Fanshawe Pre-Health Sciences Mathematics 1

FANSHAWE PRE-HEALTH SCIENCES MATHEMATICS 1

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CONTENTS

Unit 3: Solving Linear Equations, Graphs of Linear Equations, and Applications of Linear Functions

Unit 4: Systems of Linear Equations

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WELCOME TO PRE-HEALTH SCIENCES MATHEMATICS 1!

Welcome to Pre-Health Sciences (PHS), and in particular, to MATH-1024!
In this course, we will cover the following topics.

- Unit 1: The Real Numbers and Introduction to Algebra
- Unit 2: Measurement and Rounding Rules
- Unit 3: Solving Linear Equations, Graphs of Linear Equations, and Applications of Linear Functions
- Unit 4: Systems of Linear Equations
- Unit 5: Introduction to Polynomials
- Unit 6: Geometry and Trigonometry

Through our coverage of those topics, we will have the main goal of developing resilience, critical thinking skills, and self-directed learning skills. In addition to these transferable skills, we will aim to develop your attention to detail, determination, time management skills, as well as your mental mathematics skills.

In health care, making a small calculation error can be life threatening, and therefore, it is essential that we develop our intuition with numbers and mathematics throughout the PHS math program. To do this, we will often work without a calculator to improve on our mathematical abilities.

Mathematics can be challenging, and it is critical that students keep an open-mind and remain positive. Remember that it takes a lot of practice to develop these foundational skills, so if you do not succeed right away, stay patient, ask for help, and keep working hard!

ACKNOWLEDGEMENTS

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This book is primarily an adaptation of the following resource:

- Elementary Algebra 2e. by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

Additional content was included from:

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A list of changes from the previous sources can be found at Changes from Adapted Text in the About This Book section.

Collaborators

This project was a collaboration between the author and the team in the OER Design Studio at Fanshawe. The book was significantly re-formatted in Pressbooks to ensure accessibility.

The following staff and students were involved in the creation of this project:

- Catherine Steeves – *Instructional Designer*
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- Robert Armstrong – *Graphic Design*
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ABOUT THIS BOOK

Changes From Adapted Resource

Overall significant work was done on this version of the text to improve the accessibility and format from the import from Open Stax and other resources.

All mathematical notation was written in LaTeX to allow for MathJax functionality. MathJax provides the following accessibility features:

- Automatic magnification on hover
- Text-to-speech options
- Collapsible math
- Highlight, foreground, and background customization
- The ability to translate math into other syntaxes
- MathJax options in over 20 languages

See the full extent of MathJax's accessibility features in their documentation.

In addition the following formatting changes were made:

- The majority of images that included math where replaced with LaTeX
- Images were recolored for contrast
- Examples were reformatted into steps for ease of understanding

Specific content changes from the adapted resources can be found in the table below.

Unit 1	All sections from *Elementary Algebra 2e* Combined the Fractions sections into one.
Unit 2	*Section 2.1* – Combined parts from *Elementary Algebra 2e* Section 1.10 with Mathematics for Public and Occupational Health Professionals (1.3 Measurement Systems) and Business/Technical Mathematics (6.1) examples related to health care taken and included in section. Order of the parts was changed as well to put metric system first and then the U.S./Imperial system. Extended some tables to include extra metric prefixes. Edited for readability. *Section 2.2* -Added some content on exact vs inexact numbers, and rounding with mixed operations. *Section 2.3*– Added small section on scientific notation and significant figures and some examples to include how to preserve significant figures in scientific notation and decimal notation. *Section 2.4* – Examples and content from Rounding Rules When Converting Within the Same System of Measurement created.
Unit 3	Most sections are adapted from *Elementary Algebra 2e*. Some sections were combined from chapters 3 and 4 and some content was rearranged.
Unit 4	All sections adapted from Elementary algebra chapter 5. Some sections combined.
Unit 5	All sections from *Elementary Algebra 2e* chapter 6. Section 5.8 was created. Graphs created using desmos.com graphing calculator.
Unit 6	Images added to provide a health profession perspective.

Accessibility Statement

We are actively committed to increasing the accessibility and usability of the textbooks we produce. Every attempt has been made to make this OER accessible to all learners and is compatible with assistive and adaptive technologies. We have attempted to provide closed captions, alternative text, or multiple formats for on-screen and off-line access.

The web version of this resource has been designed to meet Web Content Accessibility Guidelines 2.0, level AA. In addition, it follows all guidelines in Appendix A: Checklist for Accessibility of the *Accessibility Toolkit – 2nd Edition*.

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Pedrosa, L. (2019, July 10). MathJax: now available on PressbooksEDU. Pressbooks. https://pressbooks.com/new-features/mathjax-now-available-on-pressbooksedu/

UNIT 1: THE REAL NUMBERS AND INTRODUCTION TO ALGEBRA

Chapter Outline

1.0 INTRODUCTION

Figure 1.0.1 In order to be structurally sound, the foundation of a building must be carefully constructed. Photo by Open Stax

In order to be structurally sound, the foundation of a building must be carefully constructed. Just like a building needs a firm foundation to support it, your study of algebra needs to have a firm foundation. To ensure this, we begin this book with a review of arithmetic operations with whole numbers, integers, fractions, and decimals, so that you have a solid base that will support your study of algebra.

1.1 INTRODUCTION TO WHOLE NUMBERS

Learning Objectives

By the end of this section, you will be able to:

- Use place value with whole numbers
- Identify multiples and apply divisibility tests
- Find prime factorizations and least common multiples

As we begin our study of elementary algebra, we need to refresh some of our skills and vocabulary. This chapter will focus on whole numbers, integers, fractions, decimals, and real numbers. We will also begin our use of algebraic notation and vocabulary.

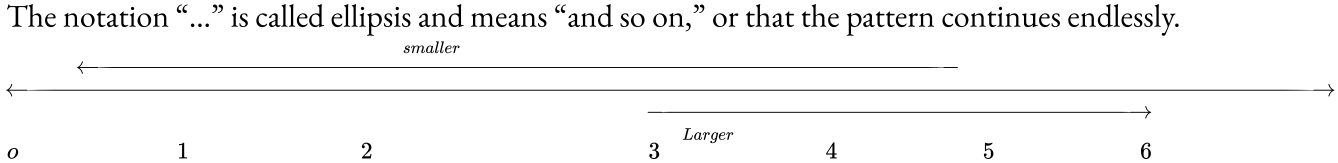
Use Place Value with Whole Numbers

The most basic numbers used in algebra are the numbers we use to count objects in our world: 1, 2, 3, 4, and so on. These are called the **counting numbers**. Counting numbers are also called *natural numbers*. If we add zero to the counting numbers, we get the set of whole numbers.

Counting Numbers: 1, 2, 3, …

Whole Numbers: 0, 1, 2, 3, …

The notation "…" is called ellipsis and means "and so on," or that the pattern continues endlessly.



The numbers on the **number line** get larger as they go from left to right and smaller as they go from right to left. While this number line shows only the whole numbers 0 through 6, the numbers keep going without end.

Our number system is called a place value system, because the value of a digit depends on its position in a number. Figure 1.1 shows the place values. The place values are separated into groups of three, which are called periods. The periods are *ones, thousands, millions, billions, trillions*, and so on. In a written number, commas separate the periods.

Place Value														
Trillions			Billions			Millions			Thousands			Ones		
Hundred trillions	Ten trillions	Trillions	Hundred billions	Ten billions	Billions	Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones
								5	2	7	8	1	9	4

Figure 1.1

Example 1.1.1

In the number 63,407,218, find the place value of each digit:

a. 7
b. 0
c. 1
d. 6
e. 3

Solution
Place the number in the place value chart:

Trillions			Billions			Millions			Thousands			Ones		
Hundred trillions	Ten trillions	Trillions	Hundred billions	Ten billions	Billions	Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones
							6	3	4	0	7	2	1	8

Figure 1.2

a. The 7 is in the thousands place.
b. The 0 is in the ten-thousands place.
c. The 1 is in the tens place.
d. The 6 is in the ten-millions place.
e. The 3 is in the millions place.

Try It

1) For the number 27,493,615, find the place value of each digit:

a. 2
b. 1
c. 4
d. 7
e. 5

Solution

a. ten millions
b. tens
c. hundred thousands
d. millions
e. ones

2) For the number 519,711,641,328, find the place value of each digit:

a. 9
b. 4
c. 2
d. 6
e. 7

Solution

a. billions
b. ten thousands
c. tens
d. hundred thousands
e. hundred millions

When you write a check, you write out the number in words as well as in digits. To write a number in words, write the number in each period, followed by the name of the period, without the *s* at the end. Start at the left, where the periods have the largest value. The ones period is not named. The commas separate the periods, so wherever there is a comma in the number, put a comma between the words (see below). The number 74,218,369 is written as seventy-four million, two hundred eighteen thousand, three hundred sixty-nine.

$$\underbrace{74}_{millions}\ ,\ \underbrace{218}_{thousands}\ ,\ \underbrace{369}_{ones} \xleftarrow{periods}$$

$$74 \rightarrow Seventy - four\ million,$$

$$218 \rightarrow two\ hundred\ eighteen\ thousand,$$

$$369 \rightarrow three\ hundred\ sixty\ nine$$

HOW TO

Name a Whole Number in Words

1. Start at the left and name the number in each period, followed by the period name.
2. Put commas in the number to separate the periods.
3. Do not name the ones period

Example 1.1.2

Name the number 8,165,432,098,710 using words.

Solution

Step 1: **Name the number in each period, followed by the period name.**

$$\underbrace{8}_{trillions},\underbrace{165}_{billions},\underbrace{432}_{millions},\underbrace{098}_{thousands},\underbrace{710}_{ones}$$

$$8 \rightarrow Eight\ trillion,$$

$$165 \rightarrow One\ hundred\ sixty - five\ billion,$$

$$432 \rightarrow Four\ hundred\ thrity - two\ million,$$

$$098 \rightarrow Ninety - eight\ thousad,$$

$$710 \rightarrow seven\ hundred\ ten$$

Step 2: **Put the commas in to separate the periods.**

So, 8,165,432,098,710 is named as eight trillion, one hundred sixty-five billion, four hundred thirty-two million, ninety-eight thousand, seven hundred ten.

Try It

3) Name the number 9,258,137,904,061 using words.

Solution
Nine trillion, two hundred fifty-eight billion, one hundred thirty-seven million, nine hundred four thousand, sixty-one.

4) Name the number 17,864,325,619,004 using words.

Solution
Seventeen trillion, eight hundred sixty-four billion, three hundred twenty-five million, six hundred nineteen thousand four.

We are now going to reverse the process by writing the digits from the name of the number. To write the number in digits, we first look for the clue words that indicate the periods. It is helpful to draw three blanks for the needed periods and then fill in the blanks with the numbers, separating the periods with commas.

HOW TO

Write a Whole Number Using Digits

1. Identify the words that indicate periods. (Remember, the ones period is never named.)
2. Draw three blanks to indicate the number of places needed in each period. Separate the periods by commas.
3. Name the number in each period and place the digits in the correct place value position.

Example 1.1.3

Write *nine billion, two hundred forty-six million, seventy-three thousand, one hundred eighty-nine* as a whole number using digits.

Solution

Step 1: **Identify the words that indicate periods.**

Except for the first period, all other periods must have three places. Draw three blanks to indicate the number of places needed in each period.

Step 2: **Separate the periods by commas.**

Then write the digits in each period.

$$\underbrace{billions}_{nine\ billion}, \underbrace{millions}_{two\ hundred\ forty-six\ million}, \underbrace{thousands}_{seventy-three\ thousand}, \underbrace{ones}_{one\ hundred\ eity-nine},$$

$$9 \qquad 246 \qquad 073 \qquad 189$$

The number is 9,246,073,189.

Try It

5) Write the number two billion, four hundred sixty-six million, seven hundred fourteen thousand, fifty-one as a whole number using digits.

Solution

2,466,714,051

6) Write the number eleven billion, nine hundred twenty-one million, eight hundred thirty thousand, one hundred six as a whole number using digits.

Solution

11,921,830,106

In 2013, the U.S. Census Bureau estimated the population of the state of New York as 19,651,127. We could say the population of New York was approximately 20 million. In many cases, you don't need the exact value; an approximate number is good enough.

The process of approximating a number is called rounding. Numbers are rounded to a specific place value, depending on how much accuracy is needed. Saying that the population of New York is approximately 20 million means that we rounded to the millions place.

Example 1.1.4

Round 23,658 to the nearest hundred.

Solution

Step 1: **Locate the given place value with an arrow. All digits to the left do not change.**

Locate the hundreds place 23,658.

{\color[rgb]{1.0, 0.0, 0.0}\text{Hundredths}}\text{ place.}\;23,{\color[rgb]{1.0, 0.0, 0.0}\underset\uparrow6}58

Step 2: **Underline the digit to the right of the given place value.**

Underline the 5, which is to the right of the hundreds place.

{\color[rgb]{1.0, 0.0, 0.0}\text{Hundredths}}\text{ place.}\;23,{\color[rgb]{1.0, 0.0, 0.0}6}{\color[rgb]{0.0, 0.0, 1.0}\underline5}8

Step 3: **Is this digit greater than or equal to 5?**

Yes – add 1 to the digit in the given place value.

Add 1 to the 6 in the hundred place, since 5 is greater than or equal to 5.

\text{Add 1.}\;23,{\color[rgb]{1.0, 0.0, 0.0}6}{\color[rgb]{0.1, 0.1, 0.1}5}8

Step 4: **Replace all digits to the right of the given place with zeros.**

Replace all digits to the right of the hundreds place with zeros.

\text{Replace with zeros.}\;23,{\color[rgb]{0.0, 0.0, 1.0}700}

Try It

7) Round to the nearest hundred: 17,852.

Solution
17,900

8) Round to the nearest hundred: 468,751.

Solution
468,800

Round Whole Numbers.

1. Locate the given place value and mark it with an arrow. All digits to the left of the arrow do not change.
2. Underline the digit to the right of the given place value.
3. Is this digit greater than or equal to 5?
    - Yes–add 1 to the digit in the given place value.
    - No–do *not* change the digit in the given place value.
4. Replace all digits to the right of the given place value with zeros.

Examples 1.1.5

Round 103,978 to the nearest:

a. hundred
b. thousand
c. ten thousand

Solution

a.

Step 1: **Locate the hundreds place in 103,978.**

{\color[rgb]{1.0, 0.0, 0.0}\text{Hundreds place:}}\;103,\boxed{\color[rgb]{1.0, 0.0, 0.0}9}78

Step 2: **Underline the digit to the right of the hundreds place.**

$$rgb]1.0, 0.0, 0.0\text{Hundreds place: } 103,\ \boxed{rgb]1.0, 0.0, 0.09}\underline{7}8$$

Step 3**: Since 7 is greater than or equal to 5, add 1 to the 9. Replace all digits to the right of the hundreds place with zeros.**

$$rgb]1.0, 0.0, 0.0\text{Hundreds place: } 103,\ \boxed{\underbrace{rgb]1.0, 0.0, 0.09}_{\substack{\textit{Add } 1 \\ 9+1=10 \\ \textit{replace } 9 \textit{ with } 0 \\ \textit{and carry the } 1}}}\ \underbrace{\underline{7}8}_{\textit{replace with } 0s} = 104,000$$

So, 104,000 is 103,978 rounded to the nearest hundred.

---

b.

Step 1: **Locate the thousands place and underline the digit to the right of the thousands place.**

$$rgb]1.0, 0.0, 0.0\text{Thousands place: } 10\ \boxed{rgb]1.0, 0.0, 0.03},\underline{9}78$$

Step 2: **Since 9 is greater than or equal to 5, add 1 to the 3. Replace all digits to the right of the hundreds place with zeros.**

$$rgb]1.0,0.0,0.0\text{Thousands place: } 10\boxed{\underbrace{rgb]1.0,0.0,0.03}_{\substack{Add\ 1\\ 3+1=4\\ replace\ 3\ with\ 4}}}\underbrace{\underline{9}78}_{replace\ with\ 0s} = 104,000$$

So, 104,000 is 103,978 rounded to the nearest thousand.

---

c.

Step 1: **Locate the ten thousands place and underline the digit to the right of the ten thousands place.**

$$rgb]1.0,0.0,0.0\text{Ten thousands place: } 1\boxed{rgb]1.0,0.0,0.00}3,978$$

Step 2: **Since 3 is less than 5, we leave the 0 as is, and then replace the digits to the right with zeros.**

$$100,000$$

So, 100,000 is 103,978 rounded to the nearest ten thousand.

Try It

9) Round 206,981 to the nearest:

a. hundred
b. thousand
c. ten thousand.

Solution

a. 207,000
b. 207,000
c. 210,000

10) Round 784,951 to the nearest:

a. hundred
b. thousand
c. ten thousand.

Solution

a. 785,000
b. 785,000
c. 780,000

Identify Multiples and Apply Divisibility Tests

The numbers 2, 4, 6, 8, 10, and 12 are called multiples of 2. A multiple of 2 can be written as the product of a counting number and 2.

2	$2 \cdot 1$
4	$2 \cdot 2$
6	$2 \cdot 3$
8	$2 \cdot 4$
10	$2 \cdot 5$
12	$2 \cdot 6$

Similarly, a multiple of 3 would be the product of a counting number and 3.

3	$3 \cdot 1$
6	$3 \cdot 2$
9	$3 \cdot 3$
12	$3 \cdot 4$
15	$3 \cdot 5$
18	$3 \cdot 6$

We could find the multiples of any number by continuing this process.

Table. 1.1 shows the multiples of 2 through 9 for the first 12 counting numbers.

Table 1.1

Counting Number	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Multiples of 2	2	4	6	8	10	12	14	16	18	20	22	24
Multiples of 3	3	6	9	12	15	18	21	24	27	30	33	36
Multiples of 4	4	8	12	16	20	24	28	32	36	40	44	48
Multiples of 5	5	10	15	20	25	30	35	40	45	50	55	60
Multiples of 6	6	12	18	24	30	36	42	48	54	60	66	72
Multiples of 7	7	14	21	28	35	42	49	56	63	70	77	84
Multiples of 8	8	16	24	32	40	48	56	64	72	80	88	96
Multiples of 9	9	18	27	36	45	54	63	72	81	90	99	108
Multiples of 10	10	20	30	40	50	60	70	80	90	100	110	120

Multiple of a Number

A number is a **multiple** of n if it is the product of a counting number and n.

Another way to say that 15 is a multiple of 3 is to say that 15 is divisible by 3. That means that when we divide 3 into 15, we get a counting number. In fact, $15 \div 3$ is 5, so 15 is 15×3.

Divisible by a Number

If a number m is a multiple of n, then m is **divisible** by n.

Look at the multiples of 5 in Table 1.1. They all end in 5 or 0. Numbers with last digit of 5 or 0 are divisible by 5. Looking for other patterns in Table 1.1 that shows multiples of the numbers 2 through 9, we can discover the following divisibility tests:

Divisibility Tests

A number is divisible by:

- 2 if the last digit is 0, 2, 4, 6, or 8.
- 3 if the sum of the digits is divisible by 3.
- 5 if the last digit is 5 or 0.
- 6 if it is divisible by both 2 and 3.
- 10 if it ends with 0.

Example 1.1.6

Is 5,625 divisible by 2? By 3? By 5? By 6? By 10?

Solution

***Step 1:* Is 5,625 divisible by 2?**

No.

***Step 2:* Does it end in 0,2,4,6, or 8?**

5,625 is not divisible by 2.

***Step 3:* Is 5,625 divisible by 3?**

Yes. 5,625 is divisible by 3.

***Step 4:* What is the sum of the digits?**

$$5+6+2+5=18$$

***Step 5:* Is the sum divisible by 3?**

Yes, 18 is divisible by 3.

***Step 6:* Is 5,625 divisible by 5 or 10?**

5,625 is divisible by 5 but not by 10.

***Step 7:* What is the last digit?**

It is 5.

Step 8: **Is 5,625 divisible by 6?**

No, it is not.

Step 9: **Is it divisible by both 2 or 3?**

No, 5,625 is not divisible by 2, but it is divisible by 3.

Try It

11) Determine whether 4,962 is divisible by 2, by 3, by 5, by 6, and by 10.

Solution
By 2, 3, and 6.

12) Determine whether 3,765 is divisible by 2, by 3, by 5, by 6, and by 10.

Solution
By 3 and 5.

Find Prime Factorizations and Least Common Multiples

In mathematics, there are often several ways to talk about the same ideas. So far, we've seen that if m is a multiple of n, we can say that m is divisible by n. For example, since 72 is a multiple of 8, we say 72 is divisible by 8. Since 72 is a multiple of 9, we say 72 is divisible by 9. We can express this still another way.

Since $8 \times 9 = 72$, we say that 8 and 9 are **factors** of 72. When we write $72 = 8 \times 9$, we say we have factored 72.

$$\underbrace{8 \times 9}_{factors} = \underbrace{72}_{product}$$

Other ways to factor 72 are 1·72, 2·36, 3·24, 4·18, and 6·12. Seventy-two has many factors: 1, 2, 3, 4, 6, 8, 9, 12, 18, 36, and 72.

Factors

If $a \cdot b = m$ then a and b are factors of m.

Some numbers, like 72, have many factors. Other numbers have only two factors.

A **prime number** is a counting number greater than 1, whose only factors are 1 and itself.

A **composite number** is a counting number that is not prime. A composite number has factors other than 1 and itself.

The **counting numbers** from 2 to 19 are listed in the below table, with their factors. Make sure to agree with the "prime" or "composite" label for each!

Number	Factors	Prime or Composite
2	1,2	Prime
3	1,3	Prime
4	1,2,4	Composite
5	1,5	Prime
6	1,2,3,6	Composite
7	1,7	Prime
8	1,2,4,8	Composite
9	1,3,9	Composite
10	1,2,5,10	Composite
11	1,11	Prime
12	1,2,3,4,6,12	Composite
13	1,13	Prime
14	1,2,7,14	Composite
15	1,3,5,15	Composite
16	1,2,4,8,16	Composite
17	1,17	Prime
18	1,2,3,6,9,18	Composite
19	1,19	Prime

The prime numbers less than 20 are 2, 3, 5, 7, 11, 13, 17, and 19. Notice that the only even prime number is 2.

A composite number can be written as a unique product of primes. This is called the prime factorization of the number. Finding the **prime factorization** of a composite number will be useful later in this course.

Prime Factorization

The prime factorization of a number is the product of prime numbers that equals the number. These prime numbers are called the prime factors.

To find the prime factorization of a composite number, find any two factors of the number and use them to create two branches. If a factor is prime, that branch is complete. Circle that prime!

If the factor is not prime, find two factors of the number and continue the process. Once all the branches have circled primes at the end, the factorization is complete. The composite number can now be written as a product of prime numbers.

Example 1.1.7

Factor 48.

Solution

4 and 8 are not prime. Break them each into two factors.

2 and 3 are prime, so circle them.

Step 1: **Find two factors whose product is the given number. Use these numbers to create two branches.**

$$48 = 2 \times 24 \qquad \overbrace{2 \qquad\qquad 24}^{48}$$

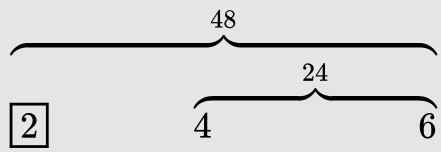
Step 2: **If a factor is prime, that branch is complete. Circle the prime.**

2 is prime. Circle the prime.

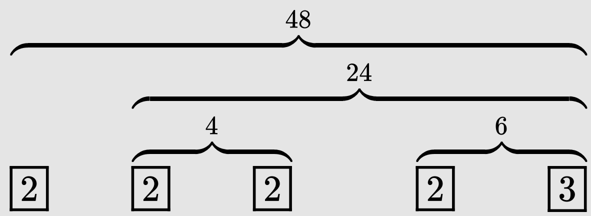
$$\overbrace{\boxed{2} \qquad\qquad 24}^{48}$$

Step 3: **If a factor is not prime, write it as the product of two factors and continue the process.**

24 is not prime break it into 2 more factors.4 and 6 are not prime. Break them each into two factors.



2 and 3 are prime, so circle them.



Step 4: **Write the composite number as the product of all the circled primes.**

$$48 = 2 \times 2 \times 2 \times 2 \times 3$$

We say 2·2·2·2·3 is the prime factorization of 48. We generally write the primes in ascending order. Be sure to multiply the factors to verify your answer!

If we first factored 48 in a different way, for example as 6·8, the result would still be the same. Finish the prime factorization and verify this for yourself.

Try It

13) Find the prime factorization of 80.

Solution

2·2·2·2·5

14) Find the prime factorization of 60.

Solution

2·2·3·5

Find the Prime Factorization of a Composite Number.

1. Find two factors whose product is the given number, and use these numbers to create two branches.
2. If a factor is prime, that branch is complete. Circle the prime, like a bud on the tree.
3. If a factor is not prime, write it as the product of two factors and continue the process.
4. Write the composite number as the product of all the circled primes.

Example 1.1.8

Find the prime factorization of 252.

Solution

Step 1: **Find two factors whose product is 252. 12 and 21 are not prime.**

Break 12 and 21 into two more factors. Continue until all primes are factored.

$$252 \begin{cases} 12 \begin{cases} 6 \begin{cases} 3 \\ 2 \end{cases} \\ 2 \end{cases} \\ 21 \begin{cases} 3 \\ 7 \end{cases} \end{cases}$$

Step 2: **Write 252 as the product of all the circled primes.**

$$252 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7$$

Try It

15) Find the prime factorization of 126.

Solution

2·3·3·7

16) Find the prime factorization of 294.

Solution

2·3·7·7

One of the reasons we look at multiples and primes is to use these techniques to find the least common multiple of two numbers. This will be useful when we add and subtract fractions with different denominators. Two methods are used most often to find the least common multiple and we will look at both of them.

The first method is the Listing Multiples Method. To find the least common multiple of 12 and 18, we list the first few multiples of 12 and 18:

12: 12, 24, 36, 48, 60, 72, 84, 96, 108...

18: 18, 36, 54, 72, 90, 108...

Common Multiples: 36, 72, 108...

Least Common Multiple: 36

Notice that some numbers appear in both lists. They are the *common multiples* of 12 and 18.

We see that the first few common multiples of 12 and 18 are 36, 72, and 108. Since 36 is the smallest of the common multiples, we call it the **least common multiple**. We often use the abbreviation LCM.

Least Common Multiple

The least common multiple (LCM) of two numbers is the smallest number that is a multiple of both numbers.

The procedure box lists the steps to take to find the LCM using the prime factors method we used above for 12 and 18.

HOW TO

Find the Least Common Multiple by Listing Multiples.

1. List several multiples of each number.
2. Look for the smallest number that appears on both lists.
3. This number is the LCM.

Example 1.1.9

Find the least common multiple of 15 and 20 by listing multiples.

Solution

***Step 1:* Make lists of the first few multiples of 15 and of 20, and use them to find the least common multiple.**

15: 15, 30, 45, 60, 75, 90, 105, 120
20: 20, 40, 60, 80, 100, 120, 140, 160

***Step 2:* Look for the smallest number that appears in both lists.**

The first number to appear on both lists is 60, so 60 is the least common multiple of 15 and 20.

Notice that 120 is in both lists, too. It is a common multiple, but it is not the *least* common multiple.

Try It

17) Find the least common multiple by listing multiples: 9 and 12.

Solution

36

18) Find the least common multiple by listing multiples: 18 and 24.

Solution

72

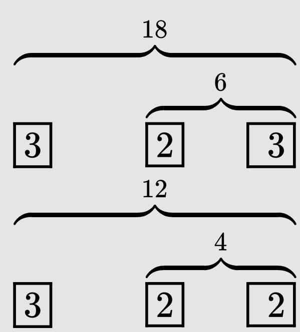
Our second method to find the least common multiple of two numbers is to use The Prime Factors Method. Let's find the LCM of 12 and 18 again, this time using their prime factors.

Example 1.1.10

Find the Least Common Multiple (LCM) of 12 and 18 using the prime factors method.

Solution

Step 1: Write each number as a product of primes.



***Step 2:* List the primes of each number. Match primes vertically when possible.**

List the primes of 12.

$$12 = 2 \times 2 \times 3$$

List the primes of 18. Line up with the primes of 12 when possible. If not create a new column.

$$18 = 2 \times 3 \times 3$$

Step 3: **Bring down the number from each column.**

$$\begin{aligned} 12 &= 2 \times 2 \times 3 \\ 18 &= 2 \times 3 \times 3 \\ LCM &= 2 \times 2 \times 3 \times 3 \end{aligned}$$

Step 4: **Multiply the factors.**

$$LCM = 36$$

Notice that the prime factors of 12 (2·2·3) and the prime factors of 18 (2·3·3) are included in the LCM (2·2·3·3). So 36 is the least common multiple of 12 and 18.

By matching up the common primes, each common prime factor is used only once. This way you are sure that 36 is the *least* common multiple.

Try It

19) Find the LCM using the prime factors method: 9 and 12.

Solution

36

20) Find the LCM using the prime factors method: 18 and 24.

Solution

72

HOW TO

Find the Least Common Multiple Using the Prime Factors Method.

1. Write each number as a product of primes.
2. List the primes of each number. Match primes vertically when possible.
3. Bring down the columns.
4. Multiply the factors.

Example 1.1.11

Find the Least Common Multiple (LCM) of 24 and 36 using the prime factors method.

Solution

***Step 1:* Find the primes of 24 and 36.**

Match primes vertically when possible.

***Step 2:* Bring down all columns.**

$$24 = 2 \times 2 \times 2 \times 3$$
$$36 = 2 \times 2 \times 3 \times 3$$

***Step 3:* Multiply the factors.**

$$LCM = 2 \times 2 \times 2 \times 3 \times 3$$
$$LCM = 72$$

The LCM of 24 and 36 is 72.

Try It

21) Find the LCM using the prime factors method: 21 and 28.

Solution

84

22) Find the LCM using the prime factors method: 24 and 32.

Solution

96

Key Concepts

- **Place Value**
- **Name a Whole Number in Words**

  1. Start at the left and name the number in each period, followed by the period name.
  2. Put commas in the number to separate the periods.
  3. Do not name the ones period.

- **Write a Whole Number Using Digits**

  1. Identify the words that indicate periods. (Remember the ones period is never named.)
  2. Draw 3 blanks to indicate the number of places needed in each period. Separate the periods by commas.
  3. Name the number in each period and place the digits in the correct place value position.

