**Solution**

$$x = \frac{\ln 5}{2} \approx 0.805$$

## Use Exponential Models in Applications

In previous sections we were able to solve some applications that were modelled with exponential equations. Now that we have so many more options to solve these equations, we are able to solve more applications.

We will again use the Compound Interest Formulas and so we list them here for reference.

## Compound Interest

For a principal,  $P$ , invested at an interest rate,  $r$ , for  $t$  years, the new balance,  $A$  is:

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{when compounded } n \text{ times a year.}$$

$$A = Pe^{rt} \quad \text{when compounded continuously.}$$

### Example 5.4.6

Jermael's parents put \$10,000 in investments for his college expenses on his first birthday. They hope the investments will be worth \$50,000 when he turns 18. If the interest compounds continuously, approximately what rate of growth will they need to achieve their goal?

#### Solution

**Step 1: Identify the variables in the formula.**

$$A = \$50,000$$

$$P = \$10,000$$

$$r = ?$$

$$t = 17 \text{ years}$$

$$A = Pe^{rt}$$

**Step 2: Substitute the values into the formula.**

$$50,000 = 10,000e^{r \cdot 17}$$

**Step 3: Solve for  $r$ . Divide each side by 10,000.**

$$5 = e^{17r}$$

**Step 4: Take the natural log of each side.**

$$\ln 5 = \ln e^{17r}$$

**Step 5: Use the Power Property.**

$$\ln 5 = 17r \ln e$$

**Step 6: Simplify.**

$$\ln 5 = 17r$$

**Step 7: Divide each side by 17.**

$$\frac{\ln 5}{17} = r$$

**Step 8: Approximate the answer.**

$$r \approx 0.095$$

**Step 9: Convert to a percentage.**

$$r \approx 9.5\%$$

They need the rate of growth to be approximately **9.5%**.

## Try It

14) Hector invests \$10,000 at age 21. He hopes the investments will be worth \$150,000 when he turns 50. If the interest compounds continuously, approximately what rate of growth will he need to achieve his goal?

**Solution**

$$r \approx 9.3\%$$

## Try It

15) Rachel invests \$15,000 at age 25. She hopes the investments will be worth \$90,000 when

she turns **40**. If the interest compounds continuously, approximately what rate of growth will she need to achieve her goal?

**Solution**

$$r \approx 11.9\%$$

We have seen that growth and decay are modelled by exponential functions. For growth and decay we use the formula  $A = A_0 e^{kt}$ . Exponential growth has a positive rate of growth or growth constant,  $k$ , and exponential decay has a negative rate of growth or decay constant,  $k$ .

## Exponential Growth and Decay

For an original amount,  $A_0$ , that grows or decays at a rate,  $k$ , for a certain time,  $t$ , the final amount,  $A$ , is:

$$A = A_0 e^{kt}$$

We can now solve applications that give us enough information to determine the rate of growth. We can then use that rate of growth to predict other situations.

### Example 5.4.7

Researchers recorded that a certain bacteria population grew from **100** to **300** in **3** hours. At this rate of growth, how many bacteria will there be **24** hours from the start of the experiment?

**Solution**

This problem requires two main steps. First we must find the unknown rate,  $k$ . Then we use that value of  $k$  to help us find the unknown number of bacteria.

**Step 1: Identify the variables in the formula.**

$$\begin{aligned}
 A &= 300 \\
 A_0 &= 100 \\
 k &= ? \\
 t &= 3 \text{ hours} \\
 A &= A_0 e^{kt}
 \end{aligned}$$

**Step 2: Substitute the values in the formula.**

$$300 = 100e^{k \cdot 3}$$

**Step 3: Solve for  $k$ . Divide each side by 100.**

$$3 = e^{3k}$$

**Step 4: Take the natural log of each side.**

$$\ln 3 = \ln e^{3k}$$

**Step 5: Use the Power Property.**

$$\ln 3 = 3k \ln e$$

**Step 6: Simplify.**

$$\ln 3 = 3k$$

**Step 7: Divide each side by 3.**

$$\frac{\ln 3}{3} = k$$

**Step 8: Approximate the answer.**

$$k \approx 0.366$$

We use this rate of growth to predict the number of bacteria there will be in **24** hours.

$$\begin{aligned}
 A &= ? \\
 A_0 &= 100 \\
 k &= \frac{\ln 3}{3} \\
 t &= 24 \text{ hours} \\
 A &= A_0 e^{kt}
 \end{aligned}$$

**Step 9: Substitute in the values.**

$$A = 100e^{\frac{\ln 3}{3} \cdot 24}$$

**Step 10: Evaluate.**

$$A \approx 656,100$$

At this rate of growth, they can expect **656,100** bacteria.

### Try It

16) Researchers recorded that a certain bacteria population grew from **100** to **500** in **6** hours. At this rate of growth, how many bacteria will there be **24** hours from the start of the experiment?

**Solution**

There will be **62,500** bacteria.

### Try It

17) Researchers recorded that a certain bacteria population declined from **700,000** to **400,000** in **5** hours after the administration of medication. At this rate of decay, how many bacteria will there be **24** hours from the start of the experiment?

**Solution**

There will be **5,870,061** bacteria.

Radioactive substances decay or decompose according to the exponential decay formula. The amount of time it takes for the substance to decay to half of its original amount is called the half-life of the substance.

Similar to the previous example, we can use the given information to determine the constant of decay, and then use that constant to answer other questions.

### Example 5.4.8

The half-life of radium-226 is 1,590 years. How much of a 100 mg sample will be left in 500 years?

#### Solution

This problem requires two main steps. First we must find the decay constant  $k$ . If we start with 100-mg, at the half-life there will be 50-mg remaining. We will use this information to find  $k$ . Then we use that value of  $k$  to help us find the amount of sample that will be left in 500 years.

**Step 1: Identify the variables in the formula.**

$$\begin{aligned} A &= 50 \\ A_0 &= 100 \\ k &= ? \\ t &= 1590 \text{ years} \\ A &= A_0 e^{kt} \end{aligned}$$

**Step 2: Substitute the values in the formula.**

$$50 = 100e^{k \cdot 1590}$$

**Step 3: Solve for  $k$ . Divide each side by 100.**

$$0.5 = e^{1590k}$$

**Step 4: Take the natural log of each side.**

$$\ln 0.5 = \ln e^{1590k}$$

**Step 5: Use the Power Property.**

$$\ln 0.5 = 1590k \ln e$$

**Step 6: Simplify.**

$$\ln 0.5 = 1590k$$

**Step 7: Divide each side by 1590.**

$$\frac{\ln 0.5}{1590} = k \text{ exact answer}$$

**Step 8: Substitute in the values.**

$$A = 100e^{\frac{\ln 0.5}{1590} \cdot 500}$$

We use this rate of growth to predict the amount that will be left in **500** years.

$$A = ?$$

$$A_0 = 100$$

$$k = \frac{\ln 0.5}{1590}$$

$$t = 500 \text{ years}$$

$$A = A_0 e^{kt}$$

**Step 9: Evaluate.**

$$A \approx 80.4 \text{ mg}$$

In **500** years there would be approximately **80.4** mg remaining.

## Try It

18) The half-life of magnesium-27 is **9.45** minutes. How much of a **10**-mg sample will be left in **6** minutes?

### Solution

There will be **6.43** mg left.



## Try It

19) The half-life of radioactive iodine is **60** days. How much of a **50**-mg sample will be left in **40** days?

### Solution

There will be **31.5** mg left.

Access these online resources for additional instruction and practice with solving exponential and logarithmic equations.

- Solving Logarithmic Equations
- Solving Logarithm Equations
- Finding the rate or time in a word problem on exponential growth or decay
- Finding the rate or time in a word problem on exponential growth or decay

## Key Concepts

- **One-to-One Property of Logarithmic Equations:**

$\begin{array}{c} \text{For } M > 0, N > 0, a > 0, \text{ and } a \neq 1 \\ \text{is any real number:} \end{array}$  If  $\log_a M = \log_a N$ , then  $M = N$

- **Compound Interest:**

For a principal,  $P$ , invested at an interest rate,  $r$ , for  $t$  years, the new balance,  $A$ , is:

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{when compounded } n \text{ times a year.}$$

$$A = Pe^{rt} \quad \text{when compounded continuously.}$$

- **Exponential Growth and Decay:**

For an original amount,  $A_0$  that grows or decays at a rate,  $r$ , for a certain time  $t$ , the final amount,  $A$ , is  $A = A_0e^{rt}$ .

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

*<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=1566#h5p-37>*

b) After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

## 5.5 UNIT 5 SOURCES

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**5.0** "Exponential and Logarithmic Functions Introduction" from Intermediate Algebra by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted

**5.1** "10.2 Evaluate and Graph Exponential Functions" from Intermediate Algebra by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted

**5.2** "10.3 Evaluate and Graph Logarithmic Functions" from Intermediate Algebra by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted

**5.3** "10.4 Use the Properties of Logarithms" from Intermediate Algebra by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted

**5.4** "10.5 Solve Exponential and Logarithmic Equations" from Intermediate Algebra by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted



# UNIT 6: INTRODUCTION TO STATISTICS

## Chapter Outline

- 6.0 Introduction
- 6.1 Basics of Statistics
- 6.2 Describing Data Using Distributions and Graphs
- 6.3 Measures of Central Tendency and Spread
- 6.4 z-Scores and the Standard Normal Distribution
- 6.5 Probability and the Normal Distribution
- 6.6 Unit 6 Sources



# 6.0 INTRODUCTION

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## What Are Statistics?



“Crack in the Ground Caused by Earthquake Katori, Kogaishima” by Katorisi CC BY-SA 3.0

This chapter provides an overview of statistics as a field of study and presents terminology that will be used throughout this unit of the course.

Statistics include numerical facts and figures. For instance:

- The largest earthquake measured 9.2 on the Richter scale.
- Men are at least 10 times more likely than women to commit murder.
- One in every eight South Africans is HIV positive.
- By the year 2050, there will be 12 people aged 65 and over for every new baby born.

The study of statistics involves math and relies upon calculations of numbers. But it also relies heavily on how the numbers are chosen and how the statistics are interpreted.

## Try It

1) Consider the following three scenarios and the interpretations based on the presented statistics. You will find that the numbers may be right, but the interpretation may be wrong. Try to identify a major flaw with each interpretation before we describe it.

1. A new advertisement for Ben & Jerry's ice cream introduced in late May of last year resulted in a 30% increase in ice cream sales for the following three months. Thus, the advertisement was effective.
2. The more churches in a city, the more crime there is. Thus, churches lead to crime.
3. Seventy-five percent more interracial marriages are occurring this year than 25 years ago. Thus, our society accepts interracial marriages.

## Solution

1. **Major flaw:** Ice cream consumption generally increases in the months of June, July, and August regardless of advertisements. This effect is called a history effect and leads people to interpret outcomes as the result of one variable when another variable (in this case, one having to do with the passage of time) is actually responsible.
2. **Major flaw:** Both increased churches and increased crime rates can be explained by larger populations. In bigger cities, there are both more churches and more crime. This problem refers to the third-variable problem. Namely, a third variable can cause both



situations; however, people erroneously believe that there is a causal relationship between the two primary variables rather than recognize that a third variable can cause both.

3. **Major flaw.** We don't have the information we need. What is the rate at which marriages are occurring? Suppose only 1% of marriages 25 years ago were interracial and so now 1.75% of marriages are interracial (1.75 is 75% higher than 1). But this latter number is hardly evidence suggesting the acceptability of interracial marriages. In addition, the statistic provided does not rule out the possibility that the number of interracial marriages has seen dramatic fluctuations over the years and this year is not the highest. Again, there is simply not enough information to understand fully the impact of the statistics.

As a whole, these examples show that statistics are not only facts and figures; they are something more than that. In the broadest sense, “statistics” refers to a range of techniques and procedures for analyzing, interpreting, displaying, and making decisions based on data.

Statistics is the language of science and data. The ability to understand and communicate using statistics enables researchers from different labs, different languages, and different fields to articulate to one another exactly what they have found in their work. It is an objective, precise, and powerful tool in science and in everyday life.

## What a Statistics Course Is Not

Many students dread the idea of taking a statistics course, and more than a few have changed majors upon learning that it is a requirement. That is because many students view statistics as a math class, which is actually not true. While many of you will not believe this or agree with it, statistics isn't math. Although math is a central component of it, statistics is a broader way of organizing, interpreting, and communicating information in an objective manner. Statistics is a way of viewing reality as it exists around us in a way that we otherwise could not.

## Why Do We Study Statistics?

Virtually every student of the behavioural sciences takes some form of statistics class. This is because statistics is how we communicate in science. It serves as the link between a research idea and usable conclusions. Without statistics, we would be unable to interpret the massive amounts of information contained in data. Even small datasets contain hundreds—if not thousands—of numbers, each representing a specific observation we made.

Without a way to organize these numbers into a more interpretable form, we would be lost, having wasted the time and money of our participants, ourselves, and the communities we serve.

Beyond its use in science, however, there is a more personal reason to study statistics. Like most people, you probably feel that it is important to “take control of your life.” But what does this mean? Partly, it means being able to properly evaluate the data and claims that bombard you every day. If you cannot distinguish good from faulty reasoning, then you are vulnerable to manipulation and to decisions that are not in your best interest. Statistics provides tools that you need in order to react intelligently to information you hear or read. In this sense, statistics is one of the most important things that you can study.

To be more specific, here are some claims that we have heard on several occasions. (We are not saying that each one of these claims is true!)

- Four out of five dentists recommend Dentine.
- Almost 85% of lung cancers in men and 45% in women are tobacco-related.
- Condoms are effective 94% of the time.
- People tend to be more persuasive when they look others directly in the eye and speak loudly and quickly.
- Women make 75 cents to every dollar a man makes when they work the same job.
- A surprising new study shows that eating egg whites can increase one’s life span.
- People predict that it is very unlikely there will ever be another baseball player with a batting average over 400.
- There is an 80% chance that in a room full of 30 people at least two people will share the same birthday.
- 79.48% of all statistics are made up on the spot.

All of these claims are statistical in character. We suspect that some of them sound familiar; if not, we bet that you have heard other claims like them. Notice how diverse the examples are. They come from psychology, health, law, sports, business, etc. Indeed, data and data interpretation show up in discourse from virtually every facet of contemporary life.

Statistics are often presented in an effort to add credibility to an argument or advice. You can see this by paying attention to television advertisements. Many of the numbers thrown about in this way do not represent careful statistical analysis. They can be misleading and push you into decisions that you might find cause to regret. For these reasons, learning about statistics is a long step toward taking control of your life. (It is not, of course, the only step needed to do so.) The purpose of this section of this course, beyond preparing you

for a career in the health sciences, is to help you learn statistical essentials. It will make you into an intelligent consumer of statistical claims.

You can take the first step right away. To be an intelligent consumer of statistics, your first reflex must be to question the statistics you encounter. The British Prime Minister Benjamin Disraeli is quoted by Mark Twain as having said, “There are three kinds of lies—lies, damned lies, and statistics.” This quote reminds us why it is so important to understand statistics. So let us invite you to reform your statistical habits from now on. No longer will you blindly accept numbers or findings. Instead, you will begin to think about the numbers, their sources, and most importantly, the procedures used to generate them.

The above section puts an emphasis on defending ourselves against fraudulent claims wrapped up as statistics, but let us look at a more positive note. Just as important as detecting the deceptive use of statistics is the appreciation of the proper use of statistics. You must also learn to recognize statistical evidence that supports a stated conclusion. Statistics are all around you, sometimes used well, sometimes not. We must learn how to distinguish the two cases. In doing so, statistics will likely be the course you use most in your day-to-day life, even if you do not ever run a formal analysis again.

# 6.1 BASICS OF STATISTICS

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## Learning Objectives

By the end of this section, you will be able to:

- Define and differentiate between the types of variables.
- Define and differentiate between the levels of measurement of variables.
- Define and differentiate between populations and samples.
- Define and identify the various types of sampling techniques and experimental designs.
- Define and differentiate between descriptive statistics and inferential statistics.
- Understand mathematical notation for statistics.

## Types of Data and How to Collect Them

In order to use **statistics**, we need **data** to analyze. Data come in an amazingly diverse range of formats, and each type gives us a unique type of information. In virtually any form, data represent the measured value of variables. A **variable** is simply a characteristic or feature of the thing we are interested in understanding. In psychology, we are interested in people, so we might get a group of people together and measure their levels of stress (one variable), anxiety (a second variable), and their physical health (a third variable). Once we have data on these three variables, we can use statistics to understand if and how they are related. Before we do so, we need to understand the nature of our data—what they represent and where they came from.

## Types of Variables

When conducting research, experimenters often manipulate variables. For example, an experimenter might compare the effectiveness of four types of antidepressants. In this case, the variable is “type of antidepressant.” When a variable is manipulated by an experimenter, it is called an **independent variable**. The experiment

seeks to determine the effect of the independent variable on relief from depression. In this example, relief from depression is called a **dependent variable**. In general, the independent variable is manipulated by the experimenter, and its effects on the dependent variable are measured.

## Try It

1) Can blueberries slow aging? A study indicates that antioxidants found in blueberries may slow down the process of aging. In this study, **19**-month-old rats (equivalent to **60**-year-old humans) were fed either their standard diet or a diet supplemented by either blueberry, strawberry, or spinach powder. After eight weeks, the rats were given memory and motor skills tests. Although all supplemented rats showed improvement, those supplemented with blueberry powder showed the most notable improvement.

- a. What is the independent variable?
- b. What are the dependent variables?

### Solution

- a. Dietary supplement: none, blueberry, strawberry, and spinach
- b. Memory test and motor skills test

## Try It

2) Does beta-carotene protect against cancer? Beta-carotene supplements have been thought to protect against cancer. However, a study published in the Journal of the National Cancer Institute suggests this is false. The study was conducted with **39,000** women aged **45** and over. These women were randomly assigned to receive a beta-carotene supplement or a placebo, and their health was studied over their lifetime. Cancer rates for women taking the beta-carotene supplement did not differ systematically from the cancer rates of those women taking the placebo.

- a. What is the independent variable?
- b. What is the dependent variable?

**Solution**

- a. Supplements: beta-carotene or placebo
- b. Occurrence of cancer

## Try It

3) How bright is right? An automobile manufacturer wants to know how bright brake lights should be in order to minimize the time required for the driver of a following car to realize that the car in front is stopping and to hit the brakes.

- a. What is the independent variable?
- b. What is the dependent variable?

**Solution**

- a. Brightness of brake lights
- b. Time to hit brakes

## Levels of an Independent Variable

If an experiment compares an **experimental** treatment with a **control** treatment, then the independent variable (type of treatment) has two levels: experimental and control. If an experiment were comparing five types of diets, then the independent variable (type of diet) would have 5 levels. In general, the number of levels of an independent variable is the number of experimental conditions.

## Qualitative and Quantitative Variables

An important distinction between variables is between **qualitative variables** and **quantitative variables**. Qualitative variables are those that express a qualitative attribute such as hair colour, eye colour, religion,

favourite movie, gender, and so on. The values of a qualitative variable do not imply a numerical ordering. Values of the variable “religion” differ qualitatively; no ordering of religions is implied. Qualitative variables are sometimes referred to as categorical or nominal variables. Quantitative variables are those variables that are measured in terms of numbers. Some examples of quantitative variables are height, weight, and shoe size.

In the study on the effect of diet discussed previously, the independent variable was type of supplement: none, strawberry, blueberry, and spinach. The variable “type of supplement” is a qualitative variable; there is nothing quantitative about it. In contrast, the dependent variable “memory test” is a quantitative variable since memory performance was measured on a quantitative scale (number correct).

## Discrete and Continuous Variables

Variables such as number of children in a household are called **discrete variables** since the possible scores are discrete points on the scale. For example, a household could have three children or six children, but not 4.53 children. Other variables such as time to respond to a question are continuous variables since the scale is continuous and not made up of discrete steps. The response time could be 1.64 seconds, or it could be 1.64237123922121 seconds. Of course, the practicalities of measurement preclude most measured variables from being truly continuous.

## Levels of Measurement

Before we can conduct a statistical analysis, we need to measure our dependent variable. Exactly how the measurement is carried out depends on the type of variable involved in the analysis. Different types are measured differently. To measure the time taken to respond to a stimulus, you might use a stop watch. Stop watches are of no use, of course, when it comes to measuring someone’s attitude toward a political candidate. A rating scale is more appropriate in this case (with labels like “very favourable,” “somewhat favourable,” etc.). For a dependent variable such as favourite colour, you can simply note the colour-word (like “red”) that the subject offers.

Although procedures for measurement differ in many ways, they can be classified using a few fundamental categories. In a given category, all of the procedures share some properties that are important for you to know about. The categories are called “scale types,” or just “scales,” and are described in this section.

## Nominal Scales

When measuring using a **nominal scale**, one simply names or categorizes responses. Gender, handedness, favourite colour, and religion are examples of variables measured on a nominal scale. The essential point about nominal scales is that they do not imply any ordering among the responses. For example, when classifying

people according to their favourite colour, there is no sense in which green is placed “ahead of” blue. Responses are merely categorized. Nominal scales embody the lowest level of measurement.

## Ordinal Scales

A researcher wishing to measure consumers’ satisfaction with their microwave ovens might ask them to specify their feelings as either “very dissatisfied,” “somewhat dissatisfied,” “somewhat satisfied,” or “very satisfied.” The items in this scale are ordered, ranging from least to most satisfied. This is what distinguishes ordinal from nominal scales. Unlike a nominal scale, an **ordinal scale** allows a comparison of the degree to which two subjects possess the dependent variable. For example, our satisfaction ordering makes it meaningful to assert that one person is more satisfied than another with their microwave oven. Such an assertion reflects the first person’s use of a verbal label that comes later in the list than the label chosen by the second person.

On the other hand, ordinal scales fail to capture important information that will be present in the other scales we examine. In particular, the difference between two levels of an ordinal scale cannot be assumed to be the same as the difference between two other levels. In our satisfaction scale, for example, the difference between the responses “very dissatisfied” and “somewhat dissatisfied” is probably not equivalent to the difference between “somewhat dissatisfied” and “somewhat satisfied.” Nothing in our measurement procedure allows us to determine whether the two differences reflect the same difference in psychological satisfaction. Statisticians express this point by saying that the differences between adjacent scale values do not necessarily represent equal intervals on the underlying scale giving rise to the measurements. (In our case, the underlying scale is the true feeling of satisfaction, which we are trying to measure.)

What if the researcher had measured satisfaction by asking consumers to indicate their level of satisfaction by choosing a number from 1 to 4? Would the difference between the responses of 1 and 2 necessarily reflect the same difference in satisfaction as the difference between the responses 2 and 3? The answer is no. Changing the response format to numbers does not change the meaning of the scale. We still are in no position to assert that the mental step from 1 to 2 (for example) is the same as the mental step from 3 to 4.

## Interval Scales

An **interval scale** is a numerical scale in which intervals have the same interpretation throughout. As an example, consider the Fahrenheit scale of temperature. The difference between 30 degrees and 40 degrees represents the same temperature difference as the difference between 80 degrees and 90 degrees. This is because each 10-degree interval has the same physical meaning (in terms of the kinetic energy of molecules).

Interval scales are not perfect, however. In particular, they do not have a true zero point even if one of the scaled values happens to carry the name “zero.” The Fahrenheit scale illustrates the issue. Zero degrees Fahrenheit does not represent the complete absence of temperature (i.e., the absence of any molecular kinetic energy). In reality, the label “zero” is applied to its temperature for quite accidental reasons connected to the



history of temperature measurement. Since an interval scale has no true zero point, it does not make sense to compute ratios of temperatures. For example, there is no sense in which the ratio of 40 to 20 degrees Fahrenheit is the same as the ratio of 100 to 50 degrees; no interesting physical property is preserved across the two ratios. After all, if the “zero” label were applied at the temperature that Fahrenheit happens to label as 10 degrees, the two ratios would instead be 30 to 10 and 90 to 40, no longer the same! For this reason, it does not make sense to say that 80 degrees is “twice as hot” as 40 degrees. Such a claim would depend on an arbitrary decision about where to “start” the temperature scale, namely, what temperature to call zero (whereas the claim is intended to make a more fundamental assertion about the underlying physical reality).

## Ratio Scales

The **ratio scale** of measurement is the most informative scale. It is an interval scale with the additional property that its zero position indicates the absence of the quantity being measured. You can think of a ratio scale as the three earlier scales rolled up in one. Like a nominal scale, it provides a name or category for each object (the numbers serve as labels). Like an ordinal scale, the objects are ordered (in terms of the ordering of the numbers). Like an interval scale, the same difference at two places on the scale has the same meaning. And in addition, the same ratio at two places on the scale also carries the same meaning.

The Fahrenheit scale for temperature has an arbitrary zero point and is therefore not a ratio scale. However, zero on the Kelvin scale is absolute zero. This makes the Kelvin scale a ratio scale. For example, if one temperature is twice as high as another as measured on the Kelvin scale, then it has twice the kinetic energy of the other temperature.

Another example of a ratio scale is the amount of money you have in your pocket right now (25 cents, 55 cents, etc.). Money is measured on a ratio scale because, in addition to having the properties of an interval scale, it has a true zero point: if you have zero money, this implies the absence of money. Since money has a true zero point, it makes sense to say that someone with 50 cents has twice as much money as someone with 25 cents (or that Bill Gates has a million times more money than you do).

## What Level of Measurement Is Used for Psychological Variables?

Rating scales are used frequently in psychological research. For example, experimental subjects may be asked to rate their level of pain, how much they like a consumer product, their attitudes about capital punishment, or their confidence in an answer to a test question. Typically these ratings are made on a 5-point or a 7-point scale. These scales are ordinal scales since there is no assurance that a given difference represents the same thing across the range of the scale. For example, there is no way to be sure that a treatment that reduces pain from a

rated pain level of **3** to a rated pain level of **2** represents the same level of relief as a treatment that reduces pain from a rated pain level of **7** to a rated pain level of **6**.

In memory experiments, the dependent variable is often the number of items correctly recalled. What scale of measurement is this? You could reasonably argue that it is a ratio scale. First, there is a true zero point; some subjects may get no items correct at all. Moreover, a difference of one represents a difference of one item recalled across the entire scale. It is certainly valid to say that someone who recalled **12** items recalled twice as many items as someone who recalled only **6** items.

But number-of-items recalled is a more complicated case than it appears at first. Consider the following example in which subjects are asked to remember as many items as possible from a list of ten. Assume that (a) there are five easy items and five difficult items, (b) half of the subjects are able to recall all the easy items and different numbers of difficult items, whereas (c) the other half of the subjects are unable to recall any of the difficult items but they do remember different numbers of easy items. Some sample data are shown in the following table.

Subject	Easy Items					Difficult Items					Score
Subject A	0	0	1	1	0	0	0	0	0	0	2
Subject B	1	0	1	1	0	0	0	0	0	0	3
Subject C	1	1	1	1	1	1	1	0	0	0	7
Subject D	1	1	1	1	1	0	1	1	0	1	8

Let's compare firstly, the difference between Subject A's score of **2** and Subject B's score of **3** and secondly, the difference between Subject C's score of **7** and Subject D's score of **8**. The former difference is a difference of one easy item; the latter difference is a difference of one difficult item. Do these two differences necessarily signify the same difference in memory? We are inclined to respond "No" to this question since only a little more memory may be needed to retain the additional easy item whereas a lot more memory may be needed to retain the additional hard item. The general point is that it is often inappropriate to consider psychological measurement scales as either interval or ratio.

## Consequences of Level of Measurement

Why are we so interested in the type of scale that measures a dependent variable? The crux of the matter is the relationship between the variable's level of measurement and the statistics that can be meaningfully computed

with that variable. For example, consider a hypothetical study in which five children are asked to choose their favourite colour from blue, red, yellow, green, and purple. The researcher codes the results as follows:

Colour	Code
Blue	1
Red	2
Yellow	3
Green	4
Purple	5

This means that if a child said her favourite colour was “Red,” then the choice was coded as “2,” if the child said her favourite colour was “Purple,” then the response was coded as 5, and so forth. Consider the following hypothetical data:

Subject	Colour	Code
Subject 1	Blue	1
Subject 2	Blue	1
Subject 3	Green	4
Subject 4	Green	4
Subject 5	Purple	5

Each code is a number, so nothing prevents us from computing the average code assigned to the children. The average happens to be 3, but you can see that it would be senseless to conclude that the average favourite colour is yellow (the colour with a code of 3). Such nonsense arises because favourite colour is a nominal scale, and taking the average of its numerical labels is like counting the number of letters in the name of a snake to see how long the beast is.

Does it make sense to compute the mean of numbers measured on an ordinal scale? This is a difficult question, one that statisticians have debated for decades. The prevailing (but by no means unanimous) opinion of statisticians is that for almost all practical situations, the mean of an ordinal variable is a meaningful statistic. However, there are extreme situations in which computing the mean of an ordinal variable can be very misleading.

## Collecting Data

We are usually interested in understanding a specific group of people. This group is known as the population of interest, or simply the **population**. The population is the collection of all people who have some characteristic in common; it can be as broad as “all people” if we have a very general research question about human psychology, or it can be extremely narrow, such as “all freshmen psychology majors at Midwestern public universities” if we have a specific group in mind.

## Populations and Samples

In statistics, we often rely on a **sample**—that is, a small subset of a larger set of data—to draw inferences about the larger set. The larger set is known as the population from which the sample is drawn.

### Try It

4) You have been hired by the National Election Commission to examine how the American people feel about the fairness of the voting procedures in the U.S. Whom will you ask?

#### **Solution**

It is not practical to ask every single American how he or she feels about the fairness of the voting procedures. Instead, we query a relatively small number of Americans and draw inferences about the entire country from their responses. The Americans actually queried constitute our sample of the larger population of all Americans.

A sample is typically a small subset of the population. In the case of voting attitudes, we would sample a few thousand Americans drawn from the hundreds of millions that make up the country. In choosing a sample, it is therefore crucial that it not over-represent one kind of citizen at the

expense of others. For example, something would be wrong with our sample if it happened to be made up entirely of Florida residents. If the sample held only Floridians, it could not be used to infer the attitudes of other Americans. The same problem would arise if the sample were comprised only of Republicans. Inferences from statistics are based on the assumption that sampling is representative of the population. If the sample is not representative, then the possibility of **sampling bias** occurs. Sampling bias means that our conclusions apply only to our sample and are not generalizable to the full population.

## Try It

5) We are interested in examining how many math classes have been taken, on average, by current graduating seniors at American colleges and universities during their four years in school.

### Solution

Whereas our population in the last example included all U.S. citizens, now it involves just the graduating seniors throughout the country. This is still a large set since there are thousands of colleges and universities, each enrolling many students. (New York University, for example, enrolls **48,000** students.) It would be prohibitively costly to examine the transcript of every college senior. We therefore take a sample of college seniors and then make inferences to the entire population based on what we find. To make the sample, we might first choose some public and private colleges and universities across the United States. Then we might sample **50** students from each of these institutions. Suppose that the average number of math classes taken by the people in our sample were **3.2**. We might speculate that **3.2** approximates the number we would find if we had the resources to examine every senior in the entire population. But we must be careful about the possibility that our sample is non-representative of the population. Perhaps we chose an overabundance of math majors, or chose too many technical institutions that have heavy math requirements. Such bad sampling makes our sample unrepresentative of the population of all seniors.

To solidify your understanding of sampling bias, consider the following examples. Try to identify the population and the sample, and then reflect on whether the sample is likely to yield the information desired.

## Try It

6) A substitute teacher wants to know how students in the class did on their last test. The teacher asks the ten students sitting in the front row to state their latest test score. He concludes from their report that the class did extremely well. What is the sample? What is the population? Can you identify any problems with choosing the sample in the way that the teacher did?

### **Solution**

The population consists of all students in the class. The sample is made up of just the ten students sitting in the front row. The sample is not likely to be representative of the population. Those who sit in the front row tend to be more interested in the class and tend to perform higher on tests. Hence, the sample may perform at a higher level than the population.

## Try It

7) A coach is interested in how many cartwheels the average college freshmen at his university can do. Eight volunteers from the freshman class step forward. After observing their performance, the coach concludes that college freshmen can do an average of **16** cartwheels in a row without stopping.

### **Solution**

The population is the class of all freshmen at the coach's university. The sample is composed of the **8** volunteers. The sample is poorly chosen because volunteers are more likely to be able to do cartwheels than the average freshman; people who can't do cartwheels probably did not volunteer! In the example, we are also not told of the gender of the volunteers. Were they all women, for example? That might affect the outcome, contributing to the non-representative nature of the sample (if the school is co-ed).

## Simple Random Sampling

Researchers adopt a variety of sampling strategies. The most straightforward is **simple random sampling**. Such sampling requires every member of the population to have an equal chance of being selected into the sample. In addition, the selection of one member must be independent of the selection of every other member. That is, picking one member from the population must not increase or decrease the probability of picking any other member (relative to the others). In this sense, we can say that simple random sampling chooses a sample by pure chance. To check your understanding of simple random sampling, consider the following example. What is the population? What is the sample? Was the sample picked by simple random sampling? Is it biased?

### Try It

8) A research scientist is interested in studying the experiences of twins raised together versus those raised apart. She obtains a list of twins from the National Twin Registry, and selects two subsets of individuals for her study. First, she chooses all those in the registry whose last name begins with Z. Then she turns to all those whose last name begins with B. Because there are so many names that start with B, however, our researcher decides to incorporate only every other name into her sample. Finally, she mails out a survey and compares characteristics of twins raised apart versus together.

#### Solution

The population consists of all twins recorded in the National Twin Registry. It is important that the researcher only make statistical generalizations to the twins on this list, not to all twins in the nation or world. That is, the National Twin Registry may not be representative of all twins. Even if inferences are limited to the Registry, a number of problems affect the sampling procedure we described. For instance, choosing only twins whose last names begin with Z does not give every individual an equal chance of being selected into the sample. Moreover, such a procedure risks over-representing ethnic groups with many surnames that begin with Z. There are other reasons why choosing just the Zs may bias the sample.

Perhaps such people are more patient than average because they often find themselves at the end of the line! The same problem occurs with choosing twins whose last name begins with B. An additional problem for the

Bs is that the every-other-one procedure disallowed adjacent names on the B part of the list from being both selected. Just this defect alone means the sample was not formed through simple random sampling.

## Sample Size Matters

Recall that the definition of a random sample is a sample in which every member of the population has an equal chance of being selected. This means that the sampling procedure rather than the results of the procedure define what it means for a sample to be random. Random samples, especially if the sample size is small, are not necessarily representative of the entire population. For example, if a random sample of 20 subjects were taken from a population with an equal number of males and females, there would be a nontrivial probability (.06) that 70% or more of the sample would be female. Such a sample would not be representative, although it would be drawn randomly. Only a large sample size makes it likely that our sample is close to representative of the population. For this reason, **inferential statistics** take into account the sample size when generalizing results from samples to populations. In later chapters, you'll see what kinds of mathematical techniques ensure this sensitivity to sample size.

## More Complex Sampling

Sometimes it is not feasible to build a sample using simple random sampling. To see the problem, consider the fact that both Dallas and Houston competed to be hosts of the 2012 Olympics. Imagine that you had been hired to assess whether most Texans preferred Houston to Dallas as the host, or the reverse. Given the impracticality of obtaining the opinion of every single Texan, you had to construct a sample of the Texas population. But notice how difficult it would have been to proceed by simple random sampling. For example, how would you have contacted those individuals who didn't vote and didn't have a phone? Even among people you found in the telephone book, how could you have identified those who had just relocated to another state (and had no reason to inform you of their move)? What would you have done about the fact that since the beginning of the study, an additional 4, 212 people took up residence in the state of Texas? As you can see, it is sometimes very difficult to develop a truly random procedure. For this reason, other kinds of sampling techniques have been devised. We now discuss two of them.

## Stratified Sampling

Since simple random sampling often does not ensure a representative sample, a sampling method called **stratified random sampling** is sometimes used to make the sample more representative of the population. This method can be used if the population has a number of distinct "strata" or groups. In stratified sampling, you first identify members of your sample who belong to each group. Then you randomly sample from each



of those subgroups in such a way that the sizes of the subgroups in the sample are proportional to their sizes in the population.

Let's take an example: Suppose you were interested in views of capital punishment at an urban university. You have the time and resources to interview 200 students. The student body is diverse with respect to age; many older people work during the day and enroll in night courses (average age is 39), while younger students generally enroll in day classes (average age of 19). It is possible that night students have different views about capital punishment than day students. If 70% of the students were day students, it makes sense to ensure that 70% of the sample consisted of day students. Thus, your sample of 200 students would consist of 140 day students and 60 night students. The proportion of day students in the sample and in the population (the entire university) would be the same. Inferences to the entire population of students at the university would therefore be more secure.

## Convenience Sampling

Not all sampling methods are perfect, and sometimes that's okay. For example, if we are beginning research into a completely unstudied area, we may sometimes take some shortcuts to quickly gather data and get a general idea of how things work before fully investing a lot of time and money into well-designed research projects with proper sampling. This is known as **convenience sampling**, named for its ease of use. In limited cases, such as the one just described, convenience sampling is okay because we intend to follow up with a representative sample. Unfortunately, sometimes convenience sampling is used due only to its convenience without the intent of improving on it in future work.

## Types of Research Designs

Research studies come in many forms, and, just like with the different types of data we have, different types of studies tell us different things. The choice of research design is determined by the research question and the logistics involved. Though a complete understanding of different research designs is the subject for at least one full class, if not more, a basic understanding of the principles is useful here. There are three types of research designs we will discuss: experimental, quasi-experimental, and non-experimental.

## Experimental Designs

If we want to know if a change in one variable causes a change in another variable, we must use a true experiment. **Experimental research** is defined by the use of random assignment to treatment conditions and manipulation of the independent variable. To understand what this means, let's look at an example:

A clinical researcher wants to know if a newly developed drug is effective in treating the flu. Working with collaborators at several local hospitals, she randomly samples 40 flu patients and randomly assigns each one

to one of two conditions: Group A receives the new drug, and Group B receives a placebo. She measures the symptoms of all participants after one week to see if there is a difference in symptoms between the groups.

In the example, the independent variable is the drug treatment; we manipulate it into two levels: new drug or placebo. Without the researcher administering the drug (i.e., manipulating the independent variable), there would be no difference between the groups. Each person, after being randomly sampled to be in the research, was then randomly assigned to one of the two groups. That is, random sampling and random assignment are not the same thing and cannot be used interchangeably. For research to be a true experiment, random assignment must be used. For research to be representative of the population, random sampling must be used. The use of both techniques helps ensure that there are no systematic differences between the groups, thus eliminating the potential for sampling bias.

The dependent variable in the example is flu symptoms. Barring any other intervention, we would assume that people in both groups, on average, get better at roughly the same rate. Because there are no systematic differences between the two groups, if the researcher does find a difference in symptoms, she can confidently attribute it to the effectiveness of the new drug.

## Quasi-Experimental Designs

**Quasi-experimental research** involves getting as close as possible to the conditions of a true experiment when we cannot meet all requirements. Specifically, a quasi-experiment involves manipulating the independent variable but not randomly assigning people to groups. There are several reasons this might be used. First, it may be unethical to deny potential treatment to someone if there is good reason to believe it will be effective and that the person would unduly suffer if they did not receive it. Alternatively, it may be impossible to randomly assign people to groups. Consider the following example:

A professor wants to test out a new teaching method to see if it improves student learning. Because he is teaching two sections of the same course, he decides to teach one section the traditional way and the other section using the new method. At the end of the semester, he compares the grades on the final for each class to see if there is a difference.

In this example, the professor has manipulated his teaching method, which is the independent variable, hoping to find a difference in student performance, the dependent variable. However, because students enroll in courses, he cannot randomly assign the students to a particular group, thus precluding using a true experiment to answer his research question. Because of this, we cannot know for sure that there are no systematic differences between the classes other than teaching style and therefore cannot determine causality.

## Non-Experimental Designs

Finally, **non-experimental research** (sometimes called correlational research) involves observing things as they occur naturally and recording our observations as data. Consider this example:

A data scientist wants to know if there is a relationship between how conscientious a person is and whether that person is a good employee. She hopes to use this information to predict the job performance of future employees by measuring their personality when they are still job applicants. She randomly samples volunteer employees from several different companies, measuring their conscientiousness and having their bosses rate their performance on the job. She analyzes this data to find a relationship.

Here, it is not possible to manipulate conscientiousness, so the researcher must gather data from employees as they are in order to find a relationship between her variables. Although this technique cannot establish causality, it can still be quite useful. If the relationship between conscientiousness and job performance is consistent, then it doesn't necessarily matter if conscientiousness causes good performance or if they are both caused by something else—she can still measure conscientiousness to predict future performance. Additionally, these studies have the benefit of reflecting reality as it actually exists since we as researchers do not change anything.

## Types of Statistical Analyses

Now that we understand the nature of our data, let's turn to the types of statistics we can use to interpret them. There are two types of statistics: descriptive and inferential.

### Descriptive Statistics

**Descriptive statistics** are numbers that are used to summarize and describe data. The word “data” refers to the information that has been collected from an experiment, a survey, a historical record, etc. (By the way, data is plural. One piece of information is called a datum.) If we are analyzing birth certificates, for example, a descriptive statistic might be the percentage of certificates issued in New York State, or the average age of the mother. Any other number we choose to compute also counts as a descriptive statistic for the data from which the statistic is computed. Several descriptive statistics are often used at one time to give a full picture of the data.

Descriptive statistics are just descriptive. They do not involve generalizing beyond the data at hand. Generalizing from our data to another set of cases is the business of inferential statistics, which you'll be studying in another section. Here we focus on (mere) descriptive statistics.

Some descriptive statistics are shown in Table 6.1.1. The table shows the average salaries for various occupations in the United States in 1999. Descriptive statistics like these offer insight into American society.

It is interesting to note, for example, that we pay the people who educate our children and who protect our citizens a great deal less than we pay people who take care of our feet or our teeth.

**Table 6.1.1. Average salaries for various U.S. occupations in 1999.**

<b>Occupation</b>	<b>Salary</b>
Pediatricians	\$112,760
Dentists	\$106,130
Podiatrists	\$100,090
Physicists	\$76,140
Architects	\$53,410
School, clinical, and counselling psychologists	\$49,720
Flight attendants	\$47,910
Elementary school teachers	\$39,560
Police officers	\$38,710
Floral designers	\$18,980

For more descriptive statistics, consider Table 6.1.2. It shows the number of unmarried men per 100 unmarried women in U.S. metro areas in 1990. From this table we see that men outnumber women most in Jacksonville, North Carolina, and women outnumber men most in Sarasota, Florida. You can see that descriptive statistics can be useful if we are looking for an opposite-sex partner! (These data come from the Information Please Almanac.)

**Table 6.1.2. Number of unmarried men per 100 unmarried women in U.S. metro areas in 1990. Note: Unmarried includes never-married, widowed, and divorced persons, 15 years or older.**

Cities with Mostly Men	Men per 100 Women	Cities with Mostly Women	Men per 100 Women
1. Jacksonville, North Carolina	224	1. Sarasota, Florida	66
2. Killeen–Temple, Texas	123	2. Bradenton, Florida	68
3. Fayetteville, North Carolina	118	3. Altoona, Pennsylvania	69
4. Brazoria, Texas	117	4. Springfield, Illinois	70
5. Lawton, Oklahoma	116	5. Jacksonville, Tennessee	70
6. State College, Pennsylvania	113	6. Gadsden, Alabama	70
7. Clarksville–Hopkinsville, Tennessee–Kentucky	113	7. Wheeling, West Virginia–Ohio	70
8. Anchorage, Alaska	112	8. Charleston, West Virginia	71
9. Salinas–Seaside–Monterey, California	112	9. St. Joseph, Missouri	71
10. Bryan–College Station, Texas	111	10. Lynchburg, Virginia	71

These descriptive statistics may make us ponder why the numbers are so disparate in these cities. One potential explanation, for instance, as to why there are more women in Florida than men may involve the fact that elderly individuals tend to move down to the Sarasota region and that women tend to outlive men. Thus, more women might live in Sarasota than men. However, in the absence of proper data, this is only speculation.

You probably know that descriptive statistics are central to the world of sports. Every sporting event produces numerous statistics, such as the shooting percentage of players on a basketball team. For the Olympic marathon (a foot race of **26.2** miles), we possess data that cover more than a century of competition. (The first modern Olympics took place in 1896.) Table 6.1.3 and Table 6.1.4 show the winning times for women and men, respectively. (Women have only been allowed to compete since 1984.)