- **Round Whole Numbers**
    1. Locate the given place value and mark it with an arrow. All digits to the left of the arrow do not change.
    2. Underline the digit to the right of the given place value.
    3. Is this digit greater than or equal to 5?
        - Yes—add 1 to the digit in the given place value.
        - No—do *not* change the digit in the given place value.
    4. Replace all digits to the right of the given place value with zeros.
- **Divisibility Tests:** A number is divisible by:
    - 2 if the last digit is 0, 2, 4, 6, or 8.
    - 3 if the sum of the digits is divisible by 3.
    - 5 if the last digit is 5 or 0.
    - 6 if it is divisible by both 2 and 3.
    - 10 if it ends with 0.
- **Find the Prime Factorization of a Composite Number**
    1. Find two factors whose product is the given number, and use these numbers to create two branches.
    2. If a factor is prime, that branch is complete. Circle the prime, like a bud on the tree.
    3. If a factor is not prime, write it as the product of two factors and continue the process.
    4. Write the composite number as the product of all the circled primes.
- **Find the Least Common Multiple by Listing Multiples**
    1. List several multiples of each number.
    2. Look for the smallest number that appears on both lists.
    3. This number is the LCM.
- **Find the Least Common Multiple Using the Prime Factors Method**
    1. Write each number as a product of primes.
    2. List the primes of each number. Match primes vertically when possible.
    3. Bring down the columns.
    4. Multiply the factors.

Self Check

After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=60#h5p-1

Glossary

composite number
A composite number is a counting number that is not prime. A composite number has factors other than 1 and itself.

counting numbers
The counting numbers are the numbers 1, 2, 3, …

divisible by a number
If a number m is a multiple of n, then m is divisible by n. (If 6 is a multiple of 3, then 6 is divisible by 3.)

factors
If $a \times b = m$, then (a and b) are factors of m. Since $3 \times 4 = 12$, then 3 and 4 are factors of 12.

least common multiple
The least common multiple of two numbers is the smallest number that is a multiple of both numbers.

multiple of a number
A number is a multiple of n if it is the product of a counting number and n.

number line
A number line is used to visualize numbers. The numbers on the number line get larger as they go from left to right, and smaller as they go from right to left.

origin
The origin is the point labelled 0 on a number line.

prime factorization
The prime factorization of a number is the product of prime numbers that equals the number.

prime number
A prime number is a counting number greater than 1, whose only factors are 1 and itself.

whole numbers
The whole numbers are the numbers 0, 1, 2, 3, ….

1.2 INTRODUCTION TO THE LANGUAGE OF ALGEBRA

Learning Objectives

By the end of this section, you will be able to:

- Use variables and algebraic symbols
- Simplify expressions using the order of operations
- Evaluate an expression
- Identify and combine like terms
- Translate an English phrase to an algebraic expression

Use Variables and Algebraic Symbols

Suppose this year Greg is 20 years old and Alex is 23. You know that Alex is 3 years older than Greg. When Greg was 12, Alex was 15. When Greg is 35, Alex will be 38. No matter what Greg's age is, Alex's age will always be 3 years more, right? In the language of algebra, we say that Greg's age and Alex's age are **variables** and the 3 is a **constant**. The ages change ("vary") but the 3 years between them always stays the same ("constant"). Since Greg's age and Alex's age will always differ by 3 years, 3 is the *constant*.

In algebra, we use letters of the alphabet to represent variables. So if we call Greg's age g, then we could use $g+3$ to represent Alex's age.

Greg's age	Alex's age
12	15
20	23
35	38
g	$g+3$

Variable

A variable is a letter that represents a number whose value may change.

Constant

A constant is a number whose value always stays the same.

To write algebraically, we need some operation symbols as well as numbers and variables. There are several types of symbols we will be using.

There are four basic arithmetic operations: addition, subtraction, multiplication, and division. We'll list the symbols used to indicate these operations below table. You'll probably recognize some of them.

Operation	**Notation**	**Say:**
Addition	$a + b$	a plus b
Subtraction	$a - b$	a minus b
Multiplication	$a \times b$	a times b
Division	$a \div b$	a divided by b

We perform these operations on two numbers. When translating from symbolic form to English, or from English to symbolic form, pay attention to the words "of" and "and."

- The *difference of* 9 and 2 means subtract 9 and 2, in other words, 9 minus 2, which we write symbolically as $9 - 2$.
- The *product of* 4 and 8 means multiply 4 and 8, in other words 4 times 8, which we write symbolically as 4×8.

In algebra, the cross symbol, $\times$, is not used to show multiplication because that symbol may cause confusion.

Does $3xy$ mean $3 \times y$ ('three times y') or $(3 \times x \times y)$ ('three times x times y') To make it clear, use · or parentheses for multiplication.

When two quantities have the same value, we say they are equal and connect them with an equal sign.

Equality Symbol

$a = b$ is read "a is equal to b"

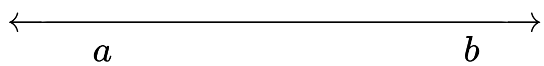
The symbol "=" is called the equal sign.

On the number line, the numbers get larger as they go from left to right. The number line can be used to explain the symbols "<" and ">".

Inequality Symbols

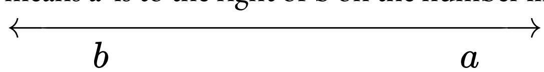
$a < b$ is read "a is less than b".

This means a is to the left of b on the number line.



$a > b$ is read a is greater than b.

This means a is to the right of b on the number line.



The expressions $a < b$ or $a > b$ can be read from left to right or right to left, though in English we usually read from left to right (Figure 1.2). In general, $a < b$ is equivalent to $a > b$. For example $7 < 11$ is equivalent to $11 > 7$. And $a > b$ is equivalent to $a < b$. For example $17 > 4$ is equivalent to $4 < 17$.

Figure 1.2

Inequality Symbols	**Words**
$a \neq b$	a is *not equal to* b
$a < b$	a is *less than* b
$a \leq b$	a is *less than or equal to* b
$a > b$	a is *greater than* b
$a \geq b$	a is *greater than or equal to* b

Example 1.2.1

Translate from algebra into English:

a. $17 < 26$
b. $8 \neq 17$
c. $12 \geq (27 \div 3)$
d. $y + 7 \leq 19$

Solution

a. $17 < 26$

17 is less than or equal to 26

b. $8 \neq 17$

8 is not equal to 17 minus 3

c. $12 \geq (27 \div 3)$

12 is greater than 27 divided by 3

d. $y + 7 \leq 19$

y plus 7 is less than 19

Try It

1) Translate from algebra into English:

a. $14 \leq 27$
b. $19 - 2 \neq 8$

c. $12 > 4 \div 2$
d. $x - 7 < 1$

Solution

a. 14 is less than or equal to 27
b. 19 minus 2 is not equal to 8
c. 12 is greater than 4 divided by 2
d. x minus 7 is less than 1

2) Translate from algebra into English:

a. $19 \geq 15$
b. $7 = 12 - 5$
c. $15 \div 3 < 8$
d. $y + 3 > 6$

Solution

a. 19 is greater than or equal to 15
b. 7 is equal to 12 minus 5
c. 15 divided by 3 is less than 8
d. y plus 3 is greater than 6

Grouping symbols in algebra are much like the commas, colons, and other punctuation marks in English. They help to make clear which expressions are to be kept together and separate from other expressions. We will introduce three types now.

Grouping Symbols

Parentheses ( )
Brackets [ ]
Braces { }

Here are some examples of expressions that include grouping symbols. We will simplify expressions like these later in this section.

$$8(14-8)$$

What is the difference in English between a phrase and a sentence? A phrase expresses a single thought that is incomplete by itself, but a sentence makes a complete statement. "Running very fast" is a phrase, but "The football player was running very fast" is a sentence. A sentence has a subject and a verb. In algebra, we have *expressions* and *equations*.

Expression

An expression is a number, a variable, or a combination of numbers and variables using operation symbols.

An **expression** is like an English phrase. Here are some examples of expressions:

Expression	Words	English Phrase
$3+5$	3 plus 5	the sum of three and five
$n-1$	n minus one	the difference of n and one
6×7	6 times 7	the product of six and seven
$\frac{x}{y}$	x divided by y	the quotient of x and y

Notice that the English phrases do not form a complete sentence because the phrase does not have a verb.

An **equation** is two expressions linked with an equal sign. When you read the words the symbols represent in an equation, you have a complete sentence in English. The equal sign gives the verb.

Equation

An equation is two expressions connected by an equal sign.

Here are some examples of equations.

Equation	English Sentence
$3+5=8$	The sum of three and five is equal to eight.
$n-1=14$	n minus one equals fourteen.
$6\times 7=42$	The product of six and seven is equal to forty-two.
$x=53$	x is equal to fifty-three.
$y+9=2y-3$	y plus nine is equal to two y minus three.

Example 1.2.2

1) Determine if each is an expression or an equation:

a. $2\left(x+3\right)=10$
b. $4\left(y-1\right)+1$
c. $x\div 25$
d. $y+8=40$

Solution

a. $2\left(x+3\right)=10$. This is an equation—two expressions are connected with an equal sign.
b. $4\left(y-1\right)+1$. This is an expression—no equal sign.

c. $x \div 25$. This is an expression—no equal sign.
d. $y + 8 = 40$. This is an equation—two expressions are connected with an equal sign.

Try It

3) Determine if each is an expression or an equation:

a. $3\left(x - 7\right) = 27$
b. $5\left(4y - 2\right) - 7$

Solution

a. equation
b. expression

4) Determine if each is an expression or an equation:

a. $y^3 \div 14$
b. $4x - 6 = 22$

Solution

a. expression
b. equation

Suppose we need to multiply 2 nine times. We could write this as $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$. This is tedious and it can be hard to keep track of all those 2s, so we use exponents. We write $2 \times 2 \times 2$ as 2^3 and $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$ as 2^9. In expressions such as 2^3 the 2 is called the *base* and the 3 is called the *exponent*. The exponent tells us how many times we need to multiply the base.

$$base \leftarrow 2^{3 \rightarrow \ exponent}$$

means multiply 2 by itself, three times, as in $2 \times 2 \times 2$

We read 2^3 as "two to the third power" or "two cubed."

We say 2^3 is in *exponential notation* and $2 \times 2 \times 2$ is in *expanded notation*.

Exponential Notation

a^n means multiply a by itself, n times.

$base \leftarrow a^{n \rightarrow\ exponent}$

$$a^n = a \times a \times \underbrace{a} \times \ldots \times a \quad n\ factors$$

The expression a^n is read a to the nth power.

While we read a^n as "a to the nth power," we usually read:

- a^2 "a squared"
- a^3 "a cubed"

We'll see later why a^2 and a^3 have special names.

The table below shows how we read some expressions with exponents.

Expression	In Words
7^2	7 to the second power or 7 squared
5^3	5 to the third power or 5 cubed
9^4	9 to the fourth power
12^5	12 to the fifth power

Example 1.2.3

Simplify: 3^4

Solution

***Step 1:* Expand the expression.**

$$3^4 = 3 \times 3 \times 3 \times 3$$

***Step 2:* Multiply left to right.**

$$9 \times 3 \times 3$$

***Step 3:* Multiply.**

$$27 \times 3$$

***Step 4:* Multiply.**

$$81$$

Try It

5) Simplify

a. 5^3
b. 1^7

Solution

a. 125
b. 1

6) Simplify:

a. 7^2
b. 0^5

Solution

a. 49
b. 0

Simplify Expressions Using the Order of Operations

To **simplify an expression** means to do all the math possible. For example, to simplify $4 \times 2 + 1$ we'd first multiply 4×2 to get 8 and then add the 1 to get 9. A good habit to develop is to work down the page, writing each step of the process below the previous step. The example just described would look like this:

$$\begin{aligned} 4 \times 2 + &\ 1 \\ 8 + &\ 1 \\ = &\ 9 \end{aligned}$$

By not using an equal sign when you simplify an expression, you may avoid confusing expressions with equation.

Simplify an Expression

To simplify an expression, do all operations in the expression.

We've introduced most of the symbols and notation used in algebra, but now we need to clarify the order of operations. Otherwise, expressions may have different meanings, and they may result in different values. For example, consider the expression:

$$4 + 3 \times 7$$

If you simplify this expression, what do you get?

Some students say 49,	
	$4+3\times7$
since $4+3$ gives 7.	7×7
7×7 is 49	49
Others say 25,	
	$4+3\times7$
Since 3×7 is 21.	$4+21$
And $21+4$ makes 25.	25

Imagine the confusion in our banking system if every problem had several different correct answers!

The same expression should give the same result. So mathematicians early on established some guidelines that are called the Order of Operations.

HOW TO

Perform the Order of Operations.

1. Parentheses and Other Grouping Symbols
    - Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
2. Exponents
    - Simplify all expressions with exponents.
3. Multiplication and Division
    - Perform all multiplication and division in order from left to right. These operations have equal priority.
4. Addition and Subtraction
    - Perform all addition and subtraction in order from left to right. These operations have equal priority.

Doing the Manipulative Mathematics activity “Game of 24” gives you practice using the order of operations.

Students often ask, “How will I remember the order?” Here is a way to help you remember: Take the first letter of each keyword and substitute the silly phrase: “Please Excuse My Dear Aunt Sally.”

Keyword	**Mnemonic**
Parentheses	Please
Exponents	Excuse
Multiplication Division	My Dear
Addition Subtraction	Aunt Sally

It’s good that “**M**y **D**ear” goes together, as this reminds us that **m**ultiplication and **d**ivision have equal priority. We do not always do multiplication before division or always do division before multiplication. We do them in order from left to right.

Similarly, “**A**unt **S**ally” goes together and so reminds us that **a**ddition and **s**ubtraction also have equal priority and we do them in order from left to right.

Let’s try an example.

Example 1.2.4

Simplify:

a. $4+3\times 7$

b. $4+(3\times 7)$

Solution

a.

Step 1: **Are there any parentheses?**

No.

Step 2: **Are there any exponents?**

No.

Step 3: **Is there any multiplication or division?**

Yes.

Step 4: **Multiply first.**

$$4 + 3 \times 7$$

Step 5: ***A*****dd.**

$$4 + 21 = 25$$

---

b.

Step 1: **Are there any parentheses?**

Yes.

$$(4 + 3) \times 7$$

Step 2: **Simplify inside the parentheses.**

$$(7) \times 7$$

Step 3: **Are there any exponents?**

No.

Step 4: **Is there any multiplication or division?**

Yes.

Step 5: ***M*****ultiply.**

$$49$$

Try It

7) Simplify:

a. $12 - 5 \times 2$
b. $(12 - 5) \times 2$

Solution

a. 2
b. 14

8) Simplify:

a. $8 + 3 \times 9$
b. $(8 + 3) \times 9$

Solution

a. 35
b. 99

Example 1.2.5

Simplify: $18 \div 6 + 4\,(5 - 2)$

Solution

Step 1: **Parentheses?**

Yes, subtract first.

$$18 \div 6 + 4\,(5 - 2)$$
$$18 \div 6 + 4\,(3)$$

Step 2: **Exponents?**

No.

Step 3: **Multiplication or division?**

Yes.

Step 4: **Divide first because we multiply and divide left to right.**

$$18 \div 6 + 4\,(3)$$

Step 5: **Any other multiplication or division?**

Yes.

Step 6: **Multiply.**

$$3+4\,(3)$$

Step 7: **Any other multiplication or division?**

No.

Step 8: **Any addition or subtraction?**

Yes, add.

$$3+12=15$$

Try It

9) Simplify: $30 \div 5 + 10(3-2)$

Solution

16

10) Simplify: $70 \div 10 + 4\,(6-2)$

Solution

23

When there are multiple grouping symbols, we simplify the innermost parentheses first and work outward.

Example 1.2.6

Simplify: $5+2^3+3[6-3(4-2)]$

Solution

Step 1: **Are there any parentheses (or other grouping symbol)?**

Yes.

Step 2: **Focus on the parentheses that are inside the brackets.**

$$5+2^3+3[6-3(4-2)]$$

Step 3: **Subtract.**

$$5+2^3+3[6-3(2)]$$

Step 4: **Continue inside the brackets and multiply.**

$$5+2^3+3[6-6]$$

Step 5: **Continue inside the brackets and subtract.**

The expression inside the brackets requires no further simplification.

$$5+2^3+3[0]$$

Step 6: **Are there any exponents?**

Yes.

Step 7: **Simplify exponents.**

$$5+(8)+3[0]$$

Step 8: **Is there any multiplication or division?**

Yes, multiply.

$$5+8+(3\times 0=0)$$

Step 9: **Is there any addition or subtraction?**

Yes, add.

$$5+8+0=13$$

Try It

11) Simplify: $9+5^3-[4(9+3)]$

Solution

86

12) Simplify: $7^2 - 2\left[4\left(5+1\right)\right]$

Solution

1

Evaluate an Expression

In the last few examples, we simplified expressions using the order of operations. Now we'll evaluate some expressions—again following the order of operations. To **evaluate an expression** means to find the value of the expression when the variable is replaced by a given number.

To evaluate an expression means to find the value of the expression when the variable is replaced by a given number.

To evaluate an expression, substitute that number for the variable in the expression and then simplify the expression.

Example 1.2.7

Evaluate $7x - 4$, when

a. $x = 5$ and,
b. $x = 1$

Solution

a.

Step 1: **Substitute 5 for x in the expression.**

Simplify the expression.

$$7 \times 5 - 4$$

Step 2: **Multiply.**

$$35 - 4$$

Step 3: **Subtract.**

$$31$$

---

b.

Step 1: **Substitute 1 for x in the expression.**

Simplify the expression.

$$7 \times 1 - 4$$

Step 2: **Multiply.**

$$7 - 4$$

Step 3: **Subtract.**

$$3$$

Try It

13) Evaluate $8x - 3$, when

a. $x = 2$
b. $x = 1$

Solution

a. 13
b. 5

14) Evaluate $4y - 4$, when

a. $y = 3$
b. $y = 5$

Solution

a. 8
b. 16

Example 1.2.8

Evaluate $x = 4$, when

a. x^2
b. 3^x

Solution

a.

Step 1: **Replace x with 4.**

$$4^2$$

Step 2: **Use definition of exponent.**

$$4 \times 4$$

Step 3: **Simplify.**

$$16$$

---

b.

Step 1: **Replace $x$with 4.**

$$3^4$$

Step 2: **Use definition of exponent.**

$$3 \times 3 \times 3 \times 3$$

Step 3: **Simplify.**

$$81$$

Try It

15) Evaluate

a. x^2
b. 4^x

when $x = 3$

Solution

a. 9
b. 64

16) Evaluate

a. x^3
b. 2^x

when $x = 6$

Solution

a. 216
b. 64

Example 1.2.9

Evaluate $2x^2 + 3x + 8$ when $x = 4$

Solution

Step 1: **Substitute** $x = 4$.

$$2(4)^2 + 3(4) + 8$$

***Step 2:* Follow the order of operations.**

$$2(16) + 12 + 8$$
$$32 + 12 + 8 = 52$$

Try It

17) Evaluate $3x^2 + 4x + 1$ when $x = 3$

Solution

40

18) Evaluate $6x^2 - 4x - 7$ when $x = 2$

Solution

9

Identify and Combine Like Terms

Term

Algebraic expressions are made up of terms. A term is a constant or the product of a constant and one or more variables.

Examples of terms are 7, y, $5x^2$, $9a$, b^5

The constant that multiplies the variable is called the **coefficient**.

Coefficient

The coefficient of a term is the constant that multiplies the variable in a term.

Think of the coefficient as the number in front of the variable. The coefficient of the term $3x$ is 3. When we write x, the coefficient is 1, since $x = 1x$.

Example 1.2.10

Identify the coefficient of each term:

a. $14y$
b. $15x^2$
c. a

Solution

1. The **coefficient** of $14y$ is 14.
2. The **coefficient** of $15x^2$ is 15.
3. The **coefficient** of a is 1 since $a = 1a$

Try It

19) Identify the coefficient of each term:

a. $17x$
b. $41b^2$
c. z

Solution

a. 14
b. 41
c. 1

20) Identify the coefficient of each term:

a. $9p$
b. $13a^3$
c. y^3

Solution

a. 9
b. 13
c. 1

Some terms share common traits. Look at the following 6 terms. Which ones seem to have traits in common?

$$5x \qquad 7 \qquad n^2 \qquad 4 \qquad 3x \qquad 9n^2$$

The 7 and the 4 are both **constant** terms.
The $5x$ and the $3x$ are both terms with x.
The n^2 and the $9n^2$ are both terms with n^2
When two terms are constants or have the same variable and exponent, we say they are **like terms**.

- 7 and 4 are like terms.
- $5x$ and $3x$ are like terms.
- x^2 and $9x^2$ are like terms.

Like Terms

Terms that are either constants or have the same variables raised to the same powers are called like terms.

Example 1.2.11

Identify the like terms: y^3, $7x^2$, 14, 23, $4y^3$, $9x$, $5x^2$

Solution

y^3 and $4y^3$ are like terms because both have y^3 the variable and the exponent match.

$7x^2$ and $5x^2$ are like terms because both have x^2 the variable and the exponent match.

14 and 23 are like terms because both are constants.

There is no other term like $9x$

Try It

21) Identify the like terms: 9, $2x^3$, y^2, $8x^3$, 15, $9y$, $11y^2$

Solution

9 and 15, y^2 and $11y^2$, $2x^3$ and $8x^3$

22) Identify the like terms: $4x^3$, $8x^2$, 19, $3x^2$, 24, $6x^3$

Solution

19 and 24, $8x^2$ and $3x^2$, $4x^3$ and $6x^3$

Adding or subtracting terms forms an expression. In the expression $2x^2+3x+8$ when $x=4$ from Example 1.2.9, the three terms are $2x^2$, $3x$, and 8.

Example 1.2.12

Identify the terms in each expression.

a. $9x^2+7x+12$
b. $8x+3y$

Solution

a. The terms of $9x^2+7x+12$ are $9x^2$, $7x$, and 12.
b. The terms of $8x+3y$ are $8x$ and $3y$

Try It

23) Identify the terms in the expression $4x^2+5x+17$

Solution

$4x^2$, $5x$, 17

24) Identify the terms in the expression $5x+2y$

Solution

$5x$, $2y$

If there are like terms in an expression, you can simplify the expression by combining the like terms. What do you think $4x + 7x + x$ would simplify to? If you thought 12, you would be right!

$$4x + 7x + x$$
$$x + x + x + x \quad + x + x + x + x + x + x + x \quad + x$$
$$12x$$

Add the coefficients and keep the same variable. It doesn't matter what x is—if you have 4 of something and add 7 more of the same thing and then add 1 more, the result is 12 of them. For example, 4 oranges plus 7 oranges plus 1 orange is 12 oranges. We will discuss the mathematical properties behind this later.

Simplify: $4x + 7x + x$.

Add the coefficients. $12x$

Example 1.2.13

How To Combine Like Terms:

Simplify: $2x^2 + 3x + 7 + x^2 + 4x + 5$

Solution

***Step 1*: Identify the like terms**

$$2x^2 + 3x + 7 + x^2 + 4x + 5$$
$$\boxed{2x^2} + \boxed{3x} + 7 + \boxed{x^2} + \boxed{4x} + 5$$

***Step 2:* Rearrange the expression so the like terms are together.**

$$\boxed{2x^2} + \boxed{x^2} + \boxed{3x} + \boxed{4x} + 7 + 5$$

***Step 3*: Combine like terms.**

$$3x^2 + 7x + 12$$

Try It

25) Simplify: $3x^2+7x+9+7x^2+9x+8$

Solution

$10x^2+16x+17$

26) Simplify: $4y^2+5y+2+8y^2+4y+5$

Solution

$12y^2+9y+7$

HOW TO

Combine Like Terms.

1. Identify like terms.
2. Rearrange the expression so like terms are together.
3. Add or subtract the coefficients and keep the same variable for each group of like terms.

Translate an English Phrase to an Algebraic Expression

In the last section, we listed many operation symbols that are used in algebra, then we translated expressions and equations into English phrases and sentences. Now we'll reverse the process. We'll translate English phrases into algebraic expressions. The symbols and variables we've talked about will help us do that. Below table summarizes them.

Operation	Phrase
Addition	a plus b the sum of a and b a increased by b b more than a the total of a and b b added to a
Subtraction	a minus b the difference of a and b a decreased by b b less than a b subtracted from a
Multiplication	a times b the product of a and b twice a
Division	a divided by b the quotient of a and b the ratio of a and b b divided into a

Look closely at these phrases using the four operations:

the sum *of* a *and* b

the difference *of* a *and* b

the product *of* a *and* b

the quotient *of* a *and* b

Each phrase tells us to operate on two numbers. Look for the words *of*, *and* to find the numbers.

Example 1.2.14

Translate each English phrase into an algebraic expression:

a. the difference of $17x$ and 5
b. the quotient of $10x^2$ and 7

Solution

a.

***Step 1:* The key word is "difference", which tells us the operation is subtraction.**

The *difference* of $17x$ and 5.

***Step 2*: Look for the words *of* and *and* find the numbers to subtract.**

The difference *of* $17x$ and 5.

***Step 3*: Find the difference of $17x$ and 5**

$$17x - 5$$

---

b.

***Step 1:* The key word is "quotient," which tells us the operation is division.**

The *quotient* of $10x^2$ and 7.

***Step 2:* Divide $10x^2$ by 7**

$$10x^2 \div 7$$

This can also be written $10x^2/7$ or $\frac{10x^2}{7}$.

Try It

27) Translate the English phrase into an algebraic expression:

a. the difference of $14x^2$ and 13

b. the quotient of $12x$ and 2

Solution

a. $14x^2 - 13$
b. $12x \div 2$

28) Translate the English phrase into an algebraic expression:

a. the sum of $17y^2$ and 19
b. the product of 7 and y

Solution

a. $17y^2 + 19$
b. $7y$

How old will you be in eight years? What age is eight more years than your age now? Did you add 8 to your present age? Eight "more than" means 8 added to your present age. How old were you seven years ago? This is 7 years less than your age now. You subtract 7 from your present age. Seven "less than" means 7 subtracted from your present age.

Example 1.2.15

Translate the English phrase into an algebraic expression:

a. Seventeen more than y
b. Nine less than $9x^2$

Solution

a.

Step 1:* The key words are *more than.

They tell us the operation is addition. *More than* means "added to."

***Step 2:* Seventeen more than y.**

***Step 3:* Seventeen added to y.**

$$y + 17$$

---

b.

***Step 1:* The key words are *less than*.**

They tell us to subtract. *Less than* means "subtracted from."

***Step 2:* Nine less than $9x^2$**

***Step 3:* Nine subtracted from $9x^2$**

$$9x^2 - 9$$

Try It

29) Translate the English phrase into an algebraic expression:

a. Eleven more than x
b. Fourteen less than $11a$

Solution

a. $x + 11$
b. $11a - 14$

30) Translate the English phrase into an algebraic expression:

a. 13 more than z

b. 18 less than $8x$

Solution

a. $z+13$
b. $8x-18$

Example 1.2.16

Translate the English phrase into an algebraic expression:

a. five times the sum of m and n
b. the sum of five times m and n

Solution

a.

Step 1: **There are two operation words—*times* tells us to multiply and *sum* tells us to add.**

Step 2: **Because we are multiplying 5 times the sum we need parentheses around the sum of m and n, $(m+n)$.**

This forces us to determine the sum first. (Remember the order of operations.)

Step 3: **Five times the sum of m and n.**

$$5(m+n)$$

---

b.

Step 1: **To take a sum, we look for the words "of" and "and" to see what is being added.**

Here we are taking the sum *of* five times m and n.

Step 2: **The sum of five times m and n.**

$$5m + n$$

Try It

31) Translate the English phrase into an algebraic expression:

a. four times the sum of p and q
b. the sum of four times p and q

Solution

a. $4(p + q)$
b. $4p + q$

32) Translate the English phrase into an algebraic expression:

a. the difference of two times x and 8
b. two times the difference of x and 8

Solution

a. $2x - 8$
b. $2(x - 8)$

Later in this course, we'll apply our skills in algebra to solving applications. The first step will be to translate an English phrase to an algebraic expression. We'll see how to do this in the next two examples.

Example 1.2.17

The length of a rectangle is 6 less than the width. Let w represent the width of the rectangle. Write an expression for the length of the rectangle.

Solution

***Step 1:* Write a phrase about the length of the rectangle.**

6 less than the width

***Step 2:* Substitute w for "the width."**

6 less than w

***Step 3:* Rewrite "less than" as "subtracted from."**

6 subtracted from w

***Step 4:* Translate the phrase into algebra.**

$$w-6$$

Try It

33) The length of a rectangle is 7 less than the width. Let w represent the width of the rectangle. Write an expression for the length of the rectangle.

Solution

$w-7$

34) The width of a rectangle is 6 less than the length. Let l represent the length of the rectangle. Write an expression for the width of the rectangle.

Solution

$l - 6$

Example 1.2.18

June has dimes and quarters in her purse. The number of dimes is three less than four times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

Solution

Step 1: **Write a phrase about the number of dimes.**

three less than four times the number of quarters

Step 2: **Substitute** q **for the number of quarters.**

3 less than 4 times q

Step 3: **Translate "4 times** q**".**

3 less than $4q$

Step 4: **Translate the phase into algebra.**

$$4q - 3$$

Try It

35) Geoffrey has dimes and quarters in his pocket. The number of dimes is eight less than four

times the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.

Solution

$4q - 8$

36) Lauren has dimes and nickels in her purse. The number of dimes is three more than seven times the number of nickels. Let n represent the number of nickels. Write an expression for the number of dimes.