**Table 6.1.3. Women's winning Olympic marathon times, 1984–2004.**

<b>Year</b>	<b>Winner</b>	<b>Country</b>	<b>Time</b>
1984	Joan Benoit	United States	2 : 24 : 52
1988	Rosa Mota	Portugal	2 : 25 : 40
1992	Valentina Yegorova	Unified Team	2 : 32 : 41
1996	Fatuma Roba	Ethiopia	2 : 26 : 05
2000	Naoko Takahashi	Japan	2 : 23 : 14
2004	Mizuki Noguchi	Japan	2 : 26 : 20

**Table 6.1.4. Men's winning Olympic marathon times, 1896–2004.**

Year	Winner	Country	Time
1896	Spyridon Louis	Greece	2 : 58 : 50
1900	Michel Théato	France	2 : 59 : 45
1904	Thomas Hicks	United States	3 : 28 : 53
1906	Billy Sherring	Canada	2 : 51 : 23
1908	Johnny Hayes	United States	2 : 55 : 18
1912	Kenneth McArthur	South Africa	2 : 36 : 54
1920	Hannes Kolehmainen	Finland	2 : 32 : 35
1924	Albin Stenroos	Finland	2 : 41 : 22
1928	Boughera El Ouafi	France	2 : 32 : 57
1932	Juan Carlos Zabala	Argentina	2 : 31 : 36
1936	Sohn Kee-chung	Japan	2 : 29 : 19
1948	Delfo Cabrera	Argentina	2 : 34 : 51
1952	Emil Zátopek	Czechoslovakia	2 : 23 : 03
1956	Alain Mimoun	France	2 : 25 : 00
1960	Abebe Bikila	Ethiopia	2 : 15 : 16
1964	Abebe Bikila	Ethiopia	2 : 12 : 11
1968	Mamo Wolde	Ethiopia	2 : 20 : 26

Year	Winner	Country	Time
1972	Frank Shorter	United States	2 : 12 : 19
1976	Waldemar Cierpinski	East Germany	2 : 09 : 55
1980	Waldemar Cierpinski	East Germany	2 : 11 : 03
1984	Carlos Lopes	Portugal	2 : 09 : 21
1988	Gelindo Bordin	Italy	2 : 10 : 32
1992	Hwang Young-cho	South Korea	2 : 13 : 23
1996	Josia Thugwane	South Africa	2 : 12 : 36
2000	Gezahegne Abera	Ethiopia	2 : 10.10
2004	Stefano Baldini	Italy	2 : 10 : 55

There are many descriptive statistics that we can compute from the data in these tables. To gain insight into the improvement in speed over the years, let us divide the men’s times into two pieces, namely, the first 13 races (up to 1952) and the second 13 (starting from 1956). The mean winning time for the first 13 races is 2 hours, 44 minutes, and 22 seconds (written 2 : 44 : 22). The mean winning time for the second 13 races is 2 : 13 : 18. This is quite a difference (over half an hour). Does this prove that the fastest men are running faster? Or is the difference just due to chance, no more than what often emerges from chance differences in performance from year to year? We can’t answer this question with descriptive statistics alone. All we can affirm is that the two means are “suggestive.”

Examining Table 6.1.3 and Table 6.1.4 leads to many other questions. We note that Takahashi (the lead female runner in 2000) would have beaten the male runner in 1956 and all male runners in the first 12 marathons. This fact leads us to ask whether the gender gap will close or remain constant. When we look at the times within each gender, we also wonder how far they will decrease (if at all) in the next century of the Olympics. Might we one day witness a sub-2 hour marathon? The study of statistics can help you make reasonable guesses about the answers to these questions.

It is also important to differentiate what we use to describe populations vs. what we use to describe samples. A population is described by a parameter; the parameter is the true value of the descriptive in the population,



but one that we can never know for sure. For example, the Bureau of Labor Statistics reports that the average hourly wage of chefs is \$23.87. However, even if this number were computed using information from every single chef in the United States (making it a parameter), it would quickly become slightly off as one chef retires and a new chef enters the job market. Additionally, as noted above, there is virtually no way to collect data from every single person in a population. In order to understand a variable, we estimate the population parameter using a sample statistic. Here, the term statistic refers to the specific number we compute from the data (e.g., the average), not the field of statistics. A sample statistic is an estimate of the true population parameter, and if our sample is representative of the population, then the statistic is considered to be a good estimator of the parameter.

Even the best sample will be somewhat off from the full population, earlier referred to as sampling bias, and as a result, there will always be a tiny discrepancy between the parameter and the statistic we use to estimate it. This difference is known as **sampling error**, and, as we will see throughout the course, understanding sampling error is the key to understanding statistics. Every observation we make about a variable, be it a full research study or observing an individual's behaviour, is incapable of being completely representative of all possibilities for that variable. Knowing where to draw the line between an unusual observation and a true difference is what statistics is all about.

## Inferential Statistics

Descriptive statistics are wonderful at telling us what our data look like. However, what we often want to understand is how our data behave. What variables are related to other variables? Under what conditions will the value of a variable change? Are two groups different from each other, and if so, are people within each group different or similar? These are the questions answered by inferential statistics, and inferential statistics are how we generalize from our sample back up to our population.

For example, we will learn how to use a t-statistic to determine whether people change over time when enrolled in an intervention. We will also use an F-statistic to determine if we can predict future values on a variable based on current known values of a variable. There are many types of inferential statistics, each allowing us insight into a different behaviour of the data we collect. This course will only touch on a small subset (or a sample) of them, but the principles we learn along the way will make it easier to learn new tests, as most inferential statistics follow the same structure and format.

## A Note about Statistical Software

Many pieces of technology support statistical analysis and quantitative data analysis done by psychologists, health care professionals and scientists. Commonly used technologies include the proprietary Statistical Package for the Social Sciences (SPSS), the free and open-source tool JASP, and the programming language R.

Several of the figures used in this text were generated using JASP, but providing an overview or introduction to these technologies is outside the scope of this work. Instruction manuals can be found on the JASP website.

## Mathematical Notation

As noted earlier, statistics is not math. It does, however, use math as a tool. Many statistical formulas involve summing numbers. Fortunately, there is a convenient notation for expressing summation. This section covers the basics of this summation notation.

Let's say we have a variable  $X$  that represents the weights (in grams) of 4 grapes:

Grape	$X$
Grape 1	4.6
Grape 2	5.1
Grape 3	4.9
Grape 4	4.4

We label the weight of Grape 1 as  $X_1$ , of Grape 2 as  $X_2$ , etc. The following formula means to sum up the weights of the four grapes:

$$\sum_{i=1}^4 X_i$$

The Greek letter  $\sum$  indicates summation. The " $i = 1$ " at the bottom indicates that the summation is to start with  $X_1$ , and the 4 at the top indicates that the summation will end with  $X_4$ . The " $X_i$ " indicates that  $X$  is the variable to be summed as  $i$  goes from 1 to 4. Therefore,

$$\sum_{i=1}^4 X_i = X_1 + X_2 + X_3 + X_4 = 4.6 + 5.1 + 4.9 + 4.4 = 19$$

The symbol

$$\sum_{i=1}^3 X_i$$

indicates that only the first 3 scores are to be summed. The index variable  $i$  goes from 1 to 3.

When all the scores of a variable (such as  $X$ ) are to be summed, it is often convenient to use the following abbreviated notation:

$$\sum X$$

Thus, when no values of  $i$  are shown, it means to sum all the values of  $X$ .

Many formulas involve squaring numbers before they are summed. This is indicated as

$$\sum X^2 = 4.6^2 + 5.1^2 + 4.9^2 + 4.4^2 = 21.16 + 26.01 + 24.01 + 19.36 = 90.54$$

Notice that:

$$\left(\sum X\right)^2 \neq \sum X^2$$

because the expression on the left means to sum up all the values of  $X$  and then square the sum ( $19^2 = 361$ ), whereas the expression on the right means to square the numbers and then sum the squares ( $90.54$ , as shown).

Some formulas involve the sum of cross products. Below are the data for variables  $X$  and  $Y$ . The cross products ( $XY$ ) are shown in the third column. The sum of the cross products is  $3 + 4 + 21 = 28$ .

$X$	$Y$	$XY$
1	3	3
2	2	4
3	7	21

In summation notation, this is written as:

$$\sum XY = 28$$

### Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

*<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=2130#h5p-38>*

b) After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

## Glossary

### **continuous variables**

Numerical variables that can take on any value in a certain range. Time and distance are continuous; gender, SAT score, and “time rounded to the nearest second” are not.

### **control (group)**

The group in an experimental study that is not receiving the treatment being tested.

### **convenience sampling**

A sampling strategy in which participants are recruited for their easy availability (e.g., college students). A sample obtained through convenience sampling should not be considered a representative sample.

### **data**

A collection of values to be used for statistical analysis. Data is the plural form of datum.

### **dependent variable**

A variable that measures the experimental outcome. In most experiments, the effects of the independent variable on the dependent variables are observed. For example, if a study investigated the effectiveness of an experimental treatment for depression, then the measure of depression would be the dependent variable.

**descriptive statistics**

A set of statistics—such as the mean, standard deviation, and skew—that describe a distribution.

**discrete variables**

A variable that exists in indivisible units. For quantitative variables, it is measured in whole numbers that are discrete points on the scale.

**experimental (group)**

The group in a study that is receiving the treatment being tested.

**experimental research**

Research that involves the use of random assignment to treatment conditions and manipulation of the independent variable.

**independent variable**

A variable that is manipulated by the experimenter, as opposed to a dependent variable. Most experiments consist of observing the effect of the independent variable(s) on the dependent variable(s).

**inferential statistics**

The branch of statistics concerned with drawing conclusions about a population from a sample.

**interval scale**

A numerical scale in which the distance between scores on the scale is consistent (equal) and for which the zero is relative (rather than absolute)

**nominal scale**

A scale in which no ordering is implied, and addition/subtraction and multiplication/division would be inappropriate for a variable. Variables measured on a nominal scale have no natural ordering, even if they are coded using numbers (e.g., for eye color 1 = blue, 2 = brown, 3 = hazel, etc.).

**non-experimental research**

Research that involves observing things as they occur naturally and recording observations as data. Also known as correlational research.

**ordinal scale**

A set of ordered values in which there is no set distance between scale values; for example, asking someone to indicate how much education they completed by asking them to circle one of the following: did not complete high school, high school diploma, some college, college degree, professional degree.

**population**

The complete set of observations a researcher is interested in. Contrast this with a sample which is a subset of a population. Inferential statistics are computed from sample data in order

to make inferences about the population.

### **qualitative variables**

Also known as categorical variables, qualitative variables cannot be quantified, or measured numerically. Instead, they are measured on a nominal or **ordinal scale**. Variables that are not qualitative are known as quantitative variables.

### **quantitative variables**

Variables that are measured on a numeric or quantitative scale or that can be ordered in some fashion. Ordinal, interval, and ratio scales are quantitative. A country's population, a person's shoe size, or a car's speed are all quantitative variables. Variables that are not quantitative are known as qualitative variables.

### **quasi-experimental research**

Research that involves manipulating the independent variable but not randomly assigning people to groups.

### **ratio scale**

A numerical scale in which the distance between scores on the scale is consistent (equal) and for which the zero is relative (rather than absolute).

### **sample**

A subset of a population, often taken for the purpose of statistical inference.

### **sampling bias**

Sampling bias occurs when participants are not selected at random or when they have an unequal probability of being selected for participation in a study.

### **sampling error**

The discrepancy between a parameter and the statistic used to estimate it.

### **simple random sampling**

A process of selecting a subset of a population for the purposes of statistical inference in which every member of the population is equally likely to be chosen.

### **statistics**

A range of techniques and procedures for analyzing, interpreting, displaying, and making decisions based on sample data.

### **stratified random sampling**

In stratified random sampling, the population is divided into a number of subgroups (or strata). Random samples are then taken from each subgroup with sample sizes proportional to the size of the subgroup in the population.

### **variable**

Something that can take on different values. For example, different subjects in an experiment weigh different amounts. Therefore "weight" is a variable in the experiment. Or, subjects may be given different doses of a drug. This would make "dosage" a variable. Variables can be

dependent or independent, qualitative or quantitative, and continuous or discrete.

## 6.2 DESCRIBING DATA USING DISTRIBUTIONS AND GRAPHS

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### Learning Objectives

By the end of this section, you will be able to:

- Construct and interpret common graphical representations of data, including histograms, bar charts, and pie charts
- Define the term frequency and calculate a frequency distribution, relative frequency distribution, and cumulative frequency distribution.
- Construct and interpret frequency tables for nominal and ordinal data.
- Construct and interpret frequency and relative frequency histograms.
- Describe distributions in terms of their shape, mode and skew.

Before we can understand our analyses, we must first understand our data. The first step in doing this is using tables, charts, graphs, plots, and other visual tools to see what our data look like.

### Graphing Qualitative Variables

When Apple Computer introduced the iMac computer in August 1998, the company wanted to learn whether the iMac was expanding Apple's market share. Was the iMac just attracting previous Macintosh owners? Or was it purchased by newcomers to the computer market and by previous Windows users who were switching over? To find out, 500 iMac customers were interviewed. Each customer was categorized as a previous Macintosh owner, a previous Windows owner, or a new computer purchaser.

This section examines graphical methods for displaying the results of the interviews. We'll learn some general lessons about how to graph data that fall into a small number of categories. A later section will consider how to graph numerical data in which each observation is represented by a number in some range. The key point



about the qualitative data that occupy us in the present section is that they do not come with a pre-established ordering (the way numbers are ordered). For example, there is no natural sense in which the category of previous Windows users comes before or after the category of previous Macintosh users. This situation may be contrasted with quantitative data, such as a person's weight. People of one weight are naturally ordered with respect to people of a different weight.

## Frequency Tables

All of the graphical methods shown in this section are derived from frequency tables. Table 6.2.1 shows a frequency table for the results of the iMac study; it shows the frequencies of the various response categories. It also shows the relative frequencies, which are the proportion of responses in each category. For example, the relative frequency for “none” of  $.17 = 85/500$ .

**Table 6.2.1. Frequency table for the iMac data.**

Previous Ownership	Frequency	Relative Frequency
None	85	0.17
Windows	60	0.12
Macintosh	355	0.71
Total	500	1.00

## Pie Charts

The pie chart in Figure 6.2.1 shows the results of the iMac study. In a pie chart, each category is represented by a slice of the pie. The area of the slice is proportional to the percentage of responses in the category. This is simply the relative frequency multiplied by 100. Although most iMac purchasers were Macintosh owners (71%), Apple was encouraged by the 12% of purchasers who were former Windows users, and by the 17% of purchasers who were buying a computer for the first time.

Pie charts are effective for displaying the relative frequencies of a small number of categories. They are not recommended, however, when you have a large number of categories. Pie charts can also be confusing when they are used to compare the outcomes of two different surveys or experiments. In an influential book on the use of graphs, Edward Tufte asserted, “The only worse design than a pie chart is several of them.”<sup>1</sup>

Here is another important point about pie charts. If they are based on a small number of observations, it can be misleading to label the pie slices with percentages. For example, if just **5** people had been interviewed by Apple Computers, and **3** were former Windows users, it would be misleading to display a pie chart with the Windows slice showing **60%**. With so few people interviewed, such a large percentage of Windows users might easily have occurred since chance can cause large errors with small samples. In this case, it is better to alert the user of the pie chart to the actual numbers involved. The slices should therefore be labelled with the actual frequencies observed (e.g., **3**) instead of with percentages.

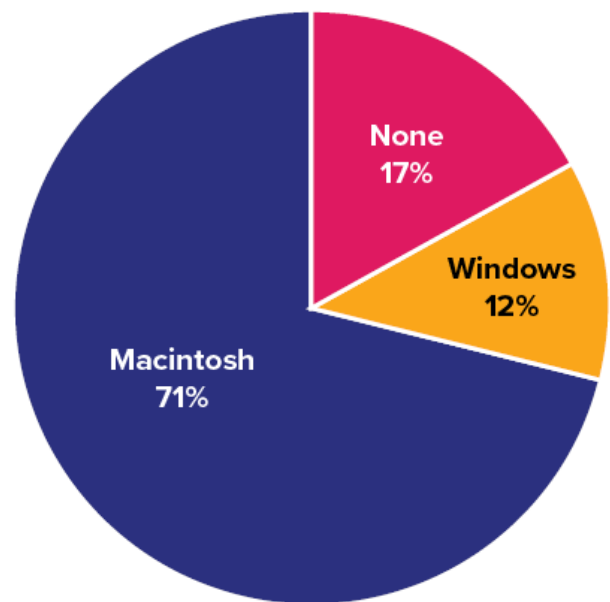


Figure 6.2.1. Pie chart of iMac purchases illustrating frequencies of previous computer ownership:  $71\%$  of purchasers owned a Macintosh before buying their iMac. (“Mac Pie Chart” by Judy Schmitt CC BY-NC-SA 4.0.)

## Try It

1) Based on the pie chart below, which was made from a sample of **300** students, construct a frequency table of college majors.

1. Tufte, E. R. (1983). The visual display of quantitative information (p. 178). Graphics Press.

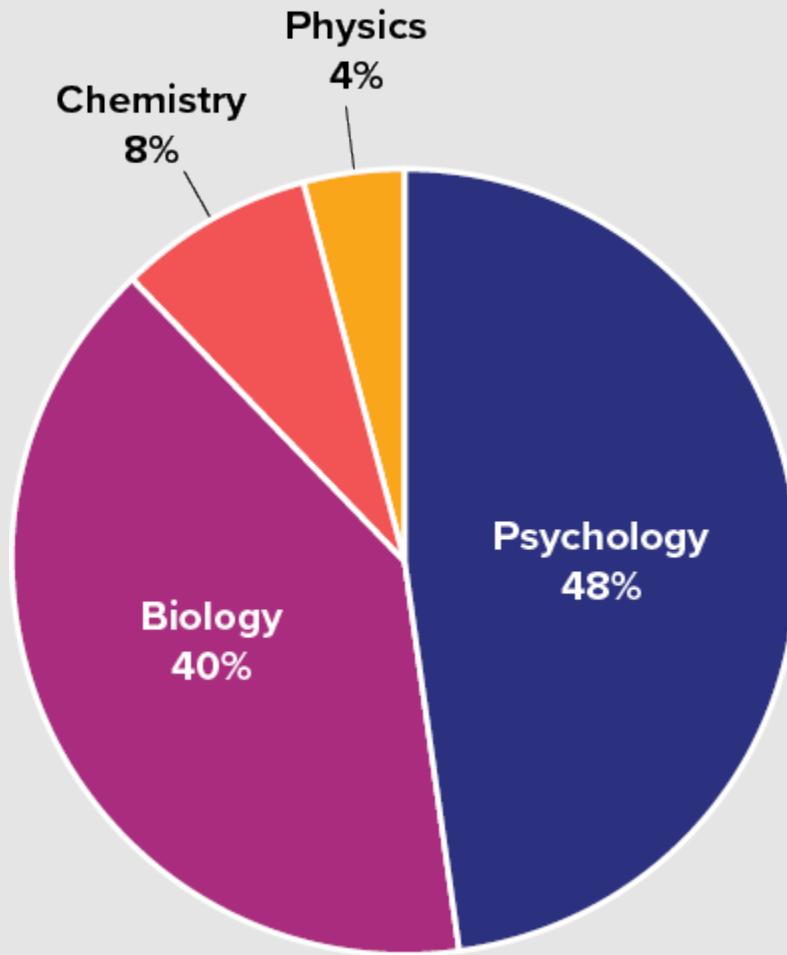


Figure 6.2.2. “College Majors Pie Chart” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.

### **Solution**

The frequency table appears below:

**Table 6.2.2**

<b>Major</b>	<b>Frequency</b>
Psychology	144
Biology	120
Chemistry	24
Physics	12

## Bar Charts

Bar charts can also be used to represent frequencies of different categories. A bar chart of the iMac purchases is shown in Figure 6.2.3. Frequencies are shown on the  $y$ -axis and the type of computer previously owned is shown on the  $x$ -axis. Typically, the  $y$ -axis shows the number of observations in each category rather than the percentage of observations in each category as is typical in pie charts.

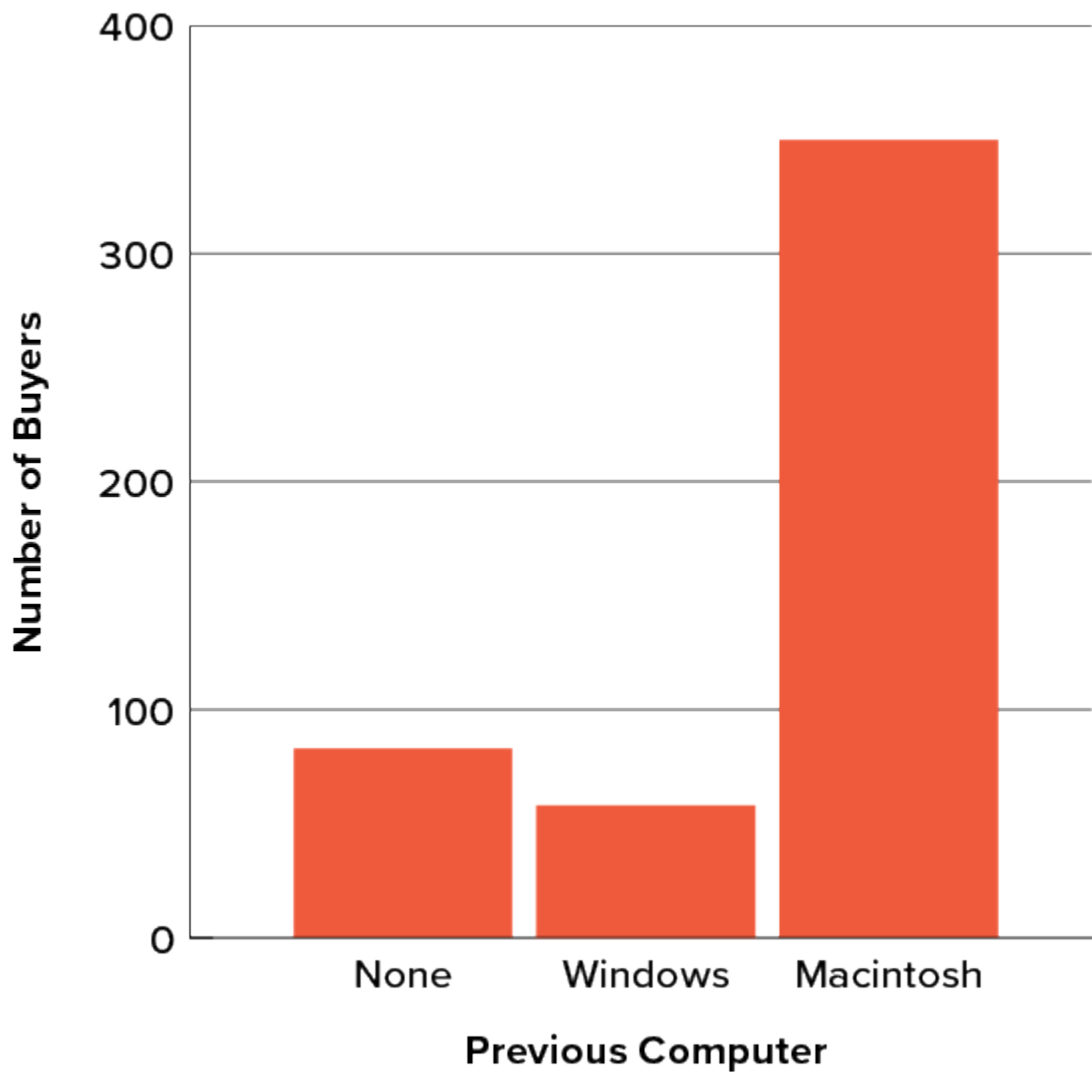


Figure 6.2.3. Bar chart of iMac purchases as a function of previous computer ownership. (“Mac Bar Chart” by Judy Schmitt CC BY-NC-SA 4.0.)

## Try It

2) Given the following data, construct a pie chart and a bar chart. Which do you think is the more appropriate or useful way to display the data?

Table 6.2.3

Favourite Movie Genre	Frequency
Comedy	14
Horror	9
Romance	8
Action	12

**Solution**

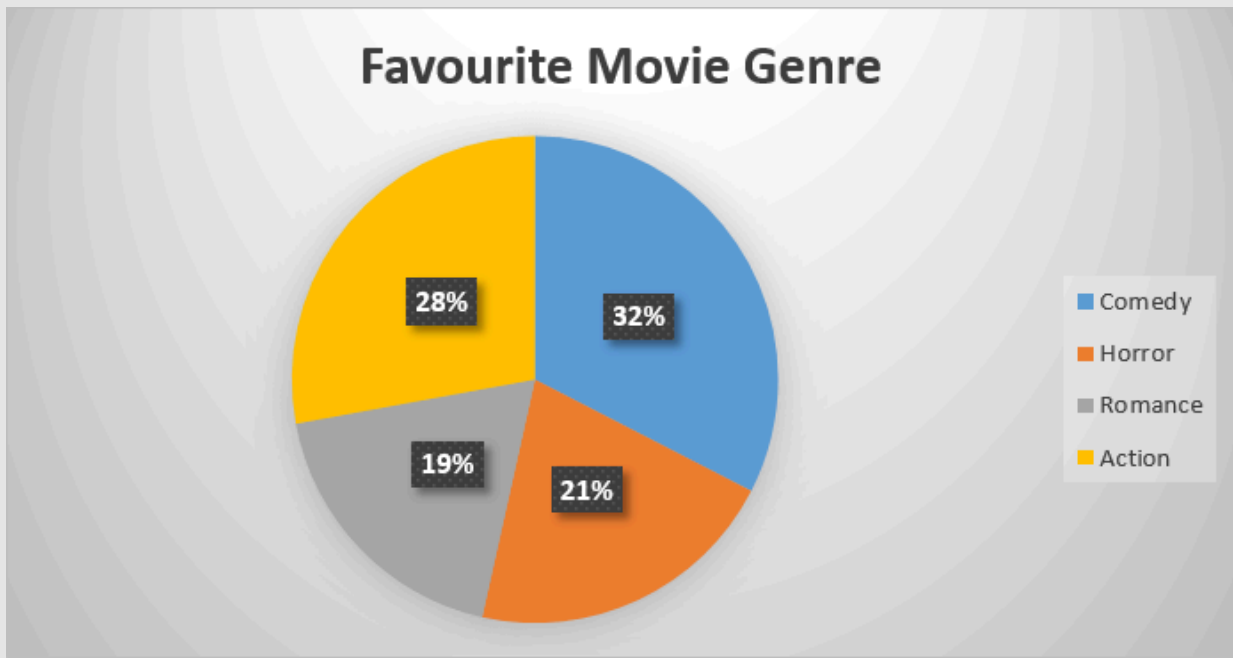


Figure 6.2.4

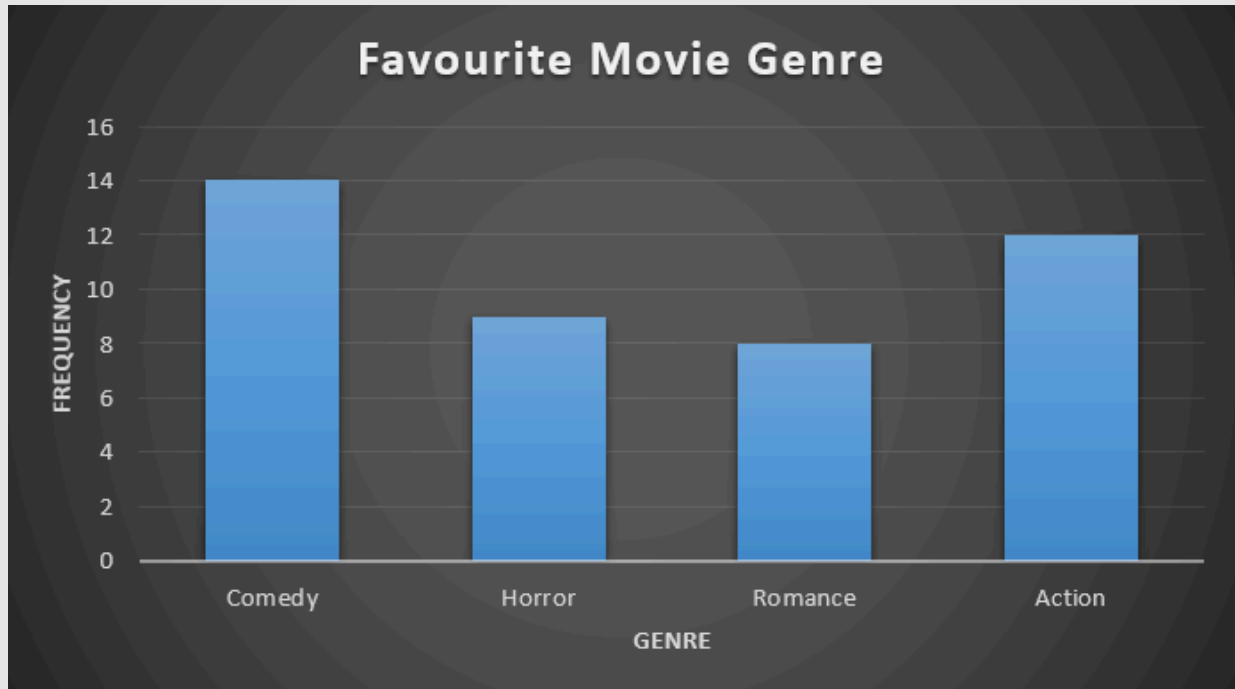


Figure 6.2.5

In this case, it seems that both of the graphs represent the data well. The bar chart might be the best way to represent the data because we can see the number of respondents that answered each genre. If the reader is provided with the frequency table or the number of participants, the pie chart could be acceptable.

## Comparing Distributions

Often we need to compare the results of different surveys, or of different conditions within the same overall survey. In this case, we are comparing the “distributions” of responses between the surveys or conditions. Bar charts are often excellent for illustrating differences between two distributions. Figure 6.2.6 shows the number of people playing card games at the Yahoo web site on a Sunday and on a Wednesday in the spring of 2001. We see that there were more players overall on Wednesday compared to Sunday. The number of people playing Pinochle was nonetheless the same on these two days. In contrast, there were about twice as many people playing Hearts on Wednesday as on Sunday. Facts like these emerge clearly from a well-designed bar chart.

The bars in Figure 6.2.6 are oriented horizontally rather than vertically. The horizontal format is useful when you have many categories because there is more room for the category labels. We’ll have more to say about bar charts when we consider numerical quantities later in this chapter.

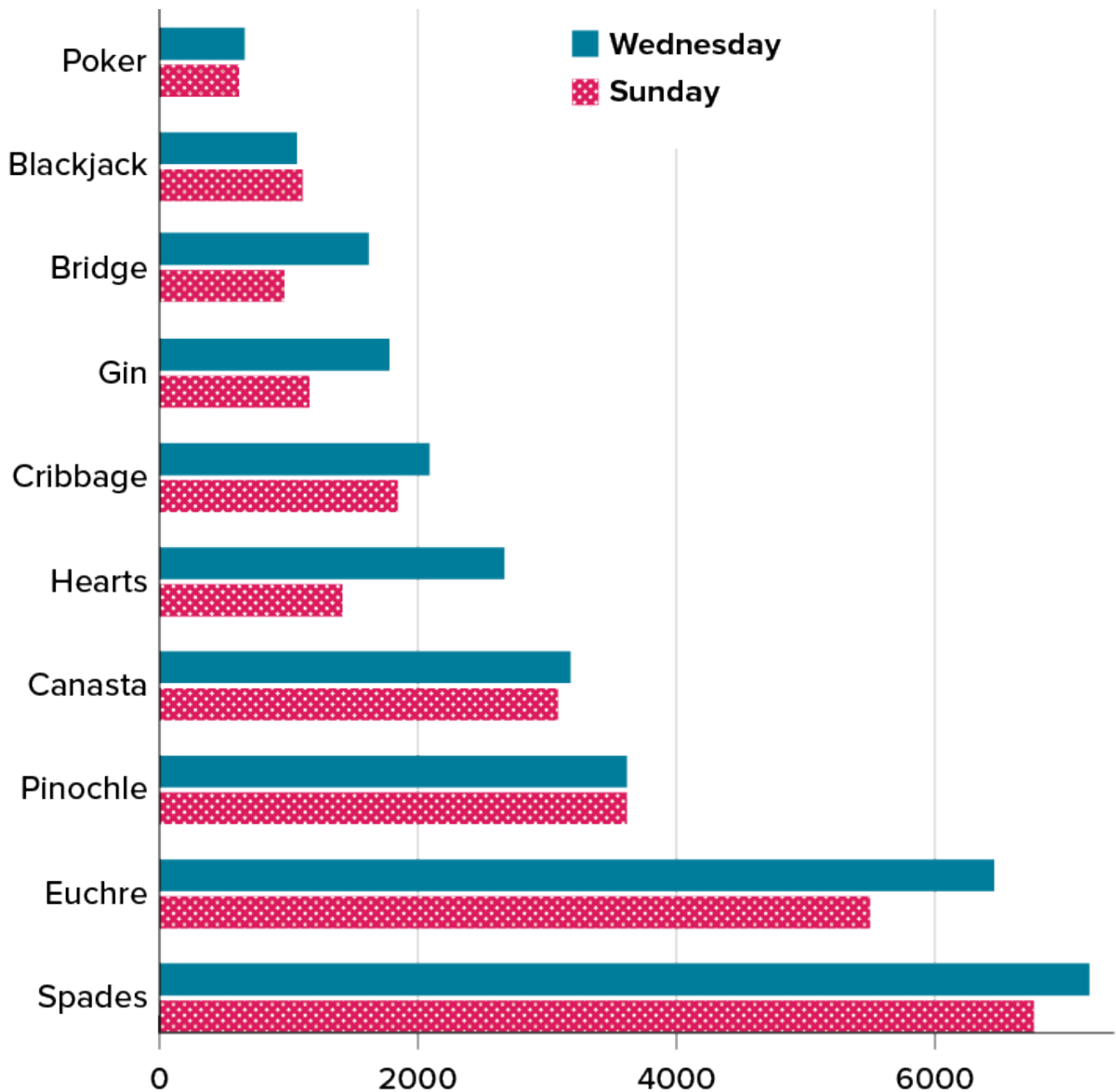


Figure 6.2.6. A bar chart of the number of people playing different card games on Sunday and Wednesday. (“Card Game Bar Chart” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

## Some Graphical Mistakes to Avoid

Don’t get fancy! People sometimes add features to graphs that don’t help to convey their information. For example, three-dimensional bar charts such as the one shown in Figure 6.2.7 are usually not as effective as their two-dimensional counterparts.



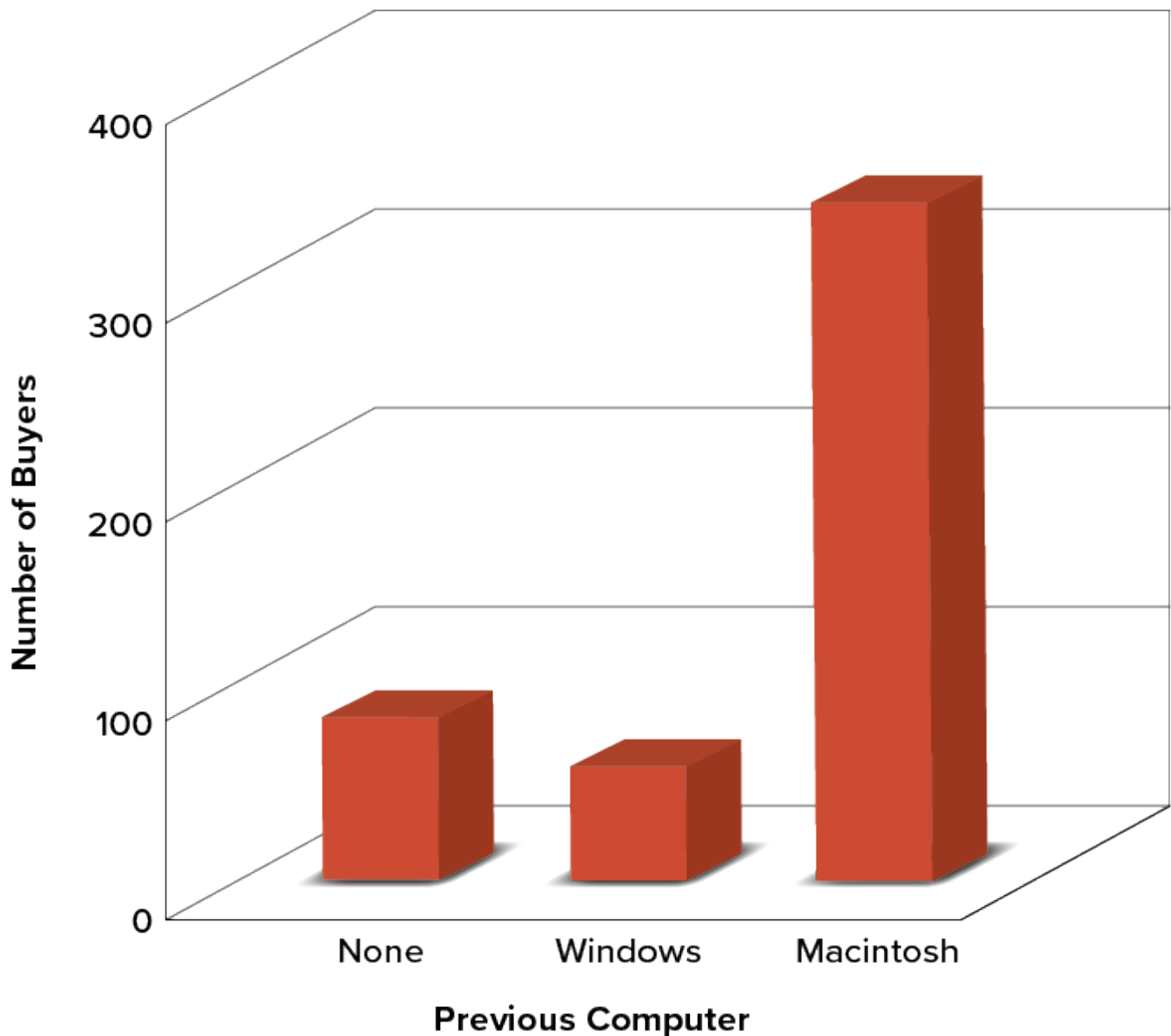


Figure 6.2.7. A three-dimensional version of Figure 6.2.3. Charts like this are less effective. (“Mac Bar Chart 3D” by Judy Schmitt is licenced under CC BY-NC-SA 4.0.)

Here is another way that fanciness can lead to trouble. Instead of plain bars, it is tempting to substitute meaningful images. For example, Figure 6.2.8 presents the iMac data using pictures of computers. The heights of the pictures accurately represent the number of buyers, yet Figure 6.2.8 is misleading because the viewer’s attention will be captured by areas. The areas can exaggerate the size differences between the groups. In terms of percentages, the ratio of previous Macintosh owners to previous Windows owners is about **6 to 1**. But the ratio of the two areas in Figure 6.2.8 is about **35 to 1**. A biased person wishing to hide the fact that many Windows owners purchased iMacs would be tempted to use Figure 6.2.8 instead of Figure 6.2.3!

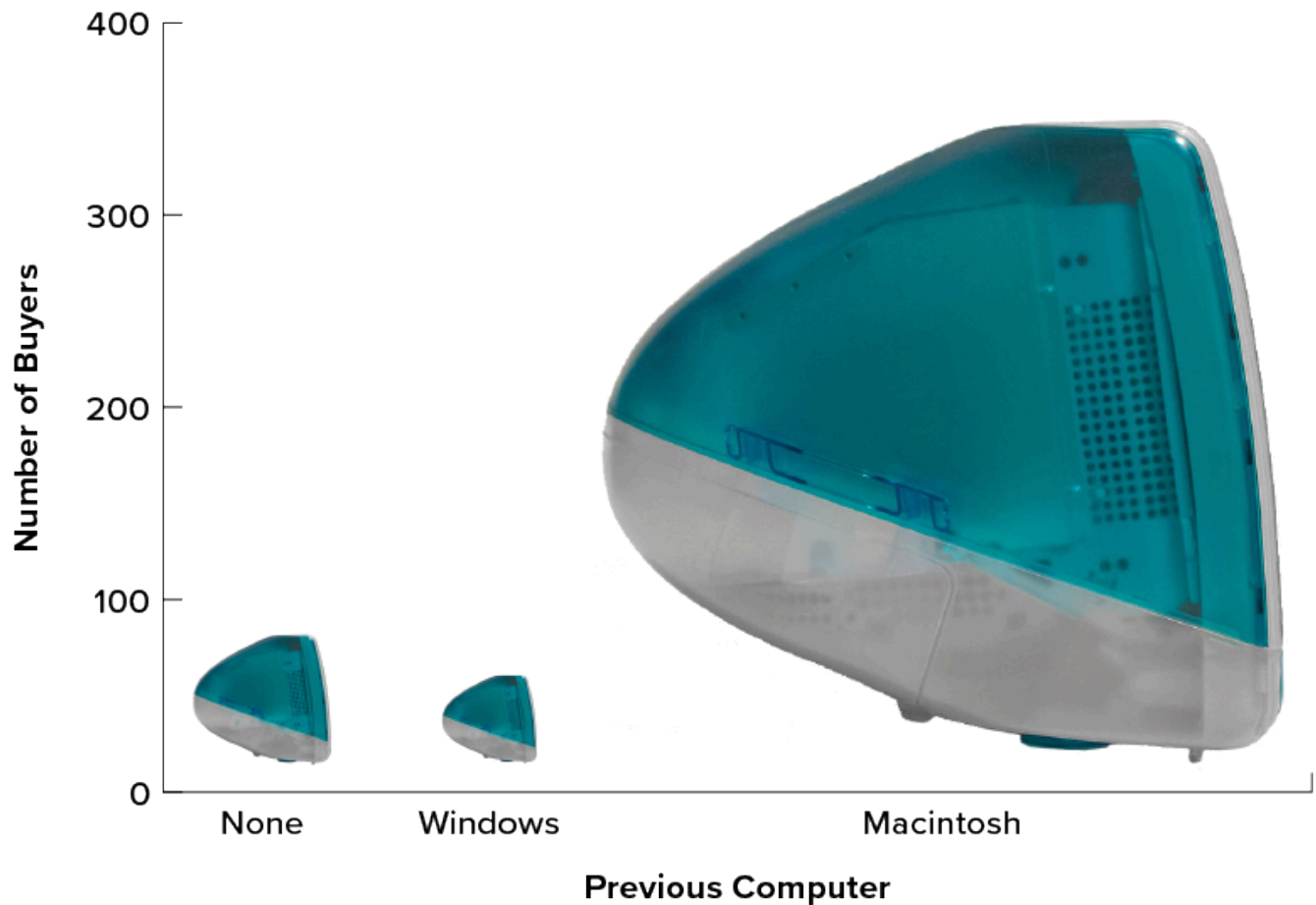


Figure 6.2.8. A redrawing of Figure 6.2.3 with a lie factor greater than  $8$ . (“Mac Bar Chart Lie Factor” by Judy Schmitt is licensed under CC BY-NC-SA 4.0. “Apple iMac G3 (1998)” by albaco/Flickr is licensed under CC BY-NC-SA 2.0; image was brightened and background was removed.)

Edward Tufte coined the term lie factor to refer to the ratio of the size of the effect shown in a graph to the size of the effect shown in the data. He suggests that lie factors greater than  $1.05$  or less than  $0.95$  produce unacceptable distortion.

Another distortion in bar charts results from setting the baseline to a value other than zero. The baseline is the bottom of the  $y$ -axis, representing the least number of cases that could have occurred in a category. Normally, but not always, this number should be zero. Figure 6.2.9 shows the iMac data with a baseline of  $50$ . Once again, the differences in areas suggests a different story than the true differences in percentages. The number of Windows-switchers seems minuscule compared to its true value of  $12\%$ .