Solution

$7n + 3$

Key Concepts

- **Notation** **The result is…**

$a + b$	the sur
$a - b$	the dif
ab, $a \times b$, $(a)(b)$, $(a)b$, $a(b)$	the pro
a/b, $a \div b$, $\frac{a}{b}$, $b\overline{)a}$	the qu

- **Inequality**

$a < b$ is read "a is less than b"	a is to the left of b on the number line
$a < b$ is read "a is greater than b"	a is to the right of b on the number line

- **Inequality Symbols** **Words**

$a \neq b$	a is not equal to b
$a < b$	a is less than b
$a \leq b$	a is less than or equal to b
$a > b$	a is greater than b
$a \geq b$	a is greater than or equal to b

- **Grouping Symbols**
  - Parentheses ()
  - Brackets []
  - Braces {}
- **Exponential Notation**
  - a^n means multiply a by itself, n times. The expression a^n is read a to the n^{th} power.
- **Order of Operations:** When simplifying mathematical expressions perform the operations in the following order:
  1. Parentheses and other Grouping Symbols: Simplify all expressions inside the parentheses or other grouping symbols, working on the innermost parentheses first.
  2. Exponents: Simplify all expressions with exponents.
  3. Multiplication and Division: Perform all multiplication and division in order from left to right. These operations have equal priority.
  4. Addition and Subtraction: Perform all addition and subtraction in order from left to right. These operations have equal priority.
- **Combine Like Terms**
  1. Identify like terms.
  2. Rearrange the expression so like terms are together.
  3. Add or subtract the coefficients and keep the same variable for each group of like terms.

Self Check:

a. Use this checklist to evaluate your mastery of the objectives of this section.

- the quotient of $10x^2$ and 7
- divide $10x^2$ by 7
- $10x^2 \div 7$

b. After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

coefficient
: The coefficient of a term is the constant that multiplies the variable in a term.

constant
: A constant is a number whose value always stays the same.

equality symbol
: The symbol "=" is called the equal sign. We read $a = b$ as "a is equal to b."

equation
: An equation is two expressions connected by an equal sign.

evaluate an expression
: To evaluate an expression means to find the value of the expression when the variable is replaced by a given number.

expression
: An expression is a number, a variable, or a combination of numbers and variables using operation symbols.

like terms
: Terms that are either constants or have the same variables raised to the same powers are called like terms.

simplify an expression

To simplify an expression, do all operations in the expression.

term

A term is a constant or the product of a constant and one or more variables.

variable

A variable is a letter that represents a number whose value may change.

1.3 INTEGERS

Learning Objectives

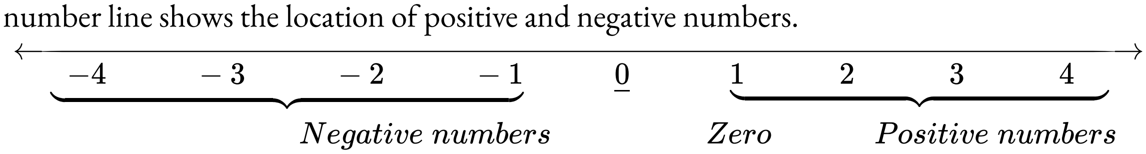
By the end of this section, you will be able to:

- Use negatives and opposites
- Simplify: expressions with absolute value
- Add integers
- Subtract integers
- Multiply integers
- Divide integers
- Simplify expressions with integers
- Evaluate variable expressions with integers
- Translate English phrases to algebraic expressions
- Use integers in applications

Use Negatives and Opposites

Our work so far has only included the counting numbers and the whole numbers. But if you have ever experienced a temperature below zero or accidentally overdrawn your checking account, you are already familiar with negative numbers. Negative numbers are numbers less than 0. The negative numbers are to the left of zero on the number line.

The number line shows the location of positive and negative numbers.

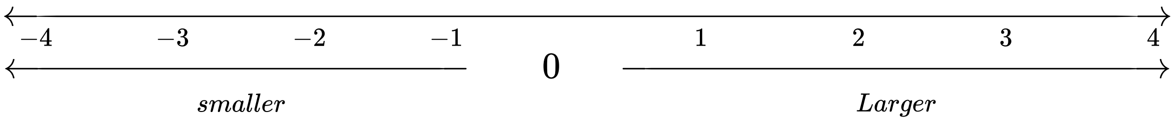


The arrows on the ends of the number line indicate that the numbers keep going forever. There is no biggest positive number, and there is no smallest negative number.

Is zero a positive or a negative number? Numbers larger than zero are positive, and numbers smaller than zero are negative. Zero is neither positive nor negative.

Consider how numbers are ordered on the number line. Going from left to right, the numbers increase in value. Going from right to left, the numbers decrease in value.

The numbers on a number line increase in value going from left to right and decrease in value going from right to left.



Remember that we use the notation:

$a < b$ (read "a is less than b") when a is to the left of b on the number line.

$a > b$ (read "a is greater than b") when a is to the right of b on the number line.

Now we need to extend the number line which showed the whole numbers to include negative numbers, too. The numbers marked by points in figure 1.3.1 are called the integers. The integers are the numbers {…}−3,−2,−1,0,1,2,3{…}

All the marked numbers are called *integers*.

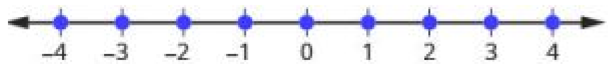


Figure 1.3.1

Example 1.3.1

Order each of the following pairs of numbers, using < or >:

a. $14\_\_\_6$
b. $-1\_\_\_9$
c. $-1\_\_\_-4$
d. $2\_\_\_-20$

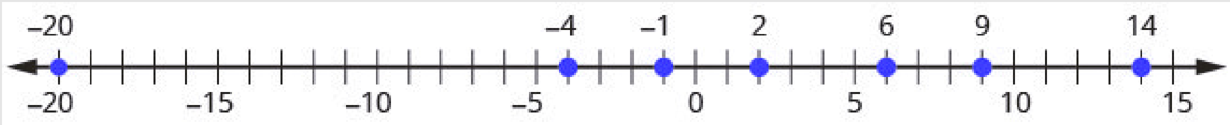


Figure 1.3.2

Solution

It may be helpful to refer to the number line shown.

a. 14 is to the right of 6 on the number line.

$14\_\_\_6 \qquad 14 > 6$

b. −1 is to the left of 9 on the number line.

$-1\_\_\_9 \qquad -1 < 9$

c. −1 is to the right of −4 on the number line.

$-1\_\_\_-4 \qquad -1 > -4$

d. 2 is to the right of −20 on the number line.

$2\_\_\_-20 \qquad 2 > -20$

Try It

1) Order each of the following pairs of numbers, using < or >:

a. $15\_\_\_7$

b. $-2\_\_\_5$

c. $-3\_\_\_-7$

d. $5\_\_\_-17$

Solution

a. >
b. <
c. >
d. >

2) Order each of the following pairs of numbers, using < or >:

a. $8\_\_\_13$
b. $3\_\_\_-4$
c. $-5\_\_\_-2$
d. $9\_\_\_-21$

Solution

a. <
b. >
c. <
d. >

You may have noticed that, on the number line, the negative numbers are a mirror image of the positive numbers, with zero in the middle. Because the numbers 2 and −2 are the same distance from zero, they are called **opposites**. The opposite of 2 is −2 and the opposite of −2 is 2.

Opposite

The opposite of a number is the number that is the same distance from zero on the number line but on the opposite side of zero.

Figure 1.3.2 illustrates the definition.

The opposite of 3 is −3.

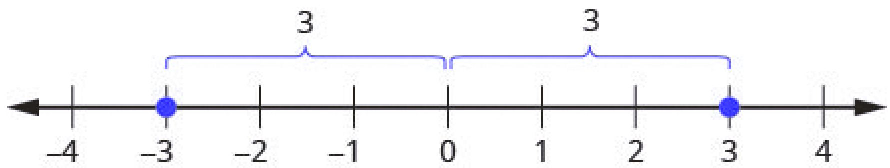


Figure 1.3.3

Sometimes in algebra, the same symbol has different meanings. Just like some words in English, the specific meaning becomes clear by looking at how it is used. You have seen the symbol "–" used in three different ways.

$10-4$	Between two numbers, it indicates the operation of subtraction. We read $10-4$ as " 10 minus 4."
-8	In front of a number, it indicates a negative number. We read –8 as "negative eight."
$-x$	In front of a variable, it indicates the opposite. We read $-x$ as "the opposite of x."
$-(-2)$	Here there are two "–" signs. The one is the parentheses tells us the number is negative 2. and the one outside the parentheses tells us to take the opposite of –2. We read (–2) as "the opposite of negative two."

Opposite Notation

$-a$ means the opposite of the number a.

The notation $-a$ is read as "the opposite of a."

Example 1.3.2

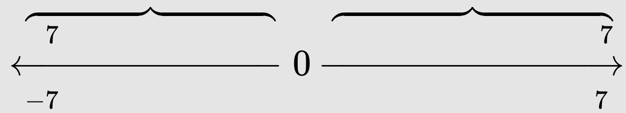
Find:

a. the opposite of 7

b. the opposite of -10
c. the opposite of -6

Solution

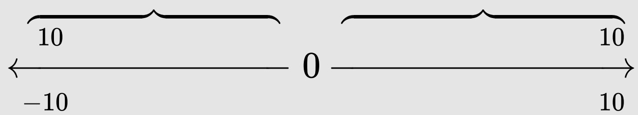
a. -7 is the same distance from 0 as 7, but on the opposite side of 0



The opposite of 7 is -7.

---

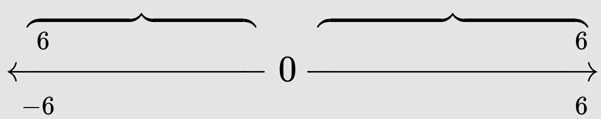
b. 10 is the same distance from 0 as -10, but on the opposite side of 0.



The opposite of -10 is 10.

---

c. $-(-6)$



The opposite of $-(-6)$ is -6.

Try It

3) Find:

a. the opposite of 4
b. the opposite of -3
c. $-(-1)$

Solution

a. -4
b. 3
c. 1

4) Find:

a. the opposite of 8
b. the opposite of -5
c. $-(-5)$

Solution

a. -8
b. 5
c. 5

Our work with opposites gives us a way to define the integers. The whole numbers and their opposites are called the **integers**. The integers are the numbers −3,−2,−1,0,1,2,3

Integers

The whole numbers and their opposites are called the integers.

The integers are the numbers −3,−2,−1,0,1,2,3

When evaluating the opposite of a variable, we must be very careful. Without knowing whether the variable represents a positive or negative number, we don't know whether x is positive or negative. We can see this in the below figure.

Example 1.3.3

Evaluate

a. $-x$, when $x = 8$
b. $-x$, when $x = -8$.

Solution

a.

To evaluate when $x = 8$ means to substitute 8 for x.

Step 1: **Substitute 8 for x.**

$$-(8)$$

Step 2: **Write the opposite of 8.**

$$-8$$

---

b.

To evaluate when $x = -8$ means to substitute -8 for x.

Step 1: **Substitute -8 for x.**

$$-(-8)$$

Step 2: **Write the opposite of -8.**

$$8$$

Try It

5) Evaluate $-n$ when

a. $n = 4$
b. $n = -4$

Solution

a. -4
b. 4

6) Evaluate $-m$ when

a. $m = 11$
b. $m = -11$

Solution

a. -11
b. 11

Simplify: Expressions with Absolute Value

We saw that numbers such as 2 and −2 are opposites because they are the same distance from 0 on the number line. They are both two units from 0. The distance between 0 and any number on the number line is called the **absolute value** of that number.

Absolute Value

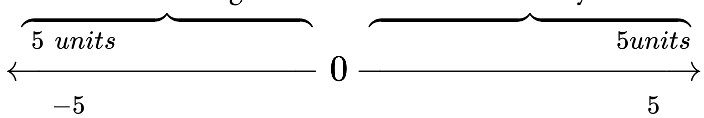
The absolute value of a number is its distance from 0 on the number line.

The absolute value of a number n is written as $|n|$.

For example,

- -5 is 5 units away from 0, so $|-5| = 5$.
- 5 is 5 units away from 0, so $|5| = 5$.

The figure below illustrates this idea. The integers 5 and −5 are 5 units away from 0.



The absolute value of a number is never negative (because distance cannot be negative). The only number with absolute value equal to zero is the number zero itself, because the distance from 0 to 0 on the number line is zero units.

Property of Absolute Value

$|n| \geq 0$ for all numbers

Absolute values are always greater than or equal to zero!

Mathematicians say it more precisely, "absolute values are always non-negative." Non-negative means greater than or equal to zero.

Example 1.3.4

7) Simplify:

a. $|3|$
b. $|-44|$
c. $|0|$

Solution
The absolute value of a number is the distance between the number and zero. Distance is never negative, so the absolute value is never negative.

a. $|3| = 3$

---

b. $|-44| = 44$

---

c. $|0| = 0$

Try It

7) Simplify:

a. $|4|$
b. $|-28|$

c. $|0|$

Solution

a. 4
b. 28
c. 0

8) Simplify:

a. $|-13|$
b. $|47|$
c. $|0|$

Solution

a. 13
b. 47
c. 0

In the next example, we'll order expressions with absolute values. Remember, positive numbers are always greater than negative numbers!

Example 1.3.5

Fill in $<,>$ or $=$ for each of the following pairs of numbers:

1. $|-5|$___ $-|-5|$
2. 8___ $-|-8|$
3. -9___ $-|-9|$
4. $-(-16)$ ___ $|-16|$

Solution

a.

Step 1: **Simplify.**

$$-5\_\_\_-{-5}$$

Step 2: **Order.**

$$5 > -5$$
$$|5| > -|-5|$$

---

b.

Step 1: **Simplify.**

$$8\_\_\_-{-8}$$

Step 2: **Order.**

$$8 > -8$$
$$|8| > -|-8|$$

---

c.

Step 1: **Simplify.**

$$-9\_\_\_-{-9}$$

Step 2: **Order.**

$$-9 = -9$$
$$|9| = -|-9|$$

---

d.

Step 1: **Simplify.**

$$--16\_\_\_-16|$$

Step 2: **Order.**

$$16 > -16$$
$$-(-16) > -|-16|$$

Try It

9) Fill in $<$, $>$, or $=$ for each of the following pairs of numbers:

a. $|-9|$ ___ $-|-9|$
b. $|2|$ ___ $-|-2|$
c. $|-8|$ ___ $|-8|$
d. $-(-9)$ ___ $-|-9|$

Solution

a. >
b. >
c. <
d. >

10) Fill in $<$, $>$, or $=$ for each of the following pairs of numbers:

a. $|7|$ ___ $-|-7|$
b. $-(-10)$ ___ $-|-10|$
c. $|-4|$ ___ $-|-4|$
d. $|-1|$ ___ $|-1|$

Solution

a. >
b. >

c. >
d. <

We now add **absolute value** bars to our list of grouping symbols. When we use the order of operations, first we simplify inside the absolute value bars as much as possible, then we take the absolute value of the resulting number.

Grouping Symbols

Parentheses ( )	Braces {}
Brackets []	Absolute value ||

In the next example, we simplify the expressions inside **absolute value** bars first, just like we do with parentheses.

Example 1.3.6

Simplify: $24-|19-3(6-2)|$

Solution

Step 1: **Work inside parentheses first: subtract 2 from 6.**

$$24-|19-3(4)|$$

Step 2: **Multiply 3(4).**

$$24-|19-12|$$

Step 3: **Subtract inside the absolute value bars.**

$$24 - |7|$$

Step 4: **Take the absolute value.**

$$24 - 7$$

Step 5: **Subtract**

$$17$$

Try It

11) Simplify: $19 - |11 - 4(3 - 1)|$

Solution

16

12) Simplify: $9 - |8 - 4(7 - 5)|$

Solution

9

Example 1.3.7

Evaluate:

a. $|x|$ when $x = -35$
b. $|-y|$ when $y = -20$

c. $-\left|u\right|$ when $u=12$
d. $-\left|p\right|$ when $p=-14$

Solution

a.

Step 1: **Substitute $-35$for x.**

$$\left|-35\right|$$

Step 2**: Take the absolute value.**

$$35$$

---

b.

Step 1: **Substitute -20 for y.**

\left|-(-20)\right|

Step 2**: Simplify.**

$$\left|20\right|$$

Step 3**: Take the absolute value.**

$$20$$

---

c.

Step 1: **Substitute 12 for u.**

$$-\left|12\right|$$

Step 2**: Take the absolute value.**

$$-12$$

---

d.

Step 1: **Substitute -14 for p.**

$$-\left|-14\right|$$

Step 2**: Take the absolute value**

$$-14$$

Try It

13) Evaluate:

a. $\left|x\right|$ when $x=-17$
b. $\left|-y\right|$ when $y=39$
c. $-\left|m\right|$ when $m=22$
d. $-\left|p\right|$ when $p=-11$.

Solution

a. 17
b. 39
c. -22
d. -11

14) Evaluate:

a. $\left|y\right|$ when $y=-23$
b. $\left|-y\right|$ when $y=-21$
c. $-\left|n\right|$ when $n=37$
d. $-\left|q\right|$ when $q=-49$

Solution

a. 23

b. 21
c. −37
d. −49

Add Integers

Most students are comfortable with the addition and subtraction facts for positive numbers. But doing addition or subtraction with both positive and negative numbers may be more challenging.

We will use two colour counters to model addition and subtraction of negatives so that you can visualize the procedures instead of memorizing the rules.

We let one colour (blue) represent positive. The other colour (red) will represent the negatives. If we have one positive counter and one negative counter, the value of the pair is zero. They form a neutral pair. The value of this neutral pair is zero.

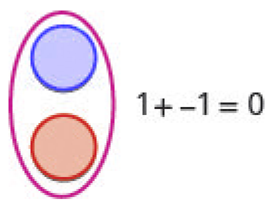


Figure 1.3.4

We will use the counters to show how to add the four addition facts using the numbers 5,−5 and 3,−3.

$5+3 \qquad -5+(-3) \qquad -5+3 \qquad 5+(-3)$

To add $5+3$, we realize that $5+3$ means the sum of 5 and 3.

We start with 5 positives.

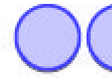


Figure 1.3.5

And then we add 3 positives.

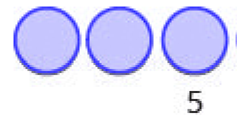


Figure 1.3.6

We now have 8 positives. The sum of 5 and 3 is 8.

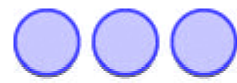


Figure 1.3.7

Now we will add $-5+(-3)$. Watch for similarities to the last example $5+3=8$.

To add $-5+(-3)$, we realize this means the sum of −5 and −3.

We start with 5 negatives.

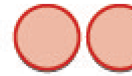


Figure 1.3.8

And then we add 3 negatives.

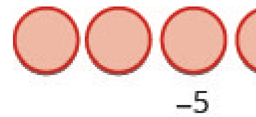


Figure 1.3.9

We now have 8 negatives. The sum of −5 and −3 is −8.

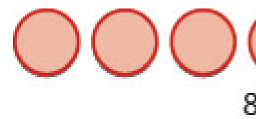


Figure 1.3.10

In what ways were these first two examples similar?

- The first example adds 5 positives and 3 positives—both positives.
- The second example adds 5 negatives and 3 negatives—both negatives.

In each case we got 8—either 8 positives or 8 negatives.

When the signs were the same, the counters were all the same colour, and so we added them.

Figure 1.3.11

Example 1.3.8

Add:

a. $1+4$
b. $-1+(-4)$

Solution

a.

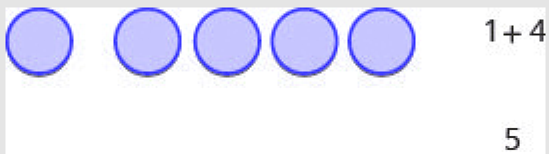


Figure 1.3.12

1 positive plus 4 positives is 5 positives.

---

b.

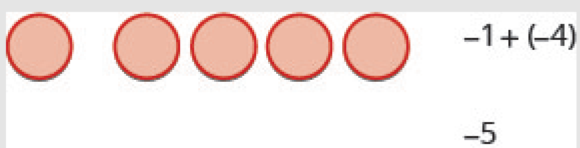


Figure 1.3.13

1 negative plus 4 negatives is 5 negatives.

Try It

15) Add:

a. $2+4$
b. $-2+(-4)$

Solution

a. 6
b. -6

16) Add:

a. $2+5$
b. $-2+(-5)$

Solution

a. 7
b. -7

So what happens when the signs are different? Let's add −5+3. We realize this means the sum of −5 and 3. When the counters were the same colour, we put them in a row. When the counters are a different colour, we line them up under each other.

−5 + 3 means the sum of −5 and

We start with 5 negatives.

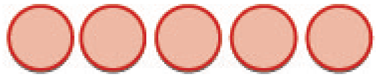


Figure 1.3.14

And then we add 3 positives.

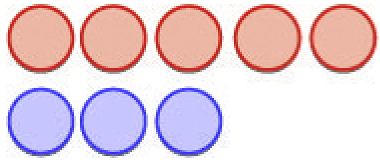


Figure 1.3.15

We remove any neutral pairs.

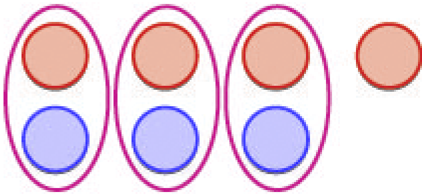


Figure 1.3.16

We have 2 negatives left.

Figure 1.3.17

The sum of −5 and 3 is −2. $-5+3=-2$

Notice that there were more negatives than positives, so the result was negative.

Let's now add the last combination, $5+(-3)$.

5 + (−3) means the sum of 5 and −3.

We start with 5 positives.

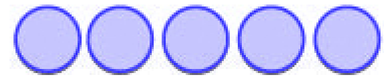


Figure 1.3.18

And then we add 3 negatives.

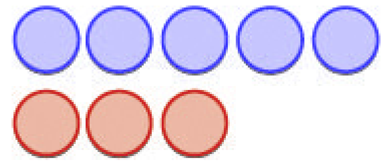


Figure 1.3.19

We remove any neutral pairs.

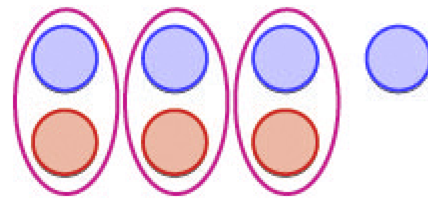


Figure 1.3.20

We have 2 positives left.

Figure 1.3.21

The sum of 5 and −3 is 2. $5 + (-3) = 2$

When we use counters to model addition of positive and negative integers, it is easy to see whether there are more positive or more negative counters. So we know whether the sum will be positive or negative.

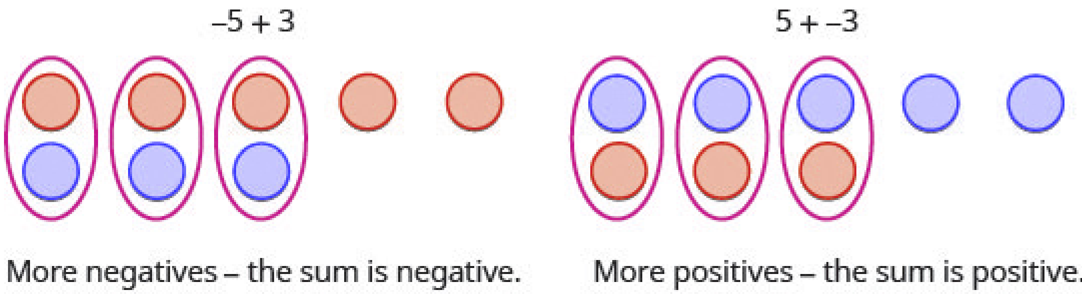


Figure 1.3.22

Example 1.3.9

Add:

a. $-1+5$
b. $1+(-5)$

Solution

a.

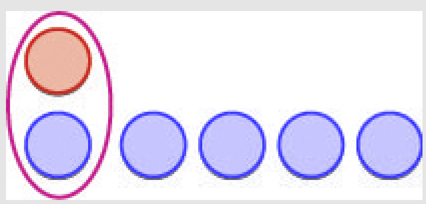


Figure 1.3.23

There are more positives, so the sum is positive.

$$-1+5=4$$

---

b.

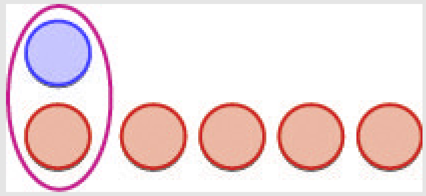


Figure 1.3.24

There are more negatives, so the sum is negative.

$$1 + (-5) = -4$$

Try It

17) Add:

a. $-2 + 4$
b. $2 + (-4)$

Solution

a. 2
b. -2

18) Add:

a. $-2 + 5$
b. $2 + (-5)$

Solution

a. 3
b. -3

Now that we have added small positive and negative integers with a model, we can visualize the model in our minds to simplify problems with any numbers.

When you need to add numbers such as $37 + (-53)$, you really don't want to have to count out 37 blue counters and 53 red counters. With the model in your mind, can you visualize what you would do to solve the problem?

Picture 37 blue counters with 53 red counters lined up underneath. Since there would be more red (negative) counters than blue (positive) counters, the sum would be *negative*. How many more red counters would there be? Because $53 - 37 = 16$, there are 16 more red counters.

Therefore, the sum of $37 + (-53)$ is −16.

$$37 + (-53) = -16$$

Let's try another one. We'll add $-74 + (-27)$. Again, imagine 74 red counters and 27 more red counters, so we'd have 101 red counters. This means the sum is −101.

$$-74 + (-27) = -101$$

Let's look again at the results of adding the different combinations of 5,−5 and 3,−3.

Addition of Positive and Negative Integers

$5 + 3$	$-5 + (-3)$
8	-8
both positive, sum positive	both negative, sum negative

When the signs are the same, the counters would be all the same colour, so add them.

$-5 + 3$	$5 + (-3)$
-2	2
different signs, more negatives, sum negative	different signs, more positives, sum positive

When the signs are different, some of the counters would make neutral pairs, so subtract to see how many are left.

Visualize the model as you simplify the expressions in the following examples.

Example 1.3.10

Simplify

a. $19+(-47)$
b. $-14+(-36)$

Solution

a. Since the signs are different, we subtract 19 from 47. The answer will be negative because there are more negatives than positives.

***Step 1:* Subtract**

$$19+(-47)$$
$$-28$$

---

b. Since the signs are the same, we add. The answer will be negative because there are only negatives.

***Step 1:* Add**

$$-14+(-36)$$
$$-50$$

Try It

19) Simplify:

a. $-31+(-19)$
b. $15+(-32)$

Solution

a. -50
b. -17

20) Simplify:

a. $-42+(-28)$
b. $25+(-61)$

Solution

a. -70
b. -36

The techniques used up to now extend to more complicated problems, like the ones we've seen before. Remember to follow the order of operations!

Example 1.3.11

Simplify: $-5+3(-2+7)$

Solution
Step 1: **Simplify inside the parentheses.**

$$-5+3(5)$$

Step 2: **Multiply.**

$$-5+15$$

Step 3: **Add left to right.**

$$10$$

Try It

21) Simplify: $-2 + 5(-4 + 7)$

Solution

13

22) Simplify: $-4 + 2(-3 + 5)$

Solution

0

Subtract Integers

We will continue to use counters to model subtraction. Remember, the blue counters represent positive numbers and the red counters represent negative numbers.

Perhaps when you were younger, you read "5–3" as "5 take away 3". When you use counters, you can think of subtraction the same way!

We will model the four subtraction facts using the numbers 5 and 3.

$$5 - 3 \qquad -5 - (-3) \qquad -5 - 3 \qquad 5 - (-3)$$

To subtract 5-3, we restate the problem as "5 takes away 3".

We start with 5 positives.

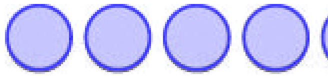


Figure 1.3.25

We 'take away' 3 positives.

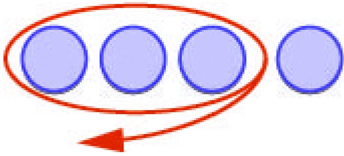


Figure 1.3.26

We have 2 positives left.

The difference of 5 and 3 is

Now we will subtract $-5-(-3)$. Watch for similarities to the last example $5-3=2$.

To subtract $-5-(-3)$, we restate this as "–5 take away –3".

We start with 5 negatives.

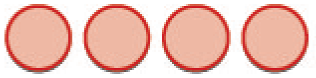


Figure 1.3.27

We 'take away' 3 negatives.

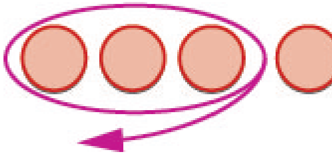


Figure 1.3.28

We have 2 negatives left.

The difference of –5 and –3

Notice that these two examples are much alike: The first example, we subtract 3 positives from 5 positives and end up with 2 positives.

In the second example, we subtract 3 negatives from 5 negatives and end up with 2 negatives.

Each example used counters of only one colour, and the "take away" model of subtraction was easy to apply.

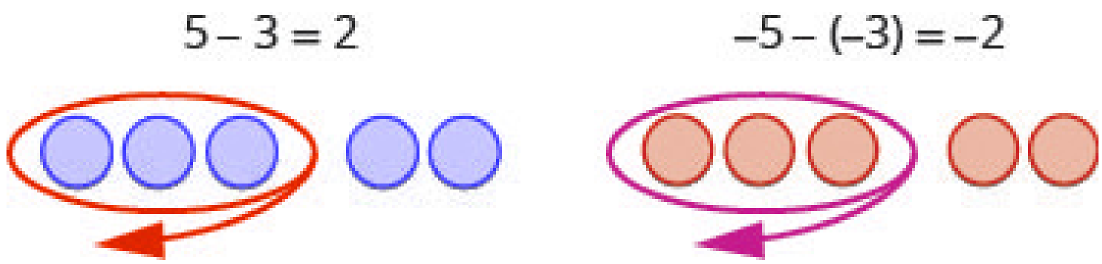


Figure 1.3.29

Example 1.3.12

Subtract:

a. $7 - 5$
b. $-7 - (-5)$

Solution

a.

Step 1: **Take 5 positive from 7 positives and get 2 positives.**

$$7 - 5 = 2$$

---

b.

Step 1: **Take 5 negatives from 7 negatives and get 2 negatives.**

$$-7 - (-5) = -2$$

Try It

23) Subtract:

a. $6 - 4$
b. $-6 - (-4)$

Solution

a. 2
b. -2

24) Subtract:

a. $7 - 4$
b. $-7 - (-4)$

Solution

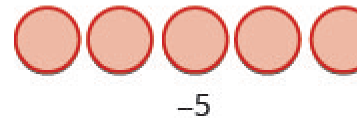
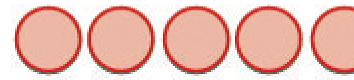
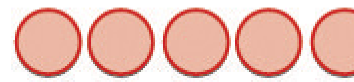
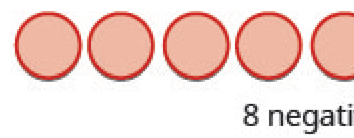
a. 3
b. -3

What happens when we have to subtract one positive and one negative number? We'll need to use both white and red counters as well as some neutral pairs. Adding a neutral pair does not change the value. It is like changing quarters to nickels—the value is the same, but it looks different.

- To subtract $-5 - 3$, we restate it as –5 take away 3.

We start with 5 negatives. We need to take away 3 positives, but we do not have any positives to take away.

Remember, a neutral pair has value zero. If we add 0 to 5 its value is still 5. We add neutral pairs to the 5 negatives until we get 3 positives to take away.

We start with 5 negatives.	 Figure 1.3.30
We now add the neutrals needed to get 3 positives.	 Figure 1.3.31
We remove the 3 positives.	 Figure 1.3.32
We are left with 8 negatives.	 Figure 1.3.33
The difference of −5 and 3 is −8.	$-5-3$

And now, the fourth case, $5-(-3)$. We start with 5 positives. We need to take away 3 negatives, but there are no negatives to take away. So we add neutral pairs until we have 3 negatives to take away.

We start with 5 positives.

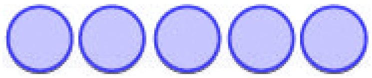


Figure 1.3.34

We now add the needed neutrals pairs.

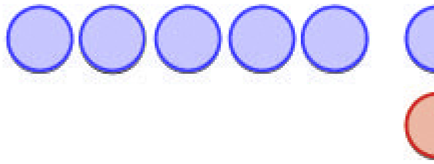


Figure 1.3.35

We remove the 3 negatives.

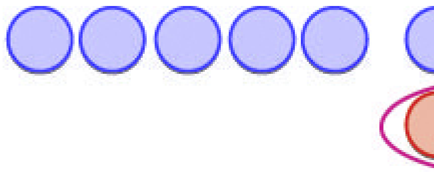


Figure 1.3.36

We are left with 8 positives.

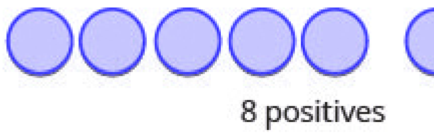


Figure 1.3.37

The difference of 5 and −3 is 8. $5-(-3)=8$

Example 1.3.13

Subtract:

a. $-3-1$
b. $3-(-1)$.

Solution

a.

Step 1: **Take 1 positive from the one added neutral pair.**

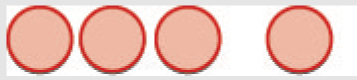
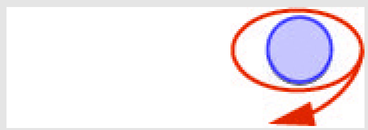


Figure 1.3.38

$-3-1$



1.3.39

-4

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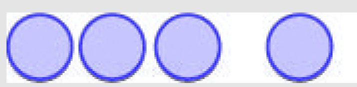
b.

Step 1: **Take 1 negative from the one added neutral pair.**



1.3.40

$3-(-1)$

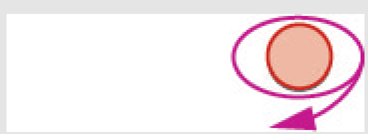


Figure 1.3.41

4

Try It

25) Subtract:

a. $-6-4$
b. $6-(-4)$

Solution

a. -10
b. 10

26) Subtract:

a. $-7-4$
b. $7-(-4)$

Solution

a. -11
b. 11

Have you noticed that *subtraction of signed numbers can be done by adding the opposite*? In example 1.3.13 (above), $-3-1$ is the same as $-3+(-1)$ and $3-(-1)$ is the same as $3+1$. You will often see this idea, the subtraction property, written as follows:

Subtraction Property

$$a-b=a+(-b)$$

Subtracting a number is the same as adding its opposite.

Look at these two examples.

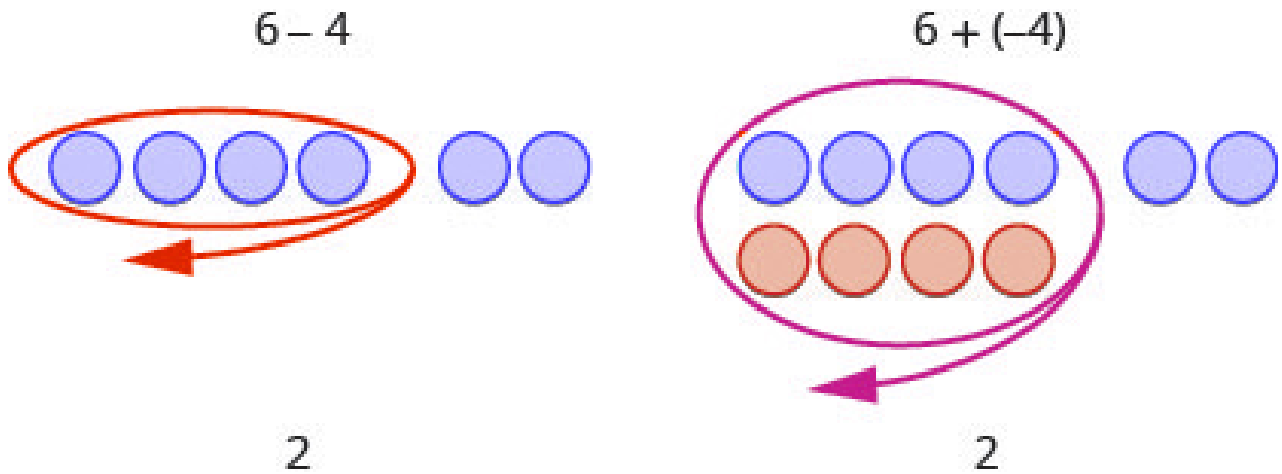


Figure 1.3.42

$6-4$ gives the same answer as $6+(-4)$.

Of course, when you have a subtraction problem that has only positive numbers, like $6-4$, you just do the subtraction. You already knew how to subtract $6-4$ long ago. But *knowing* that $6-4$ gives the same answer as $6+(-4)$ helps when you are subtracting negative numbers. Make sure that you understand how $6-4$ and $6+(-4)$ give the same results!

Example 1.3.14

Simplify:

a. $13-8$ and $13+(-8)$
b. $-17-9$ and $-17+(-9)$

Solution

a.

***Step 1:* Subtract.**

$$13 - 8 = 5$$
$$13 + (-8) = 5$$

---

b.

Step 1: **Subtract.**

$$-17 - 9 = -26$$
$$-17 + (-9) = -26$$

Try It

27) Simplify:

a. $21 - 13$ and $21 + (-13)$
b. $-11 - 7$ and $-11 + (-7)$.

Solution

a. 8
b. -18

28) Simplify:

a. $15 - 7$ and $15 + (-7)$
b. $-14 - 8$ and $-14 + (-8)$.

Solution

a. 8
b. -22

Look at what happens when we subtract a negative.

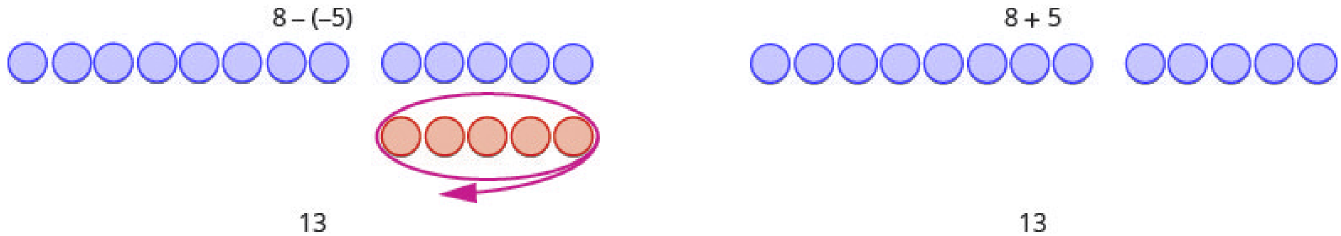


Figure 1.3.43

$8-(-5)$ gives he same answer as $8+5$

Subtracting a negative number is like adding a positive!

You will often see this written as $a-b=a+b$.

Does that work for other numbers, too? Let's do the following example and see.

Example 1.3.15

Simplify:

a. $9-(-15)$ and $9+15$
b. $-7-(-4)$ and $-7+4$

Solution

a.

***Step 1:* Subtract.**

$$9-(-15)=24$$
$$9+15=24$$

---

b.

***Step 1:* Subtract.**

$$-7-(-4)=-3$$
$$-7+4=-3$$

Try It

29) Simplify

a. $6-(-13)$ and $6+13$
b. $-5-(-1)$ and $-5+1$

Solution

a. 19
b. -4

30) Simplify:

a. $4-(-19)$ and $4+19$
b. $-4-(-7)$ and $-4+7$

Solution

a. 23
b. 3

Let's look again at the results of subtracting the different combinations of 5,−5 and 3,−3.

Subtraction of Integers

$5 - 3$ 2 5 positives take 3 positives 2 positives	$-5 - (-3)$ -2 5 negatives take away 3 negatives 2 negatives

When there would be enough counters of the colour to take away, subtract.

$-5 - 3$ -8 5 negatives, want to take away 3 positives need neutral pairs	$5 - (-3)$ 8 5 positives, want to take away 3 negatives need neutral pairs

When there would be not enough counters of the colour to take away, add.

What happens when there are more than three integers? We just use the order of operations as usual.

Example 1.3.16

Simplify: $7 - (-4 - 3) - 9$

Solution

***Step 1:* Simplify inside the parentheses first.**

$$7 - (-7) - 9$$

***Step 2:* Subtract left to right**

$$14 - 9$$

***Step 3:* Subtract**

$$5$$

Try It

31) Simplify: $8 - (-3 - 1) - 9$

Solution

3

32) Simplify: $12 - (-9 - 6) - 14$

Solution

13

Multiply Integers

Since multiplication is mathematical shorthand for repeated addition, our model can easily be applied to show multiplication of integers. Let's look at this concrete model to see what patterns we notice. We will use the same examples that we used for addition and subtraction. Here, we will use the model just to help us discover the pattern.

We remember that $a \times b$ means add a, b times. Here, we are using the model just to help us discover the pattern.

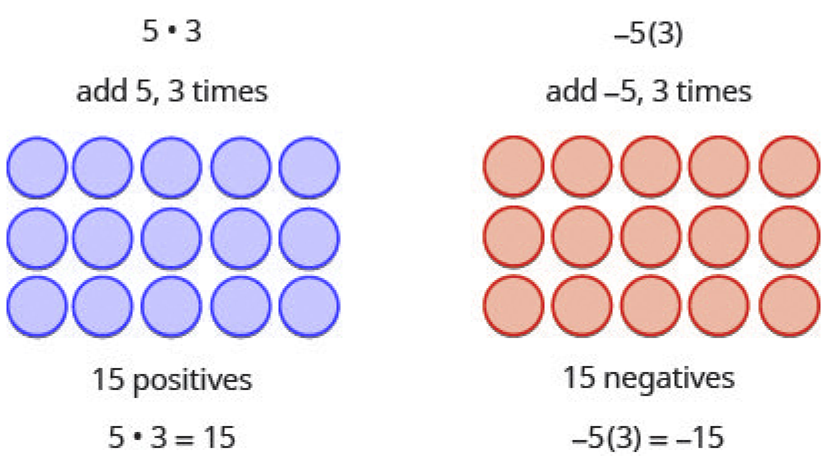


Figure 1.3.44

The next two examples are more interesting.

What does it mean to multiply 5 by −3? It means subtract 5, 3 times. Looking at subtraction as "taking away," it means to take away 5, 3 times. But there is nothing to take away, so we start by adding neutral pairs on the workspace. Then we take away 5 three times.

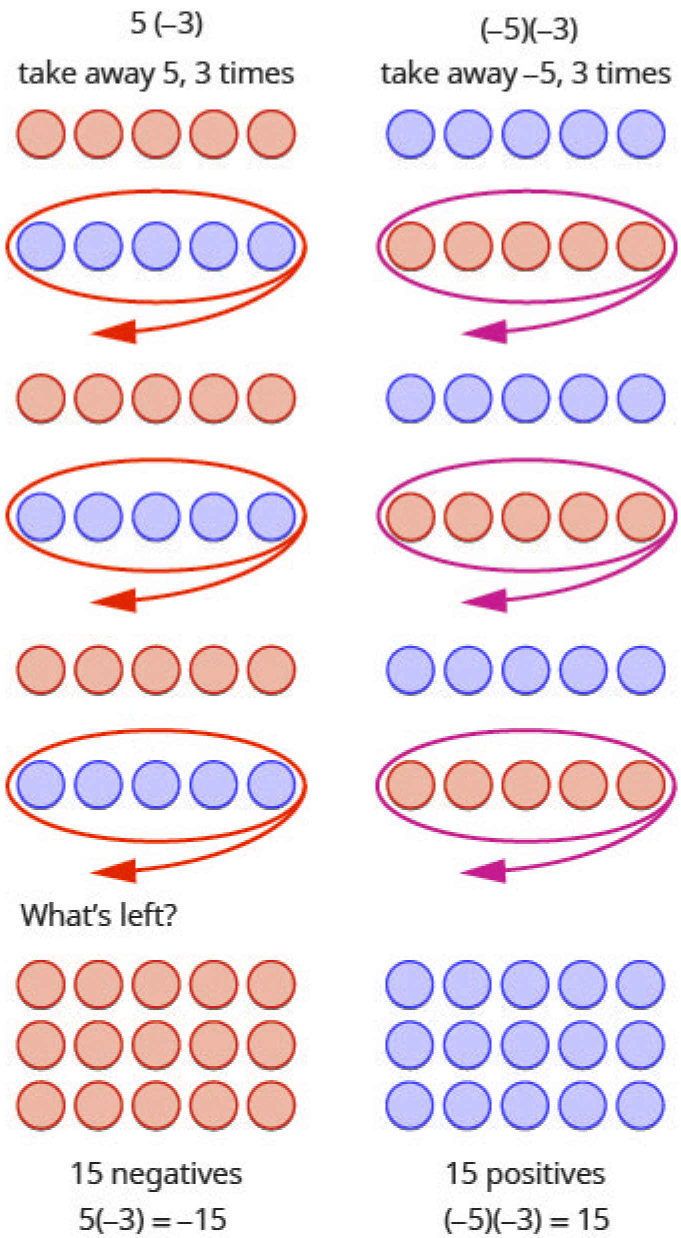


Figure 1.3.45

In summary:

$5 \times 3 = 15$	$-5 \times 3 = -15$
$5 \times -3 = -15$	$-5 \times -3 = 15$

Notice that for multiplication of two signed numbers, when the:

- signs are the *same*, the product is *positive*.
- signs are *different*, the product is *negative*.

We'll put this all together in the chart below.

Multiplication of Signed Numbers

For multiplication of two signed numbers:

Same signs	**Product**	**Example**
Two positives Two negatives	Positive Positive	$7 \times 4 = 28$ $-8 \times -6 = 48$

If the signs are the same, the result is positive.

Different signs	**Product**	**Example**
Positive and negative Negative and positive	Negative Negative	$7 \times -9 = -63$ $-5 \times 10 = -50$

If the signs are different, the result is negative.

Example 1.3.17

a. 9×3
b. $-2(-5)$

c. $4(-8)$
d. 7×6

Solution

a.

Step 1: **Multiply, noting that the signs are different so the product is negative.**

$$9 \times 3 = -27$$

---

b.

Step 1: **Multiply, noting that the signs are the same so the product is positive.**

$$-2(-5) = 10$$

---

c.

Step 1: **Multiply, with different signs.**

$$4(-8) = -32$$

---

d.

Step 1: **Multiply, with same signs.**

$$7 \times 6 = 42$$

Try It

33) Multiply:

a. -6×8
b. $-4(-7)$
c. $9(-7)$
d. 5×12

Solution

a. -48
b. 28
c. -63
d. 60

34) Multiply:

a. -8×7
b. $-6(-9)$
c. $7(-4)$
d. 3×13

Solution

a. -56
b. 54
c. -28
d. 39

When we multiply a number by 1, the result is the same number. What happens when we multiply a number by -1. Let's multiply a positive number and then a negative number by -1 to see what we get.

Each time we multiply a number by -1, we get its opposite!

Multiplication by -1

$$-1a = -a$$

Multiplying a number by -1 gives its opposite.

Example 1.3.18

Multiply:

a. -1×7
b. $-1(-11)$

Solution

a.

***Step 1:* Multiply, noting that the signs are different so the product is negative.**

$$-1 \times 7 = -7$$

-7 is the opposite of 7.

---

b.

***Step 1:* Multiply, noting that the signs are the same so the product is positive.**

$$-1(-11) = 11$$

11 is the opposite of -11.

Try It

35) Multiply:

a. -1×9
b. $-1(-17)$

Solution

a. -9
b. 17

36) Multiply:

a. -1×8
b. $-1(-16)$

Solution

a. -8
b. 16

Divide Integers

What about division? Division is the inverse operation of multiplication. So, $15 \div 3 = 5$ because $5 \times 3 = 15$ In words, this expression says that 15 can be divided into three groups of five each because adding five three times gives 15. Look at some examples of multiplying integers, to figure out the rules for dividing integers.

Division follows the same rules as multiplication!

For division of two signed numbers, when the:

- signs are the *same*, the quotient is *positive*.
- signs are *different*, the quotient is *negative*.

And remember that we can always check the answer of a division problem by multiplying.

Multiplication and Division of Signed Numbers

For multiplication and division of two signed numbers:

- If the signs are the same, the result is positive.
- If the signs are different, the result is negative.

Same signs	Result
Two positives Two negatives	Positive Positive
If the signs are the same, the result is positive.	

Different signs	Result
Positive and negative Negative and positive	Negative Negative
If the signs are different, the result is negative.	

Example 1.3.19

Divide:

a. $(-27) \div 3$
b. $-100 \div (-4)$

Solution

a.

Step 1: **Divide. With different signs, the quotient is negative.**

$$-27 \div 3 = -9$$

b.

Step 1: **Divide. With signs that are the same, the quotient is positive.**

$$-100 \div (-4) = 25$$

Try It

37) Divide:

a. $-42 \div 6$
b. $-117 \div -3$

Solution

a. -7
b. 39

38) Divide:

a. $-63 \div 7$
b. $-115 \div -5$

Solution

a. -9
b. 23

Simplify Expressions with Integers

What happens when there are more than two numbers in an expression? The order of operations still applies when negatives are included. Remember My Dear Aunt Sally?

Let's try some examples. We'll simplify expressions that use all four operations with integers—addition, subtraction, multiplication, and division. Remember to follow the order of operations.

Example 1.3.20

Simplify: $7(-2)+4(-7)-6$.

Solution

***Step 1:* Multiply first.**

$$-14+(-28)-6$$

***Step 2:* Add.**

$$-42-6$$

***Step 3:* Subtract.**

$$-48$$

Try It

39) Simplify: $8(-3)+5(-7)-4$

Solution

-63

40) Simplify: $9(-3)+7(-8)-1$

Solution

-84

Example 1.3.21

Simplify:

a. $(-2)^4$
b. -2^4

Solution

a.

Step 1: **Write in expanded from.**

$$(-2)(-2)(-2)(-2)$$

Step 2: **Multiply.**

$$4(-2)(-2)$$

Step 3: **Multiply.**

$$-8(-2)$$

Step 4: **Multiply.**

$$16$$

---

b.

Step 1: **Write in expanded form. We are asked to find the opposite of 2^4.**

$$-(2 \times 2 \times 2 \times 2)$$

Step 2: **Multiply.**

$$-(4 \times 2 \times 2)$$

Step 3: **Multiply.**

$$-(8 \times 2) = -16$$

Notice the difference in parts a and b. In part a, the exponent means to raise what is in the parentheses, the (−2) to the 4th power. In part b, the exponent means to raise just the 2 to the 4th power and then take the opposite.

Try It

41) Simplify:

a. $(-3)^4$
b. -3^4

Solution

a. 81
b. −81

42) Simplify:

a. $(-7)^2$
b. -7^2

Solution

a. 49
b. −49

The next example reminds us to simplify inside parentheses first.

Example 1.3.22

Simplify: $12 - 3(9 - 12)$

Solution

Step 1: **Subtract in parentheses first.**

$$12 - 3(-3)$$

Step 2: **Multiply.**

$$12 - (-9)$$

Step 3: **Subtract.**

$$21$$

Try It

43) Simplify: $17 - 4(8 - 11)$

Solution

29

44) Simplify: $16 - 6(7 - 13)$

Solution

52

Example 1.3.23

Simplify: $8(-9) \div (-2)^3$

Solution

Step 1: **Exponents first.**

$$8(-9) \div (-8)$$

Step 2: **Multiply**

$$-72 \div -8$$

Step 3: **Divide**.

$$9$$

Try It

45) Simplify: $12(-9) \div (-3)^3$

Solution

4

46) Simplify: $18(-4) \div (-2)^3$

Solution

9

Example 1.3.24

Simplify: $-30 \div +2 + (-3)(-7)$

Solution

Step 1: **Multiply and divide left to right, so divide first.**

$$-15 + (-3)(-7)$$

Step 2: **Add.**

$$-15 + 21 = 6$$

Try It

47) Simplify $-27 \div 3 + (-5)(-6)$

Solution

21

48) Simplify $-32 \div 4 + (-2)(-7)$

Solution

6

Evaluate Variable Expressions with Integers

Remember that to evaluate an expression means to substitute a number for the variable in the expression. Now we can use negative numbers as well as positive numbers.

Example 1.3.25

When $n = -5$, evaluate:

a. $n + 1$
b. $-n + 1$

Solution

a.

Step 1: **Substitute -5 for n.**

$$-5 + 1$$

Step 2: **Simplify.**

$$-4$$

---

b.

Step 1: **Substitute -5 for n.**

$$-(-5) + 1$$

Step 2: **Simplify.**

5+1

Step 3: **Add.**

$$6$$

Try It

49) When $n = -8$, evaluate

a. $n + 2$
b. $-n + 2$

Solution

a. -6
b. 10

50) When $y = -9$, evaluate

a. $y + 8$
b. $-y + 8$.

Solution

a. -1
b. 17

Example 1.3.26

Evaluate $(x + y)^2$ when $x = -18$ and $y = 24$.

Solution
***Step 1:* Substitute -18 for x and 24 for y.**

$$(-18 + 24)^2$$

Step 2: **Add inside parentheses.**

$$(6)^2$$

Step 3: **Simplify.**

$$36$$

Try It

51) Evaluate $(x+y)^2$ when $x=-15$ and $y=29$

Solution

196

52) Evaluate $(x+y)^3$ when $x=-8$ and $y=10$

Solution

8

Example 1.3.27

Evaluate $20-z$ when

a. $z=12$
b. $z=-12$

Solution

a.

Step 1: **Substitute 12 for z.**

$$20 - 12$$

Step 2: **Subtract.**

$$8$$

---

b.

Step 1: **Substitute -12 for z.**

$$20 - (-12)$$

Step 2: **Subtract.**

$$32$$

Try It

53) Evaluate: $17 - k$ when

a. $k = 19$
b. $k = -19$

Solution

a. -2
b. 36

54) Evaluate: $-5 - b$ when

a. $b = 14$

b. $b = -14$

Solution

a. -19
b. 9

Example 1.3.28

Evaluate: $2x^2 + 3x + 8$ when $x = 4$.

Solution
Substitute 4 for x. Use parentheses to show multiplication.

Step 1: **Substitute.**

$$2(4)^2 + 3(4) + 8$$

Step 2: **Evaluate exponents.**

$$2(16) + 3(4) + 8$$

Step 3: **Multiply.**

$$32 + 12 + 8$$

Step 4: **Add.**

$$52$$

Try It

55) Evaluate: $3x^2 - 2x + 6$ when $x = -3$.

Solution

39

56) Evaluate: $4x^2 - x - 5$ when $x = -2$.

Solution

13

Translate Phrases to Expressions with Integers

Our earlier work translating English to algebra also applies to phrases that include both positive and negative numbers.

Example 1.3.29

Translate and simplify: the sum of 8 and -12, increased by 3.

Solution

Step 1: Translate.

$$[8 + (-12)] + 3$$

***Step 2:* Simplify. Be careful not to confuse the brackets with an absolute value sign.**

$$(-4) + 3$$

***Step 3:* Add.**

$$-1$$

Try It

57) Translate and simplify the sum of 9 and -16, increased by 4.

Solution

$$(9-16)+4=3$$

58) Translate and simplify the sum of -8 and -12, increased by 7.

Solution

$$(-8-12)+7=-13$$

When we first introduced the operation symbols, we saw that the expression may be read in several ways. They are listed below.

- $a-b$
- a minus b
- the difference of a and b
- b subtracted from a
- b less than a

Be careful to get a and b in the right order!

Example 1.3.30

Translate and then simplify

a. The difference of 13 and -21
b. Subtract 24 from -19

Solution

a.

***Step 1:* Translate.**

$$13-(-21)$$

***Step 2:* Simplify.**

$$34$$

---

b.

***Step 1:* Translate. Remember, "subtract b from a" means $a-b$.**

$$-19-24$$

***Step 2:* Simplify.**

$$-43$$

Try It

59) Translate and simplify

a. the difference of 14 and -23
b. subtract 21 from -17

Solution

a. $14-(-23)=37$
b. $-17-21=-38$

60) Translate and simplify

a. the difference of 11 and -19
b. subtract 18 from -11.

Solution

a. $11-(-19)=30$
b. $-11-18=-29$

Once again, our prior work translating English to algebra transfers to phrases that include both multiplying and dividing integers. Remember that the keyword for multiplication is "product" and for division is "quotient."

Example 1.3.31

Translate to an algebraic expression and simplify if possible: the product of -2 and 14.

Solution
***Step 1:* Translate.**

$$(-2)(14)$$

***Step 2:* Simplify.**

$$-28$$

Try It

61) Translate to an algebraic expression and simplify if possible: the product of -5 and 12.

Solution

$-5(12) = -60$

62) Translate to an algebraic expression and simplify if possible: the product of 8 and -13.

Solution

$8(-13) = -104$

Example 1.3.32

Translate to an algebraic expression and simplify if possible: the quotient of -56 and -7.

Solution

***Step 1:* Translate.**

$$-56 \div -7$$

***Step 2:* Simplify.**

$$8$$

Try It

63) Translate to an algebraic expression and simplify if possible: the quotient of -63 and -9.

Solution

$-63 \div -9 = 7$

64) Translate to an algebraic expression and simplify if possible: the quotient of -72 and -9.

Solution

$-72 \div -9 = 8$

Use Integers in Applications

We'll outline a plan to solve applications. It's hard to find something if we don't know what we're looking for or what to call it! So when we solve an application, we first need to determine what the problem is asking us to find. Then we'll write a phrase that gives the information to find it. We'll translate the phrase into an expression and then simplify the expression to get the answer. Finally, we summarize the answer in a sentence to make sure it makes sense.

How to Apply a Strategy to Solve Applications with Integers

Example 1.3.33

The temperature in Urbana, Illinois one morning was 11 degrees. By mid-afternoon, the temperature had dropped to -9 degrees. What was the difference of the morning and afternoon temperatures?

Solution

***Step 1:* Read the problem. Make sure all the words and ideas are understood.**

***Step 2:* Identify what we are asked to find.**

The difference of the morning and afternoon temperatures.

***Step 3:* Write a phrase that gives the information to find it.**

The *difference of* 11 and -9.

***Step 4:* Translate the phrase to an expression.**

$$11-(-9)$$

***Step 5:* Simplify the expression.**

$$20$$

***Step 6:* Write a complete sentence that answers the question.**

The difference in temperatures was 20 degrees.

Try It

65) The temperature in Anchorage, Alaska one morning was 15 degrees. By mid-afternoon the temperature had dropped to 30 degrees below zero. What was the difference in the morning and afternoon temperatures?

Solution

The difference in temperatures was 45 degrees.

66) The temperature in Denver was -6 degrees at lunchtime. By sunset the temperature had dropped to -15 degrees. What was the difference in the lunchtime and sunset temperatures?

Solution

The difference in temperatures was 9 degrees.

HOW TO

Apply a Strategy to Solve Applications with Integers.

1. Read the problem. Make sure all the words and ideas are understood
2. Identify what we are asked to find.
3. Write a phrase that gives the information to find it.
4. Translate the phrase to an expression.
5. Simplify the expression.
6. Answer the question with a complete sentence.

Example 1.3.34

The Mustangs football team received three penalties in the third quarter. Each penalty gave them a loss of fifteen yards. What is the number of yards lost?

Solution

Step 1: **Read the problem. Make sure all the words and ideas are understood.**

Step 2**: Identify what we are asked to find.**

The number of yards lost.

Step 3**: Write a phrase that gives the information to find it.**

Three times a 15-yard penalty.

Step 4**: Translate the phrase to an expression.**

$$3(-15)$$

Step 5**: Simplify the expression.**

$$-45$$

Step 6**: Answer the question with a complete sentence.**

The team lost 45 yards.

Try It

67) The Bears played poorly and had seven penalties in the game. Each penalty resulted in a loss of 15 yards. What is the number of yards lost due to penalties?

Solution
The Bears lost 105 yards.

68) Bill uses the ATM on campus because it is convenient. However, each time he uses it he is charged a $2 fee. Last month he used the ATM eight times. How much was his total fee for using the ATM?

Solution
A $16 fee was deducted from his checking account.