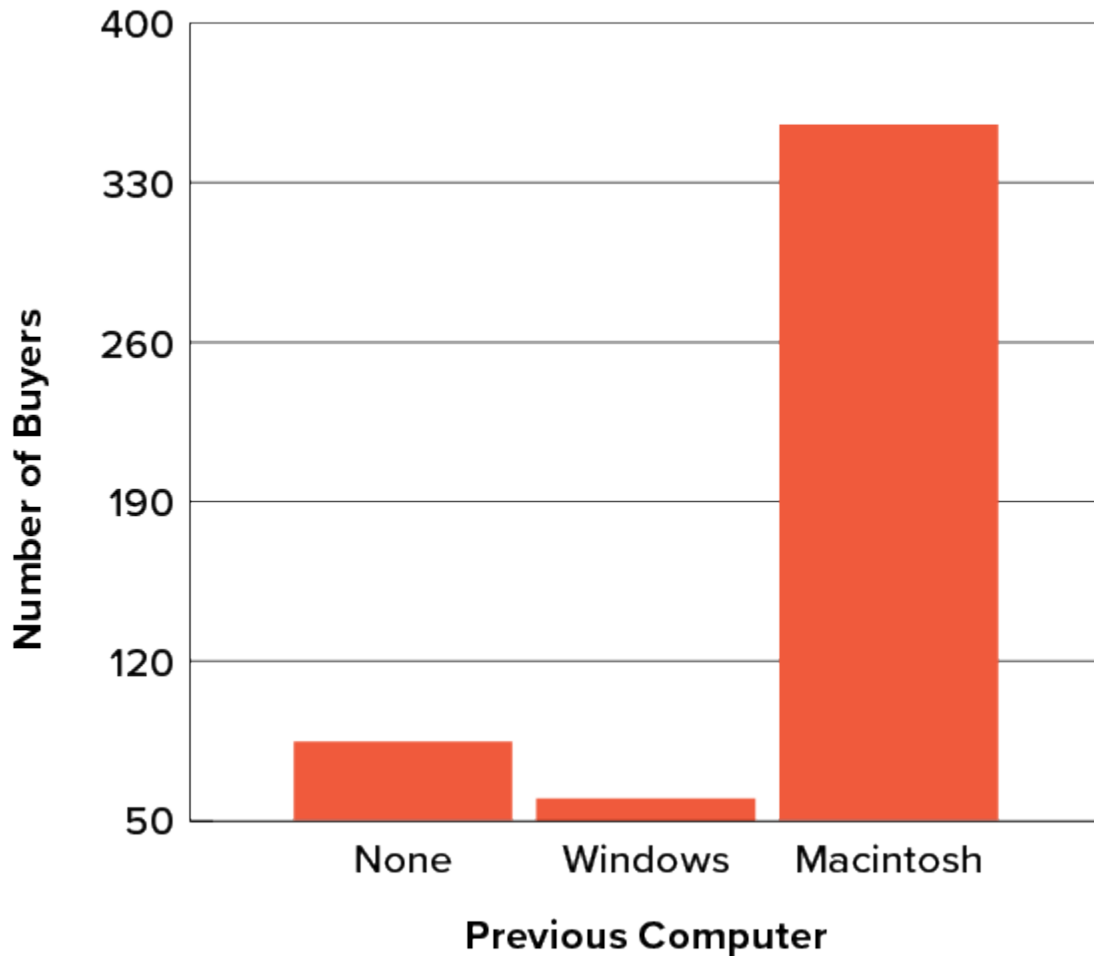


Figure 6.2.9. A redrawing of Figure 6.2.3 with a baseline of  $50$ . (“Mac Bar Chart Baseline 50” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

Finally, we note that it is a serious mistake to use a line graph when the  $x$ -axis contains merely qualitative variables. A line graph is essentially a bar graph with the tops of the bars represented by points joined by lines (the rest of the bar is suppressed). Figure 6.2.10 inappropriately shows a line graph of the card game data from Yahoo that was presented in Figure 6.2.6. The drawback to Figure 6.2.10 is that it gives the false impression that the games are naturally ordered in a numerical way when, in fact, they are ordered alphabetically.

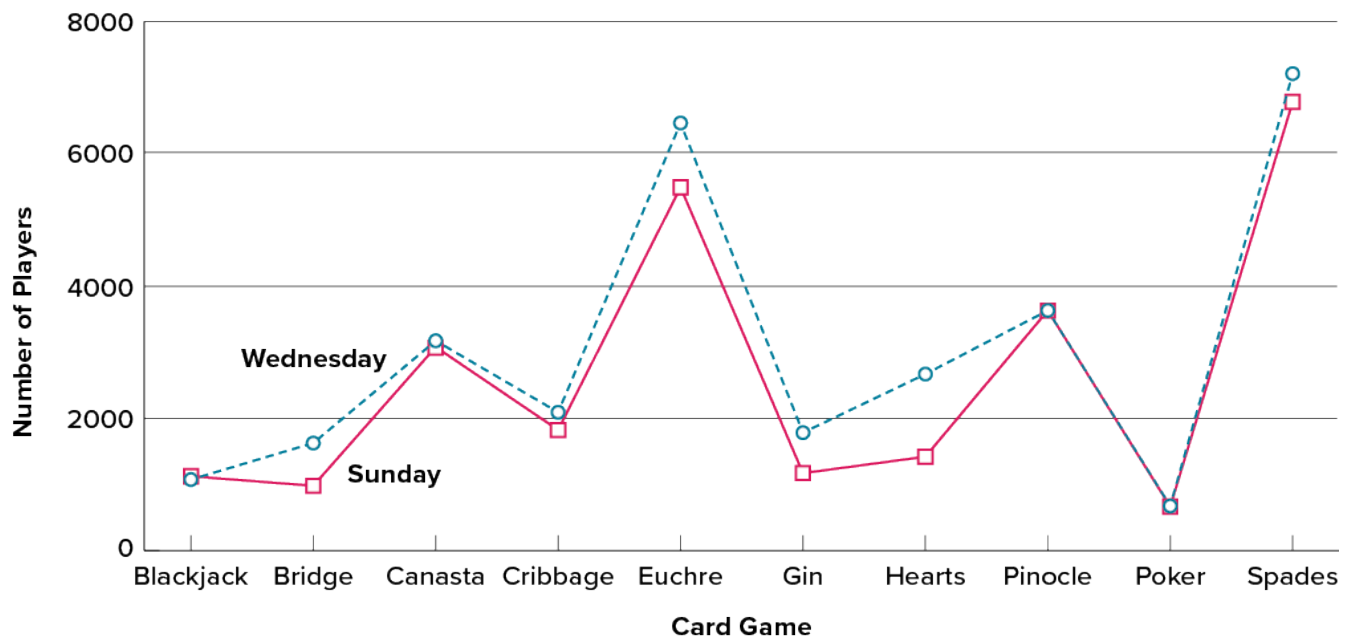


Figure 6.2.10. A line graph used inappropriately to depict the number of people playing different card games on Sunday and Wednesday. (“Line Chart Inappropriately Used” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

## Try It

3) A graph appears below showing the number of adults and children who prefer each type of soda. There were **130** adults and kids surveyed. Discuss some ways in which the graph could be improved.

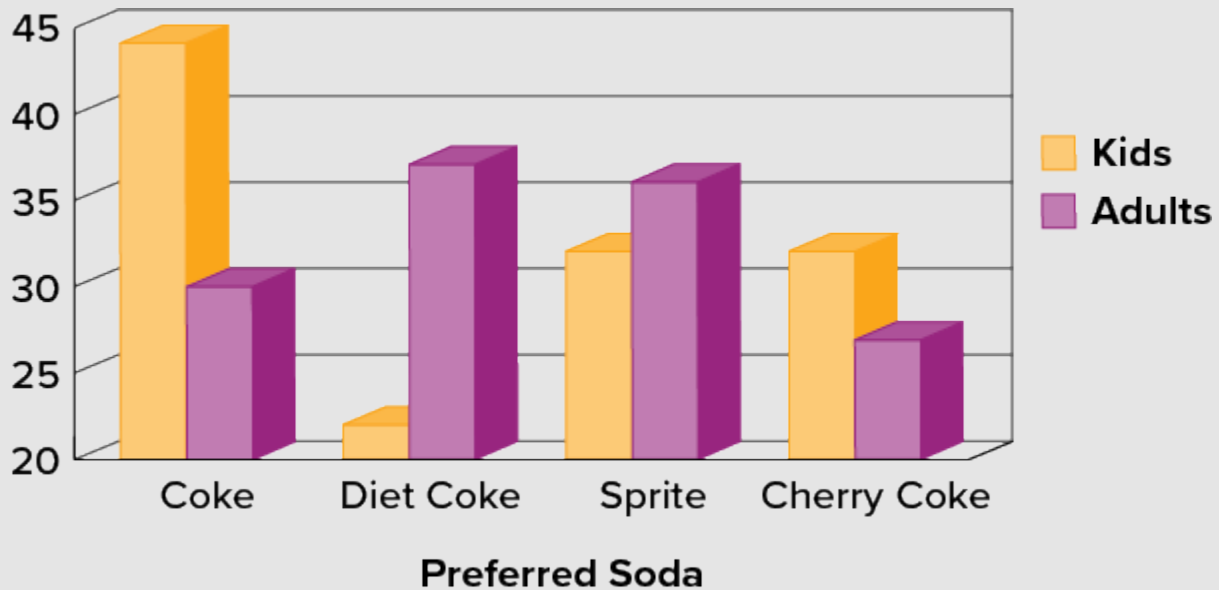


Figure 6.2.11. “Improvable Bar Chart” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.

### Solution

There are multiple answers possible for this question. Here are a few suggestions. Give the graph a title (place it at the top) and a title for the vertical axis. The legend can be placed inside the chart area. Add more graduations along the vertical axis to make the total of each bar more clear. A two-dimensional graph may be clearer to compare this data. Start the vertical axis scale at 0 or add a line break. A short description of the graph can be added at the bottom as a caption.

## Summary

Pie charts and bar charts can both be effective methods of portraying qualitative data. Bar charts are better when there are more than just a few categories and for comparing two or more distributions. Be careful to avoid creating misleading graphs.

## Graphing Quantitative Variables

As discussed in the section on variables in 6.1 Basics of Statistics, quantitative variables are variables measured on a numeric scale. Height, weight, response time, subjective rating of pain, temperature, and score on an exam are all examples of quantitative variables. Quantitative variables are distinguished from qualitative variables

(sometimes called **categorical variables** or nominal variables), such as favourite colour, religion, city of birth, and favourite sport, in which there is no ordering or measuring involved.

There are many types of graphs that can be used to portray distributions of quantitative variables. The upcoming sections cover the following types of graphs: (1) stem-and-leaf displays, (2) histograms, (3) frequency polygons, (4) box plots, (5) bar charts, (6) line graphs, (7) dot plots, and (8) scatter plots. Some graph types, such as stem-and-leaf displays, are best-suited for small to moderate amounts of data, whereas others, such as histograms, are best-suited for large amounts of data. Graph types such as box plots are good at depicting differences between distributions. Scatter plots are used to show the relationship between two variables.

## Stem-and-Leaf Displays

A **stem-and-leaf display** is a graphical method of displaying data. It is particularly useful when your data are not too numerous. In this section, we will explain how to construct and interpret this kind of graph.

As usual, we will start with an example. Consider Figure 6.2.12, which shows the number of touchdown passes (TD passes) thrown by each of the 31 teams in the National Football League during the 2000 season

37, 33, 33, 32, 29, 28,
28, 23, 22, 22, 22, 21,
21, 21, 20, 20, 19, 19,
18, 18, 18, 18, 16, 15,
14, 14, 14, 12, 12, 9, 6

Figure 6.2.12. Number of touchdown passes. (“Touchdown Passes Raw Data” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

A stem-and-leaf display of the data is shown in Figure 6.2.11. The left portion of Figure 6.2.13 contains the stems. They are the numbers 3, 2, 1, and 0, arranged as a column to the left of the bars. Think of these numbers as 10s digits. A stem of 3, for example, can be used to represent the 10s digit in any of the numbers from 30 to 39. The numbers to the right of the bar are leaves, and they represent the 1s digits. Every leaf in the graph therefore stands for the result of adding the leaf to 10 times its stem.

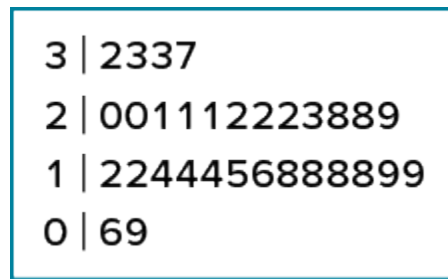


Figure 6.2.13. Stem-and-leaf display of the number of touchdown passes. (“Touchdown Passes Stem and Leaf” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

To make this clear, let us examine Figure 6.2.13 more closely. In the top row, the four leaves to the right of stem **3** are **2**, **3**, **3**, and **7**. Combined with the stem, these leaves represent the numbers **32**, **33**, **33**, and **37**, which are the numbers of TD passes for the first four teams in Figure 6.2.12. The next row has a stem of **2** and **12** leaves. Together, they represent **12** data points, namely, two occurrences of **20** TD passes, three occurrences of **21** TD passes, three occurrences of **22** TD passes, one occurrence of **23** TD passes, two occurrences of **28** TD passes, and one occurrence of **29** TD passes. We leave it to you to figure out what the third row represents. The fourth row has a stem of **0** and two leaves. It stands for the last two entries in Figure 6.2.12, namely **9** TD passes and **6** TD passes. (The latter two numbers may be thought of as **09** and **06**.)

One purpose of a stem-and-leaf display is to clarify the shape of the distribution. You can see many facts about TD passes more easily in Figure 6.2.13 than in Figure 6.2.12. For example, by looking at the stems and the shape of the plot, you can tell that most of the teams had between **10** and **29** passing TDs, with a few having more and a few having less. The precise numbers of TD passes can be determined by examining the leaves.

We can make our figure even more revealing by splitting each stem into two parts. Figure 6.2.14 shows how to do this. The top row is reserved for numbers from **35** to **39** and holds only the **37** TD passes made by the first team in Figure 6.2.12. The second row is reserved for the numbers from **30** to **34** and holds the **32**, **33**, and **33** TD passes made by the next three teams in the table. You can see for yourself what the other rows represent.

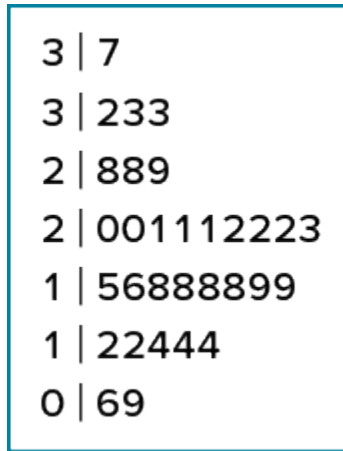


Figure 6.2.14. Stem-and-leaf display with the stems split in two. (“Touchdown Passes Split Stem and Leaf” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

Figure 6.2.14 is more revealing than Figure 6.2.13 because the latter figure lumps too many values into a single row. Whether you should split stems in a display depends on the exact form of your data. If rows get too long with single stems, you might try splitting them into two or more parts.

There is a variation of stem-and-leaf displays that is useful for comparing distributions. The two distributions are placed back to back along a common column of stems. The result is a back-to-back stem-and-leaf display, as shown in Figure 6.2.15. It compares the numbers of TD passes in the 1998 and 2000 seasons. The stems are in the middle, the leaves to the left are for the 1998 data, and the leaves to the right are for the 2000 data. For example, the second-to-last row shows that in 1998 there were teams with **11**, **12**, and **13** TD passes, and in 2000 there were two teams with **12** and three teams with **14** TD passes.



11	4	
	3	7
332	3	233
8865	2	889
44331110	2	001112223
987776665	1	56888899
321	1	22444
7	0	69

Figure 6.2.15. Back-to-back stem-and-leaf display. The left side shows the 1998 TD data and the right side shows the 2000 TD data. (“Touchdown Passes Back-to-Back Stem and Leaf” by Judy Schmitt is licensed under CC BY-NC-SA 4.0)

Figure 6.2.15 helps us see that the two seasons were similar, but that only in 1998 did any teams throw more than 40 TD passes.

There are two things about the football data that make them easy to graph with stems and leaves. First, the data are limited to whole numbers that can be represented with a one-digit stem and a one-digit leaf. Second, all the numbers are positive. If the data include numbers with three or more digits, or contain decimals, they can be rounded to two-digit accuracy. Negative values are also easily handled. Let us look at another example.

Figure 6.2.16 shows data from the Weapons and Aggression case study developed at Rice University. Each value is the mean difference over a series of trials between the times it took an experimental subject to name aggressive words (like punch) under two conditions. In one condition, the words were preceded by a non-weapon word such as bug. In the second condition, the same words were preceded by a weapon word such as gun or knife. The issue addressed by the experiment was whether a preceding weapon word would speed up (or prime) pronunciation of the aggressive word compared to a non-weapon priming word. A positive difference implies greater priming of the aggressive word by the weapon word. Negative differences imply that the priming by the weapon word was less than for a neutral word.

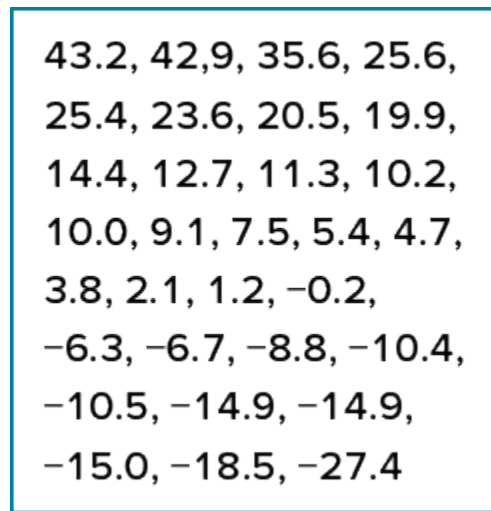


Figure 6.2.16. The effects of priming (in thousandths of a second). (“Priming Effects Raw Data” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

You see that the numbers range from **43.2** to **-27.4**. The first value indicates that one subject was **43.2** milliseconds faster pronouncing aggressive words when they were preceded by weapon words than when preceded by neutral words. The value **-27.4** indicates that another subject was **27.4** milliseconds slower pronouncing aggressive words when they were preceded by weapon words.

The data are displayed with stems and leaves in Figure 6.2.17. Since stem-and-leaf displays can only portray two whole digits (one for the stem and one for the leaf) the numbers are first rounded. Thus, the value **43.2** is rounded to **43** and represented with a stem of **4** and a leaf of **3**. Similarly, **42.9** is rounded to **43**. To represent negative numbers, we simply use negative stems. For example, the bottom row of the figure represents the number **-27**. The second-to-last row represents the numbers **-10**, **-10**, **-15**, etc. Once again, we have rounded the original values from Figure 6.2.16.

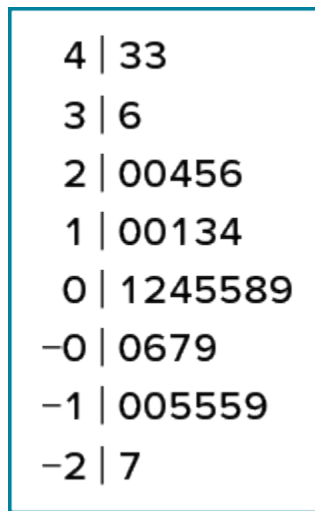


Figure 6.2.17.  
Stem-and-leaf display  
with negative numbers  
and rounding. (“Priming  
Effects Stem and Leaf”  
by Judy Schmitt is  
licensed under CC  
BY-NC-SA 4.0.)

Observe that the figure contains a row headed by “0” and another headed by “-0.” The stem of 0 is for numbers between 0 and 9, whereas the stem of -0 is for numbers between 0 and -9. For example, the fifth row of the table holds the numbers 1, 2, 4, 5, 5, 8, 9 and the sixth row holds 0, -6, -7, and -9. Values that are exactly 0 before rounding should be split as evenly as possible between the “0” and “-0” rows. In Figure 6.2.16, none of the values are 0 before rounding. The “0” that appears in the “-0” row comes from the original value of -0.2 in the table.

Although stem-and-leaf displays are unwieldy for large datasets, they are often useful for datasets with up to 200 observations. Figure 6.2.18 portrays the distribution of populations of 185 U.S. cities in 1998. To be included, a city had to have between 100,000 and 500,000 residents.



as the data in Figure 6.2.18 fit into the 10,000 and 100,000 places (for leaves and stems, respectively). Deciding what kind of graph is best suited to displaying your data thus requires good judgment. Statistics is not just recipes!

## Histograms

A **histogram** is a graphical method for displaying the shape of a distribution. It is particularly useful when there are a large number of observations. We begin with an example consisting of the scores of 642 students on a psychology test. The test consists of 197 items, each graded as “correct” or “incorrect.” The students’ scores ranged from 46 to 167.

The first step is to create a frequency table. Unfortunately, a simple frequency table would be too big, containing over 100 rows. To simplify the table, we group scores together as shown in Table 6.2.4.

**Table 6.2.4. Grouped frequency distribution of psychology test scores.**

Interval's Lower Limit	Interval's Upper Limit	Class Frequency
39.5	49.5	3
49.5	59.5	10
59.5	69.5	53
69.5	79.5	107
79.5	89.5	147
89.5	99.5	130
99.5	109.5	78
109.5	119.5	59
119.5	129.5	36
129.5	139.5	11
139.5	149.5	6
149.5	159.5	1
159.5	169.5	1

To create this table, the range of scores was broken into intervals, called class intervals. The first interval is from 39.5 to 49.5, the second from 49.5 to 59.5, etc. Next, the number of scores falling into each interval was counted to obtain the class frequencies. There are 3 scores in the first interval, 10 in the second, etc.

Class intervals of width 10 provide enough detail about the distribution to be revealing without making the graph too “choppy.” More information on choosing the widths of class intervals is presented later in this section. Placing the limits of the class intervals midway between two numbers (e.g., 49.5) ensures that every score will fall in an interval rather than on the boundary between intervals.

In a histogram, the class frequencies are represented by bars. The height of each bar corresponds to its class frequency. A histogram of these data is shown in Figure 6.2.19.

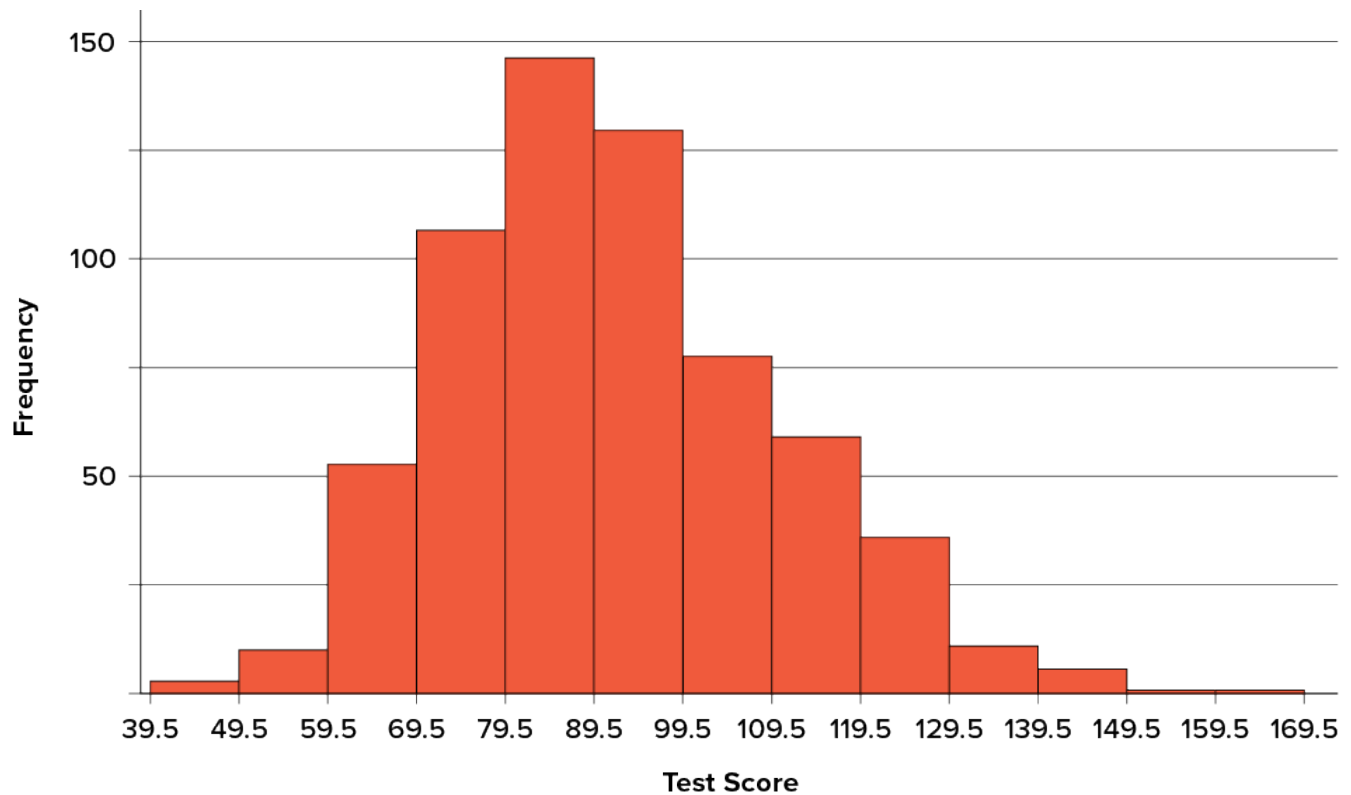


Figure 6.2.19. Histogram of scores on a psychology test. (“Psychology Test Scores Histogram” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

The histogram makes it plain that most of the scores are in the middle of the distribution, with fewer scores in the extremes. You can also see that the distribution is not symmetric: the scores extend farther to the right than they do to the left. The distribution is therefore said to be **skewed**.

In our example, the observations are whole numbers. Histograms can also be used when the scores are measured on a more continuous scale such as the length of time (in milliseconds) required to perform a task. In this case, there is no need to worry about fence sitters since they are improbable. (It would be quite a coincidence for a task to require exactly 7 seconds, measured to the nearest thousandth of a second.) We are therefore free to choose whole numbers as boundaries for our class intervals, for example, 4000, 5000, etc. The class frequency is then the number of observations that are greater than or equal to the lower bound, and strictly less than the upper bound. For example, one interval might hold times from 4000 to 4999 milliseconds. Using whole numbers as boundaries avoids a cluttered appearance, and is the practice of many computer programs that create histograms. Note also that some computer programs label the middle of each interval rather than the end points.

Histograms can be based on relative frequencies instead of actual frequencies. Histograms based on relative

frequencies show the proportion of scores in each interval rather than the number of scores. In this case, the  $y$ -axis runs from 0 to 1 (or somewhere in between if there are no extreme proportions). You can change a histogram based on frequencies to one based on relative frequencies by (a) dividing each class frequency by the total number of observations, and then (b) plotting the quotients on the  $y$ -axis (labelled as proportion).

There is more to be said about the widths of the class intervals, sometimes called **bin widths**. Your choice of bin width determines the number of class intervals. This decision, along with the choice of starting point for the first interval, affects the shape of the histogram. The best advice is to experiment with different choices of width, and to choose a histogram according to how well it communicates the shape of the distribution.

In the next example, we will show a way to create a histogram given a specific number of bins. Once again, consider Figure 6.2.20, which shows the number of touchdown passes (TD passes) thrown by each of the 31 teams in the National Football League during the 2000 season.

37, 33, 33, 32, 29, 28,
28, 23, 22, 22, 22, 21,
21, 21, 20, 20, 19, 19,
18, 18, 18, 18, 16, 15,
14, 14, 14, 12, 12, 9, 6

Figure 6.2.20. Number of touchdown passes. (“Touchdown Passes Raw Data” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

First, let’s make a frequency table using ten bins. To find the bin width, take the maximum data value and subtract the minimum data value, then divide that by the number of bins you want.

$$\text{bin width} = \frac{\text{maximum data value} - \text{minimum data value}}{\text{number of bins}} = \frac{37 - 6}{10} = 3.1$$

Always round your bin width up to the next whole number regardless of what you find.

$$\text{bin width} \approx 4$$

Start with the minimum value and create your bins so that they are mutually exclusive (no value is in two bins at once).



**Table 6.2.5. Frequency table for the Touchdown data.**

Number of Touchdowns	Frequency	Relative Frequency
6 - 11	2	0.06
12 - 17	7	0.23
18 - 23	15	0.48
24 - 29	3	0.10
30 - 35	3	0.10
36 - 41	1	0.03
<b>Total</b>	<b>31</b>	<b>1.00</b>

Using this frequency table, we can create either a frequency histogram or a relative frequency histogram. To figure out our boundaries we take the midpoint between each bin.

**Table 6.2.6. Midpoint table.**

Interval's Lower Limit	Interval's Upper Limit	Number of Touchdowns
5.5	11.5	2
11.5	17.5	7
17.5	23.5	15
23.5	29.5	3
29.5	35.5	3
35.5	41.5	1

Now, we can take the information and place it into a frequency histogram.

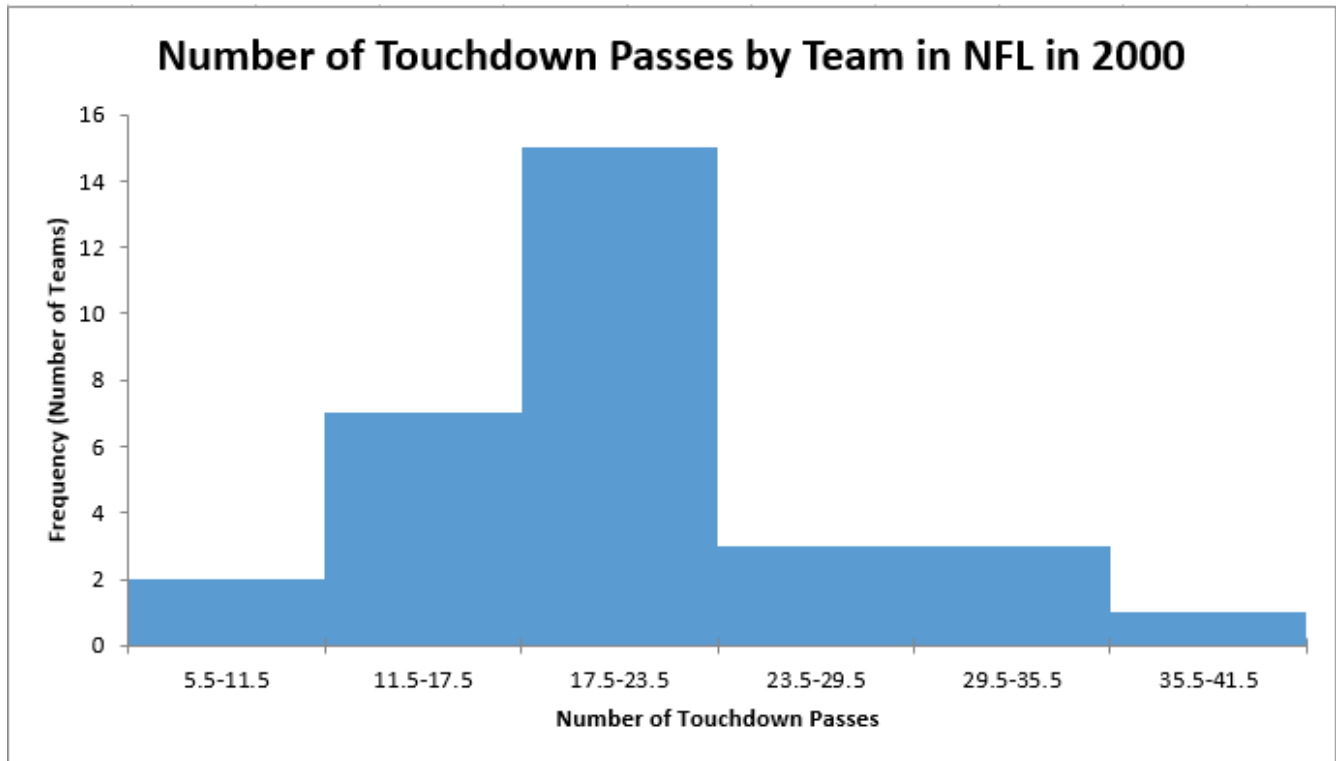


Figure 6.2.21. Histogram of number of touchdown passes by team in NFL in 2000 see table for data

This is what would be expected if we asked students to graph a histogram by hand on an assessment. That said, sometimes using technology to graph can allow us to investigate which number of bins would be most appropriate to represent the data.

## Try It

4) Pretend you are constructing a histogram for describing the distribution of salaries for individuals who are **40** years or older but not yet retired.

1. What is on the  $y$ -axis? Explain.
2. What is on the  $x$ -axis? Explain.
3. What would be the probable shape of the salary distribution? Explain why.

## Solution

[You do not need to draw the histogram, only describe it.]

1. The  $y$ -axis would show the frequency or proportion because this is always the case in histograms.
2. The  $x$ -axis would show income, because this is our quantitative variable of interest.
3. Because most income data are positively skewed, this histogram would likely be skewed positively too.

## Try It

5) Create a histogram of the following data representing how many shows children said they watch each day:

**Table 6.2.7**

<b>Number of TV Shows</b>	<b>Frequency</b>
0	2
1	18
2	36
3	7
4	3

## Solution

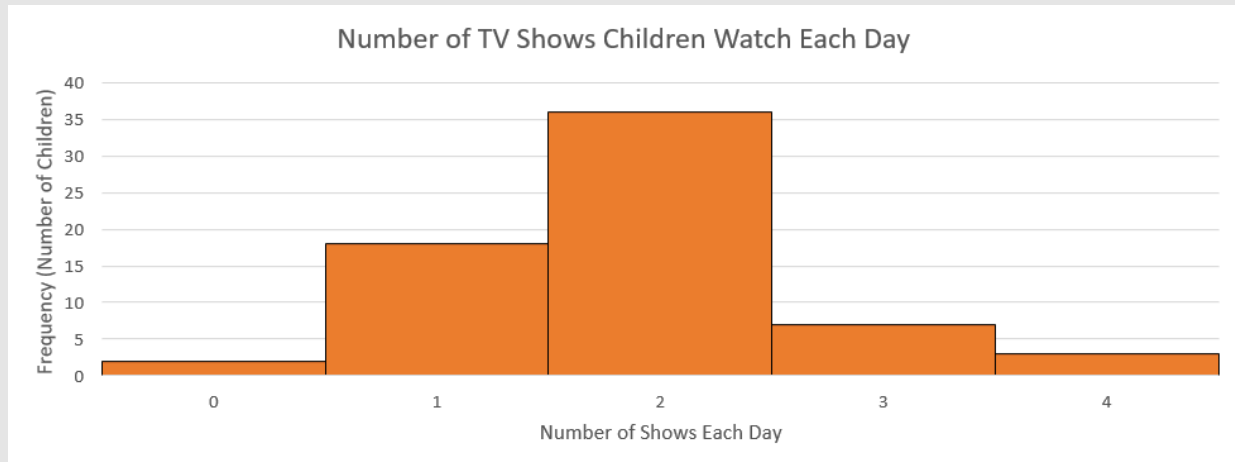


Figure 6.2.22

## Try It

6) Create a histogram of the following data. Determine if it is skewed (and direction, if so) or symmetrical.

**Table 6.2.8**

<b>Hours Worked per Week</b>	<b>Proportion</b>
0-10	4
10-20	8
20-30	11
30-40	51
40-50	12
50-60	9
60+	5

**Solution**

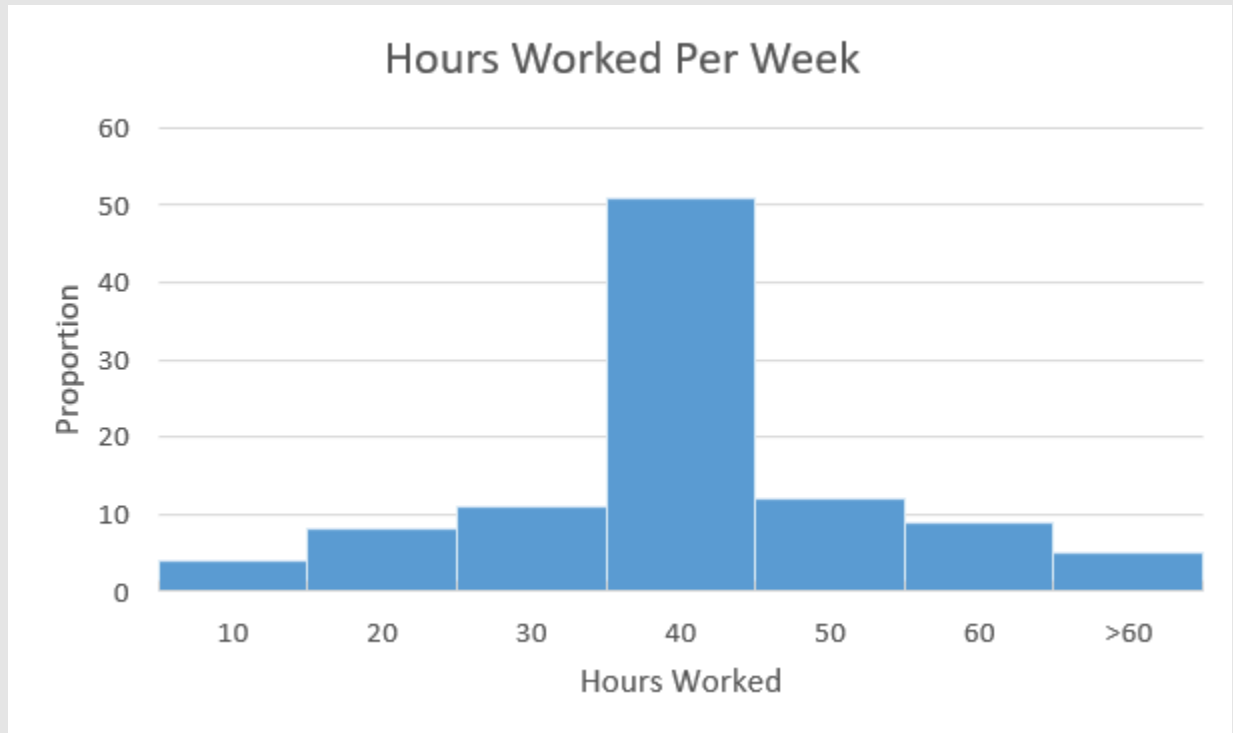


Figure 6.2.23

This distribution is symmetrical. Almost perfectly symmetrical, in fact.

## Frequency Polygons

**Frequency polygons** are a graphical device for understanding the shapes of distributions. They serve the same purpose as histograms, but are especially helpful for comparing sets of data. Frequency polygons are also a good choice for displaying cumulative frequency distributions.

To create a frequency polygon, start just as for histograms, by choosing a class interval. Then draw an  $x$ -axis representing the values of the scores in your data. Mark the middle of each class interval with a tick mark, and label it with the middle value represented by the class. Draw the  $y$ -axis to indicate the frequency of each class. Place a point in the middle of each class interval at the height corresponding to its frequency. Finally, connect the points. You should include one class interval below the lowest value in your data and one above the highest value. The graph will then touch the  $x$ -axis on both sides.

The frequency distribution of 642 psychology test scores, shown in Table 6.2.9, was used to create the frequency polygon shown in Figure 6.2.19.

**Table 6.2.9. Frequency distribution of psychology test scores**

Lower Limit	Upper Limit	Count	Cumulative Count
29.5	39.5	0	0
39.5	49.5	3	3
49.5	59.5	10	13
59.5	69.5	53	66
69.5	79.5	107	173
79.5	89.5	147	320
89.5	99.5	130	450
99.5	109.5	78	528
109.5	119.5	59	587
119.5	129.5	36	623
129.5	139.5	11	634
139.5	149.5	6	640
149.5	159.5	1	641
159.5	169.5	1	642
169.5	170.5	0	642

The first label on the  $x$ -axis is 35. This represents an interval extending from 29.5 to 39.5. Since the lowest test score is 46, this interval has a frequency of 0. The point labelled 45 represents the interval from 39.5 to 49.5. There are three scores in this interval. There are 147 scores in the interval that surrounds 85.

You can easily discern the shape of the distribution from Figure 6.2.24. Most of the scores are between 65 and 115. It is clear that the distribution is not symmetric inasmuch as good scores (to the right) trail off more gradually than poor scores (to the left). In the terminology of distribution shapes, the distribution is skewed.

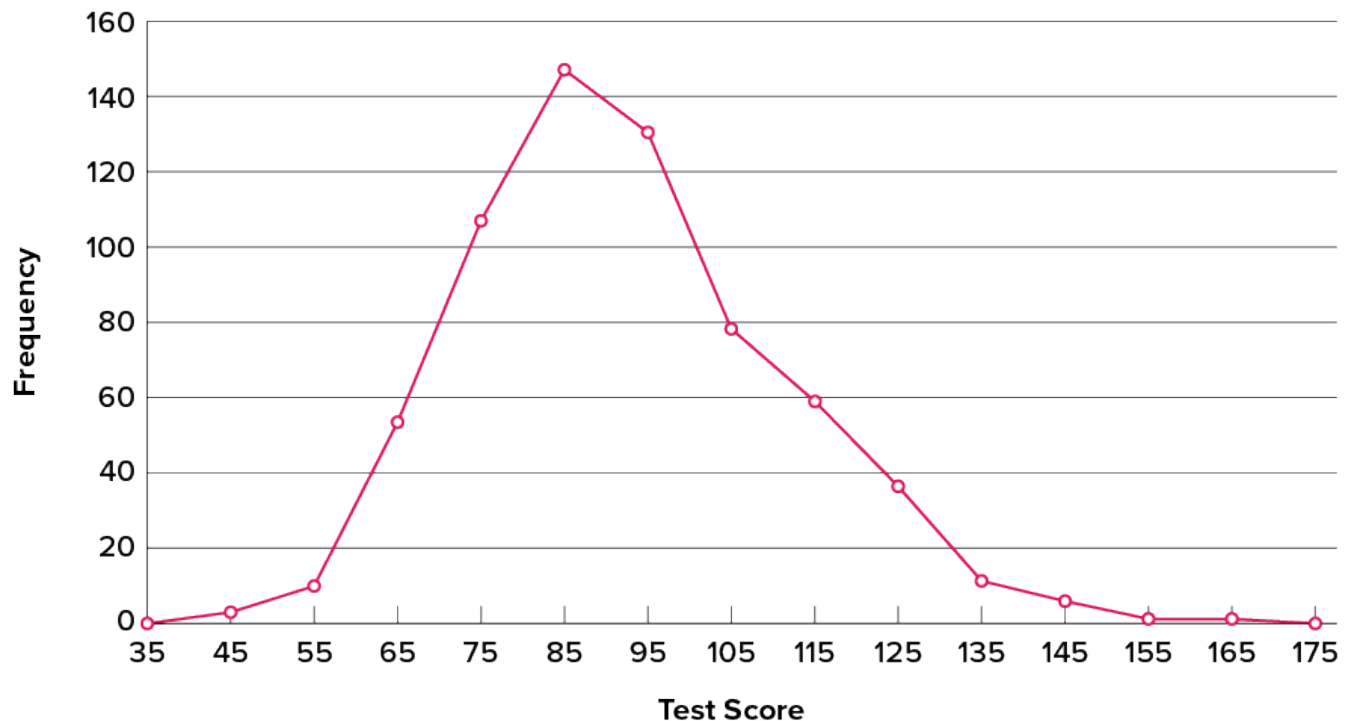


Figure 6.2.24. Frequency polygon for the psychology test scores. (“Psychology Test Scores Frequency Polygon” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

A cumulative frequency polygon for the same test scores is shown in Figure 6.2.25. The graph is the same as before except that the  $y$  value for each point is the number of students in the corresponding class interval plus all numbers in lower intervals. For example, there are no scores in the interval labelled “35,” three in the interval “45,” and 10 in the interval “55.” Therefore, the  $y$  value corresponding to “55” is 13. Since 642 students took the test, the cumulative frequency for the last interval is 642.



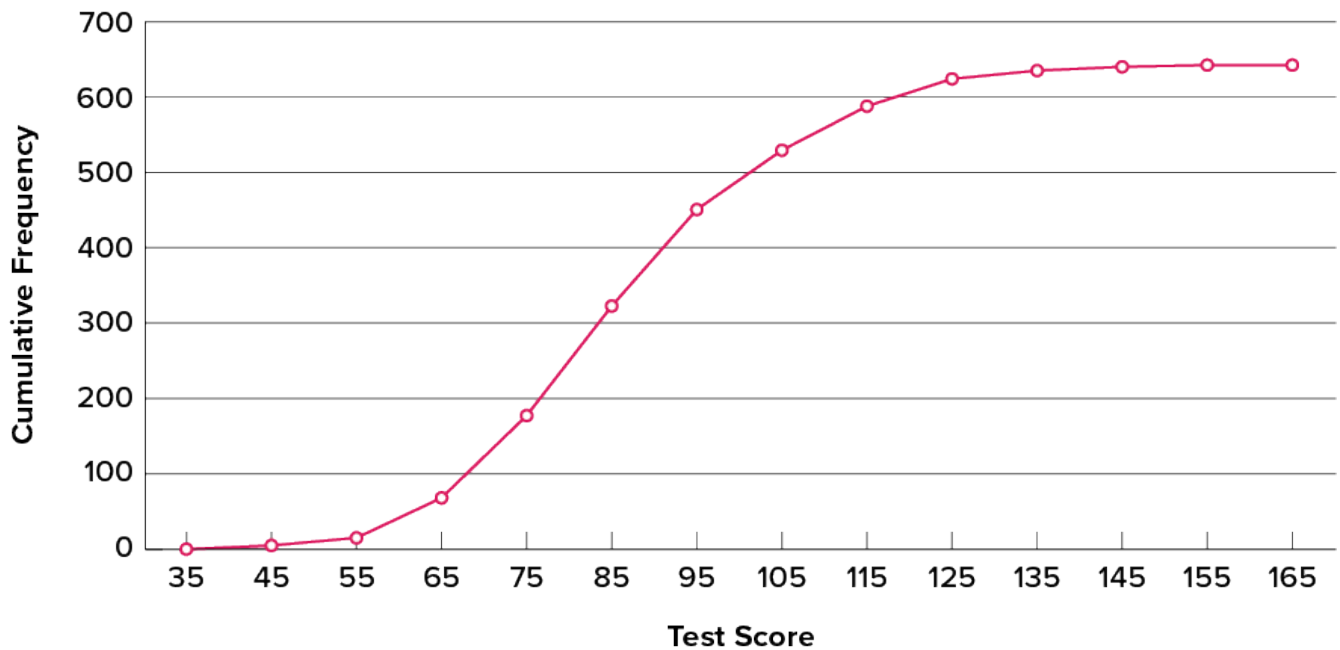


Figure 6.2.25. Cumulative frequency polygon for the psychology test scores. (“Psychology Test Scores Cumulative Frequency Polygon” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

Frequency polygons are useful for comparing distributions. This is achieved by overlaying the frequency polygons drawn for different datasets. Figure 6.2.26 provides an example. The data come from a task in which the goal is to move a computer cursor to a target on the screen as fast as possible. On **20** of the trials, the target was a small rectangle; on the other **20**, the target was a large rectangle. Time to reach the target was recorded on each trial. The two distributions (one for each target) are plotted together in Figure 6.2.26. The figure shows that, although there is some overlap in times, it generally took longer to move the cursor to the small target than to the large one.

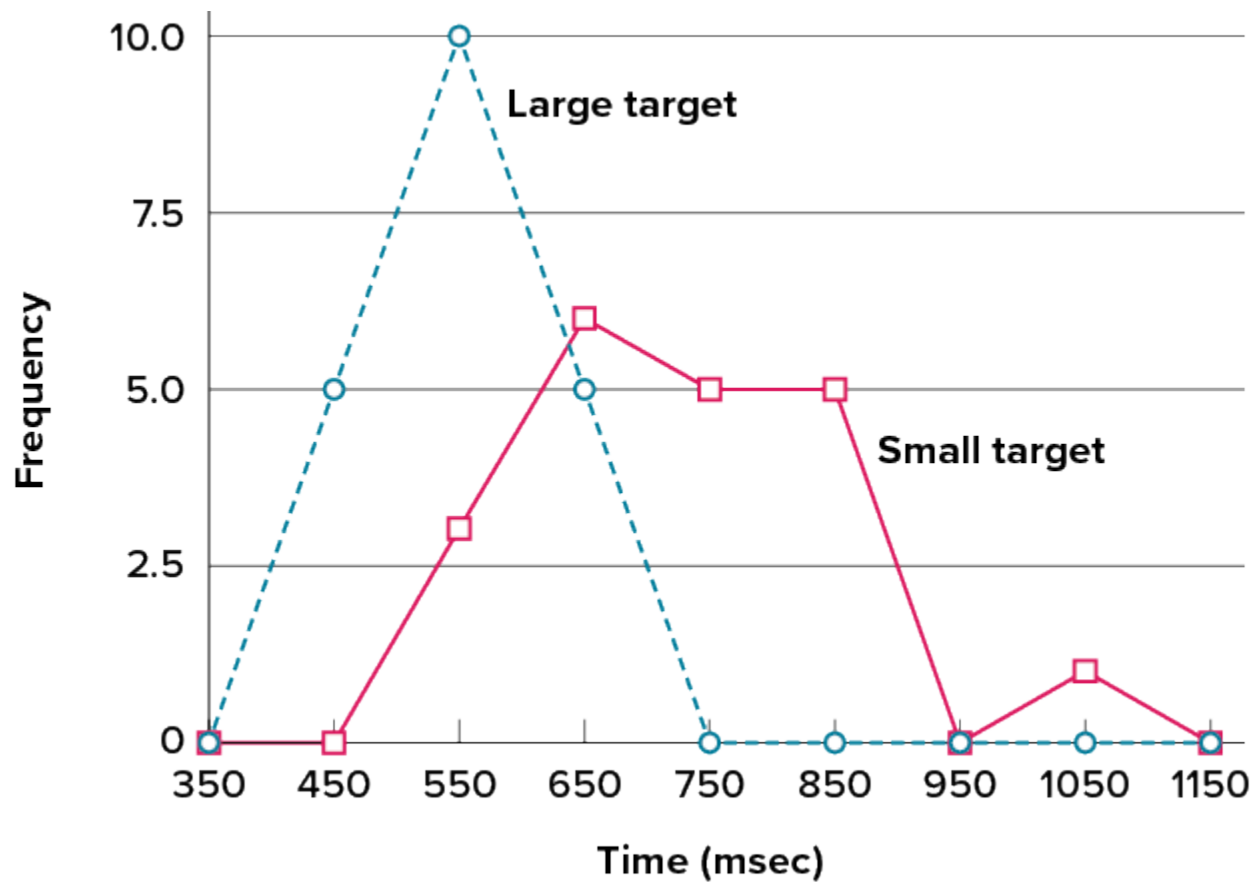


Figure 6.2.26. Overlaid frequency polygons for the cursor task. (“Cursor Task Frequency Polygons” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

It is also possible to plot two cumulative frequency distributions in the same graph. This is illustrated in Figure 6.2.27 using the same data from the cursor task. The difference in distributions for the two targets is again evident.

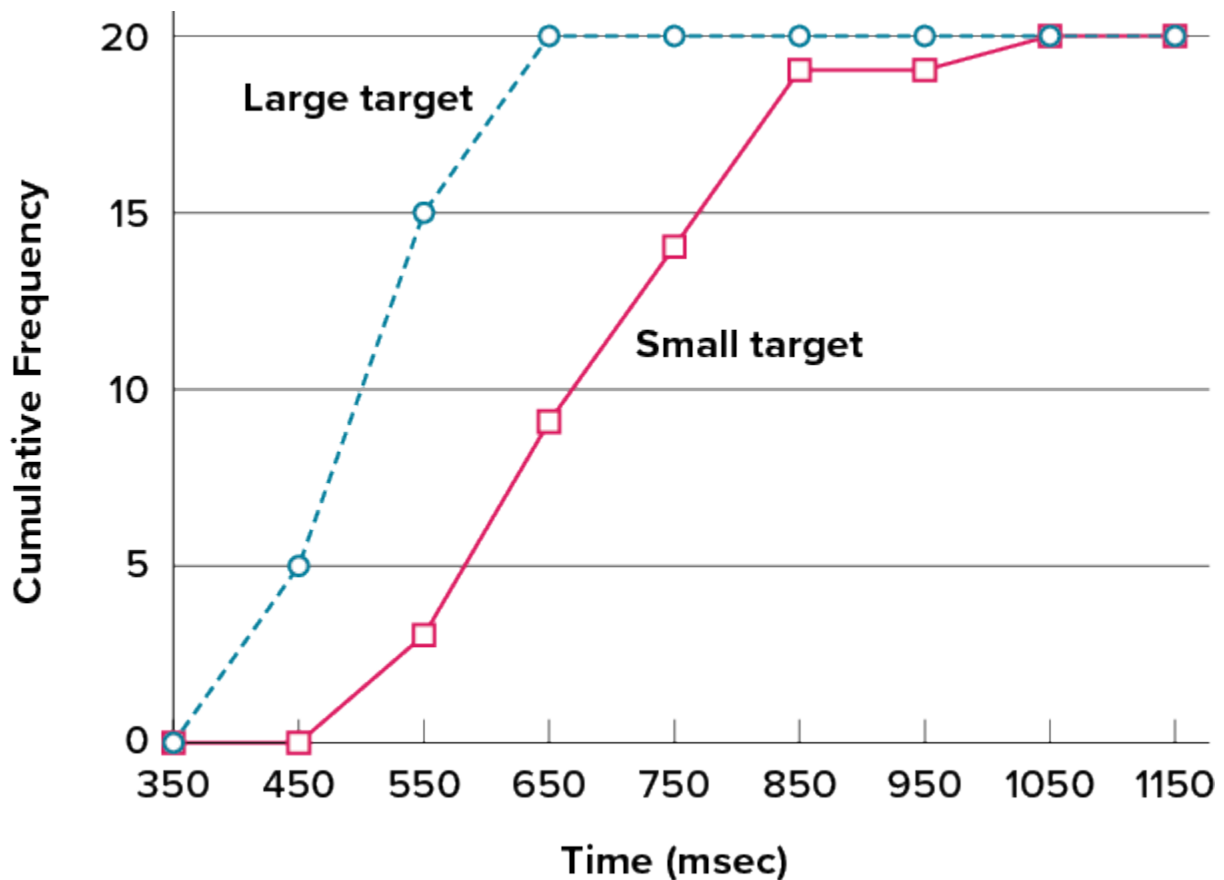


Figure 6.2.27. Overlaid cumulative frequency polygons for the cursor task. (“Cursor Task Cumulative Frequency Polygons” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

## Box Plots

We have already discussed techniques for visually representing data (see histograms and frequency polygons). In this section we present another important graph, called a box plot. **Box plots** are useful for identifying outliers and for comparing distributions. We will explain box plots with the help of data from an in-class experiment. Students in Introductory Statistics were presented with a page containing **30** coloured rectangles. Their task was to name the colours as quickly as possible. Their times (in seconds) were recorded. We’ll compare the scores for the **16** men and **31** women who participated in the experiment by making separate box plots for each gender. Such a display is said to involve parallel box plots. The data for the women in our sample are shown in Figure 6.2.28.

14, 15, 16, 16, 17, 17, 17, 17, 17, 18, 18, 18, 18, 18, 18, 19, 19, 19
20, 20, 20, 20, 20, 20, 21, 21, 22, 23, 24, 24, 29

Figure 6.2.28. Women’s times. (“Women’s Times Raw Data” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

There are several steps in constructing a box plot. The first relies on the  $25^{th}$ ,  $50^{th}$ , and  $75^{th}$  percentiles in the distribution of scores. Figure 6.2.29 shows how these three statistics are used. For each gender we draw a box extending from the  $25^{th}$  percentile to the  $75^{th}$  percentile. The  $50^{th}$  percentile is drawn inside the box. Therefore, the bottom of each box is the  $25^{th}$  percentile, the top is the  $75^{th}$  percentile, and the line in the middle is the  $50^{th}$  percentile.

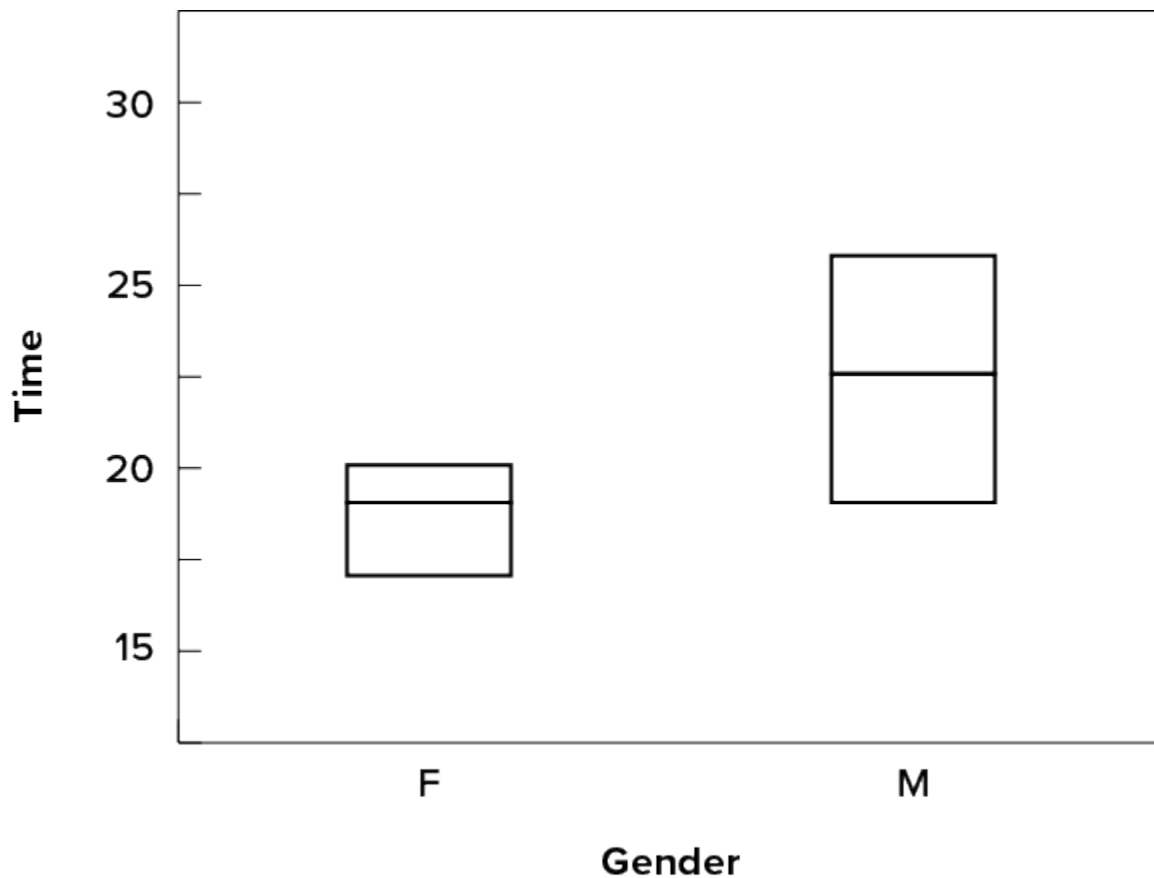


Figure 6.2.29. The first step in creating box plots. (“Box Plot First Step” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

For the data reflecting the women’s times, the  $25^{th}$  percentile is 17, the  $50^{th}$  percentile is 19, and the  $75^{th}$

percentile is 20. For the men (whose data are not shown), the 25<sup>th</sup> percentile is 19, the 50<sup>th</sup> percentile is 22.5, and the 75<sup>th</sup> percentile is 25.5.

Before proceeding, the terminology in Table 6.2.10 is helpful.

**Table 6.2.10. Box plot terms and values for women’s times.**

Name	Formula	Value
Upper Hinge	75 <sup>th</sup> percentile	20
Lower Hinge	25 <sup>th</sup> percentile	17
H-Spread	Upper Hinge – Lower Hinge	3
Step	1.5 × H-Spread	4.5
Upper Inner Fence	Upper Hinge + 1 Step	24.5
Lower Inner Fence	Lower Hinge – 1 Step	12.5
Upper Outer Fence	Upper Hinge + 2 Steps	29
Lower Outer Fence	Lower Hing – 2 Steps	8
Upper Adjacent	Largest value below Upper Inner Fence	24
Lower Adjacent	Smallest value above Lower Inner Fence	14
Outside Value	A value beyond an Inner Fence but not beyond an Outer Fence	29
Far Out Value	A value beyond an Outer Fence	None

Continuing with the box plots, we put “whiskers” above and below each box to give additional information about the spread of data. **Whiskers** are vertical lines that end in a horizontal stroke. Whiskers are drawn from the upper and lower hinges to the upper and lower adjacent values (24 and 14 for the women’s data), as shown in Figure 6.2.30.

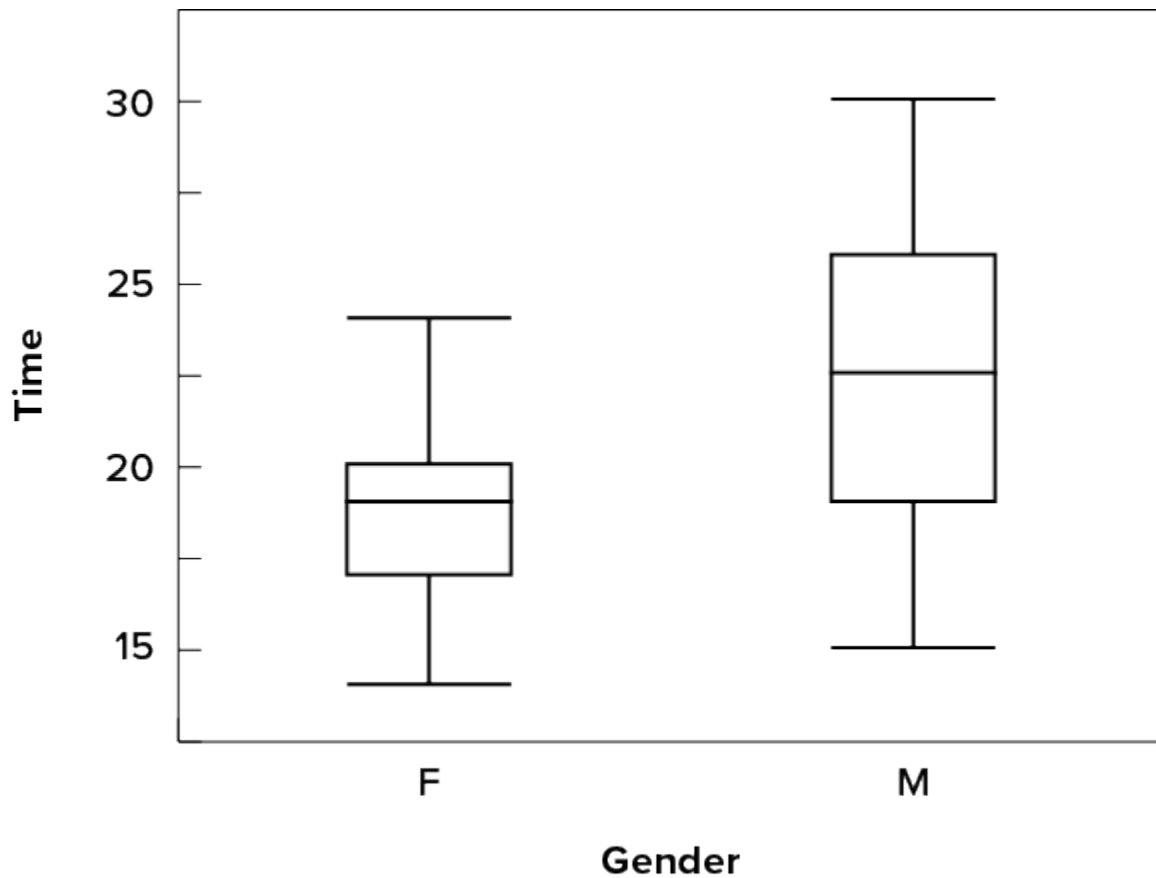


Figure 6.2.30. The box plots with the whiskers drawn. (“Box Plot Whiskers” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

Although we don’t draw whiskers all the way to outside or far out values, we still wish to represent them in our box plots. This is achieved by adding additional marks beyond the whiskers. Specifically, outside values are indicated by small circles, and far out values are indicated by asterisks (\*). In our data, there are no far-out values and just one outside value. This outside value of **29** is for the women and is shown in Figure 6.2.31.

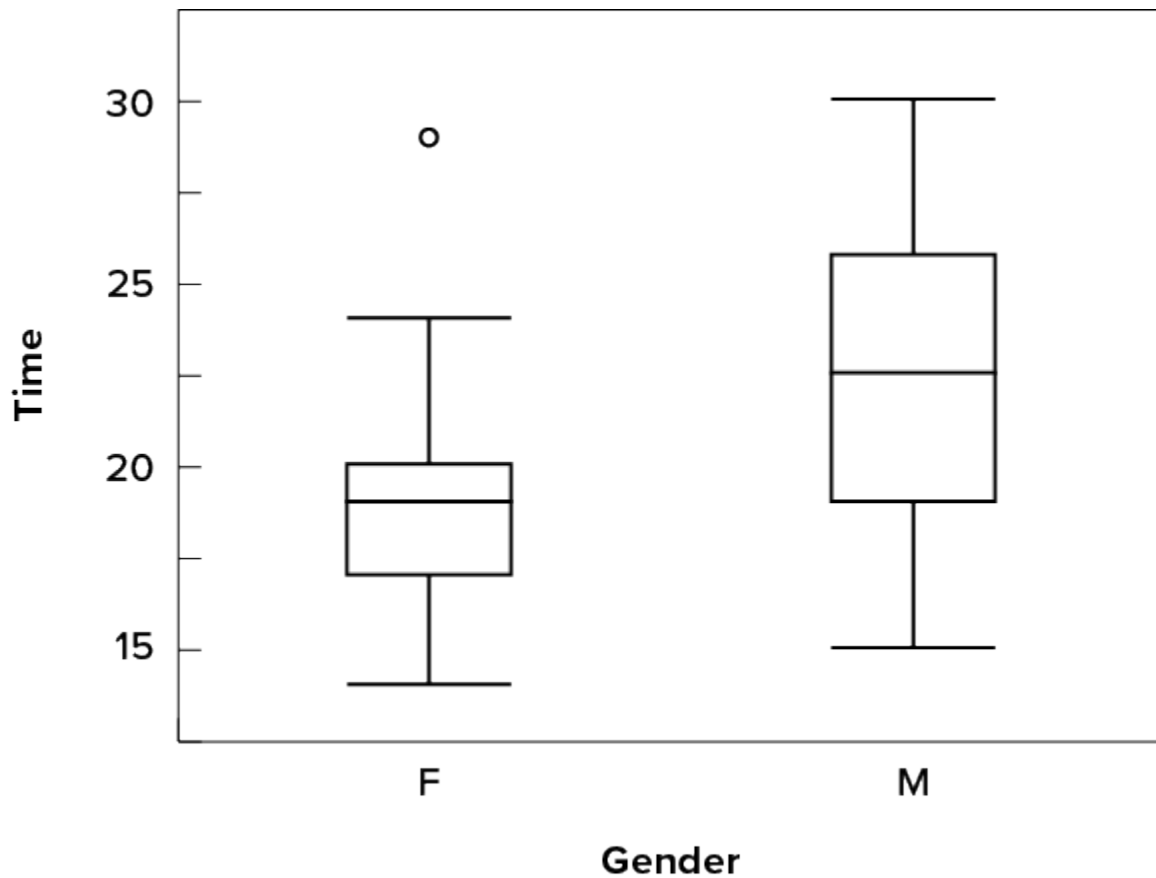


Figure 6.2.31. The box plots with the outside value shown. (“Box Plot Outside Value” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

There is one more mark to include in box plots (although sometimes it is omitted). We indicate the mean score for a group by inserting a plus sign. Figure 6.2.32 shows the result of adding means to our box plots.

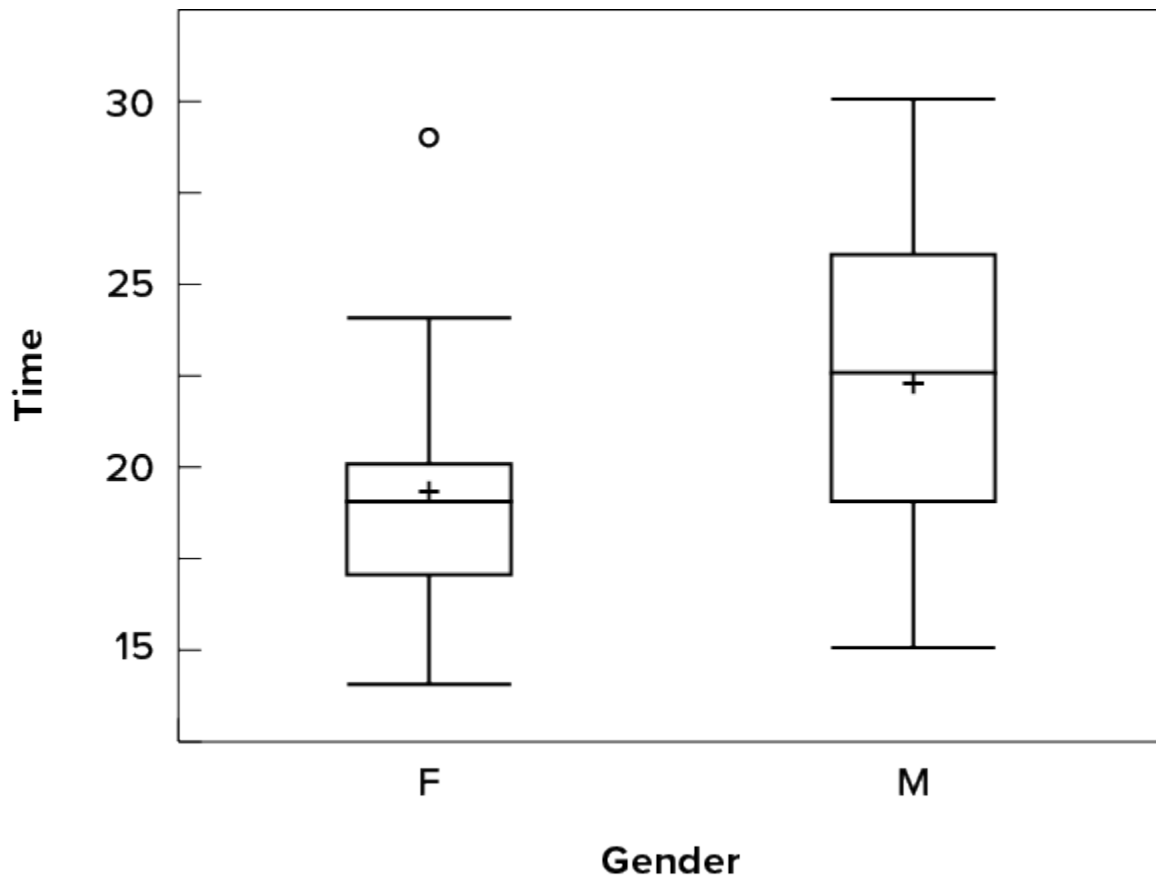


Figure 6.2.32. The completed box plots. (“Box Plot Mean Scores” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

Figure 6.2.32 provides a revealing summary of the data. Since half the scores in a distribution are between the hinges (recall that the hinges are the  $25^{th}$  and  $75^{th}$  percentiles), we see that half the women’s times are between 17 and 20 seconds whereas half the men’s times are between 19 and 25.5 seconds. We also see that women generally named the colours faster than the men did, although one woman was slower than almost all of the men. Figure 6.2.33 shows the box plot for the women’s data with detailed labels.



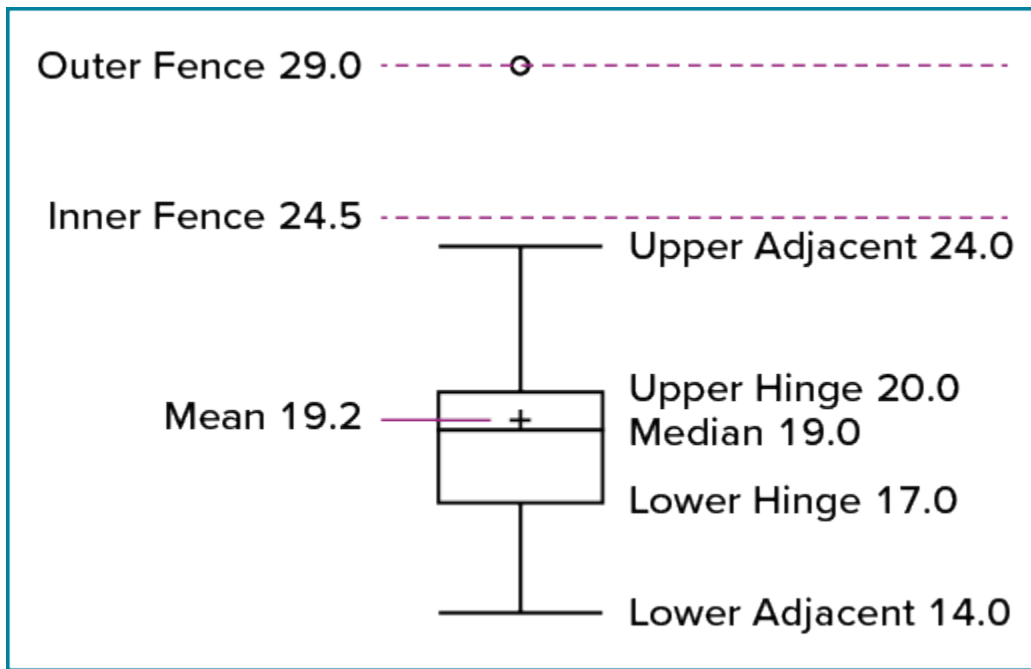


Figure 6.2.33. The box plots for the women's data with detailed labels. ("Women's Data Labeled Box Plot" by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

Box plots provide basic information about a distribution. For example, a distribution with a positive skew would have a longer whisker in the positive direction than in the negative direction. A larger mean than median would also indicate a positive skew. Box plots are good at portraying extreme values and are especially good at showing differences between distributions. However, many of the details of a distribution are not revealed in a box plot; to examine these details one should create a histogram and/or a stem-and-leaf display.

## Try It

7) Which of the box plots on the graph has a large positive skew? Which has a large negative skew?

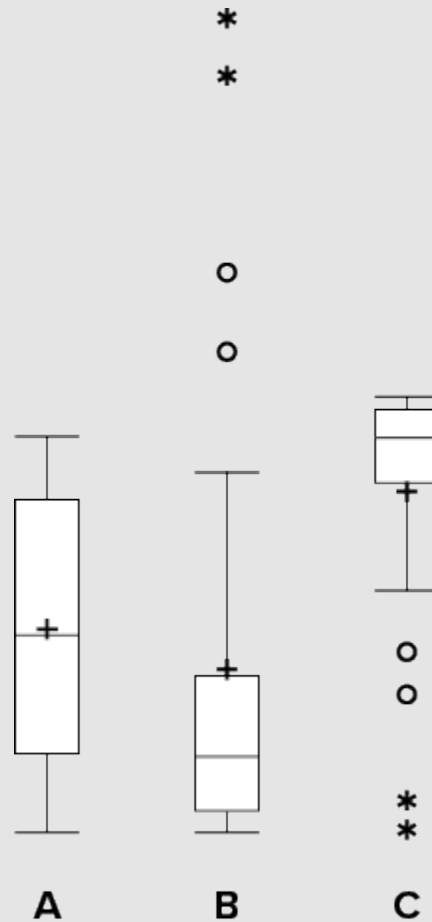


Figure 6.2.34. “Skewed Box Plots” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.

### Solution

Chart B has the positive skew because the outliers (dots and asterisks) are on the upper (higher) end; Chart C has the negative skew because the outliers are on the lower end.

## Bar Charts

In the section on qualitative variables, we saw how bar charts could be used to illustrate the frequencies of different categories. For example, as we saw earlier in this chapter, the bar chart shown in Figure 6.2.3 shows how many purchasers of iMac computers were previous Macintosh users, previous Windows users, and new computer purchasers.

In this section we show how bar charts can be used to present other kinds of quantitative information, not

just frequency counts. The bar chart in Figure 6.2.35 shows the percent increases in the Dow Jones, Standard & Poor 500 (S&P), and Nasdaq stock indexes from May 24, 2000, to May 24, 2001. Notice that both the S&P and the Nasdaq had “negative increases” which means that they decreased in value. In this bar chart, the  $y$ -axis is not frequency but rather the signed quantity percentage increase.

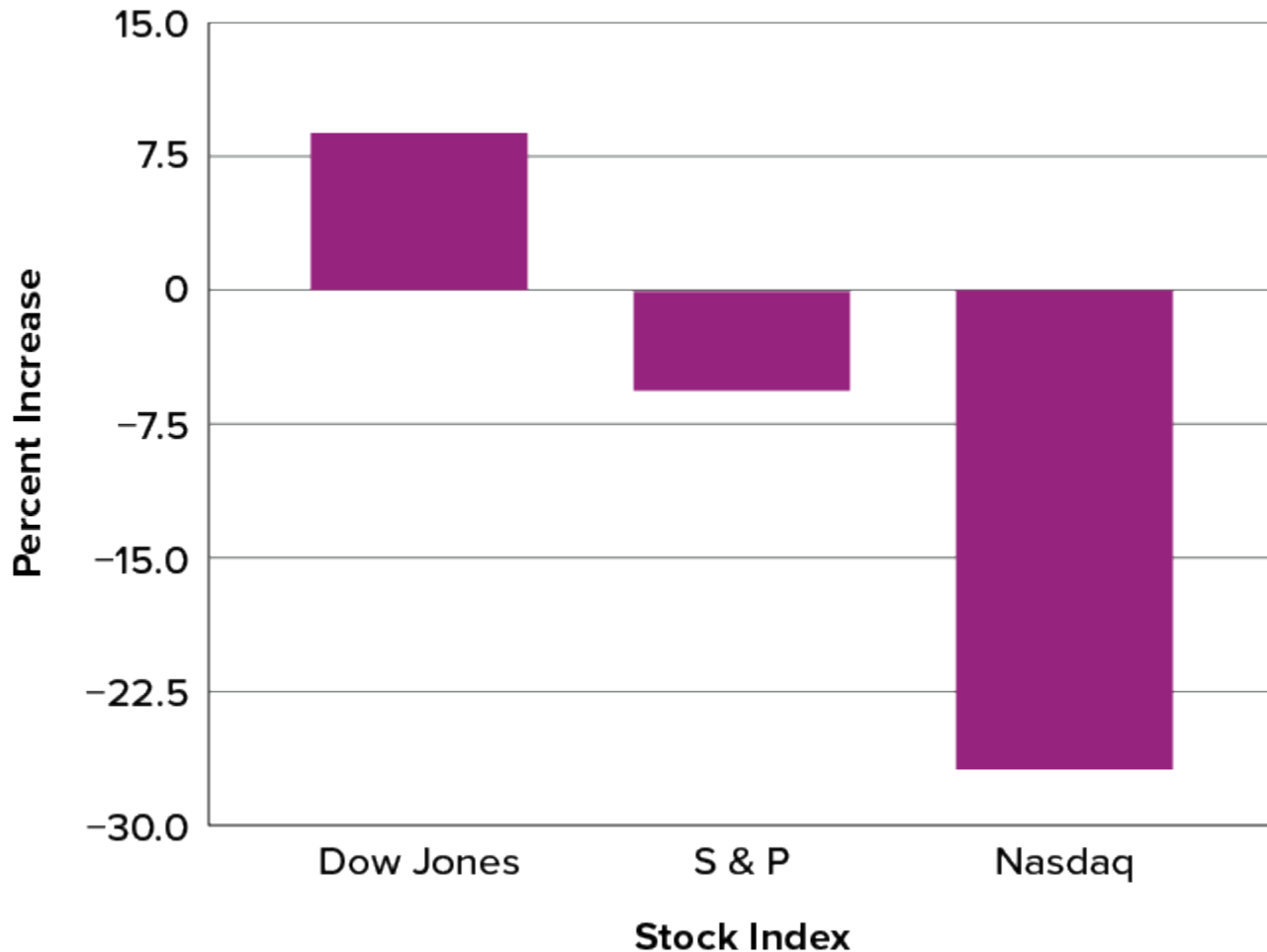


Figure 6.2.35. Percent increase in three stock indexes from May 24, 2000, to May 24, 2001. (“Percent Increase in Stock Indexes” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

Bar charts are particularly effective for showing change over time. Figure 6.2.36, for example, shows the percent increase in the Consumer Price Index (CPI) over four three-month periods. The fluctuation in inflation is apparent in the graph.

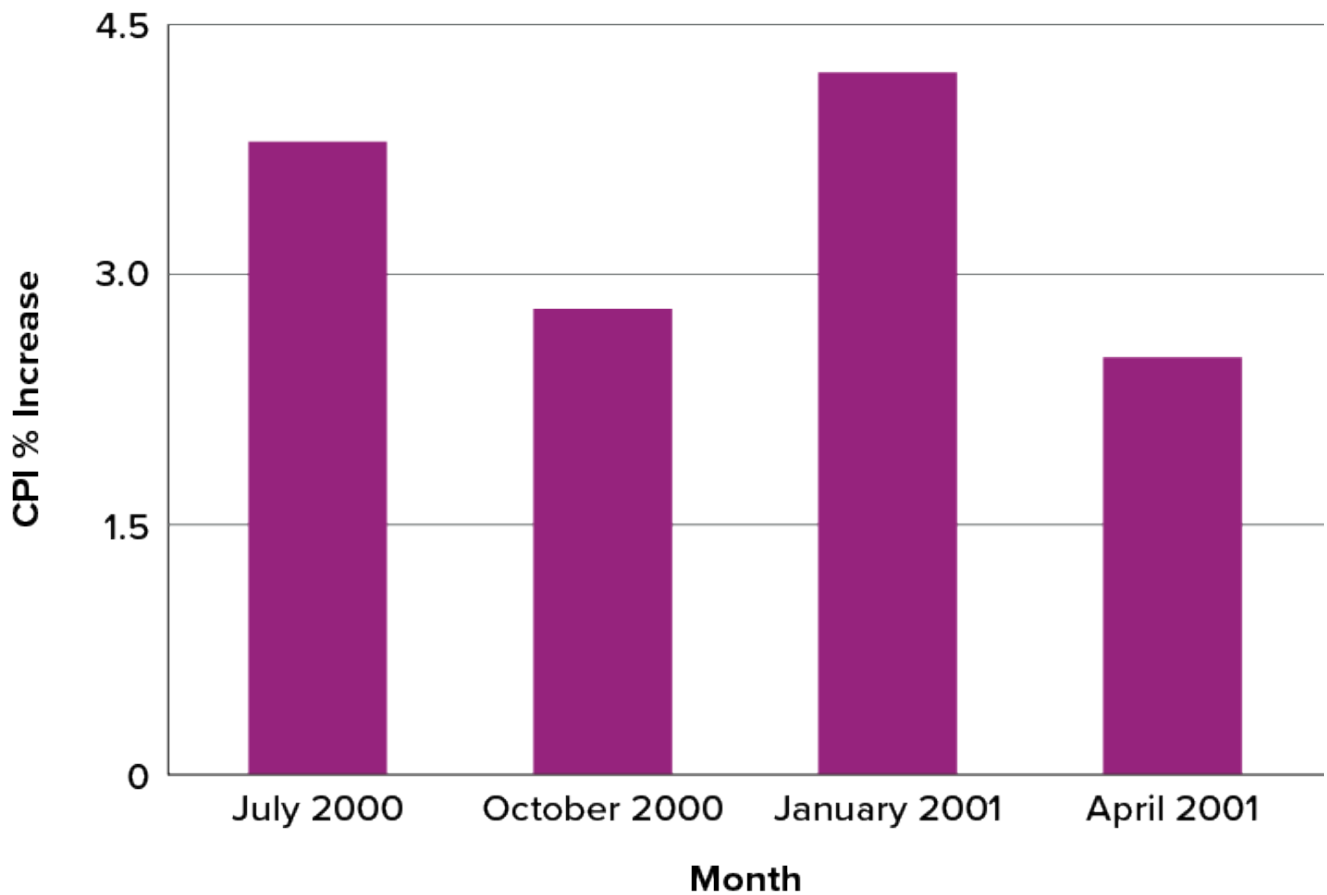


Figure 6.2.36. Percent change in the CPI over time. Each bar represents percent increase for the three months ending at the date indicated. (“Percent Change in CPI” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

Bar charts are often used to compare the means of different experimental conditions. Figure 6.2.37 shows the mean time it took one person to move the cursor to either a small target or a large target. On average, more time was required for small targets than for large ones.

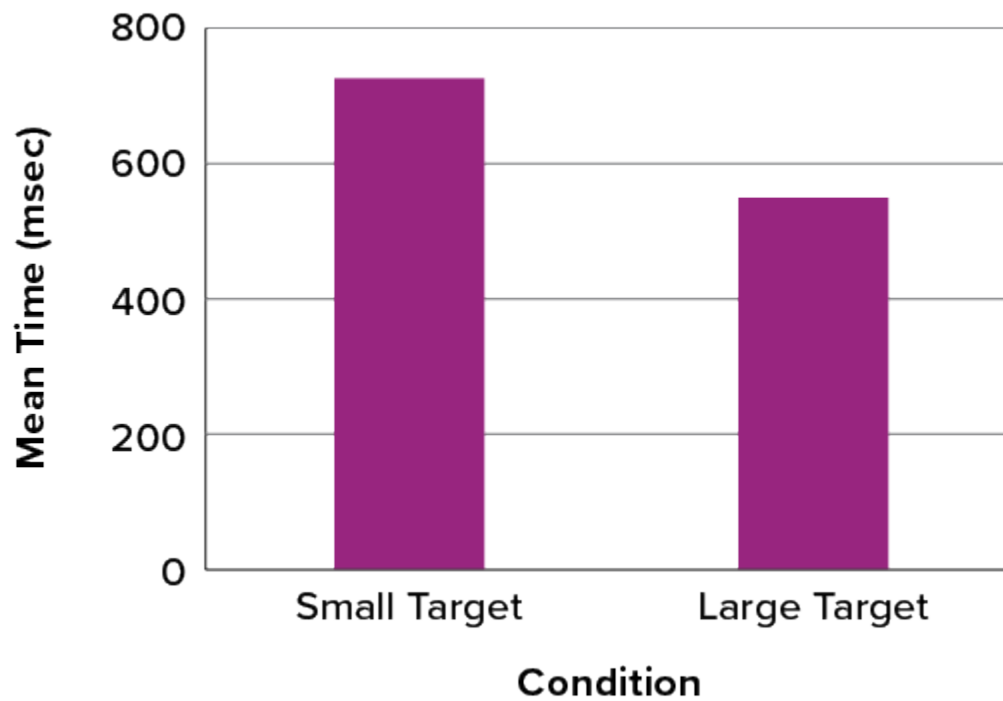


Figure 6.2.37. Bar chart showing the means for the two conditions. (“Means of Two Conditions” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

Although bar charts can display means, we do not recommend them for this purpose. Box plots should be used instead since they provide more information than bar charts without taking up more space. For example, a box plot of the cursor-movement data is shown in Figure 6.2.38. You can see that Figure 6.2.38 reveals more about the distribution of movement times than does Figure 6.2.37.

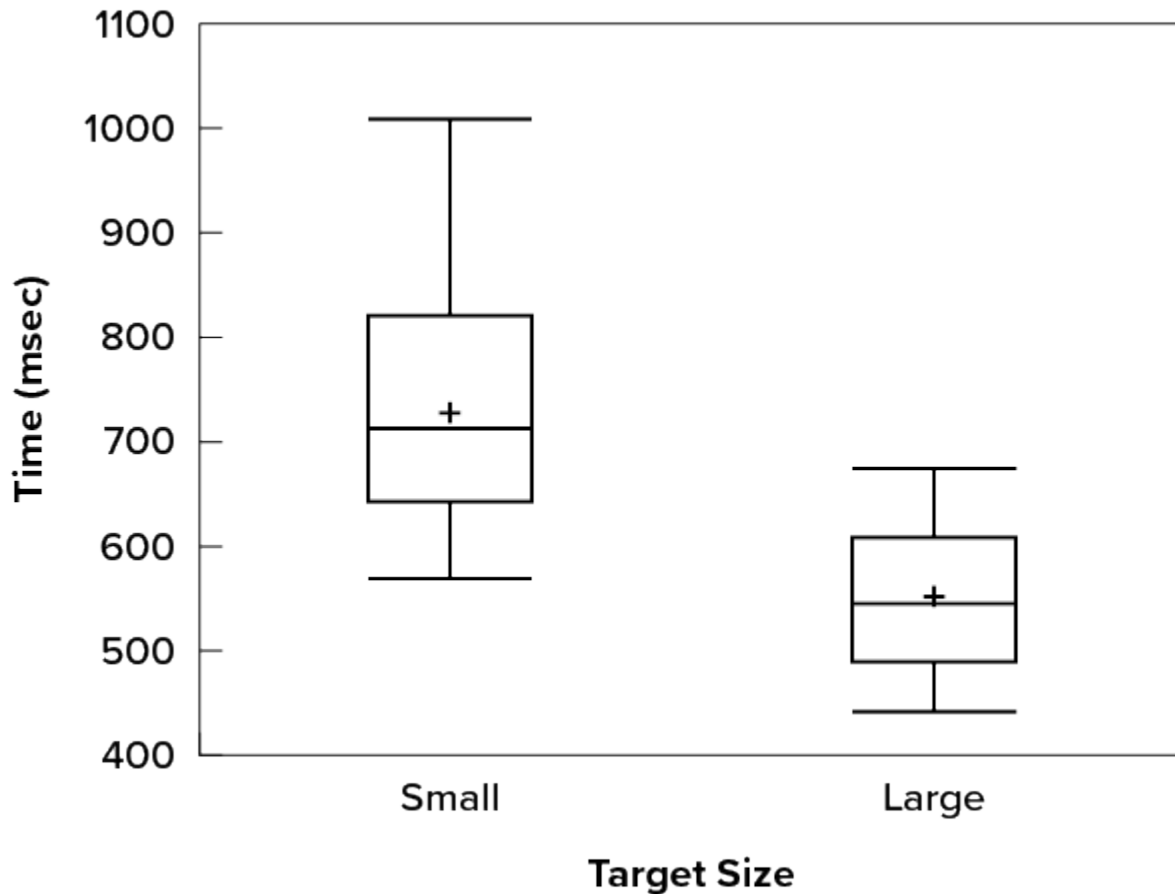


Figure 6.2.38. Box plots of times to move the cursor to the small and large targets. (“Cursor Task Box Plot” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

The section on qualitative variables presented earlier in this chapter discussed the use of bar charts for comparing distributions. Some common graphical mistakes were also noted. The earlier discussion applies equally well to the use of bar charts to display quantitative variables.

## Try It

8) Explain the differences between bar charts and histograms. When would each be used?