Key Concepts

- **Multiplication and Division of Two Signed Numbers**
    - Same signs—Product is positive
    - Different signs—Product is negative
- **Strategy for Applications**

- Identify what you are asked to find.
- Write a phrase that gives the information to find it.
- Translate the phrase to an expression.
- Simplify the expression.
- Answer the question with a complete sentence

- **Addition of Positive and Negative Integers**

$5+3=8$ both positive, sum positive	$-5+(-3)=-8$ both negative, sum negative
$-5+3=-2$ different signs, more negatives sum negative	$5+(-3)=2$ different signs, more positives sum positive

- **Property of Absolute Value**: $|n| \geq 0$ for all numbers. Absolute values are always greater than or equal to zero!
- **Subtraction of Integers**

$5-3=2$ 5 positives take away 3 positives 2 positive	$-5-(-3)=-2$ 5 negatives, take away 3 negatives 2 negatives
$-5-3=-8$ 5 negatives, want to subtract 3 positives need neutral pairs	$5-(-3)=2$ 5 positives, want to subtract 3 negatives need neutral pairs

- **Subtraction Property:** Subtracting a number is the same as adding its opposite.
- **Multiplication and Division of Two Signed Numbers**
  - Same signs—Product is positive
  - Different signs—Product is negative
- **Strategy for Applications**
  1. Identify what you are asked to find.
  2. Write a phrase that gives the information to find it.
  3. Translate the phrase to an expression.

4. Simplify the expression.
5. Answer the question with a complete sentence.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=178#h5p-3

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

absolute value
The absolute value of a number is its distance from 0 on the number line. The absolute value of a number n is written as $|n|$.

integers
The whole numbers and their opposites are called the integers: ...−3, −2, −1, 0, 1, 2, 3...

opposite
The opposite of a number is the number that is the same distance from zero on the number

line but on the opposite side of zero: $-a$ means the opposite of the number a. The notation $-a$ is read as “the opposite of a.

1.4 FRACTIONS

Learning Objectives

By the end of this section, you will be able to:

- Find equivalent fractions
- Simplify fractions
- Multiply fractions
- Divide fractions
- Simplify expressions written with a fraction bar
- Translate phrases to expressions with fractions
- Add or subtract fractions with a common denominator
- Add or subtract fractions with different denominators
- Use the order of operations to simplify complex fractions
- Evaluate variable expressions with fractions

Find Equivalent Fractions

Fractions are a way to represent parts of a whole. The fraction $\frac{1}{3}$ means that one whole has been divided into 3 equal parts and each part is one of the three equal parts. See (Figure 1.4.1). The fraction $\frac{2}{3}$ represents two of three equal parts. In the fraction $\frac{2}{3}$ the 2 is called the **numerator** and the 3 is called the **denominator**.

The circle on the left has been divided into 3 equal parts. Each part is $\frac{1}{3}$ of the 3 equal parts. In the circle on the right, $\frac{2}{3}$ of the circle is shaded (2 of the 3 equal parts).

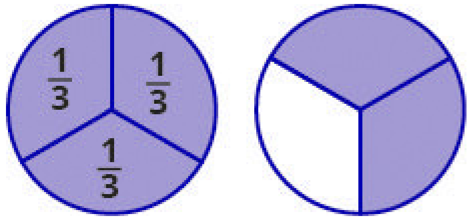


Figure 1.4.1. The circle on the left has been divided into 3 equal parts. Each part is 13 of the 3 equal parts. In the circle on the right, 23 of the circle is shaded (2 of the 3 equal parts).

Fraction

A fraction is written $\frac{a}{b}$ where $b \neq 0$ and

- a is the *numerator* and b is the *denominator*

A fraction represents parts of a whole. The denominator b is the number of equal parts the whole has been divided into, and the numerator a indicates how many parts are included.

If a whole pie has been cut into 6 pieces and we eat all 6 pieces, we ate $\frac{6}{6}$ pieces, or, in other words, one whole pie.

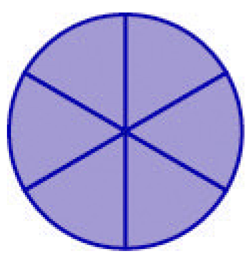


Figure 1.4.2

So $\frac{6}{6} = 1$. This leads us to the property of one that tells us that any number, except zero, divided by itself is 1.

Property of One

$\frac{a}{a} = 1 \qquad (a \neq 0)$. Any number, except zero, divided by itself is one.

If a pie was cut in 6 pieces and we ate all 6, we ate $\dfrac{6}{6}$ pieces, or, in other words, one whole pie. If the pie was cut into 8 pieces and we ate all 8, we ate $\dfrac{8}{8}$ pieces, or one whole pie. We ate the same amount—one whole pie.

The fractions $\dfrac{6}{6}$ and $\dfrac{8}{8}$ have the same value, 1, and so they are called equivalent fractions. **Equivalent fractions** are fractions that have the same value.

Let's think of pizzas this time. Figure 1.4.2 shows two images: a single pizza on the left, cut into two equal pieces, and a second pizza of the same size, cut into eight pieces on the right. This is a way to show that $\dfrac{1}{2}$ is equivalent to $\dfrac{4}{8}$. In other words, they are equivalent fractions.

Since the same amount is of each pizza is shaded, we see that $\dfrac{1}{2}$ is equivalent to $\dfrac{4}{8}$. They are equivalent fractions.

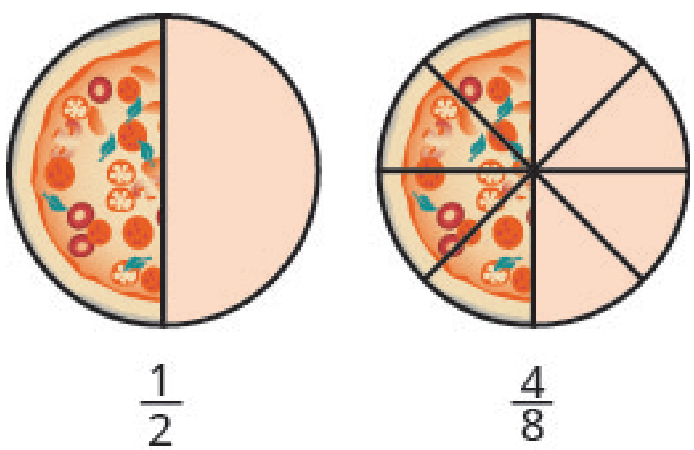


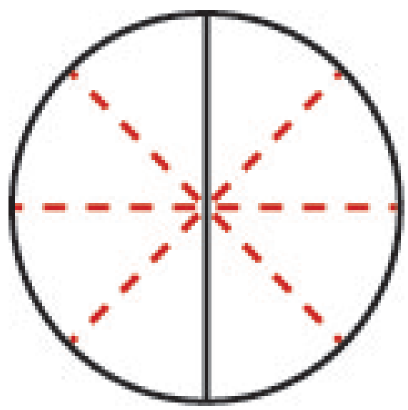
Figure 1.4.3. Since the same amount is of each pizza is shaded, we see that [latex]\frac{1}{2}[/latex] is equivalent to [latex]\frac{4}{8}[/latex]. They are equivalent fractions.

Equivalent Fractions

Equivalent fractions are fractions that have the same value.

How can we use mathematics to change $\dfrac{1}{2}$ into $\dfrac{4}{8}$? How could we take a pizza that is cut into 2 pieces and cut it into 8 pieces? We could cut each of the 2 larger pieces into 4 smaller pieces! The whole pizza would then be cut into 8 pieces instead of just 2. Mathematically, what we've described could be written like this as $\dfrac{1\times4}{2\times4}=\dfrac{4}{8}$. See Figure. 1.4.3.

Cutting each half of the pizza into 4 pieces, gives us pizza cut into 8 pieces: $\dfrac{1\times4}{2\times4}=\dfrac{4}{8}$.



1.4.4. Cutting each half of the pizza into 4 pieces, gives us pizza cut into 8 pieces.

This model leads to the following property:

Equivalent Fractions Property

If a, b, c are numbers where $b\neq0, c\neq0$, then

$$\frac{a}{b} = \frac{ac}{bc} \text{ and } \frac{ac}{bc} = \frac{a}{b}$$

If we had cut the pizza differently, we could get

$$\frac{1 \cdot 2}{2 \cdot 2} = \frac{2}{4} \text{ so } \frac{1}{2} = \frac{2}{4}$$
$$\frac{1 \cdot 3}{2 \cdot 3} = \frac{3}{6} \text{ so } \frac{1}{2} = \frac{3}{6}$$
$$\frac{1 \cdot 10}{2 \cdot 10} = \frac{10}{20} \text{ so } \frac{1}{2} = \frac{10}{20}$$

So, we say $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \text{and} \frac{10}{20}$ are equivalent fractions.

Example 1.4.1

Find three fractions equivalent to $\frac{2}{5}$

Solution

To find a fraction equivalent to $\frac{2}{5}$, we multiply the numerator and denominator by the same number. We can choose any number, except for zero. Let's multiply them by 2, 3, and then 5.

$$\frac{2 \times 2}{5 \times 2} = \frac{4}{10} \qquad \frac{2 \times 3}{5 \times 3} = \frac{6}{15} \qquad \frac{2 \times 5}{5 \times 5} = \frac{10}{25}$$

So, $\frac{4}{10}, \frac{6}{15}, \text{and} \frac{10}{25}$ are equivalent to $\frac{2}{5}$.

Try It

1) Find three fractions equivalent to $\frac{3}{5}$

Solution

$\frac{6}{10}, \frac{9}{15}, \frac{12}{20}$; answers may vary

2) Find three fractions equivalent to $\frac{4}{5}$

Solution

$\frac{8}{10}, \frac{12}{15}, \frac{16}{20}$; answers may vary

Simplify Fractions

A fraction is considered *simplified* if there are no common factors, other than 1, in its numerator and denominator.

For example,

- $\frac{2}{3}$ is simplified because there are no common factors of 2 and 3.
- $\frac{10}{15}$ is not simplified because 5 is a common factor of 10 and 15.

Simplified Fraction

A fraction is considered simplified if there are no common factors in its numerator and denominator.

The phrase *reduce a fraction* means to simplify the fraction. We simplify, or reduce, a fraction by removing the common factors of the numerator and denominator. A fraction is not simplified until all common factors have been removed. If an expression has fractions, it is not completely simplified until the fractions are simplified.

In Example 1.4.2, we used the equivalent fractions property to find equivalent fractions. Now we'll use the equivalent fractions property in reverse to simplify fractions. We can rewrite the property to show both forms together.

Equivalent Fractions Property

If a, b, c are numbers where $b \neq 0, c \neq 0$, then $\frac{a}{b} = \frac{ac}{bc}$ and $\frac{ac}{bc} = \frac{a}{b}$

Example 1.4.2

Simplify: $-\frac{32}{56}$.

Solution

Step 1: **Rewrite the numerator and denominator showing the common factors.**

$$-\frac{4 \times 8}{7 \times 8}$$

Step 2: **Simplify using the equivalent fractions property.**

$$-\frac{4}{7}$$

Notice that the fraction $-\frac{4}{7}$ is simplified because there are no more common factors.

Try It

3) Simplify: $-\frac{42}{54}$

Solution

$-\frac{7}{9}$

4) Simplify: $-\frac{45}{81}$

Solution

$-\frac{5}{9}$

Sometimes it may not be easy to find common factors of the numerator and denominator. When this happens, a good idea is to factor the numerator and the denominator into prime numbers. Then divide out the common factors using the equivalent fractions property.

Example 1.4.3

Simplify: $-\frac{210}{385}$

Solution

Step 1: **Rewrite the numerator and denominator to show the common factors. If needed, factor the numerator and denominator into prime numbers first.**

Rewrite 210 and 385 as the product of the primes.

$$= -\frac{210}{385}$$

$$= -\frac{2 \times 3 \times 5 \times 7}{5 \times 7 \times 11}$$

Step 2: **Simplify using the equivalent fractions property by dividing out common factors.**

Mark the common factors 5 and 7. Divide out the common factors.

$$= -\frac{2 \times 3 \times \cancel{5} \times \cancel{7}}{\cancel{5} \times \cancel{7} \times 11}$$

$$= -\frac{2 \times 3}{11}$$

Step 3: **Multiply the remaining factors, if necessary.**

$$-\frac{6}{11}$$

Try It

5) Simplify: $-\frac{69}{120}$

Solution

$-\frac{23}{40}$

6) Simplify: $-\frac{120}{192}$

Solution

$-\frac{5}{8}$

We now summarize the steps you should follow to simplify fractions.

HOW TO

Simplify a Fraction.

1. Rewrite the numerator and denominator to show the common factors.
   If needed, factor the numerator and denominator into prime numbers first.
2. Simplify using the equivalent fractions property by dividing out common factors.
3. Multiply any remaining factors, if needed.

Example 1.4.4

Simplify: $\dfrac{5x}{5y}$.

Solution

Step 1: Rewrite showing the common factors, then divide out the common factors.

$$\frac{\cancel{5} \times x}{\cancel{5} \times y}$$

***Step 2:* Simplify.**

$$\frac{x}{y}$$

Try It

7) Simplify: $\frac{7x}{7y}$

Solution

$\frac{x}{y}$

8)Simplify: $\frac{3a}{3b}$

Solution

$\frac{a}{b}$

Multiply Fractions

Many people find multiplying and dividing fractions easier than adding and subtracting fractions. So we will start with fraction multiplication.

We'll use a model to show you how to multiply two fractions and to help you remember the procedure. Let's start with $\frac{3}{4}$.

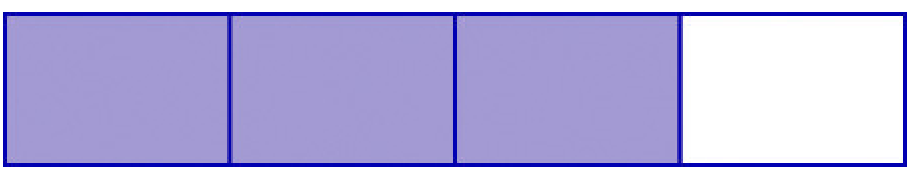


Figure 1.4.5

Now we'll take $\frac{1}{2}$ of $\frac{3}{4}$.

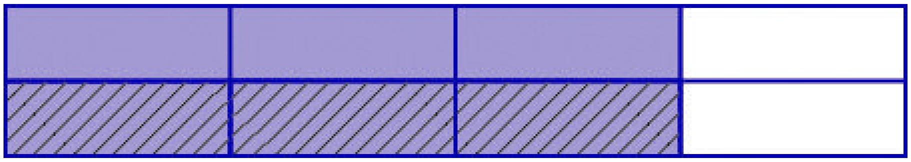


Figure 1.4.6

Notice that now, the whole is divided into 8 equal parts. So $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$

To multiply fractions, we multiply the numerators and multiply the denominators.

Fraction Multiplication

If $a, b, c \text{ and } d$ are numbers where $b \neq 0 \text{ and } d \neq 0$, then $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

To multiply fractions, multiply the numerators and multiply the denominators.

When multiplying fractions, the properties of positive and negative numbers still apply, of course. It is a good idea to determine the sign of the product as the first step. In Example 1.4.5, we will multiply negative and positive, so the product will be negative.

Example 1.4.5

Multiply: $-\frac{11}{12} \times \frac{5}{7}$

Solution

The first step is to find the sign of the product. Since the signs are the different, the product is negative.

***Step 1:* Determine the sign of the product; multiply.**

$$-\frac{11 \times 5}{12 \times 7}$$

Step 2: **Are there any common factors in the numerator and the denominator? No.**

$$-\frac{55}{84}$$

Try It

9) Multiply: $-\frac{10}{28} \times \frac{8}{15}$

Solution

$-\frac{4}{21}$

10) Multiply: $-\frac{9}{20} \times \frac{5}{12}$

Solution

$-\frac{3}{16}$

When multiplying a fraction by an integer, it may be helpful to write the integer as a fraction. Any integer, a, can be written as $\frac{a}{1}$. So, for example, $3 = \frac{3}{1}$.

Example 1.4.6

Multiply: $-\frac{12}{5}(-20x)$

Solution

Determine the sign of the product. The signs are the same, so the product is positive.

Step 1: **Write $20x$ as a fraction.**

$$\frac{12}{5}\left(\frac{20x}{1}\right)$$

Step 2: **Multiply.**

Step 3: **Rewrite 20 to show the common factor 5 and divide it out.**

$$\frac{12 \times 4 \times \cancel{5}x}{\cancel{5} \times 1}$$

Step 4: **Simplify.**

$$48x$$

Try It

11) Multiply: $\frac{11}{3}(-9a)$

Solution

$-33a$

12) Multiply: $\frac{13}{7}(-14b)$

Solution

$-2b$

Divide Fractions

Now that we know how to multiply fractions, we are almost ready to divide. Before we can do that, that we need some vocabulary.

The **reciprocal** of a fraction is found by inverting the fraction, placing the numerator in the denominator and the denominator in the numerator. The reciprocal of $\frac{2}{3}$ is $\frac{3}{20}$.

Notice that $\frac{2}{3} \cdot \frac{3}{2} = 1$. A number and its reciprocal multiply to 1.

To get a product of positive 1 when multiplying two numbers, the numbers must have the same sign. So reciprocals must have the same sign.

The reciprocal of $-\frac{10}{7}$ is $-\frac{7}{10}$, since $-\frac{10}{7}\left(-\frac{7}{10}\right) = 1$.

Reciprocal

The reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

A number and its reciprocal multiply to one $\frac{a}{b} \times \frac{b}{a} = 1$

To divide fractions, we multiply the first fraction by the reciprocal of the second.

Fraction Division

If $a, b, c \text{ and } d$ are numbers where $b \neq 0, c \neq 0 \text{ and } d \neq 0$, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

To divide fractions, we multiply the first fraction by the reciprocal of the second.

We need to say $b \neq 0, c \neq 0 \text{ and } d \neq 0$ to be sure we don't divide by zero!

Example 1.4.7

Divide: $-\frac{2}{3} \div \frac{n}{5}$.

Solution

Step 1: **To divide, multiply the first fraction by the reciprocal of the second.**

$$-\frac{2}{3} \times \frac{5}{n}$$

Step 2: **Multiply.**

$$-\frac{10}{3n}$$

Try It

13) Divide: $-\frac{3}{5} \div \frac{p}{7}$

Solution

$-\frac{21}{5p}$

14) Divide: $-\frac{5}{8} \div \frac{q}{3}$

Solution

$-\frac{15}{8q}$

Example 1.4.8

Find the quotient: $-\frac{7}{8} \div \left(-\frac{14}{27}\right)$.

Solution

Step 1: **To divide, multiply the first fraction by the reciprocal of the second.**

$$-\frac{7}{18} \times \ -\frac{27}{14}$$

Step 2: **Determine the sign of the product, and then multiply.**

$$\frac{7 \times \ 27}{18 \times \ 14}$$

Step 3: **Rewrite showing common factors.**

$$\frac{\cancel{7} \times \cancel{9} \times 3}{\cancel{9} \times 2 \times \cancel{7} \times 2}$$

Step 4: **Remove common factors.**

$$\frac{3}{2 \times 2}$$

Step 5**: Simplify.**

$$\frac{3}{4}$$

Try It

15) Find the quotient: $-\frac{7}{27} \div \left(-\frac{35}{36}\right)$

Solution

$\frac{4}{15}$

16) Find the quotient: $-\frac{5}{14} \div \left(-\frac{15}{28}\right)$

Solution

$\frac{2}{3}$

There are several ways to remember which steps to take to multiply or divide fractions. One way is to repeat the call outs to yourself. If you do this each time you do an exercise, you will have the steps memorized.

- "To multiply fractions, multiply the numerators and multiply the denominators."

- "To divide fractions, multiply the first fraction by the reciprocal of the second."

Another way is to keep two examples in mind:

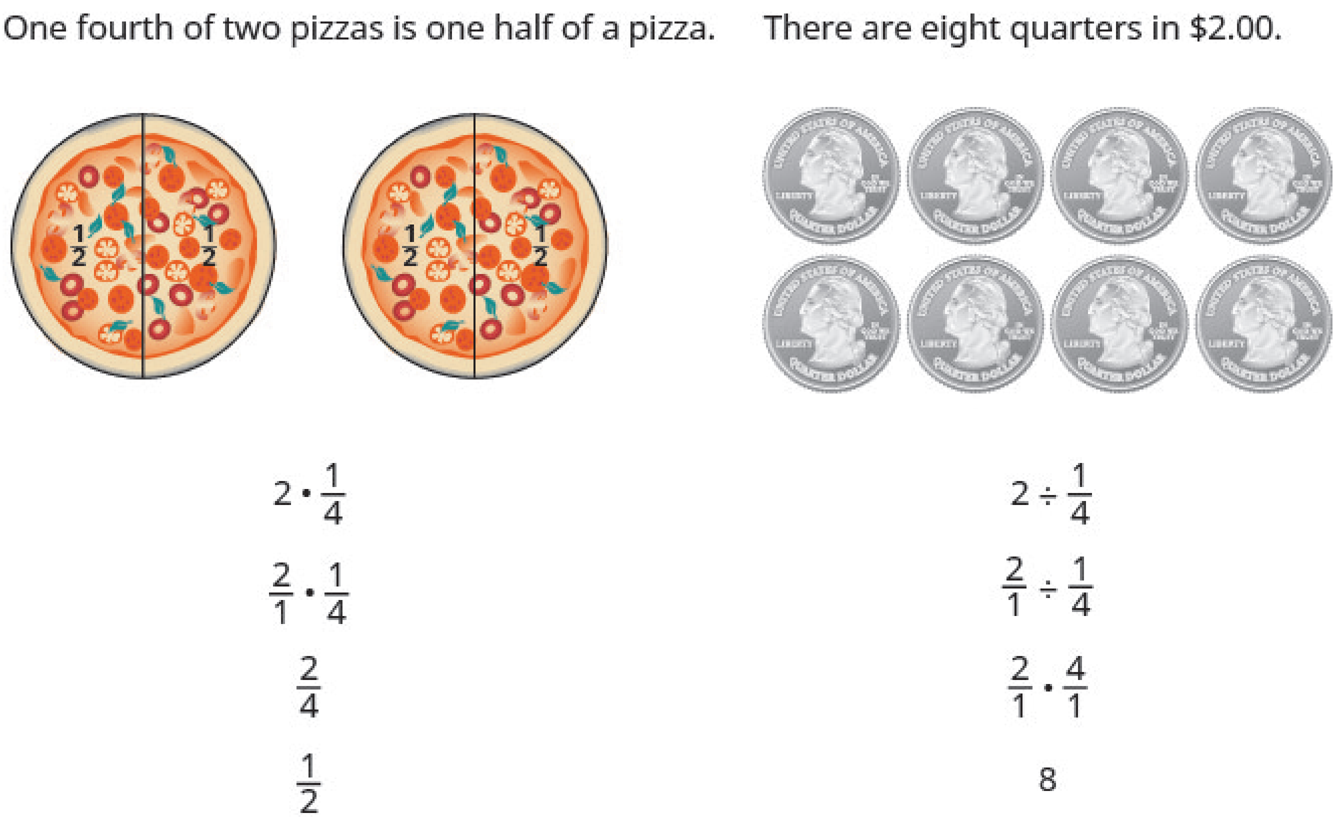


Figure 1.4.7

The numerators or denominators of some fractions contain fractions themselves. A fraction in which the numerator or the denominator is a fraction is called a **complex fraction**.

Complex Fraction

A complex fraction is a fraction in which the numerator or the denominator contains a fraction.

Some examples of complex fractions are:

$$\frac{\frac{6}{7}}{3} \qquad \frac{\frac{3}{4}}{\frac{5}{8}} \qquad \frac{\frac{x}{2}}{\frac{5}{6}}$$

To simplify a complex fraction, we remember that the fraction bar means division. For example, the complex fraction $\frac{\frac{3}{4}}{\frac{5}{8}}$ means $\frac{3}{4} \div \frac{5}{8}$.

Example 1.4.9

Simplify: $\frac{\frac{3}{4}}{\frac{5}{8}}$

Solution

Step 1: **Rewrite as division.**

$$\frac{3}{4} \div \frac{5}{8}$$

Step 2: **Multiply the first fraction by the reciprocal of the second.**

$$\frac{3}{4} \times \frac{8}{5}$$

Step 3: **Multiply.**

$$\frac{3 \times 8}{4 \times 5}$$

Step 4: **Look for common factors.**

$$\frac{3 \times \cancel{4} \times 2}{\cancel{4} \times 5}$$

Step 5: **Divide out common factors and simplify.**

$$\frac{6}{5}$$

Try It

17) Simplify: $\frac{\frac{2}{3}}{\frac{5}{6}}$

Solution

$\frac{4}{5}$

18) Simplify: $\frac{\frac{3}{7}}{\frac{6}{11}}$

Solution

$\frac{11}{14}$

Example 1.4.10

Simplify: $\frac{\frac{x}{2}}{\frac{xy}{6}}$

Solution

Step 1: **Rewrite as division.**

$$\frac{x}{2} \div \frac{xy}{6}$$

Step 2: **Multiply the first fraction by the reciprocal of the second.**

$$\frac{x}{2} \times \frac{6}{xy}$$

***Step 3:* Multiply.**

$$\frac{x \times 6}{2 \times xy}$$

***Step 4:* Look for common factors.**

$$\frac{\cancel{x} \times 3 \times \cancel{2}}{\cancel{2} \times \cancel{x} \times y}$$

***Step 5:* Divide out common factors and simplify.**

$$\frac{3}{y}$$

Try It

19) Simplify: $\frac{\frac{a}{8}}{\frac{ab}{6}}$

Solution

$\frac{3}{4b}$

20) Simplify: $\frac{\frac{p}{2}}{\frac{pq}{8}}$

Solution

$\frac{4}{2q}$

Simplify Expressions with a Fraction Bar

The line that separates the numerator from the denominator in a fraction is called a fraction bar. A fraction bar acts as grouping symbol. The order of operations then tells us to simplify the numerator and then the denominator. Then we divide.

To simplify the expression $\frac{5-3}{7+1}$, we first simplify the numerator and the denominator separately. Then we divide.

$$\begin{aligned} &= \frac{5-3}{7+1} \\ &= \frac{2}{8} \\ &= \frac{1}{4} \end{aligned}$$

HOW TO

Simplify an Expression with a Fraction Bar.

1. Simplify the expression in the numerator.
2. Simplify the expression in the denominator.
3. Simplify the fraction.

Example 1.4.11

Simplify: $\frac{4-2(3)}{2^2+2}$

Solution

***Step 1:* Use the order of operations to simplify the numerator and the denominator.**

$$\frac{4-6}{4+2}$$

Step 2: **Simplify the numerator and the denominator.**

$$\frac{-2}{6}$$

Step 3: **Simplify. A negative divided by a positive is negative.**

$$-\frac{1}{3}$$

Try It

21) Simplify: $\frac{6-3\left(5\right)}{3^2+3}$

Solution

$-\frac{3}{4}$

22) Simplify: $\frac{4-4\left(6\right)}{3^2+3}$

Solution

$-\frac{2}{3}$

Where does the negative sign go in a fraction? Usually the negative sign is in front of the fraction, but you will sometimes see a fraction with a negative numerator, or sometimes with a negative denominator. Remember that fractions represent division. When the numerator and denominator have different signs, the quotient is negative.

$$\frac{-1}{3} = -\frac{1}{3} \qquad \frac{\text{negative}}{\text{positive}} = \text{negative}$$

$$\frac{1}{-3} = -\frac{1}{3} \qquad \frac{\text{positive}}{\text{negative}} = \text{negative}$$

Placement of Negative Sign in a Fraction

For any positive numbers a and b,

$$\frac{--a}{b} = \frac{a}{--b} = -\frac{a}{b}$$

Example 1.4.12

Simplify: $\dfrac{4(-3)+6(-2)}{-3(2)-2}$

Solution

***Step 1:* Multiply.**

$$\frac{-12+(-12)}{-6-2}$$

***Step 2:* Simplify.**

$$\frac{-24}{-8}$$

***Step 3:* Divide.**

$$3$$

Try It

23) Simplify: $\frac{8(-2)+4(-3)}{-5(2)+3}$

Solution

4

24) Simplify: $\frac{7(-1)+9(-3)}{-5(3)-2}$

Solution

2

Translate Phrases to Expressions with Fractions

Now that we have done some work with fractions, we are ready to translate phrases that would result in expressions with fractions.

The English words quotient and ratio are often used to describe fractions. Remember that "quotient" means division. The quotient of a and b is the result we get from dividing a by b, or $\frac{a}{b}$.

Example 1.4.13

Translate the English phrase into an algebraic expression: the quotient of the difference of m and n, and p.

Solution

We are looking for the *quotient of* the difference of m and n, and p. This means we want to divide the difference of m and n by p.

$$\frac{m-n}{p}$$

Try It

25) Translate the English phrase into an algebraic expression: the quotient of the difference of a and b, and cd.

Solution

$$\frac{a-b}{cd}$$

26) Translate the English phrase into an algebraic expression: the quotient of the sum of p and q, and r

Solution

$$\frac{p+q}{r}$$

Add or Subtract Fractions with a Common Denominator

When we multiplied fractions, we just multiplied the numerators and multiplied the denominators right straight across. To add or subtract fractions, they must have a common denominator.

Fraction Addition and Subtraction

If $a, b, \text{ and } c$ are numbers where $c \neq 0$, then

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \quad \text{and} \quad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.

Example 1.4.14

Find the sum: $\frac{x}{3} + \frac{2}{3}$.

Solution

Step 1: **Add the numerators and place the sum over the common denominator.**

$$\frac{x+2}{3}$$

Try It

27) Find the sum: $\frac{x}{4} + \frac{3}{4}$

Solution

$$\frac{x+3}{4}$$

28) Find the sum: $\frac{y}{8}+\frac{5}{8}$

Solution

$$\frac{y+5}{8}$$

Example 1.4.15

Find the difference: $-\frac{23}{24}-\frac{13}{24}$.

Solution

Step 1: **Subtract the numerators and place the difference over the common denominator.**

$$\frac{-23-13}{24}$$

Step 2: **Simplify.**

$$\frac{-36}{24}$$

Step 3: **Simplify. Remember,** $-\frac{a}{b}=\frac{-a}{b}$**.**

$$-\frac{3}{2}$$

Try It

29) Find the difference: $-\dfrac{19}{28}-\dfrac{7}{28}$

Solution

$-\dfrac{26}{28}$

30) Find the difference: $-\dfrac{27}{32}-\dfrac{1}{32}$

Solution

$-\dfrac{7}{8}$

Example 1.4.16

Simplify: $-\dfrac{10}{x}-\dfrac{4}{x}$

Solution

Step 1: **Subtract the numerators and place the difference over the common denominator.**

$$\frac{-14}{x}$$

Step 2: **Rewrite with the sign in front of the fraction.**

$$-\frac{14}{x}$$

Try It

31) Find the difference: $-\frac{9}{x}-\frac{7}{x}$

Solution

$-\frac{16}{x}$

32) Find the difference: $-\frac{17}{a}-\frac{5}{a}$

Solution

$-\frac{22}{a}$

Now we will do an example that has both addition and subtraction.