### Solution

In bar charts, the bars do not touch; in histograms, the bars do touch. Bar charts are appropriate for qualitative variables, whereas histograms are better for quantitative variables.

## Line Graphs

A line graph is a bar graph with the tops of the bars represented by points joined by lines (the rest of the bar is suppressed). For example, Figure 6.2.36, which was presented in the section on bar charts, shows changes in the Consumer Price Index (CPI) over time. A line graph of these same data is shown in Figure 6.2.39. Although the figures are similar, the line graph emphasizes the change from period to period.



Figure 6.2.39. A line graph of the percent change in the CPI over time. Each point represents percent increase for the three months ending at the date indicated. (“Percent Change in CPI Line Graph” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

Line graphs are appropriate only when both the  $x$ - and  $y$ -axes display ordered (rather than qualitative) variables. Although bar charts can also be used in this situation, line graphs are generally better at comparing changes over time. Figure 6.2.40, for example, shows percent increases and decreases in five components of the CPI. The figure makes it easy to see that medical costs had a steadier progression than the other components. Although you could create an analogous bar chart, its interpretation would not be as easy.

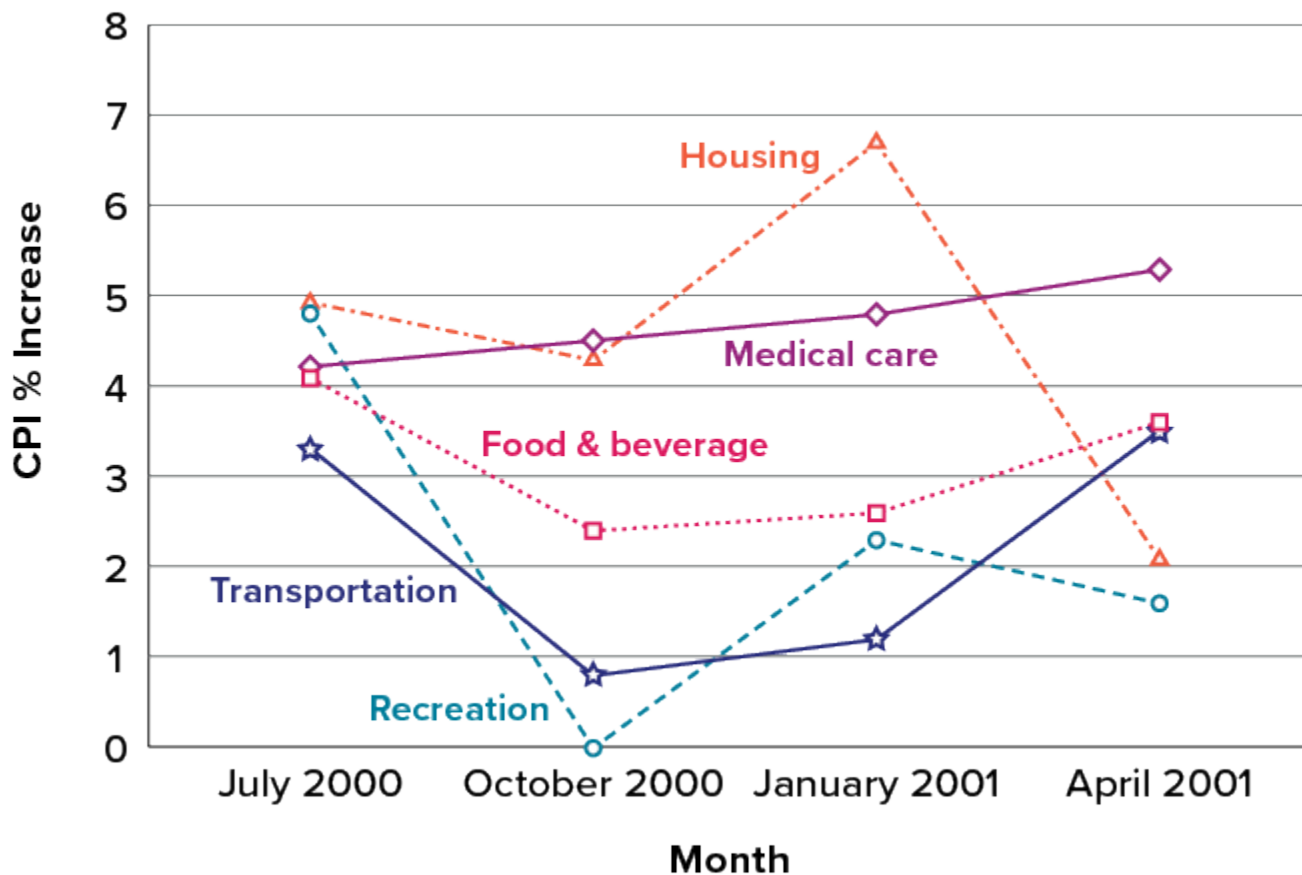


Figure 6.2.40. A line graph of the percent change in five components of the CPI over time. (“Percent Change in CPI x5 Line Graph” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

Let us stress that it is misleading to use a line graph when the  $x$ -axis contains merely qualitative variables. As we saw earlier in this chapter, Figure 6.2.8 inappropriately shows a line graph of the card game data from Yahoo, discussed in the section on qualitative variables. The defect in Figure 6.2.8 is that it gives the false impression that the games are naturally ordered in a numerical way.



## Try It

9) Name some ways to graph quantitative variables and some ways to graph qualitative variables.

### **Solution**

Qualitative variables are displayed using pie charts and bar charts. Quantitative variables are displayed as box plots, histograms, etc.

## The Shape of Distribution

Finally, it is useful to present discussion on how we describe the shapes of distributions, to learn how different shapes affect our numerical descriptors of data and distributions.

The primary characteristic we are concerned about when assessing the shape of a distribution is whether the distribution is symmetrical or skewed. A symmetrical distribution, as the name suggests, can be cut down the centre to form two mirror images. Although in practice we will never get a perfectly symmetrical distribution, we would like our data to be as close to symmetrical as possible. Many types of distributions are symmetrical, but by far the most common and pertinent distribution at this point is the normal distribution, shown in Figure 6.2.41. Notice that although the symmetry is not perfect (for instance, the bar just to the right of the centre is taller than the one just to the left), the two sides are roughly the same shape. The normal distribution has a single peak, known as the centre, and two tails that extend out equally, forming what is known as a bell shape or **bell curve**.

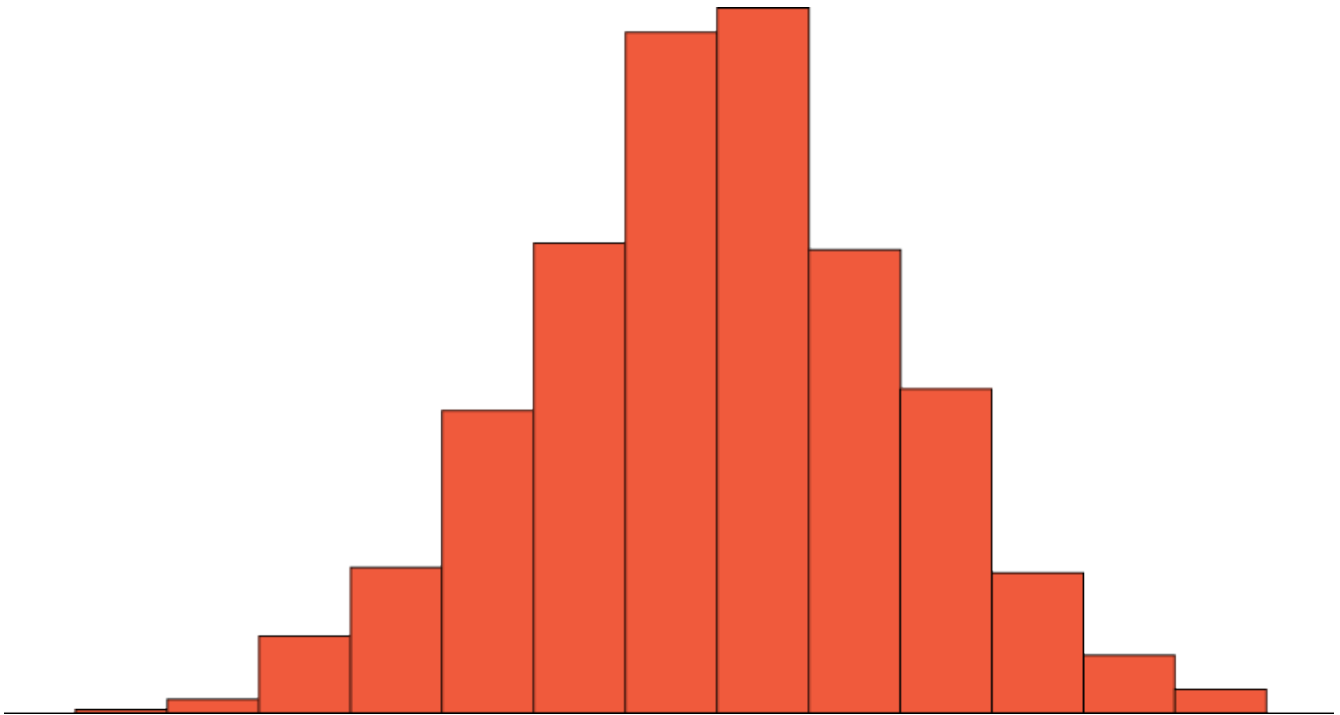


Figure 6.2.41. A symmetrical distribution. (“Symmetrical Distribution” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

Symmetrical distributions can also have multiple peaks. Figure 6.2.42 shows a **bimodal distribution**, named for the two peaks that lie roughly symmetrically on either side of the centre point. As we will see, this is not a particularly desirable characteristic of our data, and, worse, this is a relatively difficult characteristic to detect numerically. Thus, it is important to visualize your data before moving ahead with any formal analyses.

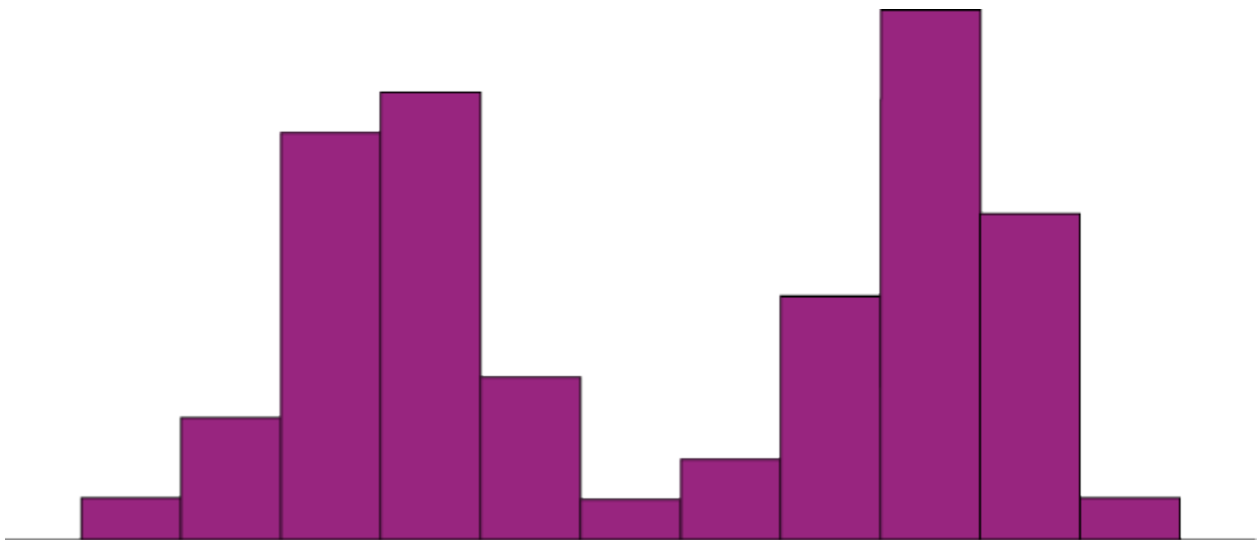
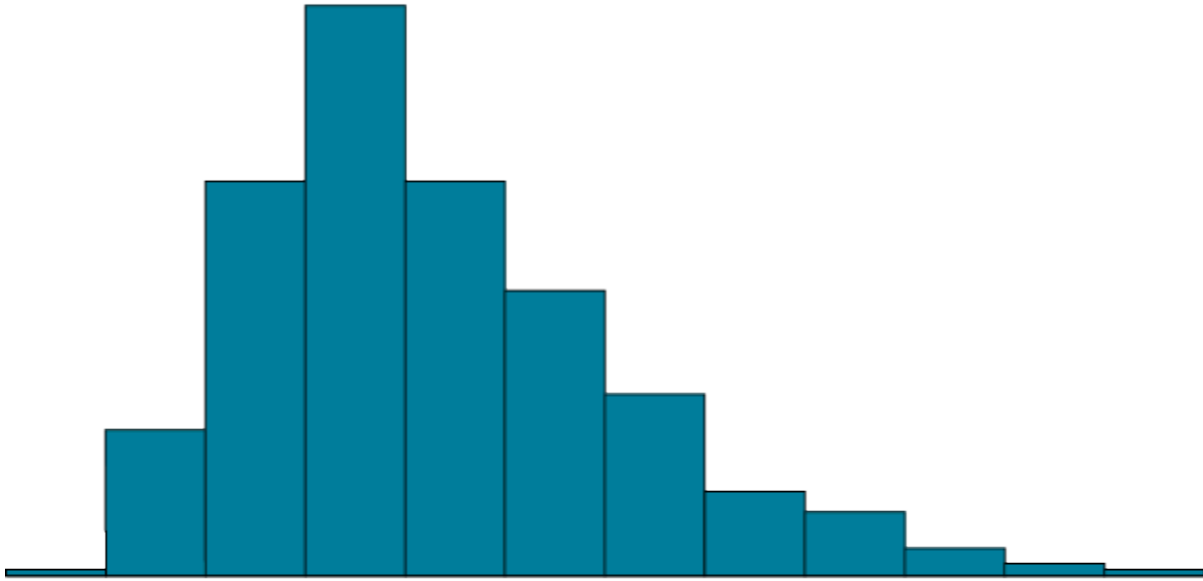


Figure 6.2.42. A bimodal distribution. (“Bimodal Distribution” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

Distributions that are not symmetrical also come in many forms, more than can be described here. The most common asymmetry to be encountered is referred to as skew, in which one of the two tails of the distribution is disproportionately longer than the other. This property can affect the value of the averages we use in our analyses and make them an inaccurate representation of our data, which causes many problems.

Skew can either be positive or negative (also known as right or left, respectively), based on which tail is longer. It is very easy to get the two confused at first; many students want to describe the skew by where the bulk of the data (larger portion of the histogram, known as the body) is placed, but the correct determination is based on which tail is longer. You can think of the tail as an arrow; whichever direction the arrow is pointing is the direction of the skew. Figure 6.2.43 shows positive (right) and negative (left) skew, respectively.

### A. Positive Skew



### B. Negative Skew

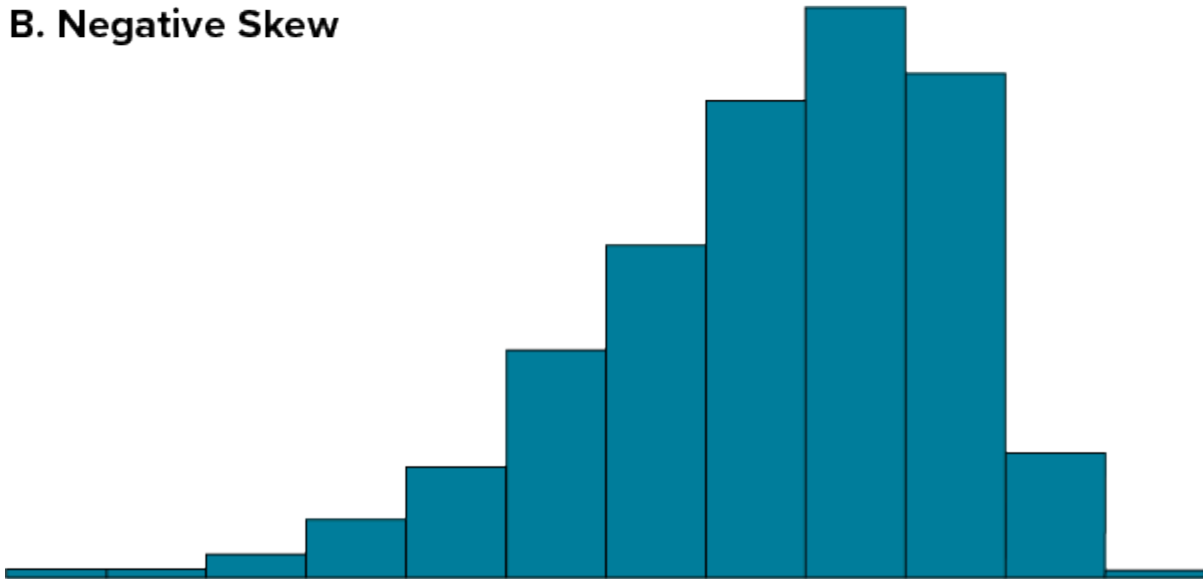


Figure 6.2.43. Positively skewed (A) and negatively skewed (B) distributions. (“Skewed Distributions” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

## Try It

10) Draw a histogram of a distribution that is

1. Negatively skewed
2. Symmetrical
3. Positively skewed

### Solution

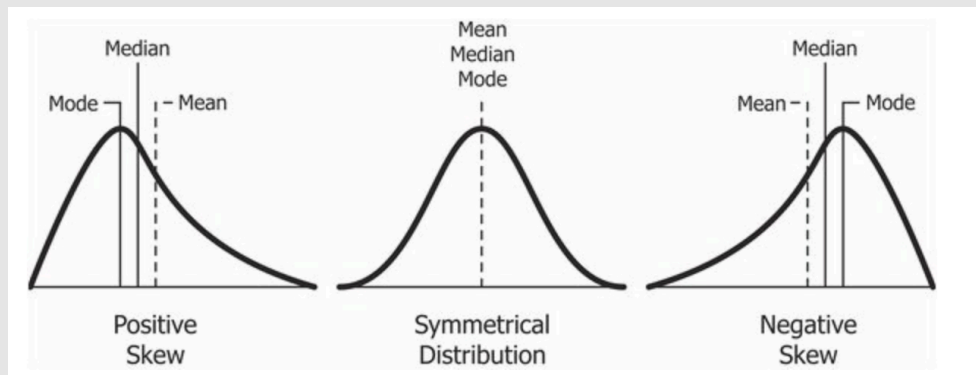


Figure 6.2.44  
“Relationship  
between mean and  
median under  
different  
skewness” by Diva  
Jain, CC BY-SA-4.0

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it*

online here:

<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=1685#h5p-39>

b) After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

## Glossary

### **bell curve**

The bell curve is a symmetrical distribution in which there is a single peak at the center and tails that extend equally out to each side. The bell curve represents a normal distribution.

### **bimodal distribution**

A distribution with two distinct peaks that lie roughly symmetrically on either side of the center point.

### **bin widths**

The widths of the class intervals. The choice of bin width determines the number of class intervals. This decision, along with the choice of starting point for the first interval, affects the shape of the histogram.

### **box plots**

One of the more effective graphical summaries of a data set, the box plot generally shows the median, 25th and 75th percentiles, and outliers.

### **categorical variables**

Also known as qualitative variables, categorical variables cannot be quantified, or measured numerically. Instead, they are measured on a nominal or ordinal scale.

### **frequency polygons**

A frequency polygon is a graphical representation of a distribution that is similar in appearance to a line graph. Frequency polygons can be grouped or ungrouped.

**histogram**

A graphical representation of a distribution that is similar in appearance to a bar chart. It partitions the variable on the x-axis into various contiguous class intervals of (usually) equal widths. The heights of the bars represent the class frequencies.

**lie factor**

The ratio of the size of the effect shown in a graph to the size of the effect shown in the data. This term was coined by Edward Tufte, who suggested that lie factors greater than 1.05 or less than 0.95 produce unacceptable distortion.

**skew**

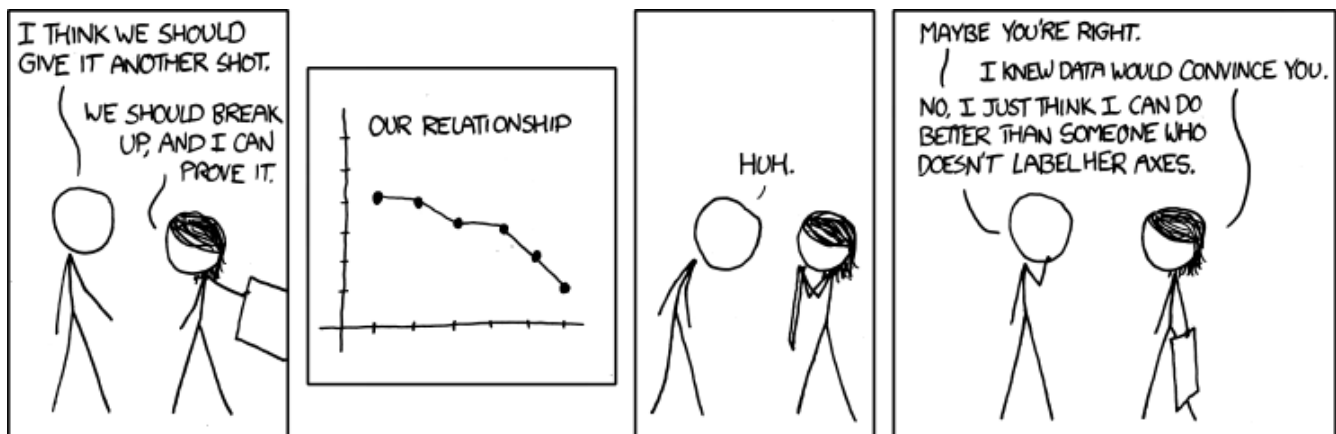
A distribution is skewed if one tail extends out further than the other, making the distribution asymmetrical. A distribution has a positive skew (is skewed to the right) if the tail to the right is longer. A distribution has a negative skew (is skewed to the left) if the tail to the left is longer.

**stem-and-leaf display**

A quasi-graphical representation of numerical data. Generally, all but the final digit of each value is a stem, and the final digit is the leaf. The stems are placed in a vertical list, with each matched leaf on one side.

**whiskers**

Vertical lines ending in a horizontal stroke that are added to box plots to indicate the spread of the data points. Whiskers are drawn from the upper and lower hinges to the upper and lower adjacent values.



“Convincing” by Randall Munroe/xkcd.com CC BY-NC 2.5.

## 6.3 MEASURES OF CENTRAL TENDENCY AND SPREAD

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### Learning Objectives

By the end of this section, you will be able to:

- Explain why central tendency and dispersion are important.
- Calculate the mean, median and mode for a set of data and explain what these measures represent.
- Identify the types of variables that the mean, median, and mode are most appropriate.
- Describe skewness and how it affects the mean.
- Define the term outlier and its impact on central tendency and dispersion.
- Calculate variance, standard deviation, range, inter-quartile range.

Now that we have visualized our data to understand its shape, we can begin with numerical analyses. The descriptive statistics presented in this chapter serve to describe the distribution of our data objectively and mathematically—our first step into statistical analysis! The topics here will serve as the basis for everything we do in the rest of the course.

### What Is Central Tendency?

What is **central tendency**, and why do we want to know the central tendency of a group of scores? Let us first try to answer these questions intuitively. Then we will proceed to a more formal discussion.

Imagine this situation: You are in a class with just four other students, and the five of you took a 5-point pop quiz. Today your instructor is walking around the room, handing back the quizzes. She stops at your desk and hands you your paper. Written in bold black ink on the front is “ $\frac{3}{5}$ .” How do you react? Are you happy with



your score of **3** or disappointed? How do you decide? You might calculate your percentage correct, realize it is **60%**, and be appalled. But it is more likely that when deciding how to react to your performance, you will want additional information. What additional information would you like?

If you are like most students, you will immediately ask your classmates, “What’d ya get?” and then ask the instructor, “How did the class do?” In other words, the additional information you want is how your quiz score compares to other students’ scores. You therefore understand the importance of comparing your score to the class distribution of scores. Should your score of **3** turn out to be among the higher scores, then you’ll be pleased after all. On the other hand, if **3** is among the lower scores in the class, you won’t be quite so happy.

This idea of comparing individual scores to a distribution of scores is fundamental to statistics. So let’s explore it further, using the same example (the pop quiz you took with your four classmates). Three possible outcomes are shown in Table 6.3.1. They are labelled “Dataset A,” “Dataset B,” and “Dataset C.” Which of the three datasets would make you happiest? In other words, in comparing your score with your fellow students’ scores, in which dataset would your score of **3** be the most impressive?

**Table 6.3.1. Three possible datasets for the 5-point make-up quiz:**

Student	Dataset A	Dataset B	Dataset C
You	<b>3</b>	<b>3</b>	<b>3</b>
Ahmed	<b>3</b>	4	2
Rosa	<b>3</b>	4	2
Tamika	<b>3</b>	4	2
Luther	<b>3</b>	5	1

In Dataset A, everyone’s score is **3**. This puts your score at the exact centre of the distribution. You can draw satisfaction from the fact that you did as well as everyone else. But of course it cuts both ways: everyone else did just as well as you.

Now consider the possibility that the scores are described as in Dataset B. This is a depressing outcome even though your score is no different than the one in Dataset A. The problem is that the other four students had higher grades, putting yours below the centre of the distribution.

Finally, let's look at Dataset C. This is more like it! All of your classmates score lower than you, so your score is above the centre of the distribution.

Now let's change the example in order to develop more insight into the centre of a distribution. Figure 6.3.1 shows the results of an experiment on memory for chess positions. Subjects were shown a chess position and then asked to reconstruct it on an empty chess board. The number of pieces correctly placed was recorded. This was repeated for two more chess positions. The scores represent the total number of chess pieces correctly placed for the three chess positions. The maximum possible score was 89.

	8	05
	7	156
	6	233
	5	168
330	4	06
9420	3	
622	2	

Figure 6.3.1. Back-to-back stem-and-leaf display. The left side shows the memory scores of the non-players. The right side shows the scores of the tournament players. ("Memory Scores Back-to-Back Stem and Leaf" by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

Two groups are compared. On the left are people who don't play chess. On the right are people who play a great deal (tournament players). It is clear that the location of the centre of the distribution for the non-players is much lower than the centre of the distribution for the tournament players.

We're sure you get the idea now about the centre of a distribution. It is time to move beyond intuition. We need a formal definition of the centre of a distribution. In fact, we'll offer you three definitions! This is not just generosity on our part. There turn out to be (at least) three different ways of thinking about the centre of a distribution, all of them useful in various contexts. In the remainder of this section we attempt to communicate the idea behind each concept. In the succeeding sections we will give statistical measures for these concepts of central tendency.

## Definitions of Centre

Now we explain the three measures of central tendency: (1) the point on which a distribution will balance, (2) the value whose average absolute deviation from all the other values is minimized, and (3) the value whose squared deviation from all the other values is minimized.

### Balance Scale

One definition of central tendency is the point at which the distribution is in balance. Figure 6.3.2 shows the distribution of the five numbers **2, 3, 4, 9, 16** placed upon a balance scale. If each number weighs one pound, and is placed at its position along the number line, then it would be possible to balance them by placing a fulcrum at a particular point.

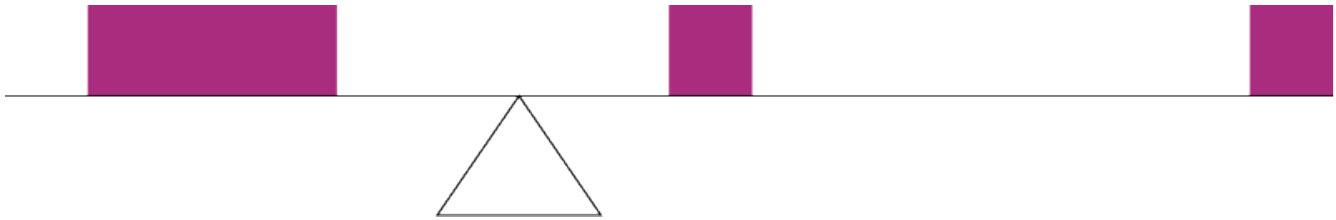


Figure 6.3.2. A balance scale. (“Balance Scale” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

For another example, consider the distribution shown in Figure 6.3.3. It is balanced by placing the fulcrum in the geometric middle.

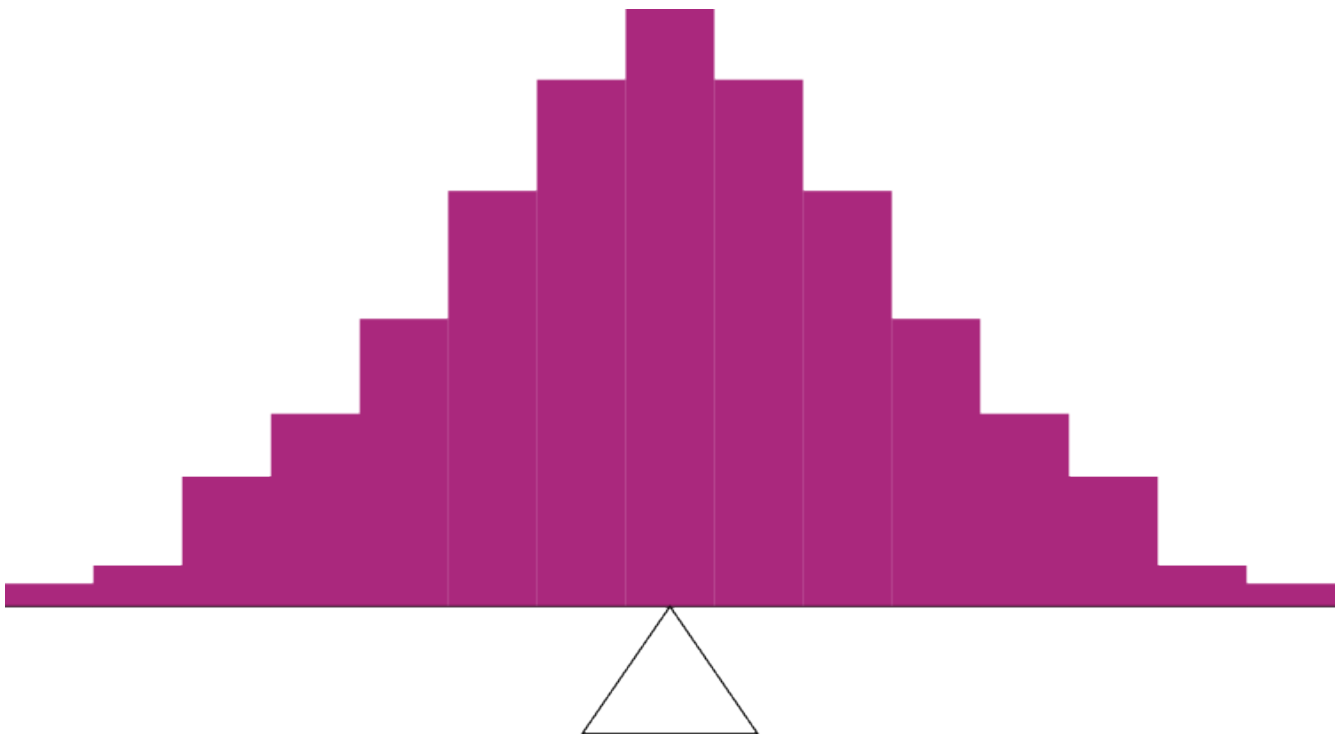


Figure 6.3.3. A distribution balanced on the tip of a triangle. (“Balanced Distribution” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

Figure 6.3.4 illustrates that the same distribution can't be balanced by placing the fulcrum to the left of centre.

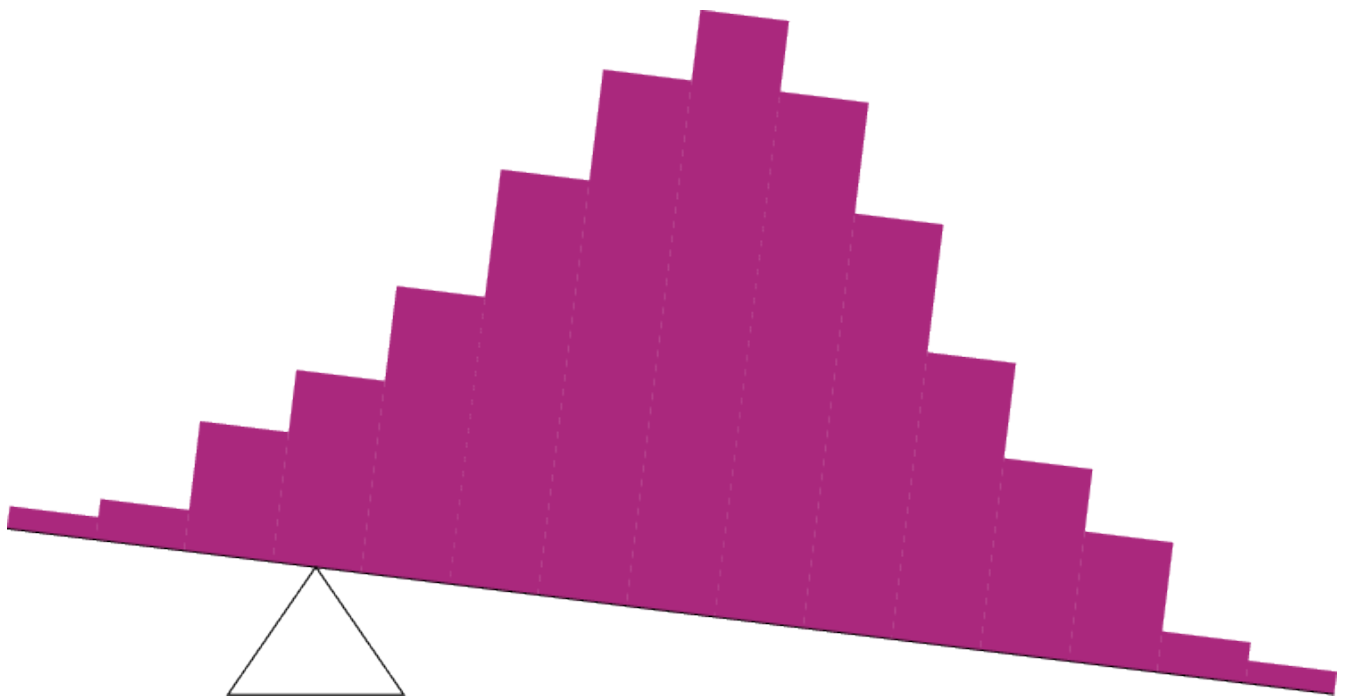


Figure 6.3.4. The distribution is not balanced. (“Unbalanced Distribution” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

Figure 6.3.5 shows an asymmetric distribution. To balance it, we cannot put the fulcrum halfway between the lowest and highest values (as we did in Figure 6.3.3). Placing the fulcrum at the “half way” point would cause it to tip towards the left.

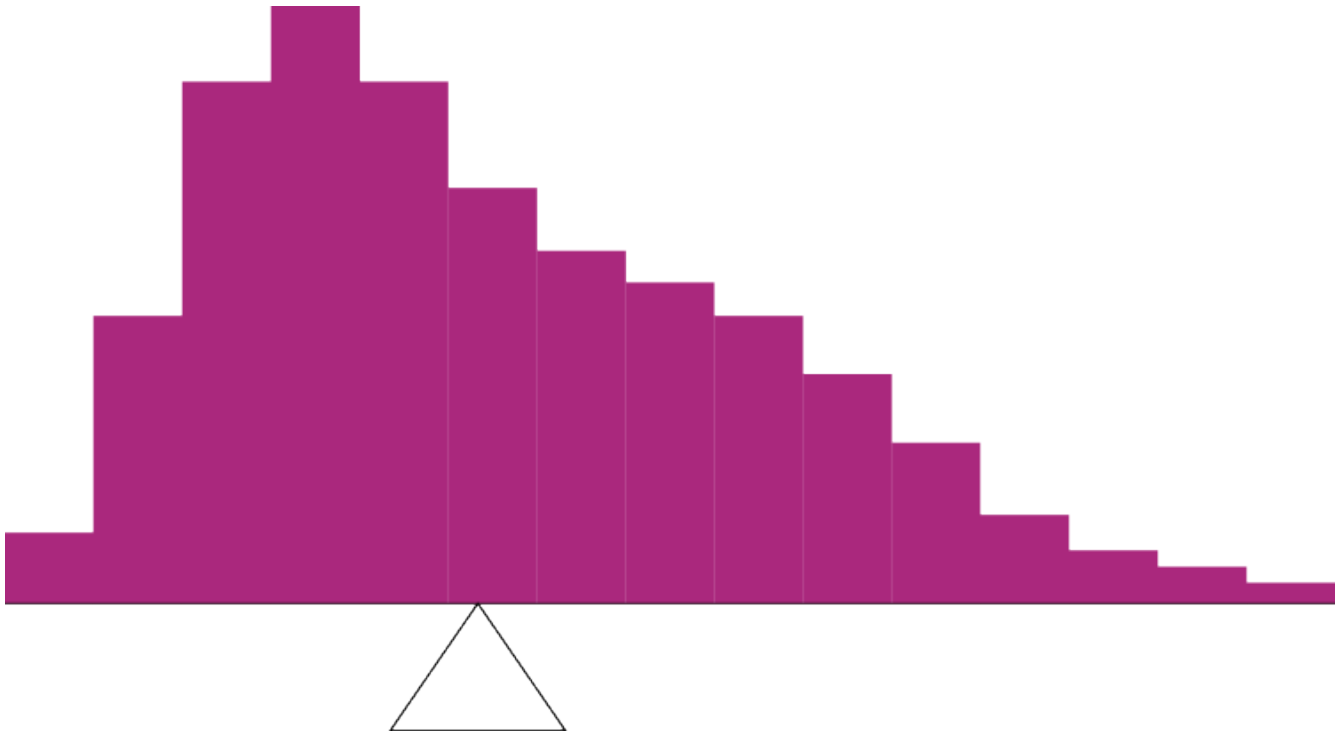


Figure 6.3.5. An asymmetric distribution balanced on the tip of a triangle. (“Asymmetric Distribution” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

## Smallest Absolute Deviation

Another way to define the centre of a distribution is based on the concept of the sum of the absolute deviations (differences). Consider the distribution made up of the five numbers **2, 3, 4, 9, 16**. Let’s see how far the distribution is from **10** (picking a number arbitrarily). Table 6.3.2 shows the sum of the absolute deviations of these numbers from the number **10**.

**Table 6.3.2. An example of the sum of absolute deviations.**

Values	Absolute Deviations from 10
2	8
3	7
4	6
9	1
16	6
<b>Sum</b>	<b>28</b>

The first row of the table shows that the absolute value of the difference between **2** and **10** is **8**; the second row shows that the absolute difference between **3** and **10** is **7**, and similarly for the other rows. When we add up the five absolute deviations, we get **28**. So, the sum of the absolute deviations from **10** is **28**. Likewise, the sum of the absolute deviations from **5** equals  $3 + 2 + 1 + 4 + 11 = 21$ . So, the sum of the absolute deviations from **5** is smaller than the sum of the absolute deviations from **10**. In this sense, **5** is closer, overall, to the other numbers than is **10**.

We are now in a position to define a second measure of central tendency, this time in terms of absolute deviations. Specifically, according to our second definition, the centre of a distribution is the number for which the sum of the absolute deviations is smallest. As we just saw, the sum of the absolute deviations from **10** is **28** and the sum of the absolute deviations from **5** is **21**. Is there a value for which the sum of the absolute deviations is even smaller than **21**? Yes. For these data, there is a value for which the sum of absolute deviations is only **20**. See if you can find it.

### Smallest Squared Deviation

We shall discuss one more way to define the centre of a distribution. It is based on the concept of the sum of squared deviations (differences). Again, consider the distribution of the five numbers **2, 3, 4, 9, 16**. Table 6.3.3 shows the sum of the squared deviations of these numbers from the number **10**.

**Table 6.3.3. An example of the sum of squared deviations.**

Values	Squared Deviations from 10
2	64
3	49
4	36
9	1
16	36
<b>Sum</b>	<b>186</b>

The first row in the table shows that the squared value of the difference between **2** and **10** is **64**; the second row shows that the squared difference between **3** and **10** is **49**, and so forth. When we add up all these squared deviations, we get **186**.

Changing the target from **10** to **5**, we calculate the sum of the squared deviations from **5** as  $9 + 4 + 1 + 16 + 121 = 151$ . So, the sum of the squared deviations from **5** is smaller than the sum of the squared deviations from **10**. Is there a value for which the sum of the squared deviations is even smaller than **151**? Yes, it is possible to reach **134.8**. Can you find the target number for which the sum of squared deviations is **134.8**?

The target that minimizes the sum of squared deviations provides another useful definition of central tendency (the last one to be discussed in this section). It can be challenging to find the value that minimizes this sum.

## Measures of Central Tendency

In the previous section we saw that there are several ways to define central tendency. This section defines the three most common measures of central tendency: the **mean**, the **median**, and the **mode**. The relationships among these measures of central tendency and the definitions given in the previous section will probably not be obvious to you.

This section gives only the basic definitions of the mean, median and mode.

## Arithmetic Mean

The **arithmetic mean**—the sum of the numbers divided by the number of numbers—is the most common measure of central tendency. The symbol “ $\mu$ ” (pronounced “mew”) is used for the mean of a population. The symbol  $M$  is used for the mean of a sample. (In advanced statistics textbooks, the symbol  $\bar{X}$ , pronounced “x bar,” may be used to represent the mean of a sample.) The formula for  $\mu$  is shown below:

$$\mu = \frac{\sum X}{N}$$

where  $\sum X$  is the sum of all the numbers in the population and  $N$  is the number of numbers in the population.

The formula for  $M$  is essentially identical:

$$M = \frac{\sum X}{n}$$

where  $\sum X$  is the sum of all the numbers in the sample and  $n$  is the number of numbers in the sample.

The only distinction between these two equations is whether we are referring to the population (in which case we use  $\mu$  and  $N$ ) or a sample of that population (in which case we use  $M$  and  $n$ ).

As an example, the mean of the numbers 1, 2, 3, 6, 8 is  $\frac{20}{5} = 4$  regardless of whether the numbers constitute the entire population or just a sample from the population.

Figure 6.3.6 shows the number of touchdown (TD) passes thrown by each of the 31 teams in the National Football League in the 2000 season. The mean number of touchdown passes thrown is 20.45, as shown below.

$$\mu = \frac{\sum X}{N} = \frac{634}{31} = 20.45$$

Although the arithmetic mean is not the only “mean” (there is also a geometric mean, a harmonic mean, and many others that are all beyond the scope of this course), it is by far the most commonly used. Therefore, if the term “mean” is used without specifying whether it is the arithmetic mean, the geometric mean, or some other mean, it is assumed to refer to the arithmetic mean.

37, 33, 33, 32, 29, 28,
28, 23, 22, 22, 22, 21,
21, 21, 20, 20, 19, 19,
18, 18, 18, 18, 16, 15,
14, 14, 14, 12, 12, 9, 6

Figure 6.3.6. Number of touchdown passes. (“Touchdown Passes Raw Data” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)



## Try It

1) If the mean time to respond to a stimulus is much higher than the median time to respond, what can you say about the shape of the distribution of response times?

### Solution

If the mean is higher, that means it is farther out into the right-hand tail of the distribution. Therefore, we know this distribution is positively skewed.

## Median

The **median** is also a frequently used measure of central tendency. The median is the midpoint of a distribution: the same number of scores is above the median as below it. For the data in Figure 6.3.6, there are 31 scores. The 16<sup>th</sup> highest score (which equals 20) is the median because there are 15 scores below the 16<sup>th</sup> score and 15 scores above the 16<sup>th</sup> score. The median can also be thought of as the 50<sup>th</sup> percentile.

When there is an odd number of numbers, the median is simply the middle number. For example, the median of 2, 4, and 7 is 4. When there is an even number of numbers, the median is the mean of the two middle numbers. Thus, the median of the numbers 2, 4, 7, 12 is:

$$\frac{4 + 7}{2} = 5.5$$

When there are numbers with the same values, each appearance of that value gets counted. For example, in the set of numbers 1, 3, 4, 4, 5, 8, and 9, the median is 4 because there are three numbers (1, 3, and 4) below it and three numbers (5, 8, and 9) above it. If we only counted 4 once, the median would incorrectly be calculated at 4.5 (4 + 5, divided by 2). When in doubt, writing out all of the numbers in order and marking them off one at a time from the top and bottom will always lead you to the correct answer.

## Mode

The **mode** is the most frequently occurring value in the dataset. For the data in Figure 6.3.6, the mode is 18 since more teams (4) had 18 touchdown passes than any other number of touchdown passes. With continuous data, such as response time measured to many decimals, the frequency of each value is one since no two scores will be exactly the same (see discussion of continuous variables in 6.1 Basics of Statistics). Therefore the mode of continuous data is normally computed from a grouped frequency distribution. Table 6.3.4 shows a

grouped frequency distribution for the target response time data. Since the interval with the highest frequency is 600 to 700, the mode is the middle of that interval (650). Although the mode is not frequently used for continuous data, it is nevertheless an important measure of central tendency as it is the only measure we can use on qualitative or categorical data.

**Table 6.3.4. Grouped frequency distribution.**

Range	Frequency
500 to 600	3
600 to 700	5
700 to 800	5
800 to 900	5
900 to 1000	0
1000 to 1100	1

## More on the Mean and Median

In the section What Is Central Tendency? above, we saw that the centre of a distribution could be defined three ways: (1) the point on which a distribution would balance, (2) the value whose average absolute deviation from all the other values is minimized, and (3) the value whose squared deviation from all the other values is minimized. The mean is the point on which a distribution would balance, the median is the value that minimizes the sum of absolute deviations, and the mean is the value that minimizes the sum of the squared deviations.

Table 6.3.5 shows the absolute and squared deviations of the numbers 2, 3, 4, 9, and 16 from their median of 4 and their mean of 6.8. You can see that the sum of absolute deviations from the median (20) is smaller than the sum of absolute deviations from the mean (22.8). On the other hand, the sum of squared deviations from the median (174) is larger than the sum of squared deviations from the mean (134.8).

**Table 6.3.5. Absolute and squared deviations from the median of 4 and the mean of 6.8.**

Value	Absolute Deviation from Median	Absolute Deviation from Mean	Squared Deviation from Median	Squared Deviation from Mean
2	2	4.8	4	23.04
3	1	3.8	1	14.44
4	0	2.8	0	7.84
9	5	2.2	25	4.84
16	12	9.2	144	84.64
<b>Total</b>	<b>20</b>	<b>22.8</b>	<b>174</b>	<b>134.80</b>

Figure 6.3.7 shows that the distribution balances at the mean of 6.8 and not at the median of 4. The relative advantages and disadvantages of the mean and median are discussed in the section Comparing Measures of Central Tendency.

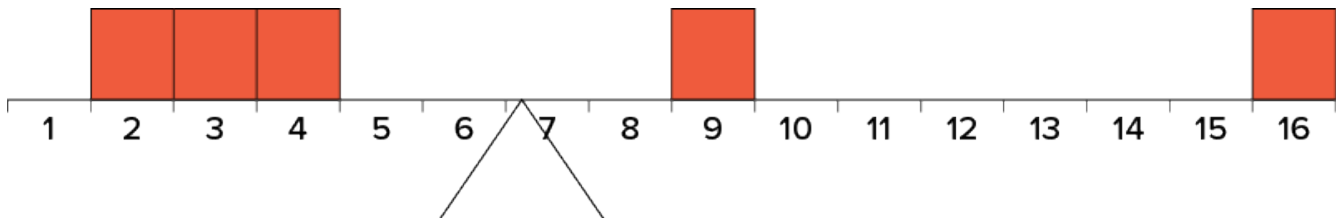


Figure 6.3.7. The distribution balances at the mean of 6.8 and not at the median of 4.0. (“Balance Scale Numbered” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

When a distribution is symmetric, then the mean and the median are the same. Consider the following distribution: 1, 3, 4, 5, 6, 7, 9. The mean and median are both 5. The mean, median, and mode are identical in the bell-shaped normal distribution.

## Try It

2) For the following problem, use the following scores:

5, 8, 8, 8, 7, 8, 9, 12, 8, 9, 8, 10, 7, 9, 7, 6, 9, 10, 11, 8.

- Create a histogram of these data. What is the shape of this histogram?
- How do you think the three measures of central tendency will compare to each other in this dataset?
- Compute the sample mean, the median, and the mode
- Draw and label lines on your histogram for each of the above values. Do your results match your predictions?

### **Solution**

a.

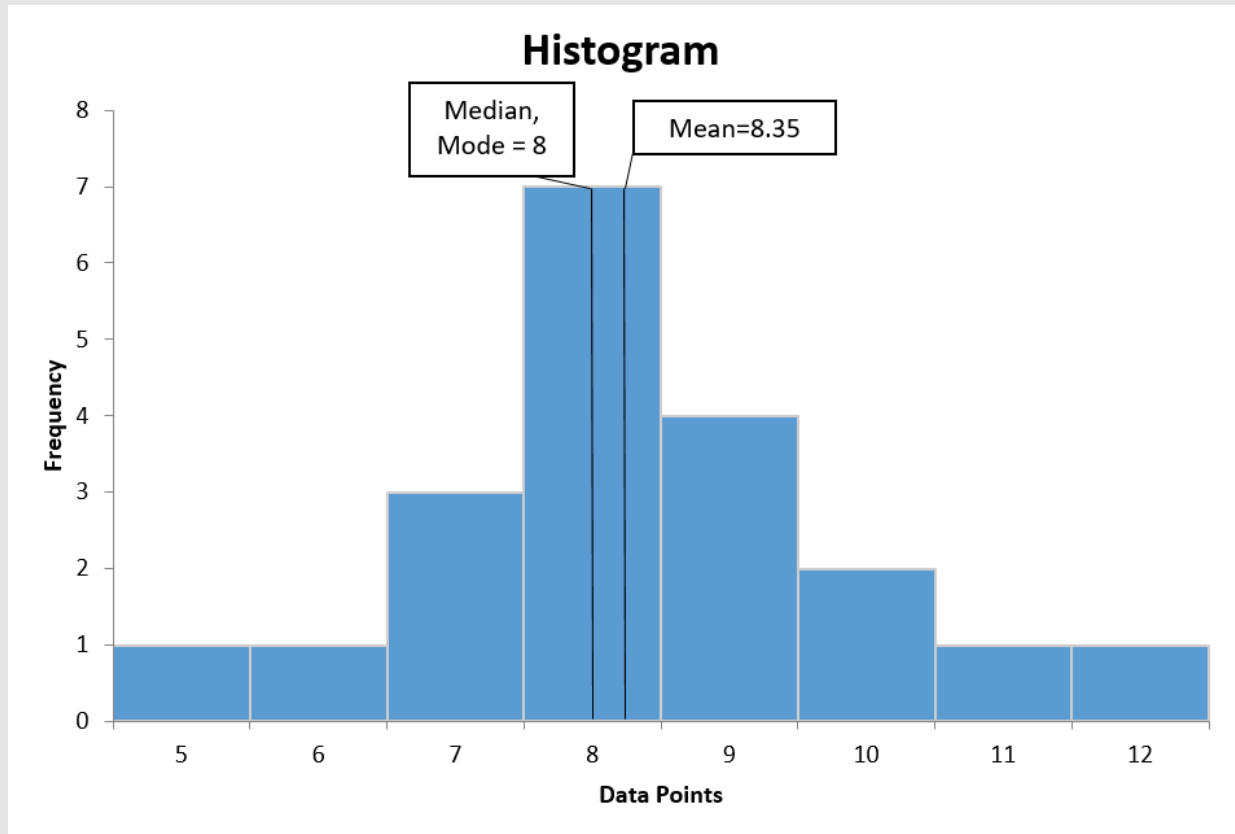


Figure 6.3.8

b. This histogram is slightly positively skewed. Since there is a positive skew, we would anticipate that the mean would be greater than the median. The mode and the median may be similar here because of the number of data points in the middle of the distribution.

c. median = 8, mode = 8, mean = 8.35

d. The mean being larger than the median was expected with the positive skew. The mode and the median being equal makes sense since the skew is minimal.

## Comparing Measures of Central Tendency

How do the various measures of central tendency compare with each other? For symmetric distributions, the mean and median are the same value, as is the mode except in bimodal distributions. However, differences among the measures occur with skewed distributions. Figure 6.3.9 shows the distribution of 642 scores on an introductory psychology test. Notice this distribution has a slight positive skew.

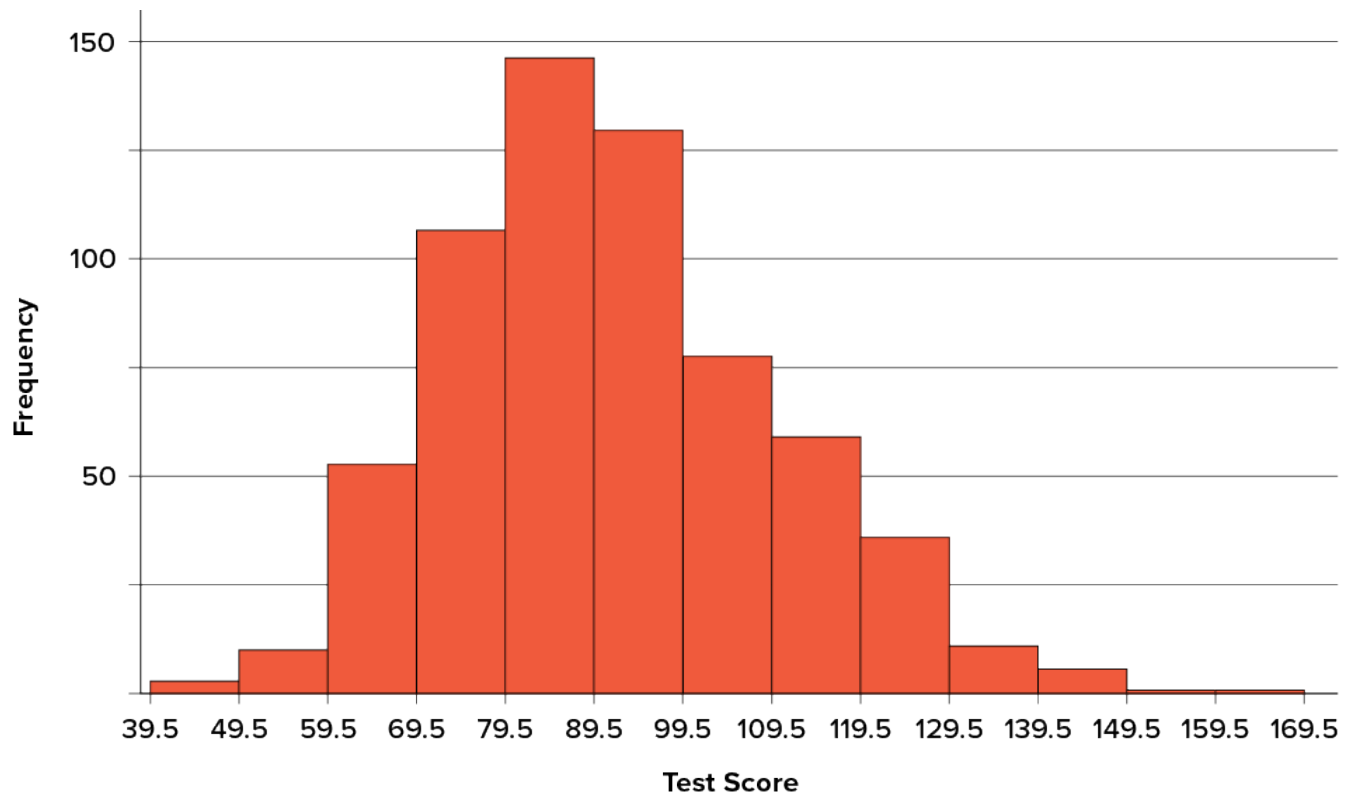


Figure 6.3.9. A distribution with a positive skew. (“Psychology Test Scores Histogram” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

Measures of central tendency are shown in Table 6.3.6. Notice they do not differ greatly, with the exception that the mode is considerably lower than the other measures. When distributions have a positive skew, the mean is typically higher than the median, although it may not be in bimodal distributions. For these data, the mean of 91.58 is higher than the median of 90. This pattern holds true for any skew: the mode will remain at the highest point in the distribution, the median will be pulled slightly out into the skewed tail (the longer end of the distribution), and the mean will be pulled the farthest out. Thus, the mean is more sensitive to skew than the median or mode, and in cases of extreme skew, the mean may no longer be appropriate to use.

**Table 6.3.6. Measures of central tendency for the test scores.**

Measure	Value
Mode	84.00
Median	90.00
Mean	91.58

The distribution of baseball salaries (in 1994) shown in Figure 6.3.10 has a much more pronounced skew than the distribution in Figure 6.3.9.

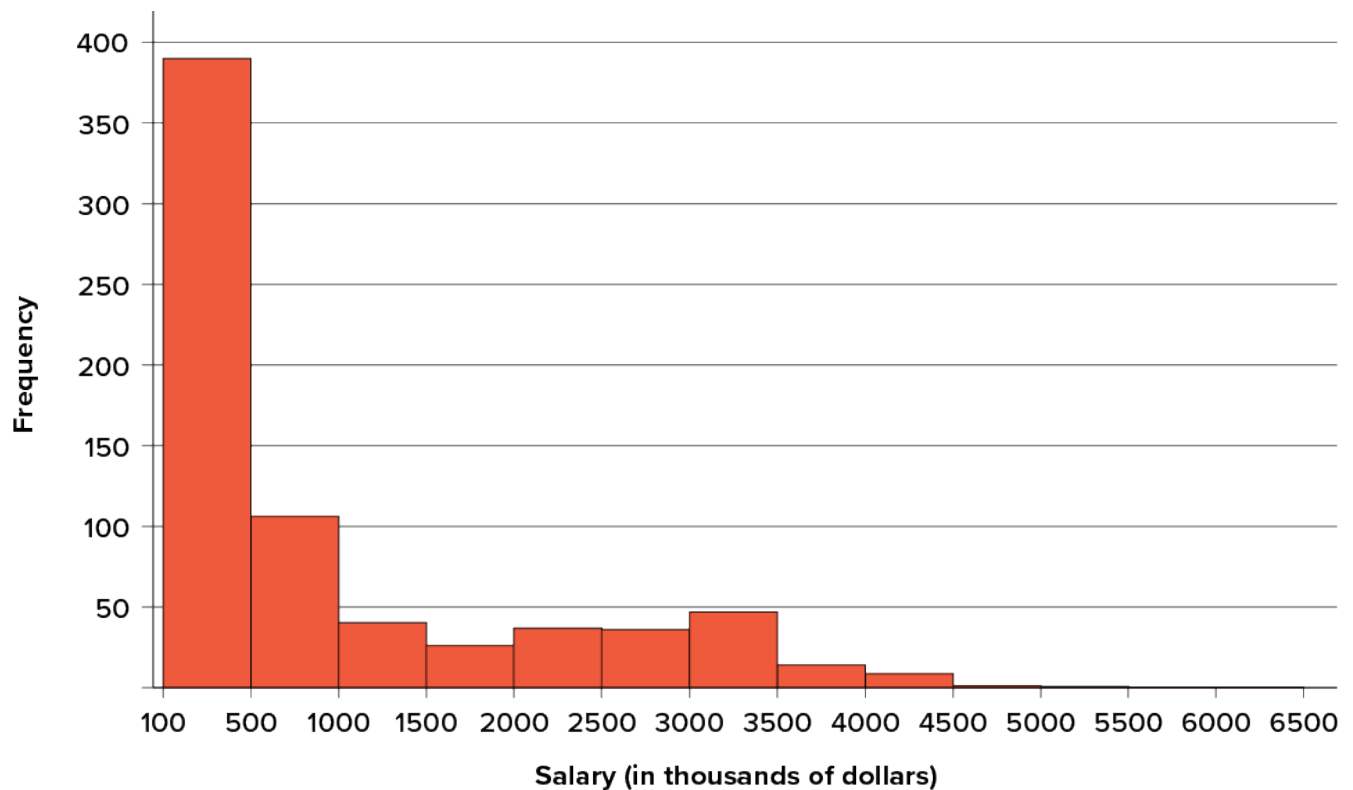


Figure 6.3.10. A distribution with a very large positive skew. This histogram shows the salaries of major league baseball players (in thousands of dollars). (“1994 MLB Salaries Histogram” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

Table 6.3.7 shows the measures of central tendency for these data. The large skew results in very different values for these measures. No single measure of central tendency is sufficient for data such as these. If you were asked

the very general question: “So, what do baseball players make?” and answered with the mean of \$1, 183, 000, you would not have told the whole story since only about one third of baseball players make that much. If you answered with the mode of \$109, 000 or the median of \$500, 000, you would not be giving any indication that some players make many millions of dollars. Fortunately, there is no need to summarize a distribution with a single number. When the various measures differ, our opinion is that you should report the mean and median. Sometimes it is worth reporting the mode as well. In the media, the median is usually reported to summarize the centre of skewed distributions. You will hear about median salaries and median prices of houses sold, etc. This is better than reporting only the mean, but it would be informative to hear more statistics.

**Table 6.3.7. Measures of central tendency for baseball salaries in 1994 (in thousands of dollars).**

Measure	Value
Mode	190
Median	500
Mean	1, 183

## Outliers

Outliers are data values that are significantly different than the other data values present. These can sometimes arise due to error in collecting the data, but can also be true data values. For example, if one baseball player made \$6, 500, 000, the mean would be pulled even more towards that value. Outliers affect the mean more than the median or mode. We say that the median and mode are more resistant to outliers.

### Try It

3) Compare the mean, median, and mode in terms of their sensitivity to extreme scores.

#### **Solution**

The mean is the most sensitive to extreme scores as all data values affect the mean. The median is



sensitive to the middle points of a data set, and so it is resistant to extreme scores. The mode is sensitive to the most frequent scores and so it is resistant to extreme scores. The best measure of central tendency when there are extreme scores is the median.

## Try It

4) Your younger brother comes home one day after taking a science test. He says someone at school told him that “60% of the students in the class scored above the median test grade.” What is wrong with this statement?

What if he had said “60% of the students scored above the mean?”

### Solution

The median is defined as the value with 50% of scores above it and 50% of scores below it; therefore, 60% of score cannot fall above the median. If 60% of scores fall above the mean, that would indicate that the mean has been pulled down below the value of the median, which means that the distribution is negatively skewed

## Spread and Variability

**Variability** refers to how “spread out” a group of scores is. To see what we mean by spread out, consider the graphs in Figure 6.3.11. These graphs represent the scores on two quizzes. The mean score for each quiz is 7.0. Despite the equality of means, you can see that the distributions are quite different. Specifically, the scores on Quiz 1 are more densely packed and those on Quiz 2 are more spread out. The differences among students were much greater on Quiz 2 than on Quiz 1.

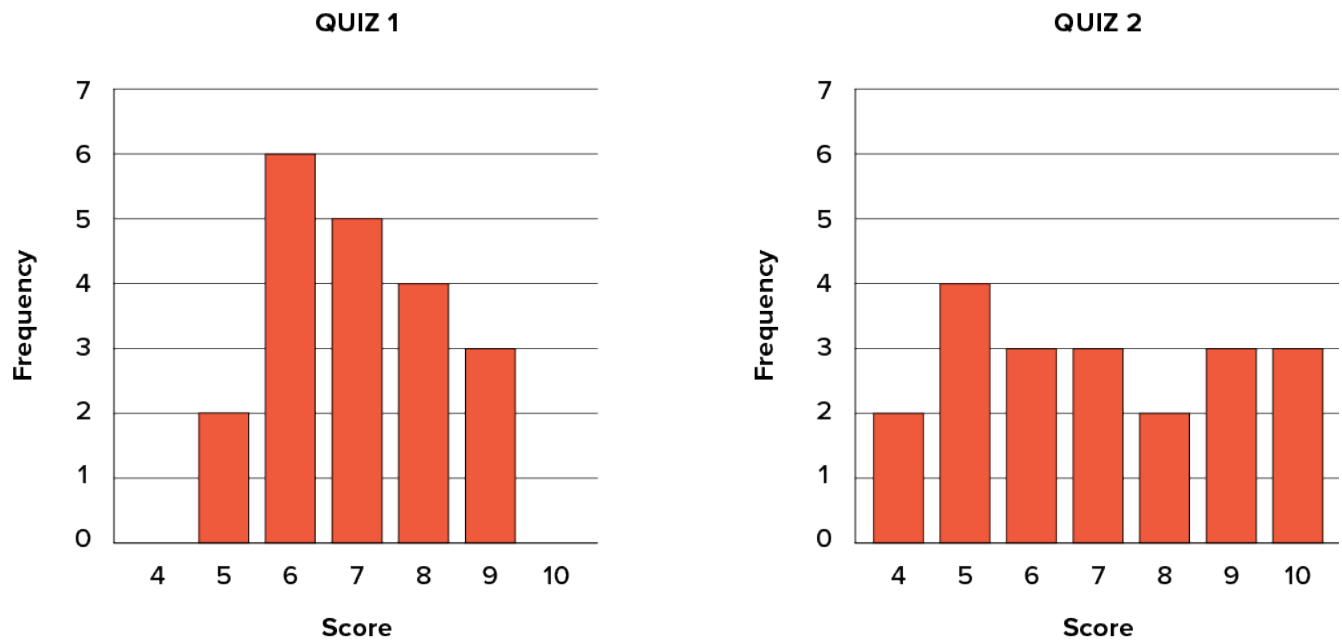


Figure 6.3.11. Bar charts of Quizzes 1 and 2. (“Quiz Score Bar Charts” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

The terms variability, **spread**, and **dispersion** are synonyms and refer to how spread out a distribution is. Just as in the section on central tendency where we discussed measures of the centre of a distribution of scores, in this section we will discuss measures of the variability of a distribution. There are three frequently used measures of variability: **range**, **variance**, and **standard deviation**. In the next few paragraphs, we will look at each of these measures of variability in more detail.

## Range

The **range** is the simplest measure of variability to calculate, and one you have probably encountered many times in your life. The range is simply the highest score minus the lowest score. Let’s take a few examples. What is the range of the following group of numbers: 10, 2, 5, 6, 7, 3, 4? Well, the highest number is 10, and the lowest number is 2, so  $10 - 2 = 8$ . The range is 8.

Let’s take another example. Here’s a dataset with 10 numbers: 99, 45, 23, 67, 45, 91, 82, 78, 62, 51. What is the range? The highest number is 99 and the lowest number is 23, so  $99 - 23 = 76$ ; the range is 76. Now consider the two quizzes shown in Figure 6.3.10. On Quiz 1, the lowest score is 5 and the highest score is 9. Therefore, the range is 4. The range on Quiz 2 was larger: the lowest score was 4 and the highest score was 10. Therefore the range is 6.

The problem with using range is that it is extremely sensitive to outliers, and one number far away from the rest of the data will greatly alter the value of the range. For example, in the set of numbers 1, 3, 4, 4, 5, 8, and

9, the range is  $8(9 - 1)$ . However, if we add a single person whose score is nowhere close to the rest of the scores, say, 20, the range more than doubles from 8 to 19.

## Interquartile Range

The **interquartile range** (IQR) is the range of the middle 50% of the scores in a distribution and is sometimes used to communicate where the bulk of the data in the distribution are located. It is computed as follows:

$$\text{IQR} = 75^{\text{th}} \text{ percentile} - 25^{\text{th}} \text{ percentile}$$

For Quiz 1, the 75<sup>th</sup> percentile is 8 and the 25<sup>th</sup> percentile is 6. The interquartile range is therefore 2. For Quiz 2, which has greater spread, the 75<sup>th</sup> percentile is 9, the 25<sup>th</sup> percentile is 5, and the interquartile range is 4. Recall that in the discussion of box plots, the 75<sup>th</sup> percentile was called the upper hinge and the 25<sup>th</sup> percentile was called the lower hinge. Using this terminology, the interquartile range is referred to as the H-spread.

## Sum of Squares

Variability can also be defined in terms of how close the scores in the distribution are to the middle of the distribution. Using the mean as the measure of the middle of the distribution, we can see how far, on average, each data point is from the centre. The data from Quiz 1 are shown in Table 6.3.8.

There are a few things to note about how Table 6.3.8 is formatted. The raw data scores ( $X$ ) are always placed in the left-most column. This column is then summed at the bottom ( $\sum X$ ) to facilitate calculating the mean by dividing the sum of  $X$  values by the number of scores in the table ( $N$ ). The mean score is  $7.0$  ( $\frac{\sum X}{N} = \frac{140}{20} = 7.0$ ). Once you have the mean, you can easily work your way down the second column calculating the deviation scores ( $X - M$ ), representing how far each score deviates from the mean, here calculated as the score ( $X$  value) minus 7. This column is also summed and has a very important property: it will always sum to 0, or close to zero if you have rounding error due to many decimal places ( $\sum (X - M) = 0$ ). This step is used as a check on your math to make sure you haven't made a mistake. If this column sums to 0, you can move on to filling in the third column, which is composed of the squared deviation scores. The deviation scores are squared to remove negative values and appear in the third column ( $X - M$ )<sup>2</sup>. When these values are summed, you have the sum of the squared deviations, or the **sum of squares** ( $SS$ ), calculated with the formula

$$SS = \sum (X - M)^2$$

**Table 6.3.8. Calculation of variance for Quiz 1 scores.**

$X$	$X - M$	$(X - M)^2$	$X^2$
9	2	4	81
9	2	4	81
9	2	4	81
8	1	1	64
8	1	1	64
8	1	1	64
8	1	1	64
7	0	0	49
7	0	0	49
7	0	0	49
7	0	0	49
7	0	0	49
6	-1	1	36
6	-1	1	36
6	-1	1	36
6	-1	1	36
6	-1	1	36

$X$	$X - M$	$(X - M)^2$	$X^2$
6	-1	1	36
5	-2	4	25
5	-2	4	25
$\sum X = 140$	$\sum (X - M) = 0$	$\sum (X - M)^2 = 30$	$\sum X^2 = 1,010$
$(\sum X)^2 = 19,600$			

$$SS = 4 + 4 + 4 + 1 + 1 + 1 + 1 + 0 + 0 + 0 + 0 + 0 + 1 + 1 + 1 + 1 + 1 + 1 + 4 + 4 = 30$$

The preceding formula is called the definitional formula, as it shows the logic behind the sum of squared deviations calculation. As mentioned earlier, there can be rounding errors in calculating the deviation scores. Also, when the set of scores is large, calculating the deviation scores, squaring the scores, and then summing those values can be tedious. To simplify the sum of squares calculation, the computational formula is used instead. The computational formula is as follows:

$$SS = \sum X^2 - \frac{(\sum X)^2}{N}$$

The last column in Table 6.3.8 represents the  $X$  values squared and then summed —  $\sum X^2$ . At the bottom of the first column, the  $\sum X$  value is squared —  $(\sum X)^2$ . These are the values used in the computational formula for the sum of squares. As you can see in the calculation below, the  $SS$  value is the same for both the definitional formula and the computational formula:

$$SS = \sum X^2 - \frac{(\sum X)^2}{N} = 1,010 - \frac{19,600}{20} = 30$$

As we will see, the sum of squares appears again and again in different formulas — it is a very important value, and using the  $X$  and  $X^2$  columns in this table makes it simple to calculate the  $SS$  without error.

## Variance

Now that we have the sum of squares calculated, we can use it to compute our formal measure of average distance from the mean—the variance. The variance is defined as the average squared difference of the scores from the mean. We square the deviation scores because, as we saw in the second column of Table 6.3.8, the sum of raw deviations is always 0, and there's nothing we can do mathematically without changing that.

The population parameter for variance is  $\sigma^2$  (“sigma-squared”) and is calculated as:

$$\sigma^2 = \frac{SS}{N}$$

We can use the value we previously calculated for  $SS$  in the numerator, then simply divide that value by  $N$  to get variance. If we assume that the values in Table 6.3.8 represent the full population, then we can take our value of sum of squares and divide it by  $N$  to get our population variance:

$$\sigma^2 = \frac{30}{20} = 1.5$$

So, on average, scores in this population are 1.5 squared units away from the mean. This measure of spread exhibits much more **robustness** (a term used by statisticians to mean resilience or resistance to outliers) than the range, so it is a much more useful value to compute. Additionally, as we will see in future chapters, variance plays a central role in inferential statistics.

The sample statistic used to estimate the variance is  $s^2$  (“s-squared”):

$$s^2 = \frac{SS}{N-1} = \frac{SS}{df}$$

This formula is very similar to the formula for the population variance with one change: we now divide by  $N - 1$  instead of  $N$ . The value  $N - 1$  has a special name: the **degrees of freedom** (abbreviated as  $df$ ). You don’t need to understand in depth what degrees of freedom are (essentially they account for the fact that we have to use a sample statistic to estimate the mean  $M$  before we estimate the variance) in order to calculate variance, but knowing that the denominator is called  $df$  provides a nice shorthand for the variance formula:

$$\frac{SS}{df}$$

Going back to the values in Table 6.3.8 and treating those scores as a sample, we can estimate the sample variance as:

$$s^2 = \frac{30}{20-1} = 1.58$$

Notice that this value is slightly larger than the one we calculated when we assumed these scores were the full population. This is because our value in the denominator is slightly smaller, making the final value larger. In general, as your sample size  $N$  gets bigger, the effect of subtracting 1 becomes less and less. Comparing a sample size of 10 to a sample size of 1000;  $10 - 1 = 9$ , or 90% of the original value, whereas  $1000 - 1 = 999$ , or 99.9% of the original value. Thus, larger sample sizes will bring the estimate of the sample variance closer to that of the population variance. This is a key idea and principle in statistics that we will see over and over again: larger sample sizes better reflect the population.

## Standard Deviation

The **standard deviation** is simply the square root of the variance. This is a useful and interpretable statistic because taking the square root of the variance (recalling that variance is the average squared difference) puts the standard deviation back into the original units of the measure we used. Thus, when reporting descriptive statistics in a study, scientists virtually always report mean and standard deviation. Standard deviation is

therefore the most commonly used measure of spread for our purposes, representing the average distance of the scores from the mean.

The population parameter for standard deviation is  $\sigma$  (“sigma”), which, intuitively, is the square root of the variance parameter  $\sigma^2$  (occasionally, the symbols work out nicely that way). The formula is simply the formula for variance under a square root sign:

$$\sigma = \sqrt{\frac{SS}{N}} = \sqrt{\frac{30}{20}} = \sqrt{1.5} = 1.22$$

The sample statistic follows the same conventions and is given as  $s$  in mathematical formulas.

$$s = \sqrt{\frac{SS}{df}} = \sqrt{\frac{30}{20-1}} = \sqrt{1.58} = 1.26$$

## Try It

5) Compute the sample mean and sample standard deviation for the following scores:

−8, −4, −7, −6, −8, −5, −7, −9, −2, 0

### Solution

Sample mean −5.6 and sample standard deviation 2.875.

The standard deviation is an especially useful measure of variability when the distribution is normal or approximately normal because the proportion of the distribution within a given number of standard deviations from the mean can be calculated. For example, 68% of the distribution is within one standard deviation (above and below) of the mean and approximately 95% of the distribution is within two standard deviations of the mean, as shown in Figure 6.3.12. Therefore, if you had a normal distribution with a mean of 50 and a standard deviation of 10, then 68% of the distribution would be between  $50 - 10 = 40$  and  $50 + 10 = 60$ . Similarly, about 95% of the distribution would be between  $50 - 2 \times 10 = 30$  and  $50 + 2 \times 10 = 70$ .

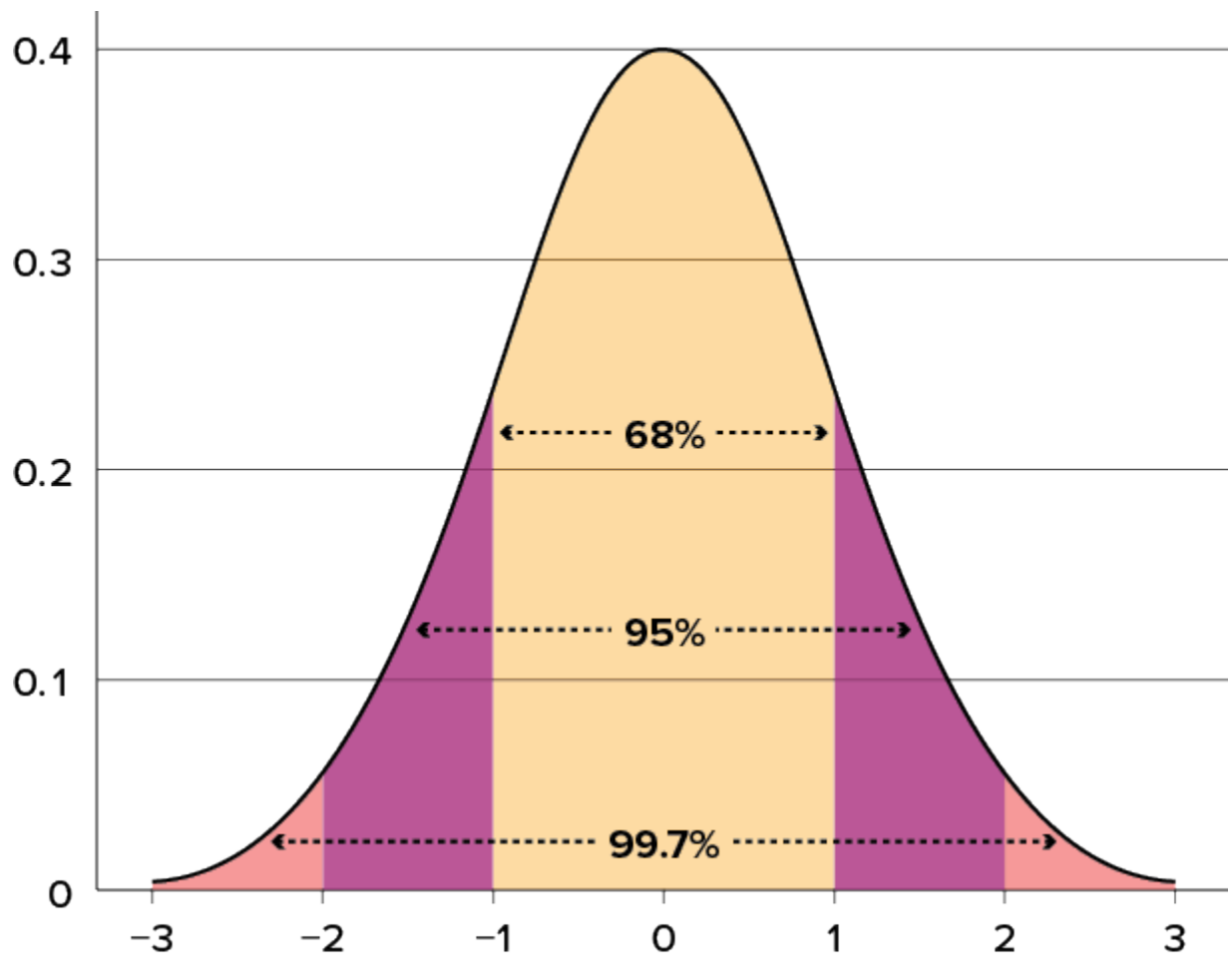


Figure 6.3.12. Percentages of the normal distribution. (“Normal Distribution Percentages” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

## Try It

6) Make up three datasets with five numbers each that have:

- The same mean but different standard deviations.
- The same mean but different medians.
- The same median but different means.

### Solution



Note, many answers are possible for this question.

- a. 1,2,3,4,5 (mean 3, standard deviation 1.41) vs 2,3,3,3,4 (mean 3, standard deviation 0.63)
- b. 1,2,3,4,5 (mean 3 median 3) vs 1,2,4,4,4 (mean 3 median 4)
- c. 1,2,3,4,5 (mean 3 median 3) vs 1,2,3,5,7 (mean 3.6 median 3)

Figure 6.3.13 shows two normal distributions. The red (left-most) distribution has a mean of 40 and a standard deviation of 5; the blue (right-most) distribution has a mean of 60 and a standard deviation of 10. For the red distribution, 68% of the distribution is between 45 and 55; for the blue distribution, 68% is between 50 and 70. Notice that as the standard deviation gets smaller, the distribution becomes much narrower, regardless of where the centre of the distribution (mean) is. Figure 6.3.14 presents several more examples of this effect.

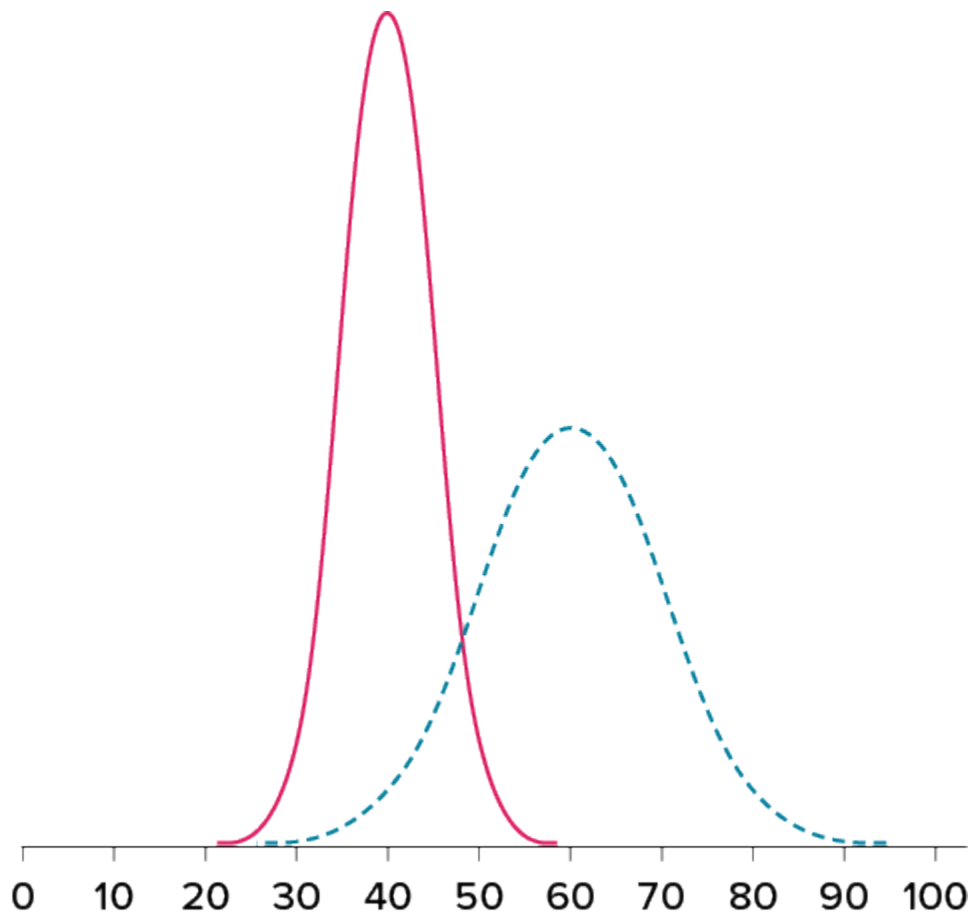


Figure 6.3.13. Normal distributions with standard deviations of  $5$  and  $10$ . (“Normal Distributions with Standard Deviations” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

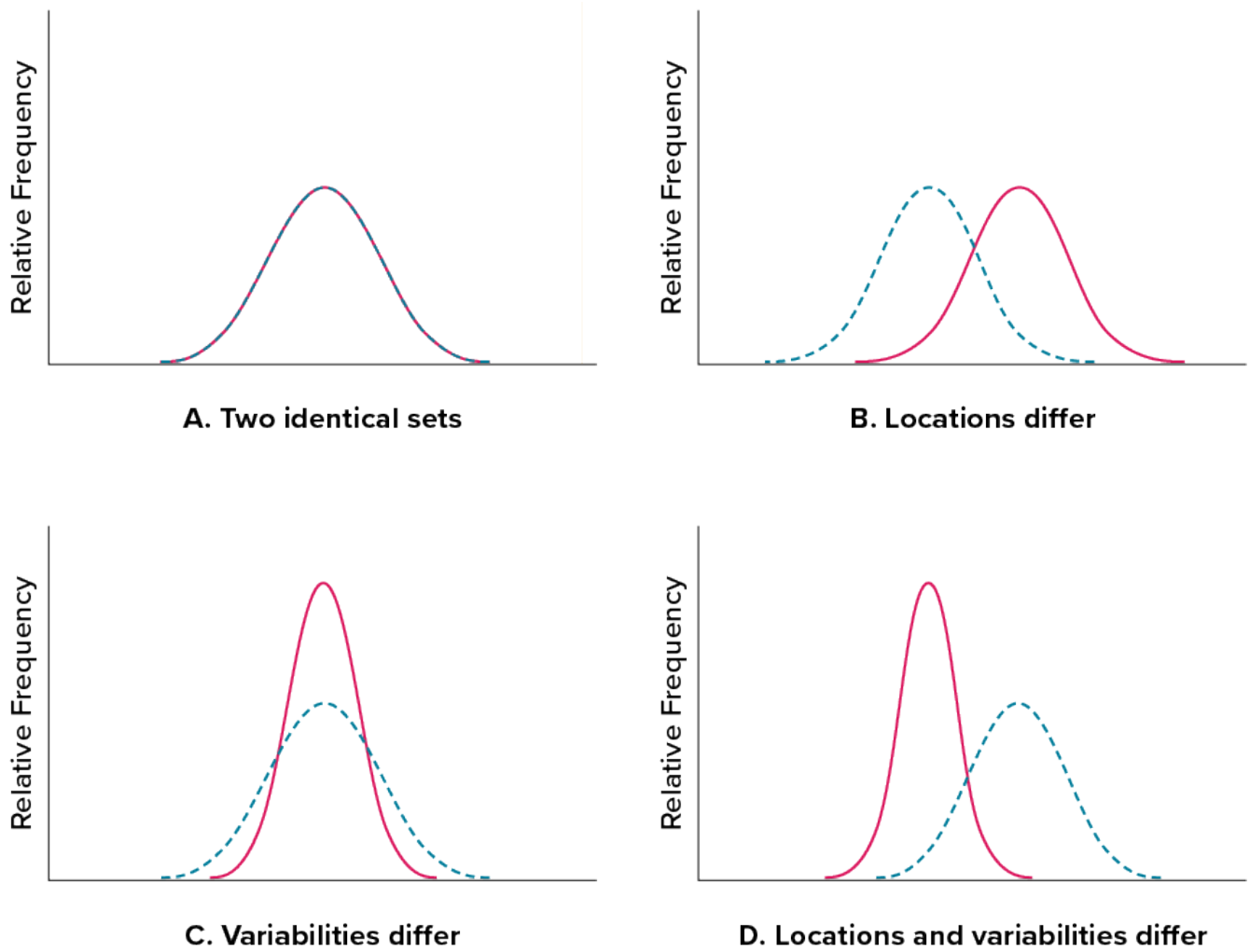


Figure 6.3.14. Differences between two datasets. (“Location and Variability Differences” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

### Try It

7) Compute the population mean and population standard deviation for the following scores (remember to use the sum of squares table): 5, 7, 8, 3, 4, 4, 2, 7, 1, 6

### Solution

$$\mu = 4.80, s^2 = 2.36$$

### Try It

8) Compute the range, sample variance, and sample standard deviation for the following scores: 25, 36, 41, 28, 29, 32, 39, 37, 34, 34, 37, 35, 30, 36, 31, 31.

**Solution**

$$\text{Range} = 16, s^2 = 18.4, s = 4.29$$

### Try It

9) Using the same values from the previous problem, calculate the range, sample variance, and sample standard deviation, but this time include **65** in the list of values. How did each of the three values change?

**Solution**

**Range** = 40,  $s^2 = 75.8$ ,  $s = 8.71$  Each of the three values increased after including the extreme value of 65.

## Try It

10) Two normal distributions have exactly the same mean, but one has a standard deviation of **20** and the other has a standard deviation of **10**. How would the shapes of the two distributions compare?

### Solution

If both distributions are normal, then they are both symmetrical, and having the same mean causes them to overlap with one another. The distribution with the standard deviation of **10** will be narrower than the other distribution.

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

*<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=1730#h5p-40>*

b) After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

## Glossary

### **arithmetic mean**

Perhaps the most common measure of central tendency, the mean is the mathematical average of the scores in a sample.

### **central tendency**

The centre or middle of a distribution. There are many measures of central tendency. The most common are the mean, median, and mode.

### **degrees of freedom**

The number of independent pieces of information that go into the estimate. In general, the degrees of freedom for an estimate is equal to the number of values minus the number of parameters estimated en route to the estimate in question.

### **dispersion**

The extent to which values differ from one another; that is, how much they vary. Dispersion is also called variability or spread.

### **interquartile range (IQR)**

The range of the middle 50% of the scores in a distribution; computed by subtracting the 25th percentile from the 75th percentile. The interquartile range is a robust measure of central tendency.

### **median**

The median is a popular measure of central tendency. It is the 50th percentile of a distribution.

### **mode**

A measure of central tendency, the mode is the most frequent value in a distribution.

### **range**

The difference between the maximum and minimum values of a variable or distribution. The range is the simplest measure of variability.

### **robustness**

Something is robust if it holds up well in the face of adversity. A measure of central tendency or variability is considered robust if it is not greatly affected by a few extreme scores. A statistical test is considered robust if it works well in spite of moderate violations of the assumptions on which it is based.

### **spread**

The extent to which values differ from one another; that is, how much they vary. Spread is also

called variability or dispersion.