Example 1.4.17

Simplify: $\frac{3}{8}+\left(-\frac{5}{8}\right)-\frac{1}{8}$

Solution

***Step 1:* Add and subtract fractions—do they have a common denominator? Yes.**

$$\frac{3}{8}+\left(-\frac{5}{8}\right)-\frac{1}{8}.$$

***Step 2:* Add and subtract the numerators and place the difference over the common denominator.**

$$\frac{3+(-5)-1}{8}$$

Step 3: **Simplify left to right.**

$$\frac{-2-1}{8}$$

Step 4: **Simplify**.

$$-\frac{3}{8}$$

Try It

33) Simplify: $\frac{-2}{9}+\left(-\frac{4}{9}\right)-\frac{7}{9}$

Solution

-1

34) Simplify: $\frac{5}{9}+\left(-\frac{4}{9}\right)-\frac{7}{9}$

Solution

$-\frac{2}{3}$

Add or Subtract Fractions with Different Denominators

As we have seen, to add or subtract fractions, their denominators must be the same. The **least common denominator** (LCD) of two fractions is the smallest number that can be used as a common denominator of the fractions. The LCD of the two fractions is the least common multiple (LCM) of their denominators.

Least Common Denominator

The least common denominator (LCD) of two fractions is the least common multiple (LCM) of their denominators.

After we find the least common denominator of two fractions, we convert the fractions to equivalent fractions with the LCD. Putting these steps together allows us to add and subtract fractions because their denominators will be the same!

Example 1.4.18

Add: $\frac{7}{12}+\frac{5}{18}$

Solution

***Step 1:* Do they have a common denominator?**

$$\left.\begin{array}{l} 12 = 2 \times 2 \times 3 \\ 18 = 2 \times 3 \times 3 \end{array}\right\} \text{LCD} = 2 \times 2 \times 3 \times 3 = 36$$

No – rewrite each fraction with the LCD (least common denominator).

$$\begin{aligned} &= \frac{7}{12}+\frac{5}{18} \\ &= \frac{7 \times 3}{12 \times 3}+\frac{5 \times 2}{18 \times 2} \\ &= \frac{21}{36}+\frac{10}{36} \text{ Do not simplify LCD fractions.} \end{aligned}$$

***Step 2:* Add or subtract the fractions.**

Add.

$$\frac{31}{36}$$

Step 3: **Simplify, if possible.**

Because 31 is a prime number, it has no factors in common with 36. The answer is simplified.

Try It

35) Add: $\frac{7}{12} + \frac{11}{15}$

Solution

$\frac{79}{60}$

36) Add: $\frac{13}{15} + \frac{17}{20}$

Solution

$\frac{103}{60}$

HOW TO

Add or Subtract Fractions.

1. Do they have a common denominator?
   - Yes—go to step 2.

- No—rewrite each fraction with the LCD (least common denominator). Find the LCD. Change each fraction into an equivalent fraction with the LCD as its denominator.

2. Add or subtract the fractions.
3. Simplify, if possible.

When finding the equivalent fractions needed to create the common denominators, there is a quick way to find the number we need to multiply both the numerator and denominator. This method works if we found the LCD by factoring into primes.

Look at the factors of the LCD and then at each column above those factors. The “missing” factors of each denominator are the numbers we need.

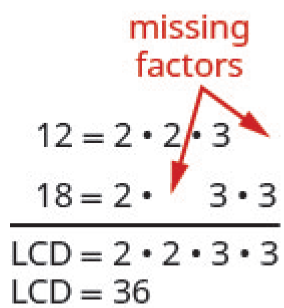


Figure 1.4.8

In the above figure the LCD, 36, has two factors of 2 and two factors of 3.

The numerator 12 has two factors of 2 but only one of 3—so it is “missing” one 3—we multiply the numerator and denominator by 3.

The numerator 18 is missing one factor of 2—so we multiply the numerator and denominator by 2.

We will apply this method as we subtract the fractions in Example 1.4.19

Example 1.4.19

Subtract: $\frac{7}{15} - \frac{19}{24}$

Solution

***Step 1:* Find the LCD.**

Do the fractions have a common denominator? No, so we need to find the LCD.

Notice, 15 is "missing" three factors of 2 and 24 is "missing" the 5 from the factors of the LCD. So we multiply 8 in the first fraction and 5 in the second fraction to get the LCD.

$$\left.\begin{array}{rcr} 15 & = & 3 \times 5 \\ 24 & = & 2 \times 2 \times 2 \times 3 \end{array}\right\} LCD = 2 \times 2 \times 2 \times 3 \times 5 = 120$$

$$= \frac{7}{15} - \frac{19}{24}$$

$$= \frac{7 \times 8}{15 \times 8} - \frac{19 \times 5}{24 \times 5}$$

***Step 2:* Rewrite as equivalent fractions with the LCD.**

$$\frac{56}{120} - \frac{95}{120}$$

***Step 3:* Subtract.**

$$\frac{39}{120}$$

***Step 4:* Check to see if the answer can be simplified.**

$$\frac{13 \times 3}{40 \times 3}$$

***Step 5:* Simplify.**

Both 39 and 120 have a factor of 3.

$$\frac{13}{40}$$

Do not simplify the equivalent fractions! If you do, you'll get back to the original fractions and lose the common denominator!

Try It

37) Subtract: $\frac{13}{24} - \frac{17}{32}$

Solution

$\frac{1}{96}$

38) Subtract: $\frac{21}{32} - \frac{9}{28}$

Solution

$\frac{75}{224}$

In the next example, one of the fractions has a variable in its numerator. Notice that we do the same steps as when both numerators are numbers.

Example 1.4.20

Add: $\frac{3}{5} + \frac{x}{8}$

Solution

The fractions have different denominators.

Step 1: **Find the LCD.**

$$\left.\begin{array}{rcr} 5 & = & 5 \\ 8 & = & 2 \times 2 \times 2 \end{array}\right\} LCD = 2 \times 2 \times 2 \times 5 = 40$$

Step 2: **Rewrite as equivalent fractions with the LCD.**

$$\frac{3 \times 8}{5 \times 8} + \frac{x \times 5}{8 \times 5}$$

Step 3: **Simplify.**

$$\frac{24}{40} + \frac{5x}{40}$$

Step 4: **Add.**

$$\frac{24 + 5x}{40}$$

Remember, we can only add like terms: 24 and $5x$ are not like terms.

Try It

39) Add: $\frac{y}{6} + \frac{7}{9}$

Solution

$$\frac{9y + 42}{54}$$

40) Add: $\frac{x}{6} + \frac{7}{15}$

Solution

$$\frac{15x + 42}{135}$$

We now have all four operations for fractions. The table below summarizes fraction operations.

Fraction Multiplication	**Fraction Division**
$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$ Multiply the numerators and multiply the denominators	Multiply
Fraction Addition	**Fraction Subtraction**
$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$ Add the numerators and place the sum over the common denominator.	Subtract the numera
To multiply or divide fractions, an LCD is NOT needed. To add or subtract fractions, an LCD is needed.	

Example 1.4.21

Simplify:

a. $\frac{5x}{6} - \frac{3}{10}$
b. $\frac{5x}{6} \times \frac{3}{10}$

Solution

First ask, "What is the operation?" Once we identify the operation that will determine whether we need a common denominator. Remember, we need a common denominator to add or subtract, but not to multiply or divide.

a. What is the operation? The operation is subtraction.

Step 1: **Do the fractions have a common denominator? No.**

$$\frac{5x}{6} - \frac{3}{10}$$

Step 2: **Rewrite each fraction as an equivalent fraction with the LCD.**

$$\frac{5x \times 5}{6 \times 5} - \frac{3 \times 3}{10 \times 3}$$

Step 3: **Subtract the numerators and place the difference over the common denominators.**

$$\frac{25x}{30} - \frac{9}{30}$$

Step 4: **Simplify, if possible There are no common factors. The fraction is simplified.**

$$\frac{25x - 9}{30}$$

---

b. What is the operation? Multiplication.

Step 1: **To multiply fractions, multiply the numerators and multiply the denominators.**

$$\frac{5x \times 3}{6 \times 10}$$

Step 2: **Rewrite, showing common factors**.

$$\frac{\cancel{5}\,x \times \cancel{3}}{2 \times \cancel{3} \times 2 \times \cancel{5}}$$

Step 3: **Remove common factors.**

Step 4: **Simplify.**

$$\frac{x}{4}$$

Notice we needed an LCD to add $\frac{5x}{6} - \frac{3}{10}$, but not to multiply $\frac{5x}{6} \times \frac{3}{10}$.

Try It

41) Simplify:

a. $\dfrac{3a}{4} - \dfrac{8}{9}$

b. $\dfrac{3a}{4} \times \dfrac{8}{9}$

Solution

a. $\dfrac{27a - 32}{36}$

b. $\dfrac{2a}{3}$

42) Simplify:

a. $\dfrac{4k}{5} - \dfrac{1}{6}$

b. $\dfrac{4k}{5} \times \dfrac{1}{6}$

Solution

a. $\dfrac{24k - 5}{30}$

b. $\dfrac{2k}{15}$

Use the Order of Operations to Simplify Complex Fractions

We have seen that a complex fraction is a fraction in which the numerator or denominator contains a fraction. The fraction bar indicates division. We simplified the complex fraction $\frac{\frac{3}{4}}{\frac{5}{8}}$ by dividing $\dfrac{3}{4}$ by $\dfrac{5}{8}$.

Now we'll look at complex fractions where the numerator or denominator contains an expression that can be simplified. So we first must completely simplify the numerator and denominator separately using the order of operations. Then we divide the numerator by the denominator.

Example 1.4.22

Simplify: $\dfrac{\left(\frac{1}{2}\right)^2}{4+3^2}$.

Solution

Step 1: Simplify the numerator.

*Remember, $\left(\dfrac{1}{2}\right)^2$ means $\dfrac{1}{2}\cdot\dfrac{1}{2}$

$$=\frac{\left(\frac{1}{2}\right)^2}{4+3^2}$$

$$=\frac{\frac{1}{4}}{4+3^2}$$

***Step 2:* Simplify the denominator.**

$$=\frac{\frac{1}{4}}{4+9}$$

$$=\frac{\frac{1}{4}}{13}$$

***Step 3:* Divide the numerator by the denominator. Simplify if possible.**

*Remember, $13=\dfrac{13}{1}$

$$= \frac{1}{4} \div 13$$
$$= \frac{1}{4} \times \frac{1}{13}$$
$$= \frac{1}{52}$$

Try It

43) Simplify: $\dfrac{\left(\frac{1}{3}\right)^2}{2^3 + 2}$

Solution

$\dfrac{1}{90}$

44) Simplify: $\dfrac{1 + 4^2}{\left(\frac{1}{4}\right)^2}$

Solution

272

Simplify Complex Fractions

1. Simplify the numerator.
2. Simplify the denominator.
3. Divide the numerator by the denominator. Simplify if possible.

Example 1.4.23

Simplify: $\dfrac{\frac{1}{2}+\frac{2}{3}}{\frac{3}{4}-\frac{1}{6}}$

Solution

It may help to put parentheses around the numerator and the denominator.

Step 1: **Simplify the numerator (LCD = 6) and simplify the denominator (LCD = 12).**

$$\frac{\left(\frac{3}{6}+\frac{4}{6}\right)}{\left(\frac{9}{12}-\frac{2}{12}\right)}$$

Step 2: **Simplify.**

$$\frac{\left(\frac{7}{6}\right)}{\left(\frac{7}{12}\right)}$$

Step 3: **Divide the numerator by the denominator.**

$$\frac{7}{6}\times\frac{12}{7}$$

Step 4: **Simplify.**

Step 5: **Divide out common factors.**

$$\frac{7\times 6\times 2}{6\times 7}$$

Step 6: **Simplify.**

$$2$$

Try It

45) Simplify: $\dfrac{\frac{1}{3}+\frac{1}{2}}{\frac{3}{4}-\frac{1}{3}}$

Solution

2

46) Simplify: $\dfrac{\frac{2}{3}-\frac{1}{2}}{\frac{1}{4}+\frac{1}{3}}$

Solution

$\dfrac{2}{7}$

Evaluate Variable Expressions with Fractions

We have evaluated expressions before, but now we can evaluate expressions with fractions. Remember, to evaluate an expression, we substitute the value of the variable into the expression and then simplify.

Example 1.4.24

Evaluate $x+\dfrac{1}{3}$ when

a. $x=-\dfrac{1}{3}$

b. $x = -\frac{3}{4}$

Solution

a. To evaluate $x + \frac{1}{3}$ when $x = -\frac{1}{3}$, substitute $-\frac{1}{3}$ for x in the expression.

Step 1: **Substitute $-\frac{1}{3}$ for x.**

$$-\frac{1}{3} + \frac{1}{3}$$

Step 2: **Simplify.**

$$0$$

---

b. To evaluate $x + \frac{1}{3}$ when $x = -\frac{3}{4}$, we substitute $-\frac{3}{4}$ for x in the expression.

Step 1: **Substitute $-\frac{3}{4}$ for x.**

$$-\frac{3}{4} + \frac{1}{3}$$

Step 2: **Rewrite as equivalent fractions with the LCD, 12.**

$$-\frac{3 \times 3}{4 \times 3} + \frac{1 \times 4}{3 \times 4}$$

Step 3:**Simplify.**

$$-\frac{9}{12} + \frac{4}{12}$$

Step 4:**Add.**

$$-\frac{5}{12}$$

Try It

47) Evaluate $x+\frac{3}{4}$ when

a. $x=-\frac{7}{4}$

b. $x=-\frac{5}{4}$

Solution

a. -1

b. $\frac{1}{2}$

48) Evaluate $y+\frac{1}{2}$ when

a. $y=\frac{2}{3}$

b. $y=-\frac{3}{4}$

Solution

a. $\frac{7}{6}$

b. $-\frac{1}{12}$

Example 1.4.25

Evaluate $-\frac{5}{6}-y$ when $y=-\frac{2}{3}$

Solution

Step 1: **Substitute $-\frac{2}{3}$ for y.**

$$-\frac{5}{6}-\left(-\frac{2}{3}\right)$$

Step 2: **Rewrite as equivalent fractions with the LCD, 6.**

$$-\frac{5}{6}-\left(-\frac{4}{6}\right)$$

Step 3: **Subtract.**

$$-\frac{5-(-4)}{6}$$

Step 4: **Simplify.**

$$-\frac{1}{6}$$

Try It

49) Evaluate $-\frac{1}{2}-y$ when $y=-\frac{1}{4}$

Solution

$-\dfrac{1}{4}$

50) Evaluate $-\dfrac{3}{8}-y$ when $x=-\dfrac{5}{2}$

Solution

$-\dfrac{17}{8}$

Example 1.4.26

Evaluate $2x^2y$ when $x=\dfrac{1}{4}$ and $y=-\dfrac{2}{3}$

Solution

Substitute the values into the expression.

***Step 1:* Substitute $\dfrac{1}{4}$ for x and $-\dfrac{2}{3}$ for y.**

$$2\left(\frac{1}{4}\right)^2\left(-\frac{2}{3}\right)$$

***Step 2:* Simplify exponents first.**

$$2\left(\frac{1}{16}\right)\left(-\frac{2}{3}\right)$$

***Step 3:* Multiply. Divide out the common factors. Notice we write 16 as $2\cdot2\cdot4$ to make it easy to remove common factors.**

$$\frac{\cancel{2}\times1\times\cancel{2}}{\cancel{2}\times\cancel{2}\times4\times3}$$

***Step 4:* Simplify.**

$$\frac{1}{12}$$

Try It

51) Evaluate $3ab^2$ when $a = -\frac{2}{3}$ and $b = -\frac{1}{2}$

Solution

$-\frac{1}{2}$

52) Evaluate $4c^3d$ when $c = -\frac{1}{2}$ and $d = -\frac{4}{3}$

Solution

$\frac{2}{3}$

The next example will have only variables, no constants.

Example 1.4.27

Evaluate $\frac{p+q}{r}$ when $p = -4, q = -2, \text{and} r = 8$

Solution

To evaluate $\frac{p+q}{r}$ when $p=-4, q=-2, \text{and} r=8$, we substitute the values into the expression.

Step 1: **Substitute -4 for p, -2 for q, and 8 for r.**

$$\frac{-4+(-2)}{8}$$

Step 2: **Add in the numerator first.**

$$\frac{-6}{8}$$

Step 3: **Simplify.**

$$-\frac{3}{4}$$

Try It

53) Evaluate $\frac{a+b}{c}$ when $a=-8, b=-7, \text{and} c=6$

Solution

$-\frac{5}{2}$

54) Evaluate $\frac{x+y}{z}$ when $x=9, y=-18, \text{and} z=-6$

Solution

$\frac{3}{2}$

Key Concepts

- **Equivalent Fractions Property:** If a, b, c are numbers where $b \neq 0, c \neq 0$, then $\frac{a}{b} = \frac{a \times c}{b \times c}$ and $\frac{a \times c}{b \times c} = \frac{a}{b}$.
- **Fraction Division:** If a, b, c and d are numbers where $b \neq 0, c \neq 0,$ and $d \neq 0$, then $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$. To divide fractions, multiply the first fraction by the reciprocal of the second.
- **Fraction Multiplication:** If a, b, c and d are numbers where $b \neq 0,$ and $d \neq 0$, then $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$. To multiply fractions, multiply the numerators and multiply the denominators.
- **Placement of Negative Sign in a Fraction:** For any positive numbers a and b, $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$.
- **Property of One:** $\frac{a}{a} = 1$; Any number, except zero, divided by itself is one.
- **Simplify a Fraction**

    1. Rewrite the numerator and denominator to show the common factors. If needed, factor the numerator and denominator into prime numbers first.
    2. Simplify using the equivalent fractions property by dividing out common factors.
    3. Multiply any remaining factors.

- **Simplify an Expression with a Fraction Bar**

    1. Simplify the expression in the numerator. Simplify the expression in the denominator.
    2. Simplify the fraction.

- **Fraction Addition and Subtraction:** If $a, b,$ and c are numbers where $c \neq 0$, then $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ and $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$.
  To add or subtract fractions, add or subtract the numerators and place the result over the common denominator.

- **Strategy for Adding or Subtracting Fractions**

  1. Do they have a common denominator?
     Yes—go to step 2.
     No—Rewrite each fraction with the LCD (Least Common Denominator). Find the LCD. Change each fraction into an equivalent fraction with the LCD as its denominator.
  2. Add or subtract the fractions.
  3. Simplify, if possible. To multiply or divide fractions, an LCD IS NOT needed. To add or subtract fractions, an LCD IS needed.

- **Simplify Complex Fractions**

  1. Simplify the numerator.
  2. Simplify the denominator.
  3. Divide the numerator by the denominator. Simplify if possible.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=230#h5p-4

b. After looking at the checklist, do you think you are well-prepared for the next chapter? Why or why not?

Glossary

complex fraction

A complex fraction is a fraction in which the numerator or the denominator contains a fraction.

denominator

The denominator is the value on the bottom part of the fraction that indicates the number of equal parts into which the whole has been divided.

equivalent fractions

Equivalent fractions are fractions that have the same value.

fraction

A fraction is written $\frac{a}{b}$, where $b \neq 0$ a is the numerator and b is the denominator. A fraction represents parts of a whole. The denominator b is the number of equal parts the whole has been divided into, and the numerator a indicates how many parts are included.

numerator

The numerator is the value on the top part of the fraction that indicates how many parts of the whole are included.

reciprocal

The reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$. A number and its reciprocal multiply to one: $\frac{a}{b} \cdot \frac{b}{a} = 1$.

simplified fraction

A fraction is considered simplified if there are no common factors in its numerator and denominator.

least common denominator

The least common denominator (LCD) of two fractions is the Least common multiple (LCM) of their denominators.

1.5 DECIMALS

Learning Objectives

By the end of this section, you will be able to:

- Name and write decimals
- Round decimals
- Add and subtract decimals
- Multiply and divide decimals
- Convert decimals, fractions, and percents

Name and Write Decimals

Decimals are another way of writing fractions whose denominators are powers of 10.

$0.1 = \frac{1}{10}$	0.1 is "one tenth"
$0.01 = \frac{1}{100}$	0.01 is "one hundredth"
$0.001 = \frac{1}{1000}$	0.001 is "one thousandth"
$0.0001 = \frac{1}{10000}$	0.0001 is "one ten-thousandth"

Notice that "ten thousand" is a number larger than one, but "one ten-thousand**th**" is a number smaller than one. The "th" at the end of the name tells you that the number is smaller than one.

When we name a whole number, the name corresponds to the place value based on the powers of ten. We read 10,000 as "ten thousand" and 10,000,000 as "ten million." Likewise, the names of the decimal places correspond to their fraction values. (Figure 1.5.1) shows the names of the place values to the left and right of the *decimal* point.

Place value of decimal numbers are shown to the left and right of the decimal point.

Place Value												
Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones		Tenths	Hundredths	Thousandths	Ten-thousandths	Hundred-thousandths
						•					

Figure 1.5.1

Example 1.5.1

Name the decimal 4.3.

Solution

***Step 1:* Name the number to the left of the decimal point.**

4 is to the left of the decimal point.

$$4.3$$
$$\text{four } \_\_\_\_\_\_\_\_\_\_$$

***Step 2:* Write 'and' for the decimal point.**

$$\text{four and } \_\_\_\_\_\_\_\_$$

***Step 3:* Name the 'number' part to the right of the decimal point as if it were a whole number.**

3 is to the right of the decimal point.

\begin{eqnarray*}\text{four and three ________}\end{eqnarray*}

Step 4: **Name the decimal place.**

Four and three tenths.

Try It

1) Name the decimal: 6.7

Solution

Six and seven tenths.

2) Name the decimal: 5.8

Solution

Five and eight tenths.

We summarize the steps needed to name a decimal below.

HOW TO

Name a Decimal.

1. Name the number to the left of the decimal point.
2. Write "and" for the decimal point.
3. Name the "number" part to the right of the decimal point as if it were a whole number.
4. Name the decimal place of the last digit.

Example 1.5.2

Name the decimal: -15.571

Solution

Step 1: **Name the number to the left of the decimal point.**

negative fifteen __________

Step 2: **Write "and" for the decimal point.**

\begin{eqnarray*}&\text{negative fifteen and ______}&\end{eqnarray*}

Step 3: **Name the number to the right of the decimal point.**

\begin{eqnarray*}&\text{negative fifteen and five hundred seventy-one __________}&\end{eqnarray*}

Step 4: **The 1 is in the thousandths place.**

Negative fifteen and five hundred seventy-one thousandths.

Try It

3) Name the decimal: -13.461

Solution

Negative thirteen and four hundred sixty-one thousandths.

4) Name the decimal: -2.053.

Solution

Negative two and fifty-three thousandths.

When we write a check we write both the numerals and the name of the number. Let's see how to write the decimal from the name.

Example 1.5.3

Write "fourteen and twenty-four thousandths" as a decimal.

Solution

Step 1: **Look for the word 'and'; it locates the decimal point. Place a decimal point under the word 'and'.**

Translate the words before 'and' into the whole number and place to the left of the decimal point.

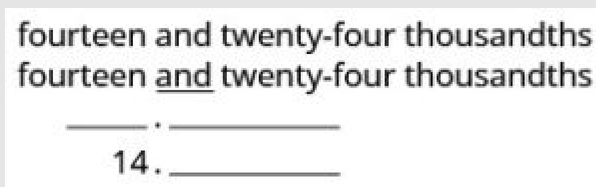


Figure 1.5.2

Step 2: **Mark the number of decimal places needed to the right of the decimal point by noting the place value indicated by the last word.**

Figure 1.5.3

Step 3: **Translate the words after 'and' into the number to the right of the decimal point.**

Write the number in the spaces — putting the final digit in the last place.

Figure 1.5.4

***Step 4:* Fill in zeros for empty place holders as needed.**

Zeros are needed in the tenths place.

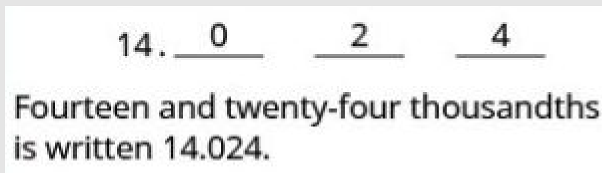


Figure 1.5.5

Try It

5) Write as a decimal: thirteen and sixty-eight thousandths.

Solution

13.68

6) Write as a decimal: five and ninety-four thousandths.

Solution

5.94

We summarize the steps to writing a decimal.

HOW TO

Write a decimal.

1. Look for the word “and”—it locates the decimal point.
   - Place a decimal point under the word “and.” Translate the words before “and” into the whole number and place it to the left of the decimal point.
   - If there is no “and,” write a “0” with a decimal point to its right.
2. Mark the number of decimal places needed to the right of the decimal point by noting the place value indicated by the last word.
3. Translate the words after “and” into the number to the right of the decimal point. Write the number in the spaces—putting the final digit in the last place.
4. Fill in zeros for place holders as needed.

Round Decimals

Rounding decimals is very much like rounding whole numbers. We will round decimals with a method based on the one we used to round whole numbers.

Example 1.5.4

Round 18.379 to the nearest hundredth.

Solution
Step 1: **Locate the given place value and mark it with an arrow.**

Figure 1.5.6

Step 2: **Underline the digit to the right of the given place value.**

Figure 1.5.7

Step 3: **Is this digit greater than or equal to 5?**

Yes: Add 1 to the digit in the given place value.
No: Do not change the digit in the given place value.

Figure 1.5.8

Step 4: **Rewrite the number, removing all digits to the right of the rounding digit.**

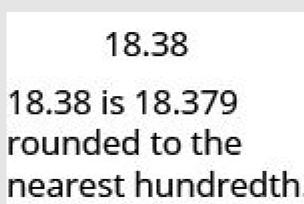


Figure 1.5.9

Try It

7) Round to the nearest hundredth: 1.047

Solution

1.05

8) Round to the nearest hundredth: 9.173

Solution

9.17

We summarize the steps for rounding a decimal here.

HOW TO

Round Decimals.

1. Locate the given place value and mark it with an arrow.
2. Underline the digit to the right of the place value.
3. Is this digit greater than or equal to 5?
    - Yes—add 1 to the digit in the given place value.
    - No—do *not* change the digit in the given place value.
4. Rewrite the number, deleting all digits to the right of the rounding digit.

Example 1.5.5

Round 18.379 to the nearest

a. tenth
b. whole number.

Solution
Round 18.379

a. to the nearest tenth

***Step 1:* Locate the tenths place with an arrow.**

Figure 1.5.10

***Step 2:* Underline the digit to the right of the given place value.**

Figure 1.5.11

***Step 3:* Because 7 is greater than or equal to 5, add 1 to the 3.**

Figure 1.5.12

***Step 4:* Rewrite the number, deleting all digits to the right of the rounding digit.**

$$18.4$$

Step 5: **Notice that the deleted digits were NOT replaced with zeros.**

So, 18.379 rounded to the nearest tenth is 18.4.

---

b. to the nearest whole number

Step 1: **Locate the ones place with an arrow.**

Figure 1.5.13

Step 2: **Underline the digit to the right of the given place value.**

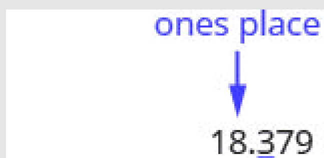


Figure 1.5.14

Step 3: **Since 3 is not greater than or equal to 5, do not add 1 to the 8.**

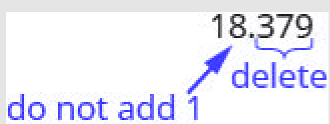


Figure 1.5.15

Step 4: **Rewrite the number, deleting all digits to the right of the rounding digit.**

So, 18.379 rounded to the nearest whole number is 18.

Try It

9) Round 6.582 to the nearest

a. hundredth
b. tenth
c. whole number.

Solution
a. 6.58
b. 6.6
c. 7

10) Round 15.2175 to the nearest

a. thousandth
b. hundredth
c. tenth.

Solution
a. 15.218
b. 15.22
c. 15.2

Add and Subtract Decimals

To add or subtract decimals, we line up the decimal points. By lining up the decimal points this way, we can add or subtract the corresponding place values. We then add or subtract the numbers as if they were whole numbers and then place the decimal point in the sum.

HOW TO

Add or Subtract Decimals.

1. Write the numbers so the decimal points line up vertically.
2. Use zeros as place holders, as needed.
3. Add or subtract the numbers as if they were whole numbers. Then place the decimal point in the answer under the decimal points in the given numbers.

Example 1.5.6

Add: $23.5 + 41.38$

Solution

Step 1: **Write the numbers so the decimal points line up vertically.**

$$\begin{array}{r} 23.50 \\ \underline{+41.38} \end{array}$$

Step 2: **Put 0 as a placeholder after the 5 in 23.5. Remember,** $\dfrac{5}{10} = \dfrac{50}{100}$ **so,** $0.5 = 0.50$.

$$\begin{array}{r} 23.50 \\ \underline{+41.38} \end{array}$$

Step 3: **Add the numbers as if they were whole numbers. Then place the decimal point in the sum.**

$$\begin{array}{r} 23.50 \\ \underline{+41.38} \\ 64.88 \end{array}$$

Try It

11) Add: $4.8 + 11.69$

Solution

16.49

12) Add: $5.123 + 18.47$

Solution

23.593

Example 1.5.7

Subtract: $20 - 14.65$

Solution

Step 1: **Write the numbers so the decimal points line up vertically.**

$$\begin{array}{r} 20. \\ \underline{-14.65} \end{array}$$

Step 2: **Put in zeros to the right as placeholders.**

Remember, 20 is a whole number, so place the decimal point after the 0.

$$\begin{array}{r} 20.00 \\ \underline{-14.65} \end{array}$$

Step 3: **Subtract and place the decimal point in the answer.**

Formula does not parse

Try It

13) Subtract $10 - 9.58$.

Solution

0.42

14) Subtract: $50 - 37.42$.

Solution

12.58

Multiply and Divide Decimals

Multiplying decimals is very much like multiplying whole numbers—we just have to determine where to place the decimal point. The procedure for multiplying decimals will make sense if we first convert them to fractions and then multiply.

So let's see what we would get as the product of decimals by converting them to fractions first. We will do two examples side-by-side. Look for a pattern!

	$(0.\underbrace{3}_{1\ place})\quad(0.\underbrace{7}_{1\ place})$
Convert to fractions.	$\frac{3}{10}\cdot\frac{7}{10}$
Multiply.	$\frac{21}{100}$
Convert to decimals.	$(0.\underbrace{21}_{2\ places})$

Notice, in the first example, we multiplied two numbers that each had one digit after the decimal point and the product had two decimal places. In the second example, we multiplied a number with one decimal place by a number with two decimal places and the product had three decimal places.

We multiply the numbers just as we do whole numbers, temporarily ignoring the decimal point. We then count the number of decimal points in the factors and that sum tells us the number of decimal places in the product.

The rules for multiplying positive and negative numbers apply to decimals, too, of course!

When *multiplying* two numbers,

- if their signs are the *same* the product is *positive*.
- if their signs are *different* the product is *negative*.

When we multiply signed decimals, first we determine the sign of the product and then multiply as if the numbers were both positive. Finally, we write the product with the appropriate sign.

HOW TO

Multiply Decimals

1. Determine the sign of the product.
2. Write in vertical format, lining up the numbers on the right. Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.
3. Place the decimal point. The number of decimal places in the product is the sum of the number of decimal places in the factors.
4. Write the product with the appropriate sign.

Example 1.5.8

Multiply: $(-3.9)(4.075)$

Solution

Step 1: **The signs are different. The product will be negative.**

Step 2: **Write in vertical format, lining up the numbers on the right.**

$$\begin{array}{r} 4.075 \\ \underline{\times\;\;3.9} \end{array}$$

Step 3: **Multiply.**

$$\begin{array}{r} 4.075 \\ \underline{\times\;\;3.9} \\ 158925 \end{array}$$

Step 4: **Add the number of decimal places in the factors** $(1+3)$**.**

$$(-3.\underbrace{9}_{1\ place})\;(4.\underbrace{075}_{3\ places})$$

Step 5: **Place the decimal point 4 places from the right.**

\begin{align*} 4.075\\ \underline{\times\;\;3.9}\\15.\underbrace{8925}&\\ \;\;\;\;\text{4 places}\end{align*}

Step 6: **The signs are different, so the product is negative.**

$$(-3.9)(4.075) = -15.8925$$

Try It

15) Multiply: $-4.5(6.107)$

Solution

-27.4815

16) Multiply: $-10.79(8.12)$

Solution

-87.6148

In many of your other classes, especially in the sciences, you will multiply decimals by powers of 10 (10, 100, 1000, etc.). If you multiply a few products on paper, you may notice a pattern relating the number of zeros in the power of 10 to number of decimal places we move the decimal point to the right to get the product.

HOW TO

Multiply a Decimal by a Power of Ten

1. Move the decimal point to the right the same number of places as the number of zeros in the power of 10.
2. Add zeros at the end of the number as needed.

Example 1.5.9

Multiply 5.63

a. by 10
b. by 100
c. by 1,000

Solution
By looking at the number of zeros in the multiple of ten, we see the number of places we need to move the decimal to the right.

a.

Step 1: **There is 1 zero in 10, so move the decimal point 1 place to the right.**

$$5.63(10)$$

Fig. 1.5.16

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b.

Step 1: **There are 2 zeros in 100, so move the decimal point 2 places to the right.**

$$5.63(100)$$

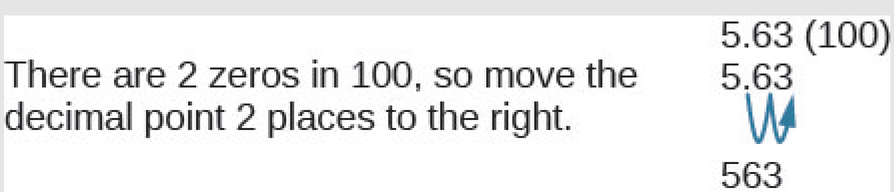

Figure 1.5.17

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c.

Step 1: **There are 3 zeros in 1,000, so move the decimal point 3 places to the right.**

$$5.63(1000)$$

Figure 1.5.18

Step 2: **A zero must be added at the end.**

$$5,630$$

Try It

17) Multiply 2.58

a. by 10
b. by 100
c. by 1,000

Solution
a. 25.8
b. 258
c. 2,580

18) Multiply 14.2
a. by 10
b. by 100
c. by 1,000

Solution
a. 142

b. 1,420
c. 14,200

Just as with multiplication, division of decimals is very much like dividing whole numbers. We just have to figure out where the decimal point must be placed.

To divide decimals, determine what power of 10 to multiply the denominator by to make it a whole number. Then multiply the numerator by that same power of 10. Because of the equivalent fractions property, we haven't changed the value of the fraction! The effect is to move the decimal points in the numerator and denominator the same number of places to the right. For example:

$$\frac{0.8}{0.4}$$
$$\frac{0.8(10)}{0.4(10)}$$
$$\frac{8}{4}$$

We use the rules for dividing positive and negative numbers with decimals, too. When dividing signed decimals, first determine the sign of the quotient and then divide as if the numbers were both positive. Finally, write the quotient with the appropriate sign.

We review the notation and vocabulary for division:

Figure 1.5.19

We'll write the steps to take when dividing decimals, for easy reference.

HOW TO

Divide Decimals.

1. Determine the sign of the quotient.
2. Make the divisor a whole number by “moving” the decimal point all the way to the right. “Move” the decimal point in the dividend the same number of places—adding zeros as needed.
3. Divide. Place the decimal point in the quotient above the decimal point in the dividend.
4. Write the quotient with the appropriate sign.

Example 1.5.10

Divide: $-25.56 \div (-0.06)$

Solution

Remember, you can “move” the decimals in the divisor and dividend because of the Equivalent Fractions Property.

Step 1: **The signs are the same.**

The quotient is positive.

Step 2: **Make the divisor a whole number by “moving” the decimal point all the way to the right.**

Step 3: **“Move” the decimal point in the dividend the same number of places.**

Figure 1.5.20

Step 4: **Divide.**

Place the decimal point in the quotient above the decimal point in the dividend.

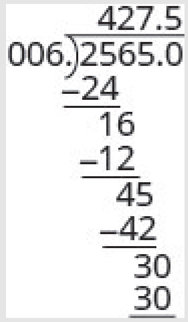


Figure 1.5.21

Step 5: **Write the quotient with the appropriate sign.**

$$-25.65 \div (-0.06) = 427.5$$

Try It

19) Divide: $-23.492 \div (-0.04)$

Solution

687.3

20) Divide: $-4.11 \div (-0.12)$

Solution

34.25

A common application of dividing whole numbers into decimals is when we want to find the price of one item that is sold as part of a multi-pack. For example, suppose a case of 24 water bottles costs $3.99. To find the price of one water bottle, we would divide $3.99 by 24. We show this division in Example 1.5.13. In calculations with money, we will round the answer to the nearest cent (hundredth).

Example 1.5.11

Divide: $\$3.99 \div 24$.

Solution

Step 1: **Place the decimal point in the quotient above the decimal point in the dividend.**

Step 2: **Divide as usual.**

When do we stop? Since this division involves money, we round it to the nearest cent (hundredth.) To do this, we must carry the division to the thousandths place.

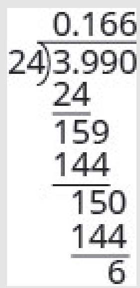


Figure 1.5.22

Step 3: **Round to the nearest cent.**

$$0.166 \approx 0.17$$
$$\$3.99 \div 24 \approx \$0.17$$

Try It

21) Divide: $\$6.99 \div 36$

Solution

0.19

22) Divide: $\$4.99 \div 12$

Solution

0.42

Convert Decimals, Fractions, and Percents

We convert decimals into fractions by identifying the place value of the last (farthest right) digit. In the decimal 0.03 the 3 is in the hundredths place, so 100 is the denominator of the fraction equivalent to 0.03.

$$0.03 = \frac{3}{100}$$

Notice, when the number to the left of the decimal is zero, we get a fraction whose numerator is less than its denominator. Fractions like this are called proper fractions.

The steps to take to convert a decimal to a fraction are summarized in the procedure box.

HOW TO

Convert a Decimal to a Proper Fraction.

1. Determine the place value of the final digit.
2. Write the fraction.
    - numerator—the “numbers” to the right of the decimal point
    - denominator—the place value corresponding to the final digit

Example 1.5.12

Write 0.374 as a fraction.

Solution

Step 1: **Determine the place value of the final digit.**

Figure 1.5.23

Step 2: **Write the fraction for 0.374:**

- The numerator is 374.
- The denominator is 1,000.

$$\frac{374}{1000}$$

Step 3: **Simplify the fraction.**

$$\frac{2 \times 187}{2 \times 500}$$

Step 4: **Divide out the common factors.**

$$\frac{187}{500}$$

So, $0.374 = \frac{187}{500}$.

Did you notice that the number of zeros in the denominator of $\frac{374}{1,000}$ is the same as the number of decimal places in 0.374?

Try It

23) Write 0.234 as a fraction.

Solution

$\frac{117}{500}$

24)Write 0.024 as a fraction.

Solution

$\frac{3}{125}$

We've learned to convert decimals to fractions. Now we will do the reverse—convert fractions to decimals. Remember that the fraction bar means division. So $\frac{4}{5}$ can be written $4 \div 5$ or $5\overline{)4}$. This leads to the following method for converting a fraction to a decimal.

HOW TO

Convert a Fraction to a Decimal.

To convert a fraction to a decimal, divide the numerator of the fraction by the denominator of the fraction.

Example 1.5.13

Write $\frac{5}{8}$ as a decimal.

Solution

Step 1: **Since a fraction bar means division, we begin by writing** $\frac{5}{8}$ **as** $8\overline{)5}$

Step 2: **Now divide.**

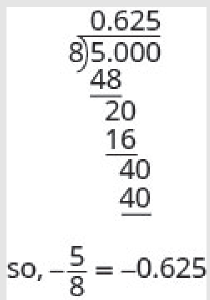


Figure 1.5.24

Try It

25) Write $\frac{-7}{8}$ as a decimal.

Solution

-0.875

26) Write $\frac{-3}{8}$ as a decimal.

Solution

-0.375

When we divide, we will not always get a zero remainder. Sometimes the quotient ends up with a decimal that repeats. A **repeating decimal** is a decimal in which the last digit or group of digits repeats endlessly. A bar is placed over the repeating block of digits to indicate it repeats.

Repeating Decimal

A **repeating decimal** is a decimal in which the last digit or group of digits repeats endlessly.

A bar is placed over the repeating block of digits to indicate it repeats.

Example 1.5.14

Write $\frac{43}{22}$ as a decimal.

Solution

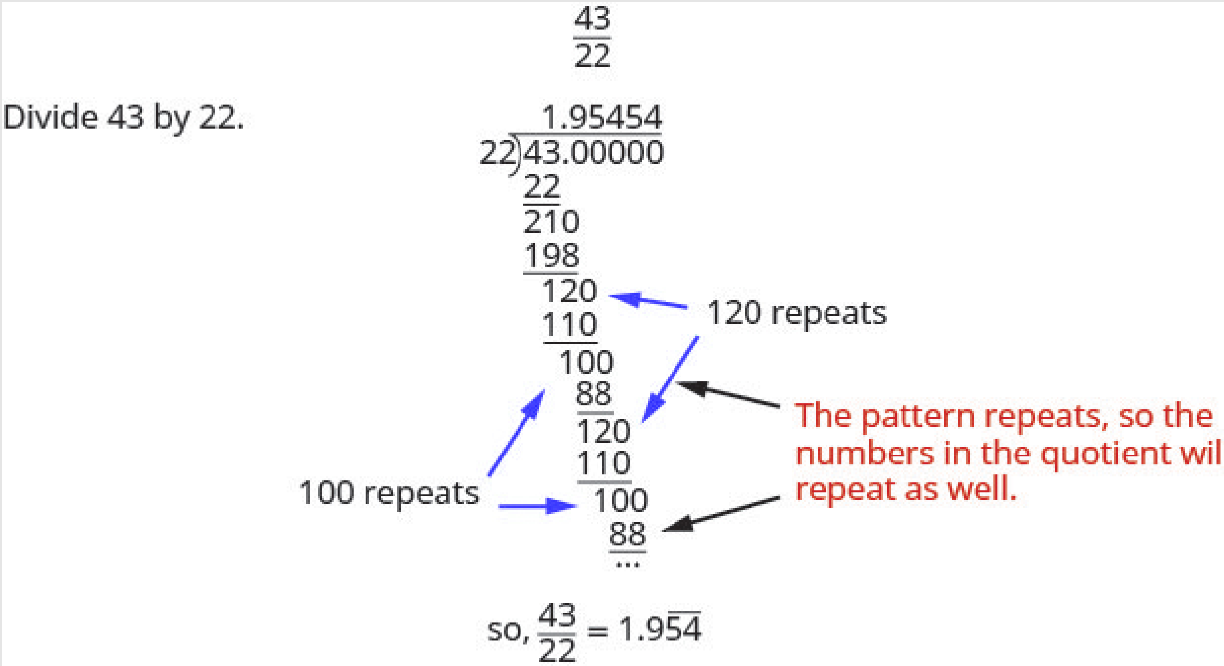


Figure 1.5.25

Try It

27) Write $\frac{27}{11}$ as a decimal.

Solution

$2.\overline{45}$

28) Write $\frac{51}{22}$ as a decimal.

Solution

$2.3\overline{18}$

Sometimes we may have to simplify expressions with fractions and decimals together.

Example 1.5.15

Simplify: $\frac{7}{8} + 6.4$

Solution

First we must change one number so both numbers are in the same form. We can change the fraction to a decimal, or change the decimal to a fraction. Usually it is easier to change the fraction to a decimal.

***Step 1:* Change $\frac{7}{8}$ to a decimal.**

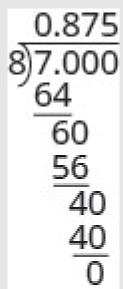


Figure 1.5.26

***Step 2:* Add.**

$$0.875 + 6.4 = 7.275$$

So, $\frac{7}{8} + 6.4 = 7.275$

Try It

29) Simplify: $\frac{3}{8} + 4.9$

Solution

5.275

30) Simplify: $5.7 + \frac{13}{20}$

Solution

6.35

A **percent** is a ratio whose denominator is 100. Percent means per hundred. We use the percent symbol, %, to show percent.

Percent

A percent is a ratio whose denominator is 100.

Since a percent is a ratio, it can easily be expressed as a fraction. Percent means per 100, so the denominator of the fraction is 100. We then change the fraction to a decimal by dividing the numerator by the denominator.

	6%	78%	135%
Write as a ratio with denominator 100.	$\frac{6}{100}$	$\frac{78}{100}$	$\frac{135}{100}$
Change the fraction to a decimal by dividing the numerator by the denominator.	0.06	0.78	1.35

Do you see the pattern? *To convert a percent number to a decimal number, we move the decimal point two places to the left.*

Figure 1.5.27

Example 1.5.16

Convert each percent to a decimal:

a. 62%
b. 135%
c. 35.7%

Solution

a.

***Step 1:* Move the decimal point two places to the left.**

Figure 1.5.28

$$0.62$$

---

b.

***Step 1:* Move the decimal point two places to the left.**

Figure 1.5.29

$$1.35$$

---

c.

***Step 1:* Move the decimal point two places to the left.**

Figure 1.5.30

$$0.057$$

Try It

31) Convert each percent to a decimal:

a. 9%
b. 87%
c. 3.9%

Solution

a. 0.09
b. 0.87
c. 0.039

32) Convert each percent to a decimal:

a. 3%
b. 91%
c. 8.3%

Solution

a. 0.03
b. 0.91
c. 0.083

Converting a decimal to a percent makes sense if we remember the definition of percent and keep place value in mind.

To convert a decimal to a percent, remember that percent means per hundred. If we change the decimal to a fraction whose denominator is 100, it is easy to change that fraction to a percent.

	0.83	1.05	0.075
Write as a fraction.	$\frac{83}{100}$	$\frac{5}{100}$	$\frac{75}{1000}$
The denominator is 100.		$\frac{105}{100}$	$\frac{7.5}{100}$
Write the ratio as a percent.	83%	105%	7.5%

Recognize the pattern? *To convert a decimal to a percent, we move the decimal point two places to the right and then add the percent sign*.

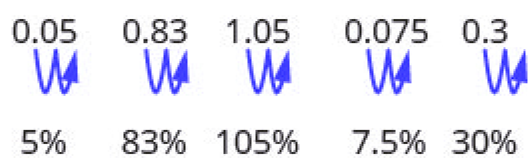


Figure 1.5.31

Example 1.5.17

Convert each decimal to a percent:

a. 0.51
b. 1.25
c. 0.093

Solution

a.

Step 1: **Move the decimal point two places to the right.**

Figure 1.5.32

$$51\%$$

---

b.

Step 1: **Move the decimal point two places to the right.**

Figure 1.5.33

$$125\%$$

---

c.

Step 1: **Move the decimal point two places to the right.**

Figure 1.5.34

$$9.3\%$$

Try It

33) Convert each decimal to a percent:

a. 0.17

b. 1.75

c. 0.0825

Solution

a. 17%

b. 175%

c. 8.25%

34) Convert each decimal to a percent:

a. 0.41

b. 2.25

c. 0.0925

Solution

a. 41%

b. 225%

c. 9.25%

Key Concepts

- **Name a Decimal**
  1. Name the number to the left of the decimal point.
  2. Write "and" for the decimal point.

3. Name the “number” part to the right of the decimal point as if it were a whole number.
4. Name the decimal place of the last digit.

- **Write a Decimal**

  1. Look for the word ‘and’—it locates the decimal point. Place a decimal point under the word ‘and.’ Translate the words before ‘and’ into the whole number and place it to the left of the decimal point. If there is no “and,” write a “0” with a decimal point to its right.
  2. Mark the number of decimal places needed to the right of the decimal point by noting the place value indicated by the last word.
  3. Translate the words after ‘and’ into the number to the right of the decimal point. Write the number in the spaces—putting the final digit in the last place.
  4. Fill in zeros for place holders as needed.

- **Round a Decimal**

  1. Locate the given place value and mark it with an arrow.
  2. Underline the digit to the right of the place value.
  3. Is this digit greater than or equal to 5? Yes—add 1 to the digit in the given place value. No—do *not* change the digit in the given place value.
  4. Rewrite the number, deleting all digits to the right of the rounding digit.

- **Add or Subtract Decimals**

  1. Write the numbers so the decimal points line up vertically.
  2. Use zeros as place holders, as needed.
  3. Add or subtract the numbers as if they were whole numbers. Then place the decimal in the answer under the decimal points in the given numbers.

- **Multiply Decimals**

  1. Determine the sign of the product.
  2. Write in vertical format, lining up the numbers on the right. Multiply the numbers as if they were whole numbers, temporarily ignoring the decimal points.
  3. Place the decimal point. The number of decimal places in the product is the sum of the decimal places in the factors.
  4. Write the product with the appropriate sign.

- **Multiply a Decimal by a Power of Ten**
    1. Move the decimal point to the right the same number of places as the number of zeros in the power of 10.
    2. Add zeros at the end of the number as needed.
- **Divide Decimals**
    1. Determine the sign of the quotient.
    2. Make the divisor a whole number by “moving” the decimal point all the way to the right. “Move” the decimal point in the dividend the same number of places – adding zeros as needed.
    3. Divide. Place the decimal point in the quotient above the decimal point in the dividend.
    4. Write the quotient with the appropriate sign.
- **Convert a Decimal to a Proper Fraction**
    1. Determine the place value of the final digit.
    2. Write the fraction: numerator—the ‘numbers’ to the right of the decimal point; denominator—the place value corresponding to the final digit.
- **Convert a Fraction to a Decimal** Divide the numerator of the fraction by the denominator

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=316#h5p-5

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

decimal

A decimal is another way of writing a fraction whose denominator is a power of ten.

percent

A percent is a ratio whose denominator is 100.

repeating decimal

A repeating decimal is a decimal in which the last digit or group of digits repeats endlessly.

1.6 THE REAL NUMBERS

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with square roots
- Identify integers, rational numbers, irrational numbers, and real numbers
- Locate fractions on the number line
- Locate decimals on the number line

Simplify Expressions with Square Roots

Remember that when a number n is multiplied by itself, we write n^2 and read it "n squared." The result is called the square of n. For example,

8^2 read \text{8 squared}

64 is called the square of 8

Similarly, 121 is the square of 11, because 11^2 is 121.

Square of a Number

If $n^2 = m,$ then m is the square of n.

Complete the following table to show the squares of the counting numbers 1 through 15.

Number	n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Square	n^2								64			121				

The numbers in the second row are called perfect square numbers. It will be helpful to learn to recognize the perfect square numbers.

The squares of the counting numbers are positive numbers. What about the squares of negative numbers? We know that when the signs of two numbers are the same, their product is positive. So the square of any negative number is also positive.

$$(-3)^2 = 9 \qquad (-8)^2 = 64 \qquad (-11)^2 = 121 \qquad (-15)^2 = 225$$

Did you notice that these squares are the same as the squares of the positive numbers?

Sometimes we will need to look at the relationship between numbers and their squares in reverse. Because $10^2 = 100$, we say 100 is the square of 10. We also say that 10 is a square root of 100. A number whose square is m is called a **square root** of m.

Square Root of a Number

If $n^2 = m$, then n is a square root of m.

Notice $(-10)^2 = 100$ also, so -10 is also a square root of 100. Therefore, both 10 and -10 are square roots of 100.

So, every positive number has two square roots—one positive and one negative. What if we only wanted the positive square root of a positive number? The **radical sign**, $\sqrt{m}$, denotes the positive square root. The positive square root is called the principal square root. When we use the radical sign that always means we want the principal square root.

We also use the radical sign for the square root of zero. Because $0^2 = 0$, $\sqrt{0} = 0$. Notice that zero has only one square root.

Square Root Notation

$\sqrt{m}$ is read "the square root of m"

$$radical\ sign \rightarrow \sqrt{m} \leftarrow radicand$$

If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$.

The square root of m, $\sqrt{m}$, is the positive number whose square is m.

Since 10 is the principal square root of 100, we write $\sqrt{100} = 10$. You may want to complete the following table to help you recognize square roots.

$\sqrt{1}$	$\sqrt{4}$	$\sqrt{9}$	$\sqrt{16}$	$\sqrt{25}$	$\sqrt{36}$	$\sqrt{49}$	$\sqrt{64}$	$\sqrt{81}$	$\sqrt{100}$	$\sqrt{121}$	$\sqrt{144}$	$\sqrt{169}$	$\sqrt{196}$	$\sqrt{225}$
									10					

Try It

Simplify:

1) $\sqrt{25}$
2) $\sqrt{121}$

Solution

1) Since $5^2 = 25$
$\sqrt{25} = 5$

2)Since $11^2 = 121$
$\sqrt{121} = 11$

Try It

Simplify:

3) $\sqrt{36}$
4) $\sqrt{169}$

Solution

3) 6
4) 13

Try It

Simplify:

5) $\sqrt{16}$
6) $\sqrt{196}$

Solution

5) 4
6) 14

We know that every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{100} = 10$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{100} = -10$. We read $-\sqrt{100}$ as "the opposite of the square root of 10."

Try It

Simplify:

7) $-\sqrt{9}$

8) $-\sqrt{144}$

Solution

7) The negative is in front of the radical sign. $-\sqrt{9} = -3$

8) The negative is in front of the radical sign. $-\sqrt{144} = -12$

Try It

Simplify:

9) $-\sqrt{4}$

10) $-\sqrt{225}$

Solution

9) -2

10) -15

Try It

Simplify:

11) $-\sqrt{81}$
12) $-\sqrt{100}$

Solution

11) -9
12) -10

Identify Integers, Rational Numbers, Irrational Numbers, and Real Numbers

We have already described numbers as *counting numbers, whole numbers,* and *integers*. What is the difference between these types of numbers?

$$\begin{array}{ll} \text{Counting numbers} & 1, 2, 3, 4, \ldots \\ \text{Whole numbers} & 0, 1, 2, 3, 4, \ldots \\ \text{Integers} & \ldots -3, -2, -1, 0, 1, 2, 3, \ldots \end{array}$$

What type of numbers would we get if we started with all the integers and then included all the fractions? The numbers we would have form the set of rational numbers. A rational number is a number that can be written as a ratio of two integers.

Rational Number

A rational number is a number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

A rational number can be written as the ratio of two integers.

All signed fractions, such as $\frac{4}{5}, -\frac{7}{8}, \frac{13}{4}, -\frac{20}{3}$ are rational numbers. Each numerator and each denominator is an integer.

Are integers rational numbers? To decide if an integer is a rational number, we try to write it as a ratio of two integers. Each integer can be written as a ratio of integers in many ways. For example, 3 is equivalent to $\frac{3}{1}, \frac{6}{2}, \frac{9}{3}, \frac{12}{4}, \frac{15}{5} \ldots$

An easy way to write an integer as a ratio of integers is to write it as a fraction with denominator one.

$$3 = \frac{3}{1} \qquad -8 = -\frac{8}{1} \qquad 0 = \frac{0}{1}$$

Since any integer can be written as the ratio of two integers, all integers are rational numbers! Remember that the counting numbers and the whole numbers are also integers, and so they, too, are rational.

What about decimals? Are they rational? Let's look at a few to see if we can write each of them as the ratio of two integers.

We've already seen that integers are rational numbers. The integer -8 could be written as the decimal -8.0. So, clearly, some decimals are rational.

Think about the decimal 7.3. Can we write it as a ratio of two integers? Because 7.3 means $7\frac{3}{10}$, we can write it as an improper fraction, $\frac{73}{10}$. So 7.3 is the ratio of the integers 73 and 10. It is a rational number.

In general, any decimal that ends after a number of digits (such as 7.3 or -1.2684) is a rational number. We can use the place value of the last digit as the denominator when writing the decimal as a fraction.

Example 1.6.1

Write as the ratio of two integers:

a. −27
b. 7.31

Solution

a.

***Step 1:* Write it as a fraction with denominator 1.**

$$-27 = \frac{-27}{1}$$

---

b.

Step 1: **Write it as a mixed number.**

Remember, 7 is the whole number and the decimal part, 0.31, indicates hundredths.

$$7\frac{31}{100}$$

Step 2: Convert to an improper fraction.

$$\frac{731}{100}$$

So we see that -27 and 7.31 are both rational numbers, since they can be written as the ratio of two integers.

Try It

Write as the ratio of two integers:

13) -24
14) 3.57

Solution

13) $\frac{-24}{1}$

14) $\frac{357}{100}$

Try It

Write as the ratio of two integers:

15) -19

16) 8.41

Solution

15) $\dfrac{-19}{1}$

16) $\dfrac{841}{100}$

Let's look at the decimal form of the numbers we know are rational.

We have seen that *every integer is a rational number*, since $a = \dfrac{a}{1}$ for any integer, a. We can also change any integer to a decimal by adding a decimal point and a zero.

Integer	-2	-1	0	1	2	3
Decimal form	-2.0	-1.0	0.0	1.0	2.0	3.0

These decimal numbers stop.

We have also seen that *every fraction is a rational number*. Look at the decimal form of the fractions we considered above.

Ratio of integers	$\frac{4}{5}$	$-\frac{7}{8}$	$\frac{13}{4}$	$-\frac{20}{3}$
The decimal form	0.8	-0.875	3.25	$-6.666\ldots$ $-6.\overline{6}$

These decimals either stop or repeat.

What do these examples tell us?

Every rational number can be written both as a ratio of integers, $\frac{p}{q}$, where p and q are integers and $q \neq 0$, *and as a decimal that either stops or repeats.*

Here are the numbers we looked at above expressed as a ratio of integers and as a decimal:

	Fractions		Integers							
Number	$\frac{4}{5}$	$-\frac{7}{8}$	$\frac{13}{4}$	$-\frac{20}{3}$	-2	-1	0	1	2	3
Ratio of Integers	$\frac{4}{5}$	$-\frac{7}{8}$	$\frac{13}{4}$	$-\frac{20}{3}$	$-\frac{2}{1}$	$-\frac{1}{1}$	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$
Decimal Form	0.8	-0.875	3.25	$-6.\overline{6}$	-2.0	-1.0	0.0	1.0	2.0	3.0

Rational Number

A rational number is a number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

Its decimal form stops or repeats.

Are there any decimals that do not stop or repeat? Yes!

The number π (the Greek letter pi, pronounced "pie"), which is very important in describing circles, has a decimal form that does not stop or repeat.

$$\pi = 3.141592654...$$

We can even create a decimal pattern that does not stop or repeat, such as

$$2.01001000100001\ldots$$

Numbers whose decimal form does not stop or repeat cannot be written as a fraction of integers. We call these numbers **irrational**.

Irrational Number

An irrational number is a number that cannot be written as the ratio of two integers.

Its decimal form does not stop and does not repeat.

Let's summarize a method we can use to determine whether a number is rational or irrational.

Rational or Irrational?

If the decimal form of a number

- *repeats or stops*, the number is *rational*.
- *does not repeat and does not stop*, the number is *irrational*.

Example 1.6.2

Given the numbers $0.58\bar{3}$, 0.47, $3.605551275\ldots$ list the

a. rational numbers
b. irrational numbers

Solution

a.

***Step 1:* Look for decimals that repeat or stop.**

The 3 repeats in $0.58\bar{3}$.

The decimal 0.47 stops after the 7.

So $0.58\overline{3}$ and 0.47 are rational.

---

b.

Step 1: **Look for decimals that repeat or stop.**

$3.605551275\ldots$ has no repeating block of digits and it does not stop.

So $3.605551275\ldots$ is irrational.