**standard deviation**

The standard deviation is a widely used measure of variability. It is computed by taking the square root of the variance.

**sum of squares**

The sum of squared deviations, or differences, between scores and the mean in a numeric dataset.

**variability**

The extent to which values differ from one another; that is, how much they vary. Variability can also be thought of as how spread out or dispersed a distribution is.

**variance**

The variance is a widely used measure of variability. It is defined as the mean squared deviation of scores from the mean.

# 6.4 Z-SCORES AND THE STANDARD NORMAL DISTRIBUTION

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## Learning Objectives

By the end of this section, you will be able to:

- Identify normal distributions and describe their characteristics.
- Find  $z$ -scores and identify their meaning.
- Understand the relationship between  $z$ -scores and area under the curve.

We now understand how to describe and present our data visually and numerically. These simple tools, and the principles behind them, will help you interpret information presented to you and understand the basics of a variable. Moving forward, we now turn our attention to how scores within a distribution are related to one another, how to precisely describe a score's location within the distribution, and how to compare scores from different distributions.

## Normal Distributions

The **normal distribution** is the most important and most widely used distribution in statistics. It is sometimes called the “bell curve,” although the tonal qualities of such a bell would be less than pleasing. It is also called the “Gaussian curve” of Gaussian distribution after the mathematician Karl Friedrich Gauss.

The Gaussian distribution is commonly used partly because of its convenient characteristics in that it is symmetric, bell-shaped and unimodal. Each distribution is given by two parameters, the mean,  $\mu$ , and the standard deviation,  $\sigma$ . For a normal distribution, we can denote its characteristics using  $N(\mu, \sigma)$ . Mathematically, there is a very complex function that represents the Gaussian distribution, and it is written as the following:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where:  $-\infty < x < \infty$ ,  $-\infty < \mu < \infty$ , and  $\sigma > 0$ .

In order to calculate the probability of an event occurring by hand, we would require a high level of mathematics knowledge and calculus skills. Luckily, there are methods that have been developed so that we can find the probability of a certain event occurring without these techniques and advanced mathematics. In this section and the next, we will learn how to use a  $z$ -table to calculate certain probabilities.

Strictly speaking, it is not correct to talk about “the normal distribution” since there are many normal distributions. Normal distributions can differ in their means and in their standard deviations. Figure 6.4.1 shows three normal distributions. The blue (left-most) distribution has a mean of  $-3$  and a standard deviation of  $0.5$ , the distribution in red (the middle distribution) has a mean of  $0$  and a standard deviation of  $1$ , and the black (right-most) distribution has a mean of  $2$  and a standard deviation of  $3$ . These as well as all other normal distributions are symmetric with relatively more values at the centre of the distribution and relatively few in the tails. What is consistent about all normal distribution is the shape and the proportion of scores within a given distance along the  $x$ -axis. We will focus on the **standard normal distribution** (also known as the unit normal distribution), which has a mean of  $0$  and a standard deviation of  $1$  (i.e., the red distribution in Figure 6.4.1).



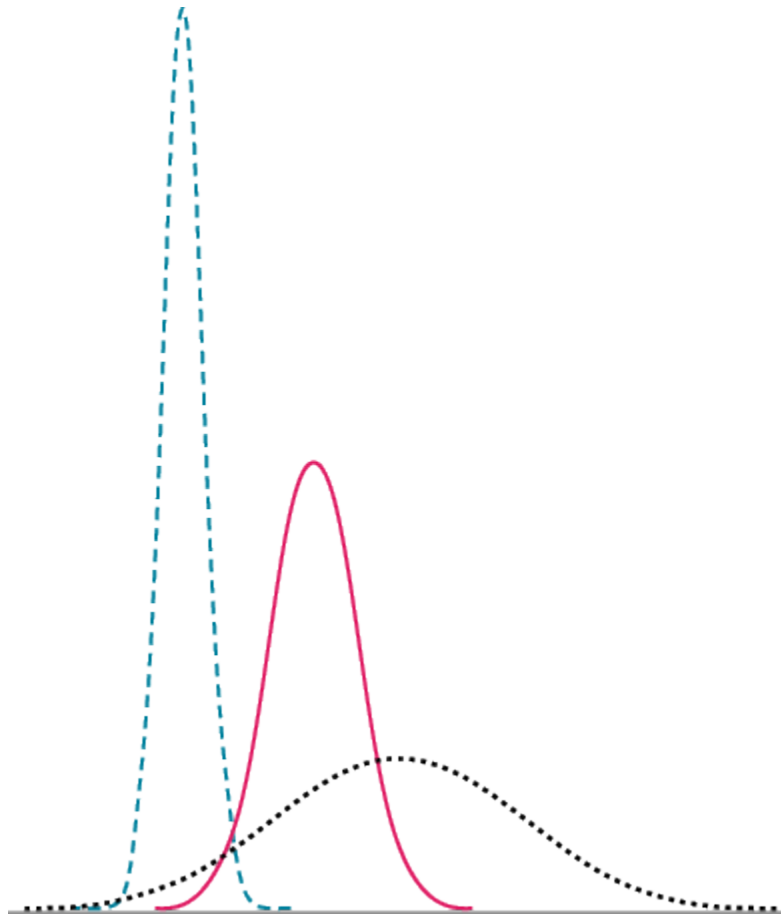


Figure 6.4.1. Normal distributions differing in mean and standard deviation. (“Normal Distributions with Different Means and Standard Deviations” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

Seven features of normal distributions are listed below.

1. Normal distributions are symmetric around their mean.
2. The mean, median, and mode of a normal distribution are equal.
3. The area under the normal curve is equal to **1.0**.
4. Normal distributions are denser in the centre and less dense in the tails.
5. Normal distributions are defined by two parameters, the mean ( $\mu$ ) and the standard deviation ( $s$ ).
6. **68%** of the area of a normal distribution is within one standard deviation of the mean.
7. Approximately **95%** of the area of a normal distribution is within two standard

deviations of the mean.

These properties enable us to use the normal distribution to understand how scores relate to one another within and across a distribution. But first, we need to learn how to calculate the standardized score that makes up a standard normal distribution.

## Try It

1) True or false:

- a. All normal distributions are symmetrical.
- b. All normal distributions have a mean of **1.0**.
- c. All normal distributions have a standard deviation of **1.0**.
- d. The total area under the curve of all normal distributions is equal to **1**.

### Solution

- a. True
- b. False. Normal distributions can have a mean of any value.
- c. False. Normal distributions can have any standard deviation.
- d. True.

## The Empirical Rule

If  $x$  is a random variable and has a normal distribution with mean  $\mu$  and standard deviation  $\sigma$ , then the **Empirical Rule** states the following:

- About **68%** of the values of  $x$  lie between a  $-1\sigma$  and  $+1\sigma$  of the mean  $\mu$  (within one standard deviation of the mean).
- About **95%** of the values of  $x$  lie between  $-2\sigma$  and  $+2\sigma$  of the mean  $\mu$  (within two standard deviations of the mean).

- About 99.7% of the values of  $x$  lie between  $-3\sigma$  and  $+3\sigma$  of the mean  $\mu$  (within three standard deviations of the mean).
- The  $z$ -scores for  $+1\sigma$  and  $-1\sigma$  are  $+1$  and  $-1$ , respectively.
- The  $z$ -scores for  $+2\sigma$  and  $-2\sigma$  are  $+2$  and  $-2$ , respectively.
- The  $z$ -scores for  $+3\sigma$  and  $-3\sigma$  are  $+3$  and  $-3$ , respectively.

The Empirical Rule is also known as the 68–95–99.7 rule.

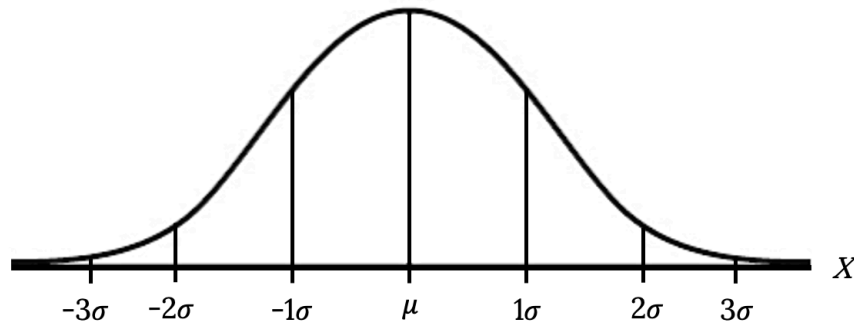


Figure 6.4.2: Empirical Rule

### Example 6.4.1

Suppose  $x$  has a normal distribution with mean 50 and standard deviation 6.

#### Solution

**Step 1: About 68% of the  $x$  values lie within one standard deviation of the mean.**

Therefore, about 68% of the  $x$  values lie between  $-1\sigma = (-1)(6) = -6$  and  $1\sigma = (1)(6) = 6$  of the mean 50. The values  $50 - 6 = 44$  and  $50 + 6 = 56$  are within one standard deviation from the mean 50. The  $z$ -scores are  $-1$  and  $+1$  for 44 and 56, respectively.

**Step 2: About 95% of the  $x$  values lie within two standard deviations of the mean.**

Therefore, about 95% of the  $x$  values lie between  $-2\sigma = (-2)(6) = -12$  and

$2\sigma = (2)(6) = 12$ . The values  $50 - 12 = 38$  and  $50 + 12 = 62$  are within two standard deviations from the mean 50. The  $z$ -scores are  $-2$  and  $+2$  for 38 and 62, respectively.

**Step 3: About 99.7% of the  $x$  values lie within three standard deviations of the mean.**

Therefore, about 95% of the  $x$  values lie between  $-3\sigma = (-3)(6) = -18$  and  $3\sigma = (3)(6) = 18$  from the mean 50. The values  $50 - 18 = 32$  and  $50 + 18 = 68$  are within three standard deviations of the mean 50. The  $z$ -scores are  $-3$  and  $+3$  for 32 and 68, respectively.

## Try It

2) From 1984 to 1985, the mean height of 15 to 18-year-old males from Chile was  $172.36\text{cm}$ , and the standard deviation was  $6.34\text{cm}$ . Let  $Y$  = the height of 15 to 18-year-old males in 1984 to 1985. Then  $Y \sim N(172.36, 6.34)$ .

- About 68% of the  $y$  values lie between what two values? These values are \_\_\_\_\_ and \_\_\_\_\_. The  $z$ -scores are \_\_\_\_\_ and \_\_\_\_\_, respectively.
- About 95% of the  $y$  values lie between what two values? These values are \_\_\_\_\_ and \_\_\_\_\_. The  $z$ -scores are \_\_\_\_\_ and \_\_\_\_\_, respectively.
- About 99.7% of the  $y$  values lie between what two values? These values are \_\_\_\_\_ and \_\_\_\_\_. The  $z$ -scores are \_\_\_\_\_ and \_\_\_\_\_, respectively.

### Solution

- About 68% of the values lie between  $166.02\text{cm}$  and  $178.7\text{cm}$ . The  $z$ -scores are  $-1$  and  $1$ .
- About 95% of the values lie between  $159.68\text{cm}$  and  $185.04\text{cm}$ . The  $z$ -scores are  $-2$  and  $2$ .
- About 99.7% of the values lie between  $153.34\text{cm}$  and  $191.38\text{cm}$ . The  $z$ -scores are  $-3$  and  $3$ .

## Z Scores

A  **$z$ -score** is a standardized version of a raw score ( $x$ ) that gives information about the relative location of that score within its distribution. The formula for converting a raw score into a  $z$ -score is

$$z = \frac{x - \mu}{\sigma} \text{ for values from a population and}$$

$$z = \frac{x - M}{s} \text{ for values from a sample.}$$

As you can see,  $z$ -scores combine information about where the distribution is located (the mean/centre) with how wide the distribution is (the standard deviation/spread) to interpret a raw score ( $x$ ). Specifically,  $z$ -scores will tell us how far the score is away from the mean in units of standard deviations and in what direction.

The value of a  $z$ -score has two parts: the sign (positive or negative) and the magnitude (the actual number). The sign of the  $z$ -score tells you in which half of the distribution the  $z$ -score falls: a positive sign (or no sign) indicates that the score is above the mean and on the right-hand side or upper end of the distribution, and a negative sign tells you the score is below the mean and on the left-hand side or lower end of the distribution. The magnitude of the number tells you, in units of standard deviations, how far away the score is from the centre or mean. The magnitude can take on any value between negative and positive infinity, but for reasons we will see soon, they generally fall between  $-3$  and  $3$ .

Let's look at some examples. A  $z$ -score value of  $-1.0$  tells us that this  $z$ -score is  $1$  standard deviation (because of the magnitude  $1.0$ ) below (because of the negative sign) the mean. Similarly, a  $z$ -score value of  $1.0$  tells us that this  $z$ -score is  $1$  standard deviation above the mean. Thus, these two scores are the same distance away from the mean but in opposite directions. A  $z$ -score of  $-2.5$  is two-and-a-half standard deviations below the mean and is therefore farther from the centre than both of the previous scores, and a  $z$ -score of  $0.25$  is closer than all of the ones before. For now, we will use a rough cut-off of  $1.5$  standard deviations in either direction as the difference between close scores (those within  $1.5$  standard deviations or between  $z = -1.5$  and  $z = 1.5$ ) and extreme scores (those farther than  $1.5$  standard deviations — below  $z = -1.5$  or above  $z = 1.5$ ).

## Try It

3) What are the two pieces of information contained in a  $z$ -score?

### Solution

The location above or below the mean (from the sign of the number) and the distance in standard deviations away from the mean (from the magnitude of the number)

## Try It

4) A  $z$ -score takes a raw score and standardizes it into units of?

### Solution

A  $z$ -score is unitless. The  $z$ -score represents how many “standard deviations from the mean” the raw score is.

## Try It

5) Assume the following five scores represent a sample: 2, 3, 5, 5, 6. Transform these scores into  $z$ -scores.

### Solution

$$M = 4.2, s = 1.64, z = -1.34, -0.73, 0.49, 0.49, 1.10$$

We can also convert raw scores into  $z$ -scores to get a better idea of where in the distribution those scores fall. Let's say we get a score of 68 on an exam. We may be disappointed to have scored so low, but perhaps it was just a very hard exam. Having information about the distribution of all scores in the class would be helpful to put some perspective on ours. We find out that the class got an average score of 54 with a standard deviation of 8. To find out our relative location within this distribution, we simply convert our test score into a  $z$ -score.

$$z = \frac{X - \mu}{\sigma} = \frac{68 - 54}{8} = 1.75$$

We find that we are **1.75** standard deviations above the average, above our rough cut-off for close and far. Suddenly our **68** is looking pretty good!

Figure 6.4.3 shows both the raw score and the  $z$ -score on their respective distributions. Notice that the red line indicating where each score lies is in the same relative spot for both. This is because transforming a raw score into a  $z$ -score does not change its relative location, it only makes it easier to know precisely where it is.

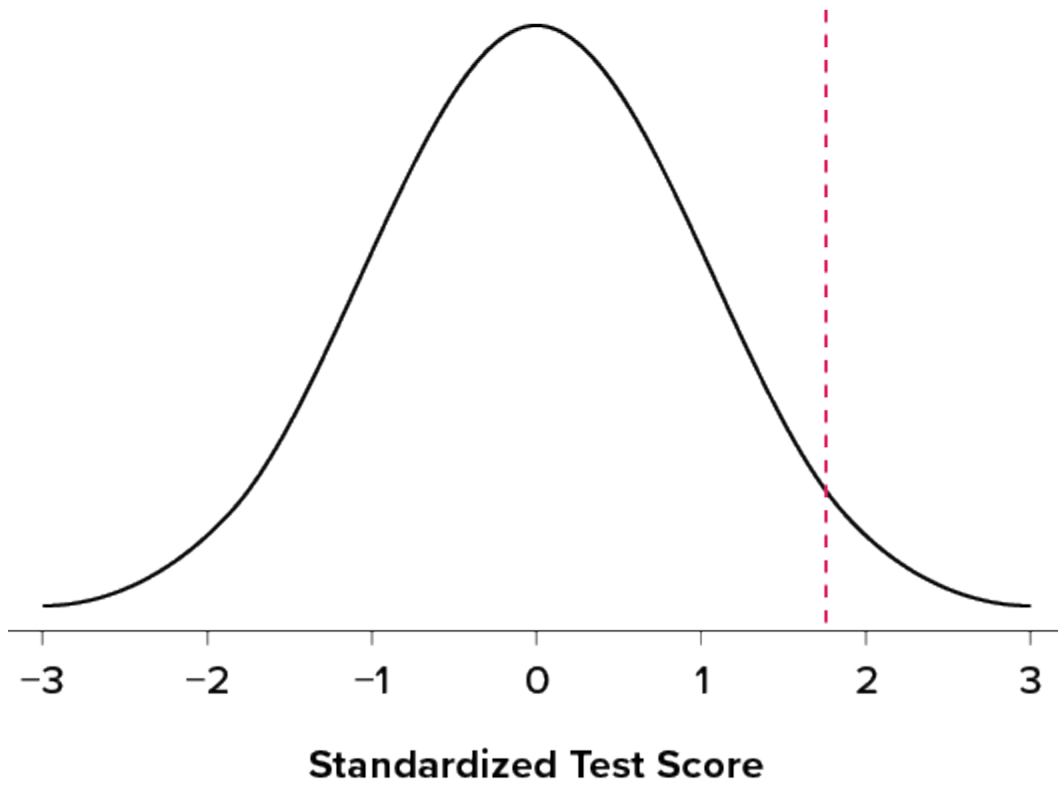
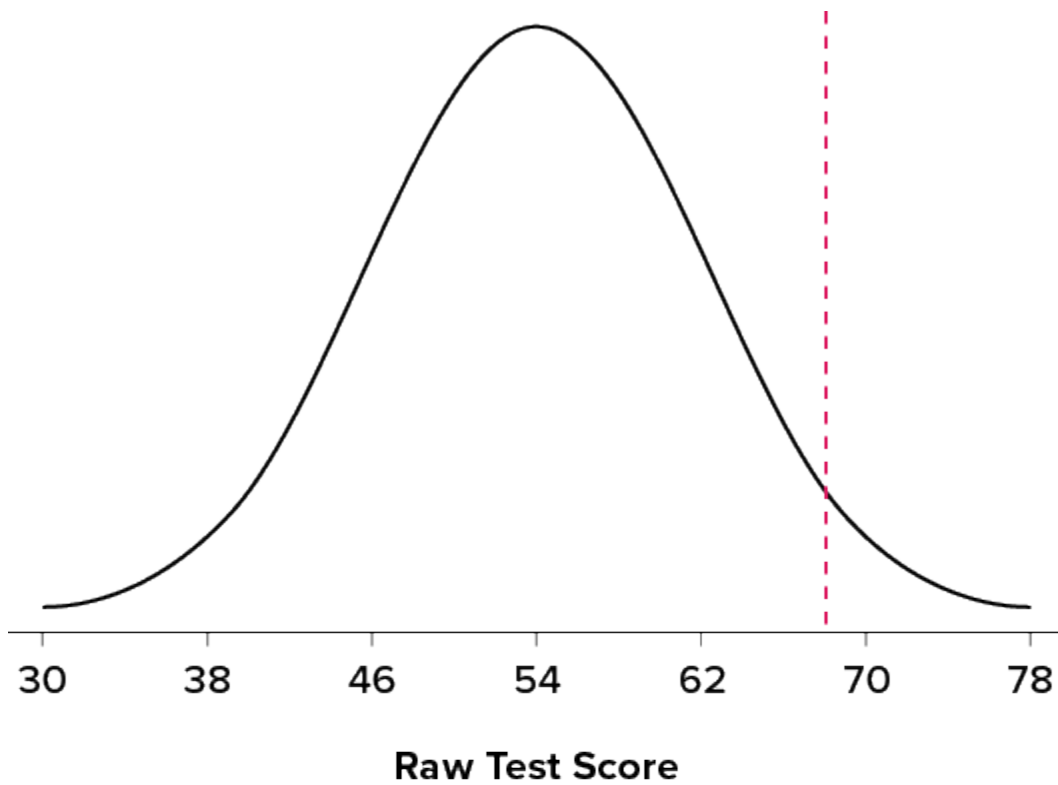


Figure 6.4.3. Raw and standardized versions of a single score. (“Raw and Standardized Versions of a Score” by Judy Schmitt is licensed under CC BY-NC-SA)



4.0.)

$z$ -Scores are also useful for comparing scores from different distributions. Let's say we take the SAT and score 501 on both the math and critical reading sections. Does that mean we did equally well on both? Scores on the math portion are distributed normally with a mean of 511 and standard deviation of 120, so our  $z$ -score on the math section is

$$z_{\text{math}} = \frac{501 - 511}{120} = -0.08$$

which is just slightly below average (note the use of "math" as a subscript; subscripts are used when presenting multiple versions of the same statistic in order to know which one is which and have no bearing on the actual calculation). The critical reading section has a mean of 495 and standard deviation of 116, so

$$z_{\text{CR}} = \frac{501 - 495}{116} = 0.05$$

So even though we were almost exactly average on both tests, we did a little bit better on the critical reading portion relative to other people.

Finally,  $z$ -scores are incredibly useful if we need to combine information from different measures that are on different scales. Let's say we give a set of employees a series of tests on things like job knowledge, personality, and leadership. We may want to combine these into a single score we can use to rate employees for development or promotion, but look what happens when we take the average of raw scores from different scales, as shown in Table 6.4.1.

**Table 6.4.1. Raw test scores on different scales (ranges in parentheses).**

Employee	Job Knowledge (0–100)	Personality (1–5)	Leadership (1–5)	Average
Employee 1	98	4.2	1.1	34.43
Employee 2	96	3.1	4.5	34.53
Employee 3	97	2.9	3.6	34.50

Because the job knowledge scores were so big and the scores were so similar, they overpowered the other scores and removed almost all variability in the average. However, if we standardize these scores into  $z$ -scores, our averages retain more variability and it is easier to assess differences between employees, as shown in Table 6.4.2.

**Table 6.4.2. Standardized scores.**

Employee	Job Knowledge (0–100)	Personality (1–5)	Leadership (1–5)	Average
Employee 1	1.00	1.14	−1.12	0.34
Employee 2	−1.00	−0.43	0.81	−0.20
Employee 3	0.00	−0.71	0.30	−0.14

## Try It

6) Interpret the location, direction, and distance (near or far) of the following  $z$ -scores:

- −2.00
- 1.25
- 3.50
- −0.34

### Solution

- 2 standard deviations below the mean, far
- 1.25 standard deviations above the mean, near
- 3.5 standard deviations above the mean, far
- 0.34 standard deviations below the mean, near

## Try It

7) Calculate  $z$ -scores for the following raw scores taken from a population with a mean of **100** and standard deviation of **16**: **112, 109, 56, 88, 135, 99**

### Solution

$$z = 0.75, 0.56, -2.75, -0.75, 2.19, -0.06$$

## Setting the Scale of a Distribution

Another convenient characteristic of  $z$ -scores is that they can be converted into any “scale” that we would like. Here, the term scale means how far apart the scores are (their spread) and where they are located (their central tendency). This can be very useful if we don’t want to work with negative numbers or if we have a specific range we would like to present. The formulas for transforming  $z$  to  $x$  are:

$$x = z\sigma + \mu \text{ for a population and}$$

$$x = zs + M \text{ for a sample.}$$

Notice that these are just simple rearrangements of the original formulas for calculating  $z$  from raw scores.

Let’s say we create a new measure of intelligence, and initial calibration finds that our scores have a mean of **40** and standard deviation of **7**. Three people who have scores of **52, 43, and 34** want to know how well they did on the measure. We can convert their raw scores into  $z$ -scores:

$$z = \frac{52 - 40}{7} = 1.71$$

$$z = \frac{43 - 40}{7} = 0.43$$

$$z = \frac{34 - 40}{7} = -0.80$$

A problem is that these new  $z$ -scores aren’t exactly intuitive for many people. We can give people information about their relative location in the distribution (for instance, the first person scored well above average), or we can translate these  $z$ -scores into the more familiar metric of IQ scores, which have a mean of **100** and standard deviation of **16**:

$$IQ = 1.71(16) + 100 = 127.6$$

$$IQ = 0.43(16) + 100 = 106.88$$

$$IQ = -0.80(16) + 100 = 87.20$$

We would also likely round these values to 127, 107, and 87, respectively, for convenience.

### Try It

8) Calculate the raw score for the following  $z$ -scores from a distribution with a mean of 10 and standard deviation of 2:  $-1.75, 2.20, 1.65, -0.95$

#### Solution

Using the formula  $x = \mu + z\sigma$ , we get the raw scores of 6.5, 14.4, 13.3, 8.1.

### Try It

9) Calculate the raw score for the following  $z$ -scores from a distribution with a mean of 15 and standard deviation of 3:

- a. 4.0
- b. 2.2
- c.  $-1.3$
- d. 0.46

#### Solution

Using the formula  $x = \mu + z\sigma$ , we get the raw scores of:

- a. 27
- b. 21.6

- c. 11.1
- d. 16.38

## Z Scores and the Area under the Curve

$z$ -Scores and the standard normal distribution go hand-in-hand. A  $z$ -score will tell you exactly where in the standard normal distribution a value is located, and any normal distribution can be converted into a standard normal distribution by converting all of the scores in the distribution into  $z$ -scores, a process known as **standardization**.

We saw in the previous section, that standard deviations can be used to divide the normal distribution: 68% of the distribution falls within 1 standard deviation of the mean, 95% within (roughly) 2 standard deviations, and 99.7% within 3 standard deviations. Because  $z$ -scores are in units of standard deviations, this means that 68% of scores fall between  $z = -1.0$  and  $z = 1.0$  and so on. We call this 68% (or any percentage we have based on our  $z$ -scores) the proportion of the area under the curve. Any area under the curve is bounded by (defined by, delineated by, etc.) by a single  $z$ -score or pair of  $z$ -scores.

An important property to point out here is that, by virtue of the fact that the total area under the curve of a distribution is always equal to 1.0 (see section on Normal Distributions at the beginning of this section), these areas under the curve can be added together or subtracted from 1 to find the proportion in other areas. For example, we know that the area between  $z = -1.0$  and  $z = 1.0$  (i.e., within one standard deviation of the mean) contains 68% of the area under the curve, which can be represented in decimal form as 0.6800. (To change a percentage to a decimal, simply move the decimal point 2 places to the left.) Because the total area under the curve is equal to 1.0, that means that the proportion of the area outside  $z = -1.0$  and  $z = 1.0$  is equal to  $1.0 - 0.6800 = 0.3200$  or 32% (see Figure 6.4.4). This area is called the **area in the tails of the distribution**. Because this area is split between two tails and because the normal distribution is symmetrical, each tail has exactly one-half, or 16%, of the area under the curve.

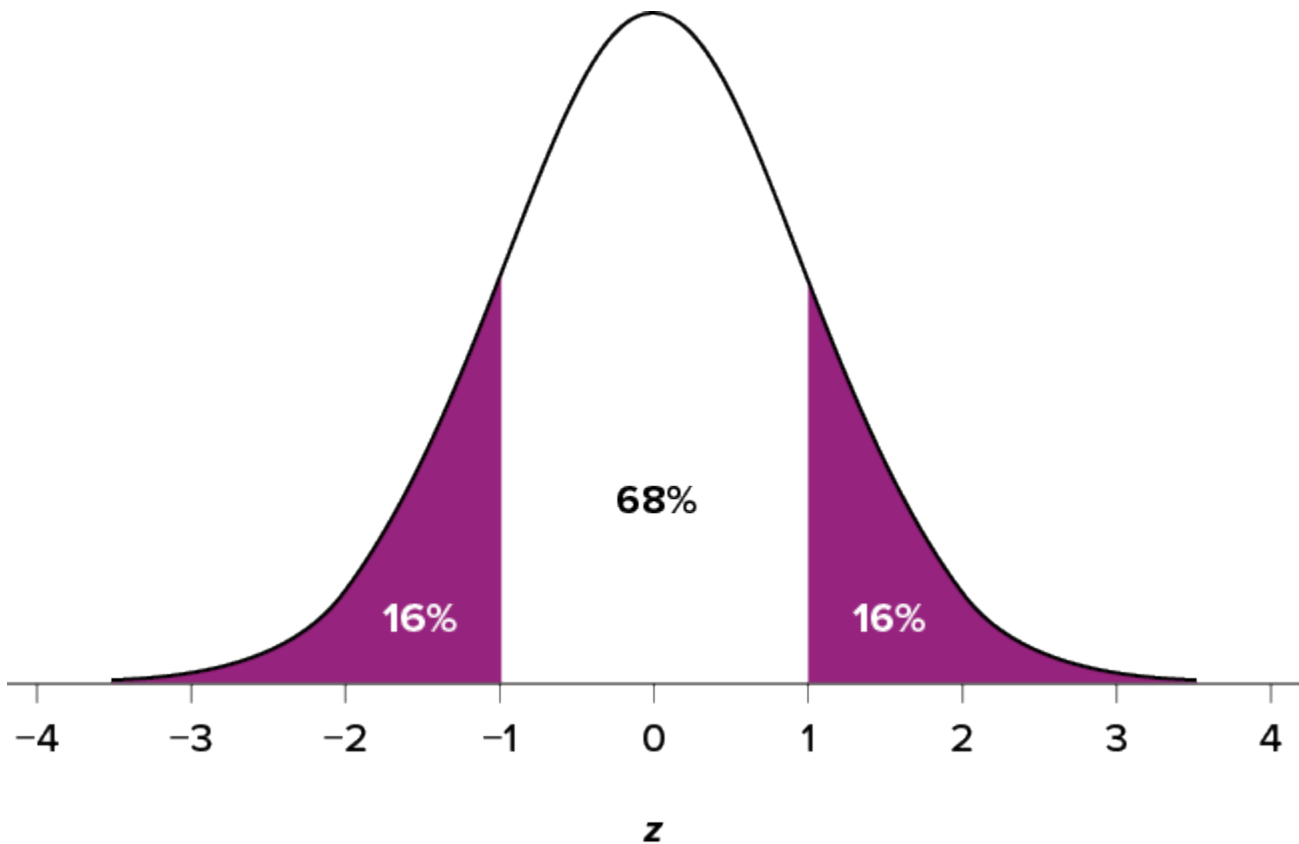


Figure 6.4.4. Shaded areas represent the area under the curve in the tails. (“Area under the Curve in the Tails” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

We will have much more to say about this concept in the coming sections. As it turns out, this is a quite powerful idea that enables us to make statements about how likely an outcome is and what that means for research questions we would like to answer and hypotheses we would like to test. But first, we need to make a brief foray into some ideas about probability in the next section.

### Try It

10) What does a  $z$ -score of 0.00 represent?

#### **Solution**

A  $z$ -score of 0.00 means that the raw value is equal to the mean of the distribution.

## Try It

11) For a distribution with a standard deviation of **20**, find  $z$ -scores that correspond to:

- a. One-half of a standard deviation below the mean
- b. **5** points above the mean
- c. Three standard deviations above the mean
- d. **22** points below the mean

### Solution

- a. **-0.50**
- b. **0.25**
- c. **3.00**
- d. **1.10**

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=1746#h5p-41>

b) After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

## Glossary

### **area in the tails of the distribution**

The proportion of the distribution that falls in the tails of a normal curve. The area in the tail of the distribution associated with a particular  $z$  score can be found in Appendix A, column C.

### **area under the curve**

The proportion of the distribution that is bounded by a single  $z$  score or a pair of  $z$  scores. The area under the curve bounded by a single  $z$  score can be found in Appendix A, column B.

### **Empirical Rule**

Roughly 68% of values are within 1 standard deviation of the mean, roughly 95% of values are within 2 standard deviations of the mean, and 99.7% of values are within 3 standard deviations of the mean.

### **normal distribution**

One of the most common continuous distributions, a normal distribution is sometimes referred to as a bell-shaped distribution, a bell curve, or a Gaussian curve. If the mean is 0 and the standard deviation is 1, the distribution is referred to as the “standard normal distribution.”

### **standardization**

The process of transforming any normal distribution into a standard normal distribution by converting all of the raw scores in the distribution into standard scores ( $z$  scores).

### **standard normal distribution**

A normal distribution that has a mean of 0 and a standard deviation of 1; also known as the unit normal distribution.

### **$z$ score**

The number of standard deviations a score is from the mean of its population. When the scores (or sample means) in the population are normally distributed, the  $z$  table can be used to find probabilities for obtaining a given  $z$  score.



# 6.5 PROBABILITY AND THE NORMAL DISTRIBUTION

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## Learning Objectives

By the end of this section, you will be able to:

- Find the probability of a discrete event.
- Explain the concept of the standard normal distribution and its importance for inference.
- Calculate event probabilities based on transforming raw scores to  $z$ -scores and percentiles and understand how they are applied to decision-making situations.
- Transform  $z$ -scores into raw scores given an event probability.

Probability can seem like a daunting topic for many students. In a mathematical statistics course this might be true, as the meaning and purpose of probability gets obscured and overwhelmed by equations and theory. In this chapter we will focus only on the principles and ideas necessary to lay the groundwork for future inferential statistics. We accomplish this by quickly tying the concepts of probability to what we already know about normal distributions and  $z$ -scores.

## What Is Probability?

When we speak of the **probability** of something happening, we are talking how likely it is that “thing” will happen based on the conditions present. For instance, what is the probability that it will rain? That is, how likely do we think it is that it will rain today under the circumstances or conditions today? To define or understand the conditions that might affect how likely it is to rain, we might look out the window and say, “It’s sunny outside, so it’s not very likely that it will rain today.” Stated using probability language: given that it is sunny outside, the probability of rain is low. “Given” is the word we use to state what the conditions are. As

the conditions change, so does the probability. Thus, if it were cloudy and windy outside, we might say, “Given the current weather conditions, there is a high probability that it is going to rain.”

In these examples, we spoke about whether or not it is going to rain. Raining is an example of an **event**, which is the catch-all term we use to talk about any specific thing happening; it is a generic term that we specified to mean “rain” in exactly the same way that “conditions” is a generic term that we specified to mean “sunny” or “cloudy and windy.”

It should also be noted that the terms “low” and “high” are relative and vague, and they will likely be interpreted different by different people (in other words: given how vague the terminology was, the probability of different interpretations is high). Most of the time we try to use more precise language or, even better, numbers to represent the probability of our event. Regardless, the basic structure and logic of our statements are consistent with how we speak about probability using numbers and formulas.

Let’s look at a slightly deeper example. Say we have a regular, six-sided die (note that die is singular and dice is plural) and want to know how likely it is that we will roll a **1**. That is, what is the probability of rolling a **1**, given that the die is not weighted (which would introduce what we call a bias, though that is beyond the scope of this section). We could roll the die and see if it is a **1** or not, but that won’t tell us about the probability, it will only tell us a single result. We could also roll the die hundreds or thousands of times, recording each outcome and seeing what the final list looks like, but this is time consuming, and rolling a die that many times may lead down a dark path to gambling or, worse, playing Dungeons & Dragons. What we need is a simple equation that represents what we are looking for and what is possible.

To calculate the probability of an event, which here is defined as rolling a **1** on an unbiased die, we need to know two things: how many outcomes satisfy the criteria of our event (stated differently, how many outcomes would count as what we are looking for) and the total number of outcomes possible. In our example, only a single outcome, rolling a **1**, will satisfy our criteria, and there are a total of six possible outcomes (rolling a **1**, rolling a **2**, rolling a **3**, rolling a **4**, rolling a **5**, and rolling a **6**). Thus, the probability of rolling a **1** on an unbiased die is **1** in **6** or  $\frac{1}{6}$ . Put into an equation using generic terms, we get:

$$\text{Probability of an event} = \frac{\text{Number of outcomes that satisfy our criteria}}{\text{Total number of possible outcomes}}$$

We can also use  $P()$  as shorthand for probability and  $A$  as shorthand for an event:

$$P(A) = \frac{\text{Number of outcomes that count as } A}{\text{Total number of possible outcomes}}$$

Using this equation, let’s now calculate the probability of rolling an even number on this die:

$$P(\text{even number}) = \frac{2, 4, \text{ or } 6}{1, 2, 3, 4, 5, \text{ or } 6} = \frac{3}{6} = \frac{1}{2}$$

So we have a **50%** chance of rolling an even number of this die. The principles laid out here operate under a certain set of conditions and can be elaborated into ideas that are complex yet powerful and elegant. However, such extensions are not necessary for a basic understanding of statistics, so we will end our discussion on the math of probability here. Now, let’s turn back to more familiar topics.

## Probability in Graphs and Distributions

We will see shortly that the normal distribution is the key to how probability works for our purposes. To understand exactly how, let's first look at a simple, intuitive example using pie charts.

### Probability in Pie Charts

Recall that a pie chart represents how frequently a category was observed and that all slices of the pie chart add up to 100%, or 1. This means that if we randomly select an observation from the data used to create the pie chart, the probability of it taking on a specific value is exactly equal to the size of that category's slice in the pie chart.

Take, for example, the pie chart in Figure 6.5.1 representing the favourite sports of 100 people. If you put this pie chart on a dart board and aimed blindly (assuming you are guaranteed to hit the board), the likelihood of hitting the slice for any given sport would be equal to the size of that slice. So, the probability of hitting the baseball slice is the highest at 36%. The probability is equal to the proportion of the chart taken up by that section.

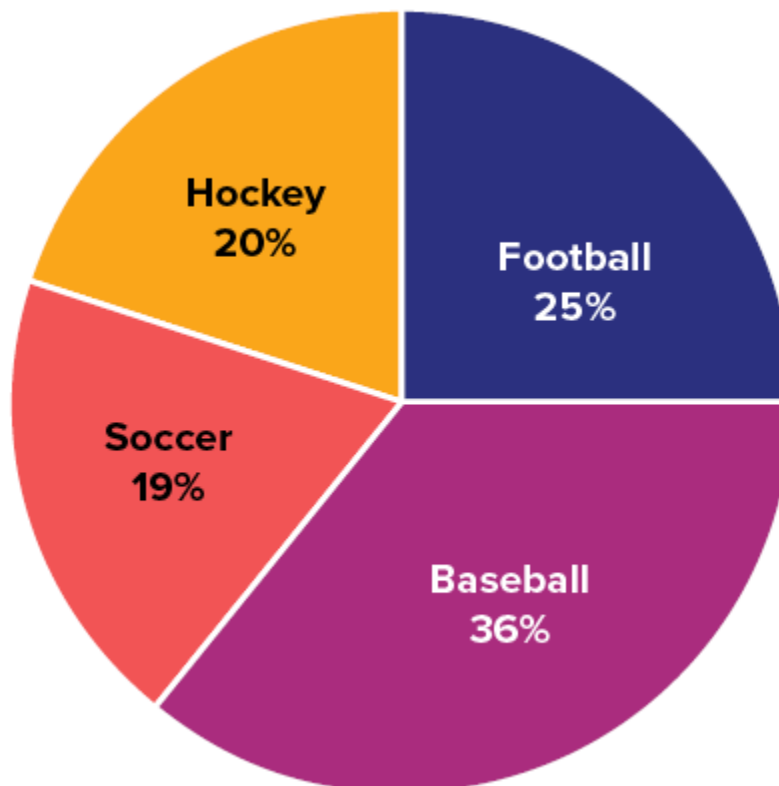


Figure 6.5.1. Favourite sports. ("Favorite Sports Pie Chart" by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

We can also add slices together. For instance, maybe we want to know the probability to finding someone

whose favourite sport is usually played on grass. The outcomes that satisfy this criterion are baseball, football, and soccer. To get the probability, we simply add their slices together to see what proportion of the area of the pie chart is in that region:  $36\% + 25\% + 20\% = 81\%$ . We can also add sections together even if they do not touch. If we want to know the likelihood that someone's favourite sport is not called football somewhere in the world (i.e., baseball and hockey), we can add those slices even though they aren't adjacent or contiguous in the chart itself:  $36\% + 20\% = 56\%$ . We are able to do all of this because (1) the size of the slice corresponds to the area of the chart taken up by that slice, (2) the percentage for a specific category can be represented as a decimal (this step was skipped for ease of explanation above), and (3) the total area of the chart is equal to  $100\%$  or  $1.0$ , which makes the size of the slices interpretable.

### Try It

1) In your own words, what is probability?

#### **Solution**

Your answer should include information about an event happening under certain conditions given certain criteria. You could also discuss the relationship between probability and the area under the curve or the proportion of the area in a chart.

### Try It

2) There is a bag with 5 red blocks, 2 yellow blocks, and 4 blue blocks. If you reach in and grab one block without looking, what is the probability it is red?

#### **Solution**

$$\frac{5}{11} = 0.4545$$

## Probability in Normal Distributions

If the language at the end of the last section sounded familiar, that's because it's exactly the language used in Section 6.4 to describe the normal distribution. Recall that the normal distribution has an area under its curve that is equal to 1 and that it can be split into sections by drawing a line through it that corresponds to a given  $z$ -score. Because of this, we can interpret areas under the normal curve as probabilities that correspond to  $z$ -scores.

First, let's look at the area between  $z = -1.00$  and  $z = 1.00$  presented in Figure 6.5.2. We were told earlier that this region contains 68% of the area under the curve. Thus, if we randomly chose a  $z$ -score from all possible  $z$ -scores, there is a 68% chance that it will be between  $z = -1.00$  and  $z = 1.00$  because those are the  $z$ -scores that satisfy our criteria.

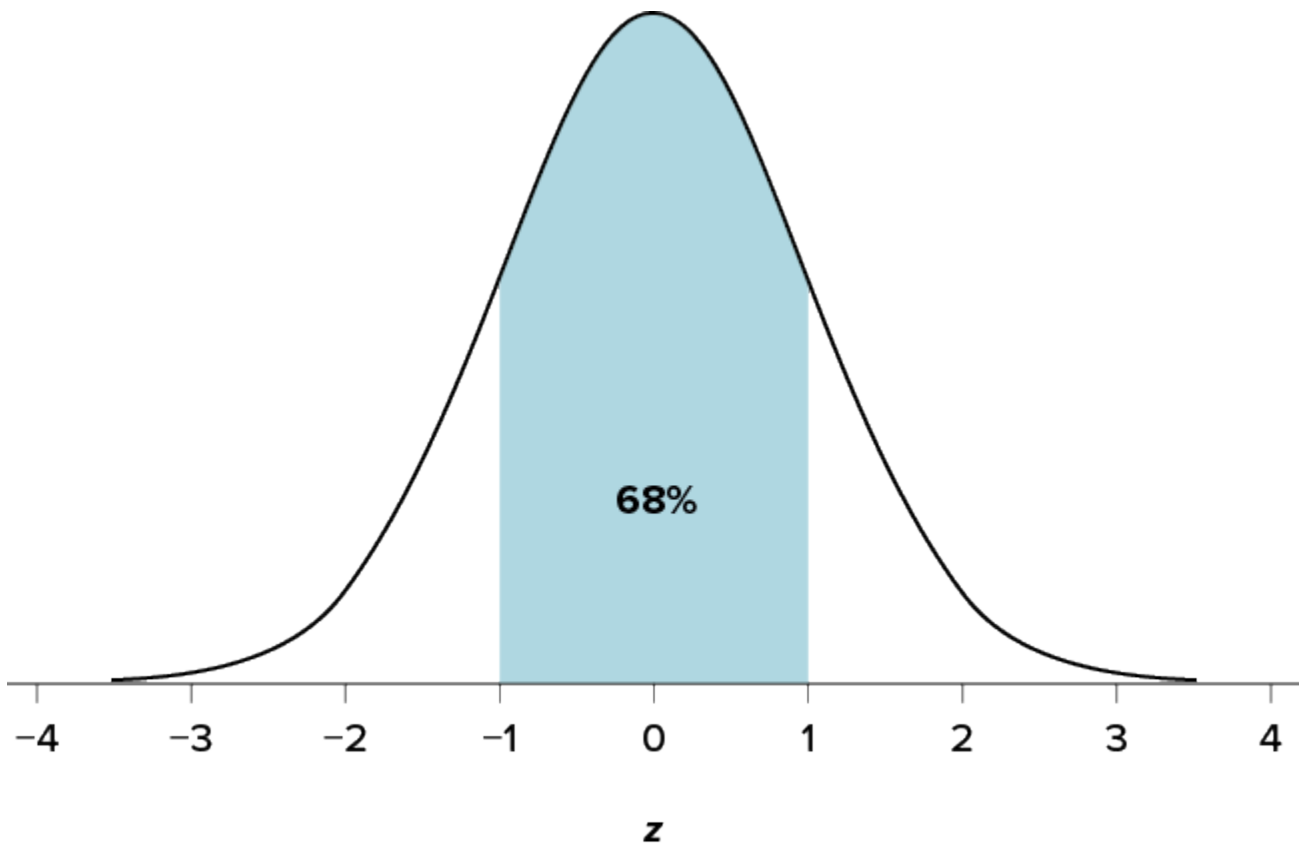


Figure 6.5.2. There is a 68% chance of selecting a  $z$ -score from the blue-shaded region. (“68 Percent of the Area under the Curve” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

Just like a pie chart is broken up into slices by drawing lines through it, we can also draw a line through the normal distribution to split it into sections. Take a look at the normal distribution in Figure 6.5.3, which has a line drawn through it at  $z = 1.25$ . This line creates two sections of the distribution: the smaller section called the tail and the larger section called the body. Differentiating between the body and the tail does not depend

on which side of the distribution the line is drawn. All that matters is the relative size of the pieces: bigger is always body.

As you can see, we can break up the normal distribution into **3** pieces (lower tail, body, and upper tail) as in Figure 6.5.2 or into **2** pieces (body and tail) as in Figure 6.5.3. We can then find the proportion of the area in the body and tail based on where the line was drawn (i.e., at what  $z$ -score). Mathematically, this is done using calculus. Fortunately, the exact values are given to you in the Standard Normal Distribution Table, also known as the  $z$ -table. A portion of this table is shown in table 6.5.1 (the entire  $z$ -score table). Using the  $z$  values in the table (A), we can find the area under the normal curve in any body and can calculate the area under the normal curve in any tail.

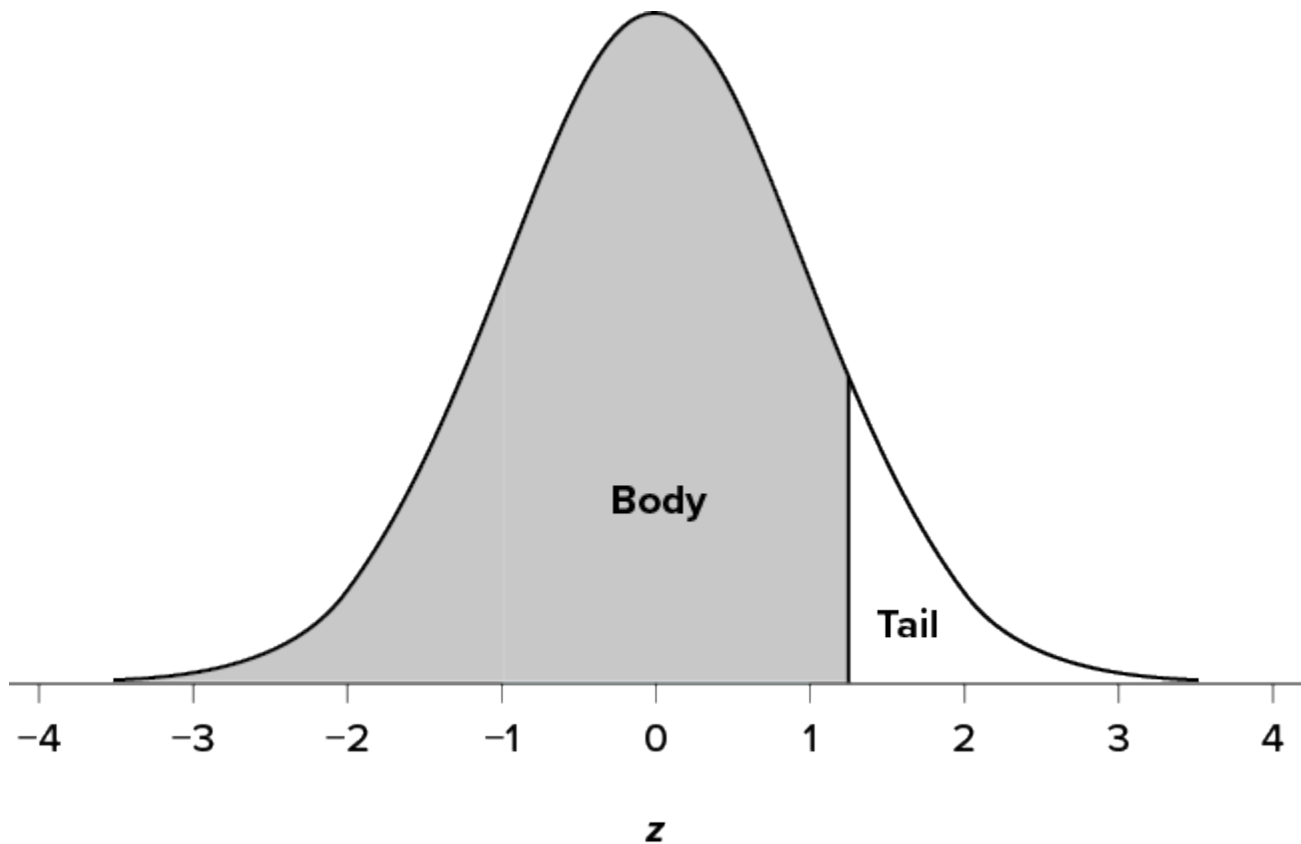


Figure 6.5.3. Body and tail of the normal distribution. (“Normal Distribution Body and Tail” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

For example, suppose we want to find the area in the body for a  $z$ -score of **1.62**. As shown in Table 6.5.1, the row for **1.62** corresponds with a value of **0.9474** for the proportion in the body of the distribution. This cell has been highlighted in yellow to help locate it. Thus, the odds of randomly selecting someone with a  $z$ -score less than (to the left of)  $z = 1.62$  is **94.74%** because that is the proportion of the area taken up by values that satisfy our criteria.

Mathematically, we present this solution in the following way:

$$P(z < 1.62) = 0.9474$$

Table 6.5.1 Area Under The Standard Normal Curve ( $Z > 0$ )

Z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633

Some versions of the  $z$ -table only present the area in the body for positive  $z$ -scores because the normal distribution is symmetrical. The area in the body of  $z = 1.62$  is equal to the area in the body for  $z = -1.62$ , though now — as illustrated in the middle distribution at the top of Table 6.5.1 — the body will be the shaded area to the right of  $z$ . (When in doubt, drawing out your distribution and shading the area you need to find will always help.) Because the total area under the normal curve is always equal to 1.00, the area in the tail is simply the area in the body subtracted from 1.00 ( $1.00 - 0.9474 = 0.0526$ ). We will also provide the negative side of the  $z$ -score table in our course, so we will be able to use it to find probabilities associated with negative  $z$ -scores.

Let's look at another example. This time, let's find the area corresponding to  $z$ -scores more extreme than  $z = -1.96$  and  $z = 1.96$ . That is, let's find the area in the tails of the distribution for values less than  $z = -1.96$  (farther negative and therefore more extreme) and greater than  $z = 1.96$  (farther positive and therefore more extreme). This region is illustrated in Figure 6.5.4.

Mathematically, we are looking for

$$P(z < -1.96) \text{ and } P(z > 1.96)$$

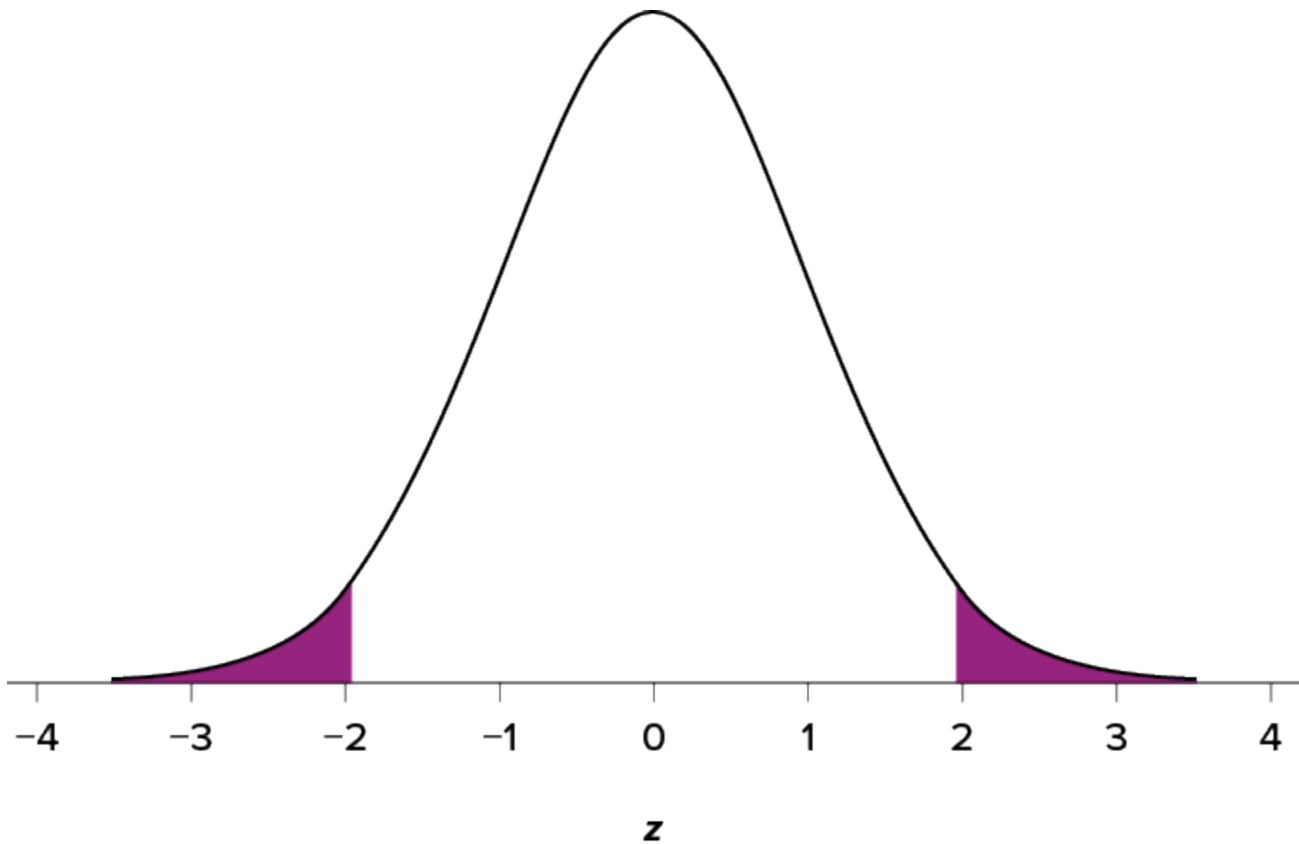


Figure 6.5.4. Area in the tails beyond  $z = -1.96$  and  $z = 1.96$ . (“Area in the Tails  $z > 1.96$ ” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

Let’s start with the tail for  $z = 1.96$ . Our  $z$ -table only provides the area under the curve to the left of a particular  $z$ -score. As such, finding  $P(z < -1.96)$  is straightforward, and we need only look to the negative side of the  $z$ -score table to find it. Looking at the negative side of the  $z$ -score table, we find  $P(z < -1.96) = 0.0250$ . Since the distribution is symmetric, this means we can deduce that  $P(z > 1.96) = 0.0250$  as well. Finally, to get the total area in the shaded region, we simply add the areas together to get 0.0500. Thus, there is a 5% chance of randomly getting a value more extreme than  $z = -1.96$  or  $z = 1.96$ .

Finally, we can find the area between two  $z$ -scores by shading and subtracting. Figure 6.5.5 shows the area between  $z = 0.50$  and  $z = 1.50$ . Because this is a subsection of a body (rather than just a body or a tail), we must first find the larger of the two bodies, in this case the body for  $z = 1.50$ , and subtract the smaller of the two bodies, or the body for  $z = 0.50$ . Aligning the distributions vertically, as in Figure 6.5.5, makes this



clearer. From the complete  $z$ -table, we see that the area in the body for  $z \geq 1.50$  is **0.9332**, and the area in the body for  $z = 0.50$  is **0.6915**. Subtracting these gives us  **$0.9332 - 0.6915 = 0.2417$** .

Mathematically, we write:

$$P(0.50 \leq z \leq 1.50) = P(z \leq 1.50) - P(z \leq 0.50)$$

$$P(0.50 \leq z \leq 1.50) = 0.9332 - 0.6915$$

$$P(0.50 \leq z \leq 1.50) = 0.2417$$

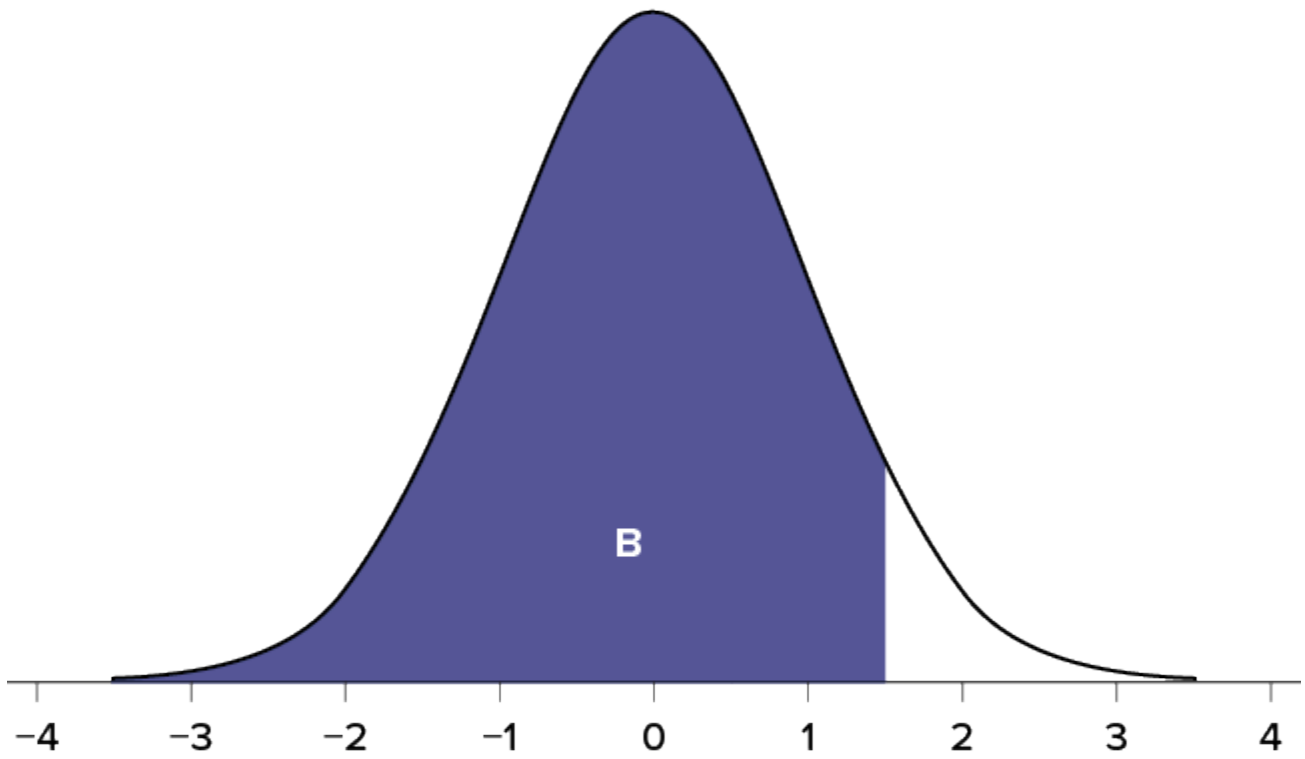
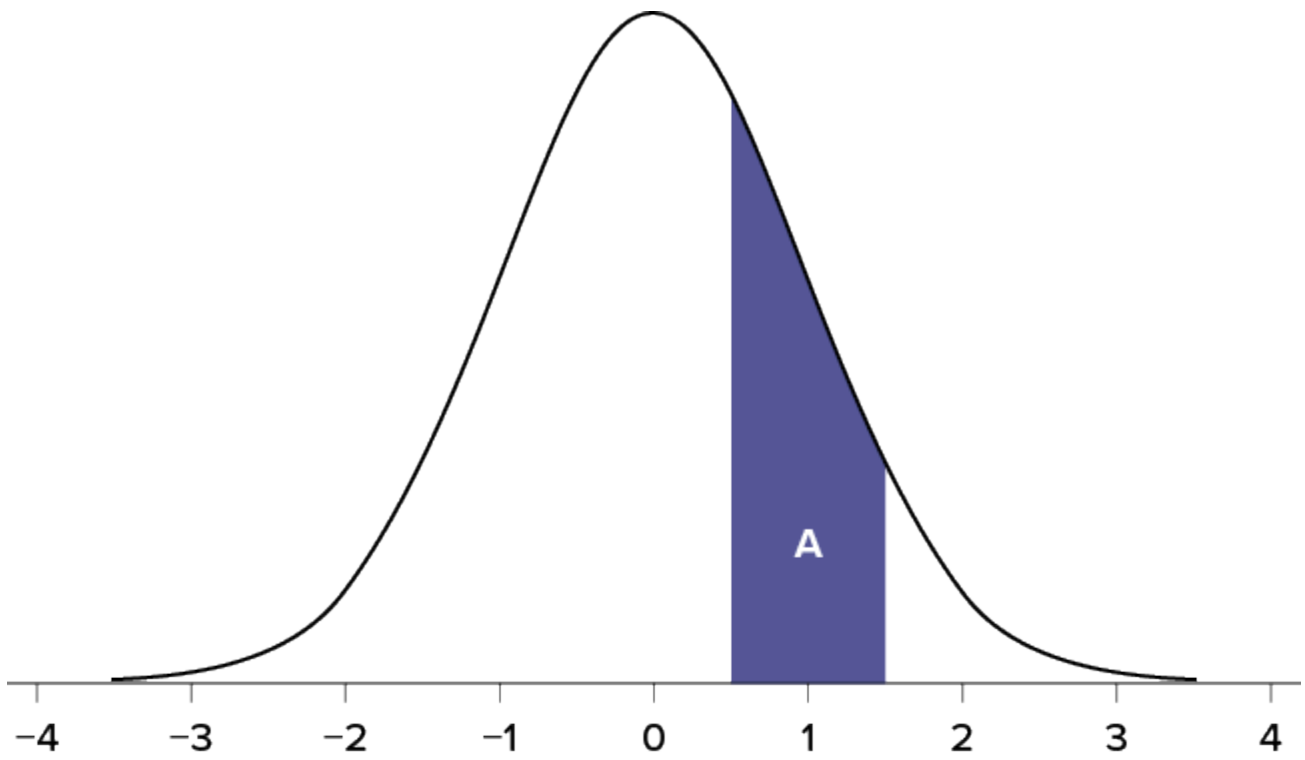


Figure 6.5.5. Area between  $z = 0.50$  and  $z = 1.50$  (A), along with the corresponding areas in the body (B and C). (“Area between  $z = 0.50$  and  $z = 1.50$ ” by Judy Schmitt is licensed under CC BY-NC-SA 4.0.)

## Example 6.5.1

Use the  $z$ -table to find the following probabilities:

- $P(z \leq 2)$
- $P(z \geq 2)$
- $P(-2 \leq z \leq 2)$

### Solution

a.

**Step 1: Access the full  $z$ -table.**

**Step 2: Determine the probability for  $P(z < 2)$**

$$P(z < 2) = 0.9772$$


---

b.

**Step 1: Write an equation to find the probability for  $P(z \geq 2)$**

$$P(z \geq 2) = 1 - P(z < 2)$$

**Step 2: Access the full  $z$ -table.**

**Step 3: Determine the probability for  $P(z < 2)$  and substitute into the equation.**

You can also use your answer from part a.

$$P(z \geq 2) = 1 - 0.9772$$

**Step 4: Solve.**

$$P(z \geq 2) = 0.0228$$


---

c.

**Step 1: We can find the area between 2 z-scored by subtracting.**

$$P(-2 \leq z \leq 2) = P(z \leq 2) - P(z \leq -2)$$

**Step 2: Access the full z-table.****Step 3: Substitute the values into the equation.**

$$P(-2 \leq z \leq 2) = 0.9772 - 0.0228$$

**Step 4: Solve**

$$P(-2 \leq z \leq 2) = 0.9544$$

## Try It

3) Use the z-table to find the following probabilities:

a.  $P(z \leq -0.73)$

b.  $P(z \geq 2.65)$

c.  $P(-1.8 \leq z \leq 0.82)$

**Solution**

a.  $P(z \leq -0.73) = 0.2327$

b.  $P(z \geq 2.65) = 0.0040$

c.  $P(-1.8 \leq z \leq 0.82) = 0.7580$

## Try It

4) Under a normal distribution, which of the following is more likely? (Note: this question can be answered without any calculations if you draw out the distributions and shade properly.)

- Getting a  $z$ -score greater than  $z = 2.75$
- Getting a  $z$ -score less than  $z = -1.50$

### Solution

Getting a  $z$ -score less than  $z = -1.50$  is more likely.  $z = 2.75$  is farther out into the right tail than  $z = -1.50$  is into the left tail; therefore, there are fewer more extreme scores beyond  $2.75$  than  $-1.50$ , regardless of the direction.

## Standardizing Normal Distributions

In the examples above, we have been given the  $z$ -scores directly. However, most of the time, we are studying a random variable  $X$  where the mean and standard deviation are different from zero and one respectively. Thus, we need to use a process called standardization, sometimes referred to in mathematical notation as  $X \sim N(\mu, \sigma)$  to  $Z \sim N(0, 1)$ , in order to be able to find the probabilities using  $z$ -scores. To do this, use the formula,

$$z = \frac{x - \mu}{\sigma}$$

Using this formula standardizes the random variable and finds the  $z$ -score associated with the values of interest so that we can find the probabilities in the  $z$ -table.

## Example 6.5.2

If the mean birth weight of a population of babies were **3,370** grams and the standard deviation was **150** grams.

- Find the probability that a baby from that population has a birth weight of less than **3000** grams.
- Find the probability that a baby from that population has a birth weight of more than **3500** grams.
- Find the probability that a baby from that population has a birth weight between **3300** grams and **3600** grams.

### Solution

a.

**Step 1: Identify what you are looking for.**

$$P(X < 3000)$$

**Step 2: Use the formula  $z = \frac{x - \mu}{\sigma}$  to find the  $z$ -score.**

$$P(X < 3000) = P\left(z < \frac{3000 - 3370}{150}\right)$$

$$P(X < 3000) = P(z < -2.47)$$

**Step 3: Find the probability in the  $z$ -table.**

$$P(X < 3000) = 0.0068$$

b.

**Step 1: Identify what you are looking for.**

$$P(X > 3500)$$

**Step 2: Use the formula  $z = \frac{x - \mu}{\sigma}$  to find the  $z$ -score.**

Remember to always round your  $z$ -score to two decimal places.

$$P(X > 3500) = P\left(z > \frac{3500 - 3370}{150}\right)$$

$$P(X > 3500) = P(z > 0.87)$$

**Step 3: Write the equation to find the  $z$ -score more than  $z$ .**

$$P(X > 3500) = 1 - P(z < 0.87)$$

**Step 4: Replace with the probability in the  $z$ -table.**

$$P(X > 3500) = 1 - 0.8078$$

**Step 5: Subtract.**

$$P(X > 3500) = 0.1922$$


---

c.

**Step 1: Identify what you are looking for.**

$$P(3300 < X < 3600)$$

**Step 2: Use the formula  $z = \frac{x - \mu}{\sigma}$  to find the  $z$ -scores.**

$$P(3300 < X < 3600) = P\left(\frac{3300 - 3370}{150} < z < \frac{3600 - 3370}{150}\right)$$

$$P(3300 < X < 3600) = P(-0.47 < z < 1.53)$$

$$P(3300 < X < 3600) = P(z < 1.53) - P(z < -0.47)$$

**Step 3: Replace with the probabilities in the  $z$ -table and solve.**

$$P(3300 < X < 3600) = 0.9370 - 0.3192$$

$$P(3300 < X < 3600) = 0.6178$$

### Example 6.5.3

The scores for a test were normally distributed with a mean of 66% and a standard deviation of 4%.

- Find the probability that a randomly selected student scored less than 55%.
- Find the probability that a student scored between 70% and 75%.

#### Solution

a.

**Step 1: Identify what you are looking for.**

$$P(X < 55)$$

**Step 2: Use the formula  $z = \frac{x - \mu}{\sigma}$  to find the  $z$ -score.**

$$P(X < 55) = P\left(z < \frac{55 - 66}{4}\right)$$

$$P(X < 55) = P(z < -2.75)$$

**Step 3: Find the probability in the  $z$ -table.**

$$P(X < 55) = 0.0030$$


---

b.

**Step 1: Identify what you are looking for.**

$$P(70 < X < 75)$$

**Step 2: Use the formula  $z = \frac{x - \mu}{\sigma}$  to find the  $z$ -scores.**



$$P(70 < X < 75) = P\left(\frac{70 - 66}{4} < z < \frac{75 - 66}{4}\right)$$

$$P(70 < X < 75) = P(1 < z < 2.25)$$

$$P(70 < X < 75) = P(z < 1) - P(z < 2.25)$$

**Step 3: Replace with the probabilities in the z-table and solve.**

$$P(70 < X < 75) = 0.9878 - 0.8413$$

$$P(70 < X < 75) = 0.1465$$

## Try It

5) The heights of women in the United States are normally distributed with a mean of **63.7** inches and a standard deviation of **2.7** inches. If you randomly select a woman in the United States, what is the probability that she will be between **65** and **67** inches tall?

### Solution

The probability that a randomly selected woman is between **65** inches and **67** inches tall is **0.2044** or **20.4%**.

## Try It

6) The heights of men in the United States are normally distributed with a mean of **69.1** inches and a standard deviation of **2.9** inches. What proportion of men are taller than **6** feet (**72** inches)?

**Solution**

15.87% or 0.1587

## Try It

7) You know you need to score at least **82** points on the final exam to pass your class. After the final, you find out that the average score on the exam was **78** with a standard deviation of **7**. How likely is it that you pass the class?

**Solution**

It is not likely that you pass the class since the probability of scoring at least 82 points on the final exam is **0.2843** or **28.43%**.

## Using Percentiles and Z-scores to Find Original Value

A useful tool when studying normal distributions is being able to track backwards to find the value on the original distribution that is associated with a particular  $z$ -score or percentile. We can do this by a process sometimes called “un-standardizing”. Essentially, we are just rearranging the standardizing formula to solve for  $x$ .

$$z = \frac{x - \mu}{\sigma}$$

$$\sigma z = x - \mu$$

$$\sigma z + \mu = x$$

So we can find the original data value by using the formula:  $x = \mu + z\sigma$ .

### Example 6.5.4

If the mean birth weight of a population of babies were **3,370** grams and the standard deviation was **150** grams.

- Find the birth weight of a baby that has a  $z$ -score of **2.34**.
- Find the birth weight of a baby in the **90<sup>th</sup>** percentile.

#### Solution

a.

**Step 1: Use the equation  $x = \mu + z\sigma$**

**Step 2: Substitute the values.**

$$x = 3370 + 2.34(150)$$

**Step 3: Solve.**

$$x = 3721$$

b.

**Step 1: Look at the  $z$ -table to find the closest to 0.9000 as we can find.**

We can find **1.28** as this  $z$ -score.

**Step 2: Substitute the  $z$ -score into the equation.**

$$x = \mu + z\sigma$$

$$x = 3370 + 1.28(150)$$

$$x = 3520$$

### Example 6.5.5

A citrus farmer who grows mandarin oranges finds that the diameters of mandarin oranges harvested on his farm follow a normal distribution with a mean diameter of **5.85**cm and a standard deviation of **0.24**cm.

- Find the **90<sup>th</sup>** percentile for the diameters of mandarin oranges:
- The middle **20%** of mandarin oranges from this farm have diameters between \_\_\_\_\_ and \_\_\_\_\_.

#### Solution

a.

**Step 1: Look at the z-table to find the closest to 0.9000 as we can find.**

We can find **1.28** as this *z*-score.

**Step 2: Substitute the *z*-score into the equation.**

$$x = \mu + z\sigma$$

$$x = 5.85 + 1.28(0.24)$$

$$x = 6.16$$

b.

**Step 1: Find the area of the tails for the middle 20%.**

Each tail will have half the value.

$$1 - 0.20 = 0.80$$

$$0.80 \div 2 = 0.40$$

**Step 2: Find the value for the 40<sup>th</sup> percentile.**

Use the z-table to find the closest to 0.4000.

$$x = \mu + z\sigma$$

$$x = 5.85 + (-0.25)(0.24)$$

$$x = 5.79$$

**Step 3: Find the value for the 60<sup>th</sup> percentile, (0.40 + 0.20 = 0.60).**

Use the z-table to find the closest to 0.6000.

$$x = \mu + z\sigma$$

$$x = 5.85 + (0.26)(0.24)$$

$$x = 5.91$$

**Step 4: Answer the question in a complete sentence.**

The middle 20% of mandarin oranges from this farm have diameters between 5.79 cm and 5.91 cm.

## Try It

8) What proportion of the area under the normal curve is greater than  $z = 1.65$ ?

**Solution**

4.95% or 0.0495

## Try It

9) Find the  $z$ -score that bounds **25%** of the lower tail of the distribution.

**Solution**

$$z = -0.67$$

## Try It

10) Find the  $z$ -score that bounds the top **9%** of the distribution.

**Solution**

$z = 1.34$  (The top **9%** means **9%** of the area is in the upper tail and **91%** is in the body to the left; the value in the normal table closest to **0.9100** is **0.9099**, which corresponds to  $z = 1.34$ .)

## Probability: The Bigger Picture

The concepts and ideas presented in this chapter are likely not intuitive at first. Probability is a tough topic for everyone, but the tools it gives us are incredibly powerful and enable us to do amazing things with data analysis. They are the heart of how inferential statistics work.

To summarize, the probability that an event happens is the number of outcomes that qualify as that event (i.e., the number of ways the event could happen) compared to the total number of outcomes (i.e., how many things are possible). This extends to graphs like a pie chart, where the biggest slices take up more of the area and are therefore more likely to be chosen at random. This idea then brings us back around to our normal distribution, which can also be broken up into regions or areas, each of which is bounded by one or two  $z$ -scores and corresponds to all  $z$ -scores in that region. The probability of randomly getting one of those  $z$ -scores in the specified region can then be found on the Standard Normal Distribution Table. Thus, the larger the

region, the more likely an event is, and vice versa. Because the tails of the distribution are, by definition, smaller and we go farther out into the tail, the likelihood or probability of finding a result out in the extremes becomes small.

## Try It

11) In a distribution with a mean of **70** and standard deviation of **12**, what proportion of scores are lower than **55**?

### Solution

In this distribution, the proportion of scores that are lower than **55** is  $\frac{66}{625}$  or the probability is **0.1056**.

## Self Check

a) After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.



*An interactive H5P element has been excluded from this version of the text. You can view it online here:*

<https://ecampusontario.pressbooks.pub/prehealthsciencesmath2/?p=1757#h5p-42>

b) After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

## Glossary

**event**

Any specific outcome that could happen.

**probability**

The likelihood of a statistical result or the number of outcomes that satisfy specific criteria divided by the total number of possible outcomes.



## 6.6 UNIT 6 SOURCES

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6.0 “Math Models and Geometry Introduction” from Prealgebra 2e. by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

6.1 “Chapter 1 Introduction” from Introduction to Statistics in the Psychological Sciences by Linda R. Cote Ph.D.; Rupa G. Gordon Ph.D.; Chrislyn E. Randell Ph.D.; Judy Schmitt; and Helena Marvin is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License, except where otherwise noted.

6.2 “Chapter 2 Describing Data Using Distributions and Graphs” from Introduction to Statistics in the Psychological Sciences by Linda R. Cote Ph.D.; Rupa G. Gordon Ph.D.; Chrislyn E. Randell Ph.D.; Judy Schmitt; and Helena Marvin is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License, except where otherwise noted.

6.3 “Chapter 3 Measures of Central Tendency and Spread” from Introduction to Statistics in the Psychological Sciences by Linda R. Cote Ph.D.; Rupa G. Gordon Ph.D.; Chrislyn E. Randell Ph.D.; Judy Schmitt; and Helena Marvin is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License, except where otherwise noted.

6.4 “Chapter 4 Z Scores and the Standard Normal Distribution” from Introduction to Statistics in the Psychological Sciences by Linda R. Cote Ph.D.; Rupa G. Gordon Ph.D.; Chrislyn E. Randell Ph.D.; Judy Schmitt; and Helena Marvin is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License, except where otherwise noted.

6.4 “Empirical Rule – 6.1 Standard Normal Distribution ” from Introductory Statistics by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

6.5 “Chapter 5 Probability” from Introduction to Statistics in the Psychological Sciences by Linda R. Cote Ph.D.; Rupa G. Gordon Ph.D.; Chrislyn E. Randell Ph.D.; Judy Schmitt; and Helena Marvin is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License, except where otherwise noted.

6.5 “6.2 Using the Normal Distribution ” from Introductory Statistics by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.



# GLOSSARY TERMS

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## area in the tails of the distribution

The proportion of the distribution that falls in the tails of a normal curve. The area in the tail of the distribution associated with a particular z score can be found in Appendix A, column C.

## area under the curve

The proportion of the distribution that is bounded by a single z score or a pair of z scores. The area under the curve bounded by a single z score can be found in Appendix A, column B.

## arithmetic mean

Perhaps the most common measure of central tendency, the mean is the mathematical average of the scores in a sample.

## asymptote

A line which a graph of a function approaches closely but never touches.

## bell curve

The bell curve is a symmetrical distribution in which there is a single peak at the center and tails that extend equally out to each side. The bell curve represents a normal distribution.

## bimodal distribution

A distribution with two distinct peaks that lie roughly symmetrically on either side of the center point.

## bin widths

The widths of the class intervals. The choice of bin width determines the number of class intervals. This decision, along with the choice of starting point for the first interval, affects the shape of the histogram.

## box plots

One of the more effective graphical summaries of a data set, the box plot generally shows the median, 25th and 75th percentiles, and outliers.

## categorical variables

Also known as qualitative variables, categorical variables cannot be quantified, or measured numerically. Instead, they are measured on a nominal or ordinal scale.

### central tendency

The center or middle of a distribution. There are many measures of central tendency. The most common are the mean, median, and mode.

### common logarithmic function

The function  $f(x) = \log x$  is the common logarithmic function with base 10, where  $x > 0$  (Note:  $y = \log x$  is equivalent to  $x = 10^y$ ).

### complex conjugate pair

A complex conjugate pair is of the form  $a + bi, a - bi$ .

### complex number

A complex number is of the form  $a + bi$ , where  $a$  and  $b$  are real numbers. We call  $a$  the real part and  $b$  the imaginary part.

### complex number system

The complex number system is made up of both the real numbers and the imaginary numbers.

### complex rational expression

A complex rational expression is a rational expression in which the numerator and/or denominator contains a rational expression.

### continuous variables

Variables whose possible values fall along a continuous scale (not broken up into discrete steps), eg. the time it takes to answer a question.

### control (group)

The group in an experimental study that is not receiving the treatment being tested.

### convenience sampling

A sampling strategy in which participants are recruited for their easy availability (e.g., college students). A sample obtained through convenience sampling should not be considered a representative sample.

### data

A collection of values to be used for statistical analysis. Data is the plural form of datum.

### degree of the polynomial equation

The degree of the polynomial equation is the degree of the polynomial.

### degrees of freedom

The number of independent pieces of information that go into the estimate. In general, the degrees of

freedom for an estimate is equal to the number of values minus the number of parameters estimated en route to the estimate in question.

### dependent variable

A variable that measures the experimental outcome. In most experiments, the effects of the independent variable on the dependent variables are observed. For example, if a study investigated the effectiveness of an experimental treatment for depression, then the measure of depression would be the dependent variable.

### descriptive statistics

A set of statistics—such as the mean, standard deviation, and skew—that describe a distribution.

### discrete variables

A variable that exists in indivisible units. For quantitative variables, it is measured in whole numbers that are discrete points on the scale.

### discriminant

In the Quadratic Formula,  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ , the quantity  $b^2 - 4ac$  is called the discriminant.

### dispersion

The extent to which values differ from one another; that is, how much they vary. Dispersion is also called variability or spread.

### event

Any specific outcome that could happen.

### experimental (group)

The group in a study that is receiving the treatment being tested.

### experimental research

Research that involves the use of random assignment to treatment conditions and manipulation of the independent variable.

### exponential function

An exponential function, where  $a > 0$  and  $a \neq 1$ , is a function of the form  $f(x) = a^x$ .

### extraneous solution to a rational equation

An extraneous solution to a rational equation is an algebraic solution that would cause any of the

expressions in the original equation to be undefined.

### factoring

Splitting a product into factors is called factoring.

### frequency polygons

A frequency polygon is a graphical representation of a distribution that is similar in appearance to a line graph. Frequency polygons can be grouped or ungrouped.

### greatest common factor

The greatest common factor (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

### histogram

A graphical representation of a distribution that is similar in appearance to a bar chart. It partitions the variable on the x-axis into various contiguous class intervals of (usually) equal widths. The heights of the bars represent the class frequencies.

### imaginary unit

The imaginary unit  $i$  is the number whose square is  $-1$ .  $i^2 = -1$  or  $i = \sqrt{-1}$

### independent variable

A variable that is manipulated by the experimenter, as opposed to a dependent variable. Most experiments consist of observing the effect of the independent variable(s) on the dependent variable(s).

### inferential statistics

The branch of statistics concerned with drawing conclusions about a population from a sample.

### interquartile range

### interquartile range (IQR)

The range of the middle 50% of the scores in a distribution; computed by subtracting the 25th percentile from the 75th percentile. The interquartile range is a robust measure of central tendency.

### interval scale

A numerical scale in which the distance between scores on the scale is consistent (equal) and for which the zero is relative (rather than absolute)

### lie factor

The ratio of the size of the effect shown in a graph to the size of the effect shown in the data. This term was coined by Edward Tufte, who suggested that lie factors greater than 1.05 or less than 0.95 produce

unacceptable distortion.

### like radicals

Like radicals are radical expressions with the same index and the same radicand.

### logarithmic function

The function  $f(x) = \log_a x$  is the logarithmic function with base  $a$ , where  $a > 0$ ,  $x > 0$ , and  $a \neq 1$ .

$y = \log_a x$  is equivalent to  $x = a^y$

### median

The median is a popular measure of central tendency. It is the 50th percentile of a distribution.

### mode

A measure of central tendency, the mode is the most frequent value in a distribution.

### natural base

The number  $e$  is defined as the value of  $\left(1 + \frac{1}{n}\right)^n$ , as  $n$  gets larger and larger. We say, as  $n$  increases without bound,  $e \approx 2.718281827\dots$

### natural exponential function

The natural exponential function is an exponential function whose base is  $e$ :  $f(x) = e^x$ . The domain is  $(-\infty, \infty)$  and the range is  $(0, \infty)$ .

### natural logarithmic function

The function  $f(x) = \ln x$  is the natural logarithmic function with base  $e$ , where  $x > 0$ .

$y = \ln x$  is equivalent to  $x = e^y$

### nominal scale

A scale in which no ordering is implied, and addition/subtraction and multiplication/division would be inappropriate for a variable. Variables measured on a nominal scale have no natural ordering, even if they are coded using numbers (e.g., for eye color 1 = blue, 2 = brown, 3 = hazel, etc.).

### non-experimental research

Research that involves observing things as they occur naturally and recording observations as data. Also known as correlational research.

### normal distribution

One of the most common continuous distributions, a normal distribution is sometimes referred to as a bell-shaped distribution, a bell curve, or a Gaussian curve. If the mean is 0 and the standard deviation is 1, the distribution is referred to as the “standard normal distribution.”

### ordinal scale

A set of ordered values in which there is no set distance between scale values; for example, asking someone to indicate how much education they completed by asking them to circle one of the following: did not complete high school, high school diploma, some college, college degree, professional degree.

### polynomial equation

A polynomial equation is an equation that contains a polynomial expression.

### population

The complete set of observations a researcher is interested in. Contrast this with a sample which is a subset of a population. Inferential statistics are computed from sample data in order to make inferences about the population.

### probability

The likelihood of a statistical result or the number of outcomes that satisfy specific criteria divided by the total number of possible outcomes.

### proportion

When two rational expressions are equal, the equation relating them is called a proportion.

### quadratic equation

Polynomial equations of degree two are called quadratic equations.

### quadratic function

A quadratic function, where  $a$ ,  $b$ , and  $c$  are real numbers and  $a \neq 0$ , is a function of the form  $f(x) = ax^2 + bx + c$ .

### qualitative variables

Also known as categorical variables, qualitative variables cannot be quantified, or measured numerically. Instead, they are measured on a nominal or ordinal scale. Variables that are not qualitative are known as quantitative variables.

### quantitative variables

Variables that are measured on a numeric or quantitative scale or that can be ordered in some fashion. Ordinal, interval, and ratio scales are quantitative. A country's population, a person's shoe size, or a car's



speed are all quantitative variables. Variables that are not quantitative are known as qualitative variables.

### quasi-experimental research

Research that involves manipulating the independent variable but not randomly assigning people to groups.

### radical equation

An equation in which a variable is in the radicand of a radical expression is called a radical equation.

### range

The difference between the maximum and minimum values of a variable or distribution. The range is the simplest measure of variability.

### ratio scale

A numerical scale in which the distance between scores on the scale is consistent (equal) and for which the zero is relative (rather than absolute).

### rational equation

A rational equation is an equation that contains a rational expression.

### rational expression

A rational expression is an expression of the form  $\frac{p}{q}$ , where  $p$  and  $q$  are polynomials and  $q \neq 0$ .

### rational function

A rational function is a function of the form  $R(x) = \frac{p(x)}{q(x)}$  where  $p(x)$  and  $q(x)$  are polynomial functions and  $q(x)$  is not zero.

### rationalizing the denominator

Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

### robustness

Something is robust if it holds up well in the face of adversity. A measure of central tendency or variability is considered robust if it is not greatly affected by a few extreme scores. A statistical test is considered robust if it works well in spite of moderate violations of the assumptions on which it is based.

### sample

A subset of a population, often taken for the purpose of statistical inference.

### sampling bias

Sampling bias occurs when participants are not selected at random or when they have an unequal probability of being selected for participation in a study.

### sampling error

The discrepancy between a parameter and the statistic used to estimate it.

### similar figures

Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides have the same ratio.

### simple random sampling

A process of selecting a subset of a population for the purposes of statistical inference in which every member of the population is equally likely to be chosen.

### simplified rational expression

A simplified rational expression has no common factors, other than **1**, in its numerator and denominator.

### skew

A distribution is skewed if one tail extends out further than the other, making the distribution asymmetrical. A distribution has a positive skew (is skewed to the right) if the tail to the right is longer. A distribution has a negative skew (is skewed to the left) if the tail to the left is longer.

### spread

The extent to which values differ from one another; that is, how much they vary. Spread is also called variability or dispersion.

### square of a number

If  $n^2 = m$ , then  $m$  is the square of  $n$ .

### square root of a number

If  $n^2 = m$ , then  $n$  is a square root of  $m$ .

### standard deviation

The standard deviation is a widely used measure of variability. It is computed by taking the square root of the variance.

### standard form

A complex number is in standard form when written as  $a + bi$ , where  $a$ ,  $b$  are real numbers.

### standard normal distribution

A normal distribution that has a mean of 0 and a standard deviation of 1; also known as the unit normal distribution.

### standardization

The process of transforming any normal distribution into a standard normal distribution by converting all of the raw scores in the distribution into standard scores (z scores).

### statistics

A range of techniques and procedures for analyzing, interpreting, displaying, and making decisions based on sample data.

### stem-and-leaf display

A quasi-graphical representation of numerical data. Generally, all but the final digit of each value is a stem, and the final digit is the leaf. The stems are placed in a vertical list, with each matched leaf on one side.

### stratified random sampling

In stratified random sampling, the population is divided into a number of subgroups (or strata). Random samples are then taken from each subgroup with sample sizes proportional to the size of the subgroup in the population.

### sum of squares

The sum of squared deviations, or differences, between scores and the mean in a numeric dataset.

### variability

The extent to which values differ from one another; that is, how much they vary. Variability can also be thought of as how spread out or dispersed a distribution is.

### variable

Something that can take on different values. For example, different subjects in an experiment weigh different amounts. Therefore “weight” is a variable in the experiment. Or, subjects may be given different doses of a drug. This would make “dosage” a variable. Variables can be dependent or independent, qualitative or quantitative, and continuous or discrete.

### variance

The variance is a widely used measure of variability. It is defined as the mean squared deviation of scores from the mean.

### whiskers

Vertical lines ending in a horizontal stroke that are added to box plots to indicate the spread of the data points. Whiskers are drawn from the upper and lower hinges to the upper and lower adjacent values.

### Z SCORE

The number of standard deviations a score is from the mean of its population. When the scores (or sample means) in the population are normally distributed, the z table can be used to find probabilities for obtaining a given z score.

### zero of the function

A value of  $x$  where the function is 0, is called a zero of the function.

### Zero Product Property

The Zero Product Property says that if the product of two quantities is zero, then at least one of the quantities is zero.

# VERSIONING HISTORY

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This page provides a record of edits and changes made to this book since its initial publication. Whenever edits or updates are made in the text, we provide a record and description of those changes here. If the change is minor, the version number increases by 0.1. If the edits involve a number of changes, the version number increases to the next full number.

The files posted alongside this book always reflect the most recent version.

<b>Version</b>	<b>Date</b>	<b>Change</b>	<b>Affected Web Page</b>
1.0	February, 2023	First Publication	N/A