Try It

For the given numbers list the:

17) rational numbers
18) irrational numbers

0.29, $0.81\overline{6}$, $2.515115111\ldots$

Solution

17) $0.29, 0.81\overline{6}$
18) $2.515115111\ldots$

Try It

For the given numbers list the:

19) rational numbers
20) irrational numbers

$2.6\overline{3}$, 0.125, $0.418302\ldots$

Solution
19) $2.6\overline{3}$, 0.125
20) $0.418302\ldots$

Try It

For each number given, identify whether it is rational or irrational:

21) $\sqrt{36}$
22) $\sqrt{44}$

Solution
21) Recognize that 36 is a perfect square, since $6^2 = 36$. So $\sqrt{36} = 6,$ therefore $\sqrt{36}$ is rational.
22) Remember that $6^2 = 36$ and $7^2 = 49,$ so 44 is not a perfect square. Therefore, the decimal form of $\sqrt{44}$ will never repeat and never stop, so $\sqrt{44}$ is irrational.

Try It

For each number given, identify whether it is rational or irrational:

23) $\sqrt{81}$
24) $\sqrt{17}$

Solution

23) rational
24) irrational

Try It

For each number given, identify whether it is rational or irrational:

25) $\sqrt{116}$
26) $\sqrt{121}$

Solution

25) irrational
26) rational

We have seen that all counting numbers are whole numbers, all whole numbers are integers, and all integers are rational numbers. The irrational numbers are numbers whose decimal form does not stop and does not repeat. When we put together the rational numbers and the irrational numbers, we get the set of **real numbers**.

Real Number

A real number is a number that is either rational or irrational.

All the numbers we use in elementary algebra are real numbers. Figure 1.6.1 illustrates how the number sets we've discussed in this section fit together.

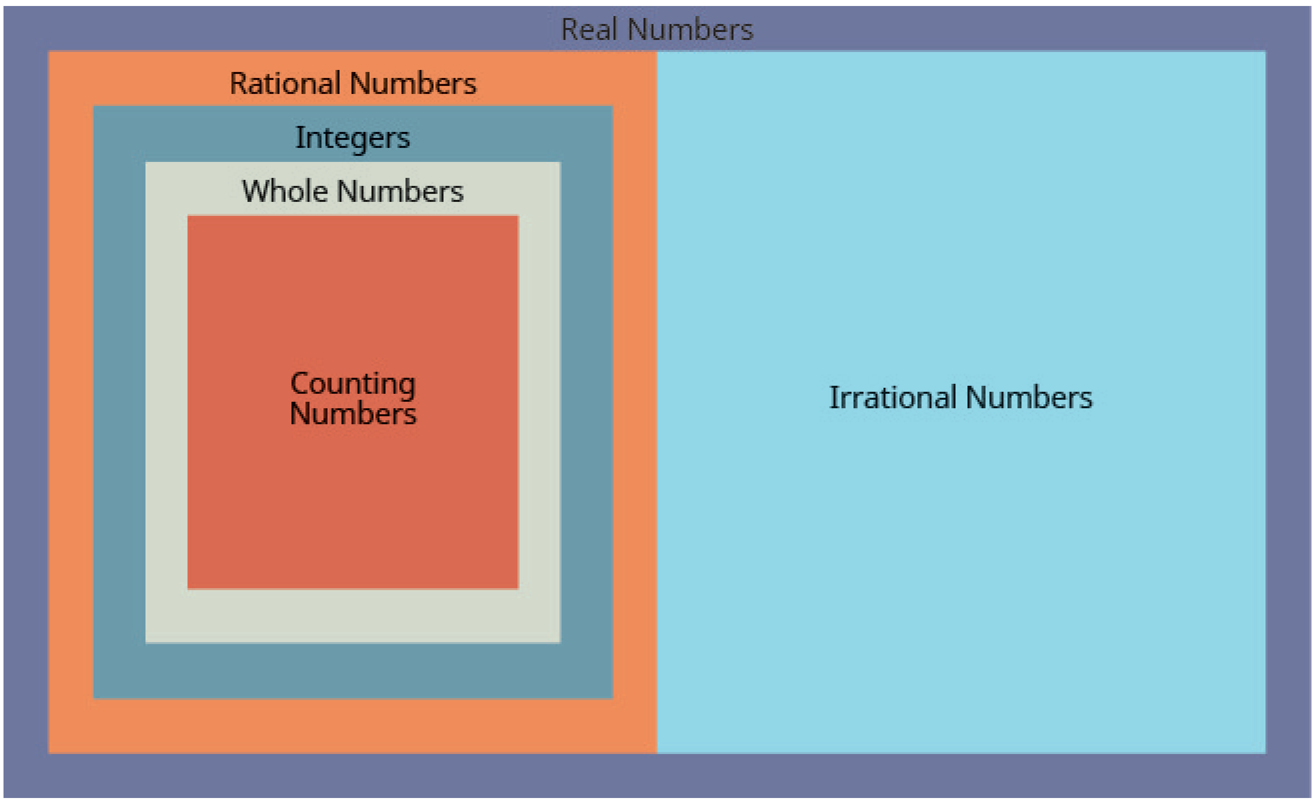


Figure 1.6.1. This chart shows the number sets that make up the set of real numbers. Does the term "real numbers" seem strange to you? Are there any numbers that are not "real," and, if so, what could they be?

This chart shows the number sets that make up the set of real numbers. Does the term "real numbers" seem strange to you? Are there any numbers that are not "real," and, if so, what could they be?

Can we simplify $\sqrt{-25}$? Is there a number whose square is -25?

$$(\ \ )^2 = -25?$$

None of the numbers that we have dealt with so far has a square that is -25. Why? Any positive number squared is positive. Any negative number squared is positive. So we say there is no real number equal to $\sqrt{-25}$.

The square root of a negative number is not a real number.

Try It

For each number given, identify whether it is a real number or not a real number:

27) $\sqrt{-169}$

28) $-\sqrt{64}$

Solution

27) There is no real number whose square is -169. Therefore, $\sqrt{-169}$ is not a real number.

28) Since the negative is in front of the radical, $-\sqrt{64}$ is -8. Since -8 is a real number, $-\sqrt{64}$ is a real number.

Try It

For each number given, identify whether it is a real number or not a real number:

29) $\sqrt{-196}$

30) $-\sqrt{81}$

Solution

29) not a real number

30) real number

Try It

For each number given, identify whether it is a real number or not a real number:

31) $-\sqrt{49}$
32) $\sqrt{-121}$

Solution

31) real number

32) not a real number

Example 1.6.3

Given the numbers $-7, \frac{14}{5}, 8, \sqrt{5}, 5.9, -\sqrt{64}$, list the:

a. whole numbers
b. integers
c. rational numbers
d. irrational numbers
e. real numbers

Solution

a. Remember, the whole numbers are 0, 1, 2, 3, … and 8 is the only whole number given.

---

b. The integers are the whole numbers, their opposites, and 0. So the whole number 8 is an integer, and -7 is the opposite of a whole number so it is an integer, too. Also, notice that 64 is the square of 8 so $-\sqrt{64} = -8$. So the integers are $-7, 8, -\sqrt{64}$.

---

c. Since all integers are rational, then $-7, 8, -\sqrt{64}$ are rational. Rational numbers also include fractions and decimals that repeat or stop, so $\frac{14}{5}$ and 5.9 are rational. So the list of rational numbers is $-7, \frac{14}{5}, 8, 5.9, -\sqrt{64}$.

---

d. Remember that 5 is not a perfect square, so $\sqrt{5}$ is irrational.

---

e. All the numbers listed are real numbers.

Try It

For the given numbers, list the:

33) whole numbers
34) integers
35) rational numbers
36) irrational numbers
37) real numbers

$-3, -\sqrt{2}, 0.\bar{3}, \frac{9}{5}, 4, \sqrt{49}$.

Solution

33) $4, \sqrt{49}$

34) $-3, 4, \sqrt{49}$

35) $-3, 0.\bar{3}, \frac{9}{5}, 4, \sqrt{49}$

36) $-\sqrt{2}$

37) $-3, -\sqrt{2}, 0.\bar{3}, \frac{9}{5}, 4, \sqrt{49}$

Try It

For the given numbers, list the:

38) whole numbers
39) integers
40) rational numbers
41) irrational numbers
42) real numbers

$-\sqrt{25}, -\frac{3}{8}, -1, 6, \sqrt{121}, 2.041975...$

Solution

38) $6, \sqrt{121}$

39) $-\sqrt{25}, -1, 6, \sqrt{121}$

40) $-\sqrt{25}, -\frac{3}{8}, -1, 6, \sqrt{121}$

41) $2.041975...$

42) $-\sqrt{25}, -\frac{3}{8}, -1, 6, \sqrt{121}, 2.041975...$

Locate Fractions on the Number Line

The last time we looked at the number line, it only had positive and negative integers on it. We now want to include fractions and decimals on it.

Let's start with fractions and locate $\frac{1}{5}, -\frac{4}{5}, 3, \frac{7}{4}, -\frac{9}{2}, -5, \text{ and } \frac{8}{3}$ on the number line.

We'll start with the whole numbers 3 and -5, because they are the easiest to plot. See Figure 1.6.2

The proper fractions listed are $\frac{1}{5} \text{ and } -\frac{4}{5}$. We know the proper fraction $\frac{1}{5}$ has value less than one and so would be located between $0 \text{ and } 1$. The denominator is 5, so we divide the unit from 0 to 1 into 5 equal parts $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$. We plot $\frac{1}{5}$. See Figure 1.6.2.

Similarly, $-\frac{4}{5}$ is between 0 and -1. After dividing the unit into 5 equal parts we plot $-\frac{4}{5}$. See Figure 1.6.2

Finally, look at the improper fractions $\frac{7}{4}, -\frac{9}{2}, \frac{8}{3}$. These are fractions in which the numerator is greater than the denominator. Locating these points may be easier if you change each of them to a mixed number. See Figure. 1.6.2.

$$\frac{7}{4} = 1\frac{3}{4} \qquad -\frac{9}{2} = -4\frac{1}{2} \qquad \frac{8}{3} = 2\frac{2}{3}$$

Figure 1.6.2 shows the number line with all the points plotted.

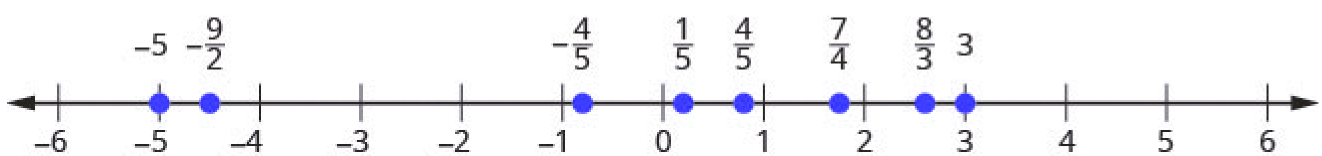


Figure 1.6.2

Example 1.6.4

Locate and label the following on a number line: $4, \frac{3}{4}, -\frac{1}{4}, -3, \frac{6}{5}, -\frac{5}{2}, \text{ and } \frac{7}{3}$.

Solution

Locate and plot the integers, $4, -3$

Step 1: **Locate the proper fraction $\frac{3}{4}$ first.**

The fraction $\frac{3}{4}$ is between 0 and 1.

Step 2: **Divide the distance between 0 and 1 into four equal parts then, we plot $\frac{3}{4}$.**

Similarly plot $-\frac{1}{4}$.

Step 3: **Now locate the improper fractions $\frac{6}{5}, -\frac{5}{2}, \frac{7}{3}$.**

$$\frac{6}{5} = 1\frac{1}{5}$$
$$-\frac{5}{2} = -2\frac{1}{2}$$
$$\frac{7}{3} = 2\frac{1}{3}$$

It is easier to plot them if we convert them to mixed numbers and then plot them as described above:

Figure 1.6.3

Try It

43) Locate and label the following on a number line: $-1, \frac{1}{3}, \frac{6}{5}, -\frac{7}{4}, \frac{9}{2}, 5, -\frac{8}{3}$.

Solution

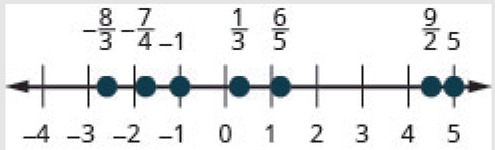


Figure 1.6.4

Try It

44) Locate and label the following on a number line: $-2, \frac{2}{3}, \frac{7}{5}, -\frac{7}{4}, \frac{7}{2}, 3, -\frac{7}{3}$.

Solution

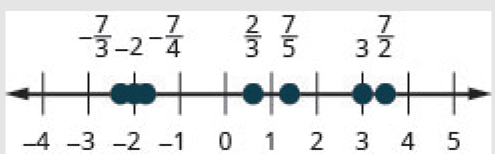


Figure 1.6.5

In Example 1.6.5, we'll use the inequality symbols to order fractions. In previous chapters we used the number line to order numbers.

- $a < b$ "a is less than b" when a is to the left of b on the number line
- $a > b$ "a is greater than b" when a is to the right of b on the number line

As we move from left to right on a number line, the values increase.

Example 1.6.5

Order each of the following pairs of numbers, using $<$ or $>$. It may be helpful to refer Figure 1.6.6.

a. $-\frac{2}{3}$ ___ -1
b. $-3\frac{1}{2}$ ___ -3
c. $-\frac{3}{4}$ ___ $-\frac{1}{4}$
d. -2 ___ $-\frac{8}{3}$

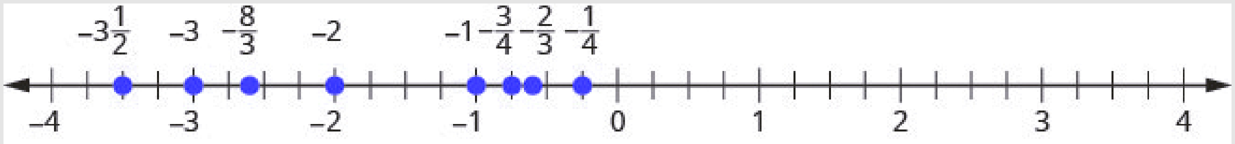


Figure 1.6.6

Solution

a.

Step 1: **$-\frac{2}{3}$ is to the right of -1 on the number line.**

$$-\frac{2}{3} \_\_\_ -1 = -\frac{2}{3} > -1$$

b.

Step 1: **$-3\frac{1}{2}$ is to the right of -3 on the number line.**

$$-3\frac{1}{2} \_\_\_ -3 = -3\frac{1}{2} < -3$$

c.

Step 1: **$-\frac{3}{4}$ is to the right of $-\frac{1}{4}$ on the number line.**

$$-\frac{3}{4}\_\_\_-\frac{1}{4} = -\frac{3}{4} < -\frac{1}{4}$$

---

d.

Step 1: **-2 is to the right of $-\frac{8}{3}$ on the number line.**

$$-2\ \_\_\_ -\frac{8}{3} = -2 > -\frac{8}{3}$$

Try It

Order each of the following pairs of numbers, using < or >:

45) $-\frac{1}{3}\_\_\_-1$

46) $-1\frac{1}{2}\_\_\_-2$

47) $-\frac{2}{3}\_\_\_-\frac{1}{3}$

48) $-3\_\_\_-\frac{7}{3}$

Solution

45) $>$

46) $>$

47) $<$

48) $<$

Try It

Order each of the following pairs of numbers, using < or >:

49) $-1\_\_\_ -\frac{2}{3}$

50) $-2\frac{1}{4}\_\_\_ -2$

51) $-\frac{3}{5}\_\_\_ -\frac{4}{5}$

52) $-4\_\_\_ -\frac{10}{3}$

Solution

49) $<$

50) $<$

51) $>$

52) $<$

Locate Decimals on the Number Line

Since decimals are forms of fractions, locating decimals on the number line is similar to locating fractions on the number line.

Example 1.6.6

Locate 0.4 on the number line.

Solution

Step 1: A proper fraction has value less than one.

The decimal number 0.4 is equivalent to $\frac{4}{10}$, a proper fraction, so 0.4 is located between 0 and 1.

Step 2: **On a number line, divide the interval between 0 and 1 into 10 equal parts.**

Step 3: **Now label the parts 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0.**

We write 0 as 0.0 and 1 and 1.0, so that the numbers are consistently in tenths.

Step 4: **Finally, mark 0.4 on the number line.** See Figure 1.6.7.

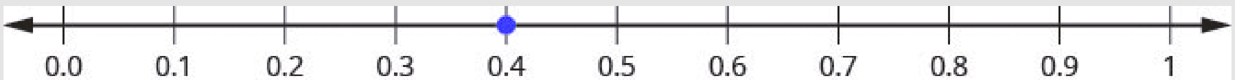


Figure 1.6.7

Try It

53) Locate on the number line: 0.6.

Solution

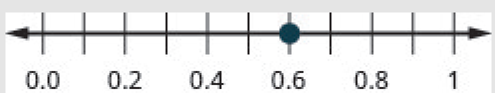


Figure 1.6.8

Try It

54) Locate on the number line: 0.9.

Solution

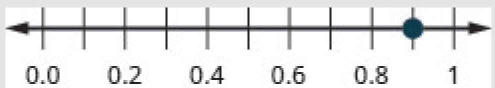


Figure 1.6.9

Example 1.6.7

Locate -0.74 on the number line.

Solution

***Step 1:* The decimal -0.74 is equivalent to $-\dfrac{74}{100}$, so it is located between 0 and -1.**

***Step 2:* On a number line, mark off and label the hundredths in the interval between 0 and -1.** See Figure 1.6.10.

Figure 1.6.10

Try It

55) Locate on the number line: -0.6.

Solution

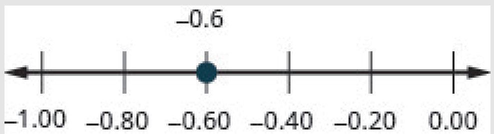


Figure 1.6.11

Try It

56) Locate on the number line: -0.7.

Solution

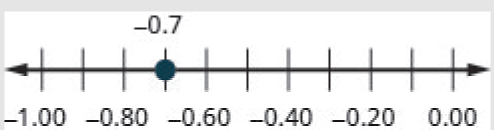


Figure 1.6.12

Which is larger, 0.04 or 0.40? If you think of this as money, you know that $\$0.40$ (forty cents) is greater than $\$0.04$ (four cents). So, $0.40 > 0.04$.

Again, we can use the number line to order numbers.

- $a < b$ "a is less than b" when a is to the left of b on the number line

- $a > b$ "a is greater than b" when a is to the right of b on the number line

Where are 0.04 and 0.40 located on the number line? See Figure 1.6.13.

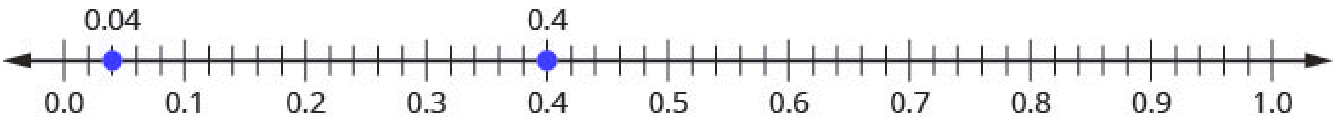


Figure 1.6.13

We see that 0.40 is to the right of 0.04 on the number line. This is another way to demonstrate that $0.40 > 0.04$.

How does 0.31 compare to 0.308? This doesn't translate into money to make it easy to compare. But if we convert 0.31 and 0.308 into fractions, we can tell which is larger.

Steps	**0.31**	**0.308**
Convert to fractions.	$\frac{31}{100}$	$\frac{308}{1000}$
We need a common denominator to compare them.	$\frac{31 \times 10}{100 \times 10}$	$\frac{308}{1000}$
	$\frac{310}{1000}$	$\frac{308}{1000}$

Because $310 > 308$, we know that $\frac{310}{1000} > \frac{308}{1000}$. Therefore, $0.31 > 0.308$.

Notice what we did in converting 0.31 to a fraction—we started with the fraction $\frac{31}{100}$ and ended with the equivalent fraction $\frac{310}{1000}$. Converting $\frac{310}{1000}$ back to a decimal gives 0.310. So 0.31 is equivalent to 0.310. Writing zeros at the end of a decimal does not change its value!

$$\frac{31}{100} = \frac{310}{1000} \quad \text{and} \quad 0.31 = 0.310$$

We say 0.31 and 0.310 are **equivalent decimals**.

Equivalent Decimals

Two decimals are equivalent if they convert to equivalent fractions.

We use equivalent decimals when we order decimals.

The steps we take to order decimals are summarized here.

HOW TO

Order Decimals

Step 1: Write the numbers one under the other, lining up the decimal points.
Step 2: Check to see if both numbers have the same number of digits. If not, write zeros at the end of the one with fewer digits to make them match.
Step 3: Compare the numbers as if they were whole numbers.
Step 4: Order the numbers using the appropriate inequality sign.

Example 1.6.8

Order $0.64\_\_\_0.6$ using $<$ or $>$.

Solution

***Step 1:* Write the numbers one under the other, lining up the decimal points.**

$$\begin{array}{r}0.64\\0.6\end{array}$$

Step 2: **Add a zero to 0.6 to make it a decimal with 2 decimal places.**

Now they are both hundredths.

$$0.64$$
$$0.60$$

Step 3: **64 is greater than 60.**

Formula does not parse

Step 4: **64 hundredths is greater than 60 hundredths.**

Formula does not parse

Try It

57) Order each of the following pairs of numbers, using $< \text{or} > : 0.42\_\_\_0.4$.

Solution

$>$

Try It

58) Order each of the following pairs of numbers, using $< \text{or} > : 0.18\_\_\_0.1$.

Solution

$>$

Example 1.6.9

Order $0.83\_\_\_0.803$ using $<$ or $>$.

Solution

Step 1: **Write the numbers one under the other, lining up the decimals.**

Step 2: **They do not have the same number of digits.**

$$\begin{array}{l} 0.\ 83 \\ 0.\ 803 \end{array}$$

Step 3: **Write one zero at the end of 0.83.**

$$\begin{array}{l} 0.\ 830 \\ 0.\ 803 \end{array}$$

Step 4: **Since $830 > 803$, 830 thousandths is greater than 803 thousandths.**

$$0.830 > 0.803$$
$$0.83 > 0.803$$

Try It

59) Order the following pair of numbers, using $< \text{or} > : 0.76\_\_\_0.706$.

Solution

$>$

Try It

60) Order the following pair of numbers, using $<$ or $>$:0.305___0.35.

Solution

$<$

When we order negative decimals, it is important to remember how to order negative integers. Recall that larger numbers are to the right on the number line. For example, because -2 lies to the right of -3 on the number line, we know that $-2 > -3$. Similarly, smaller numbers lie to the left on the number line. For example, because -9 lies to the left of -6 on the number line, we know that $-9 < -6$. See Figure 1.6.14.

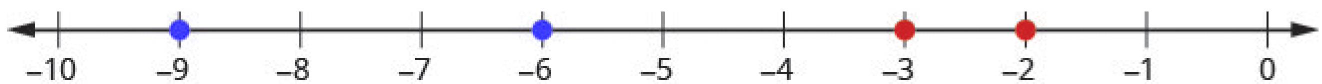


Figure 1.6.14

If we zoomed in on the interval between 0 and -1, as shown in Figure 1.6.6, we would see in the same way that $-0.2 > -0.3$ and $-0.9 < -0.6$.

Example 1.6.10

Use $<$ or $>$ to order -0.1___ -0.8.

Solution

Step 1: **Write the numbers one under the other, lining up the decimal points.**

They have the same number of digits. $\begin{array}{l} -0.1 \\ -0.8 \end{array}$

Since $-1 > -8$, -1 tenth is greater than -8 tenths. $-0.1 > -0.8$

Try It

61) Order the following pair of numbers, using < or >: $-0.3\_\_\_-0.5$.

Solution

$>$

Try It

62) Order the following pair of numbers, using < or >: $-0.6\_\_\_-0.7$.

Solution

$>$

Key Concepts

Square Root Notation

$\sqrt{m}$ is read 'the square root of m.' If $m = n^2$, then $\sqrt{m} = n$, for $n \geq 0$.

Order Decimals

1. Write the numbers one under the other, lining up the decimal points.
2. Check to see if both numbers have the same number of digits. If not, write zeros at the end of the one with fewer digits to make them match.
3. Compare the numbers as if they were whole numbers.
4. Order the numbers using the appropriate inequality sign.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objective of this section.

b. On a scale of $1 - 10,$ how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

equivalent decimals
Two decimals are equivalent if they convert to equivalent fractions.

irrational number
An irrational number is a number that cannot be written as the ratio of two integers. Its decimal form does not stop and does not repeat.

rational numberA rational number is a number of the form $\frac{p}{q}$, where p and q are integers and

$q \neq 0$. A rational number can be written as the ratio of two integers. Its decimal form stops or repeats.

radical sign
A radical sign is the symbol $\sqrt{m}$ that denotes the positive square root.

real number
A real number is a number that is either rational or irrational.

square and square root
If $n^2 = m$, then m is the square of n and n is a square root of m.

1.7 PROPERTIES OF REAL NUMBERS

Learning Objectives

By the end of this section, you will be able to:

- Use the commutative and associative properties
- Use the identity and inverse properties of addition and multiplication
- Use the properties of zero
- Simplify expressions using the distributive property

Use the Commutative and Associative Properties

Think about adding two numbers, say 5 and 3. The order we add them doesn't affect the result, does it?

$5+3$	$3+5$
8	8
$5+3$	$3+5$

The results are the same.

As we can see, the order in which we add does not matter!

What about multiplying 5 and 3?

5×3	3×5
15	15

$$5\times 3=3\times 5$$

Again, the results are the same!

The order in which we multiply does not matter!

These examples illustrate the *commutative property*. When adding or multiplying, changing the *order* gives the same result.

Commutative Property

of Addition	if a, b are real numbers, then $a + b = b + a$
of Multiplication	if a, b are real numbers, then $a \times b = b \times a$

When adding or multiplying, changing the order gives the same result.

The commutative property has to do with order. If you change the order of the numbers when adding or multiplying, the result is the same.

What about subtraction? Does order matter when we subtract numbers? Does $7 - 3$ give the same result as $3 - 7$?

$7 - 3$	$3 - 7$
4	-4

$$4 \neq -4$$
$$7 - 3 \neq -3 - 7$$

The results are not the same.

Since changing the order of the subtraction did not give the same result, we know that *subtraction is not commutative*.

Let's see what happens when we divide two numbers. Is division commutative?

$12 \div 4$	$14 \div 12$
$\frac{12}{4}$	$\frac{4}{12}$
3	$\frac{1}{3}$

$$3 \neq \frac{1}{3}$$
$$12 \div 4 \neq 4 \div 12$$

The results are not the same. Since changing the order of the division did not give the same result, *division is not commutative*.

The commutative properties only apply to addition and multiplication!

- Addition and Multiplication *are* commutative.
- Subtraction and Division *are not* commutative.

If you were asked to simplify this expression, how would you do it and what would your answer be?

$$7 + 8 + 2$$

Some people would think $7 + 8$ is 15 and then $15 + 2$ is 17. Others might start with $8 + 2$ makes 10 and then $7 + 10$ makes 17.

Either way gives the same result. Remember, we use parentheses as grouping symbols to indicate which operation should be done first.

$$\begin{array}{lr} \text{Add } 7+8 & (7+8)+2 \\ \text{Add} & 15+2 \\ & 17 \\ \text{Add } 8+2 & 7+(8+2) \\ \text{Add} & 7+10 \\ & 17 \end{array}$$

$$(7+8)+2 = 7+(8+2)$$

When adding three numbers, changing the grouping of the numbers gives the same result.

This is true for multiplication, too.

$$\begin{array}{lr} \text{Multiply } 5 \cdot \frac{1}{3} & \left(5 \times \frac{1}{3}\right) \times 3 \\ \text{Multiply.} & \frac{5}{3} \times 3 \\ & 5 \end{array}$$

$$\begin{array}{ll} \text{Multiply } \frac{1}{3}\cdot 3 & 5\times\left(\frac{1}{3}\times 3\right) \\ \text{Multiply.} & 5\times 1 \\ & 5 \end{array}$$

$$(5\times\frac{1}{3})\times 3 = 5\times(\frac{1}{3}\times 3)$$

When multiplying three numbers, changing the grouping of the numbers gives the same result.

You probably know this, but the terminology may be new to you. These examples illustrate the associative property.

Associative Property

of Addition	If a, b, c are real numbers, then $(a+b)+c = a+(b+c)$
of Multiplication	If a, b, c are real numbers, then $(a\times b)\times c = a\times(b\times c)$

When adding or multiplying, changing the *grouping* gives the same result.

Let's think again about multiplying $5\times\frac{1}{3}\times 3$. We got the same result both ways, but which way was easier? Multiplying $\frac{1}{3}$ and 3 first, as shown above on the right side, eliminates the fraction in the first step. Using the associative property can make the math easier!

The associative property has to do with grouping. If we change how the numbers are grouped, the result will be the same. Notice it is the same three numbers in the same order—the only difference is the grouping.

We saw that subtraction and division were not commutative. They are not associative either.

When simplifying an expression, it is always a good idea to plan what the steps will be. In order to combine like terms in the next example, we will use the commutative property of addition to write the like terms together.

Example 1.7.1

Simplify: $18p + 6q + 15p + 5q$

Solution
Step 1: **Use the commutative property of addition to re-order so that like terms are together.**

Step 2: **Add like terms.**

$$33p + 11q$$

Try It

1) Simplify: $23r + 14s + 9r + 15s$

Solution
$32r + 29s$

2) Simplify: $37m + 21n + 4m - 15n$

Solution
$41m + 6n$

When we have to simplify algebraic expressions, we can often make the work easier by applying the commutative or associative property first, instead of automatically following the order of operations. When adding or subtracting fractions, combine those with a common denominator first.

Example 1.7.2

Simplify: $(\frac{5}{13}+\frac{3}{4})+\frac{1}{4}$

Solution

Step 1: **Notice that the last 2 terms have a common denominator, so change the grouping.**

$$\frac{5}{13}+(\frac{3}{4}+\frac{1}{4})$$

Step 2: **Add in parentheses first.**

$$\frac{5}{13}+(\frac{4}{4})$$

Step 3: **Simplify the fraction.**

$$\frac{5}{13}+(1)$$

Step 4: **Add.**

$$1\frac{5}{13}$$

Step 5: **Convert to an improper fraction.**

$$\frac{18}{13}$$

Try It

3) Simplify: $(\frac{7}{15}+\frac{5}{8})+\frac{3}{8}$

Solution

$1\dfrac{7}{15}$

4) Simplify: $(\frac{2}{9}+\frac{7}{12})+\frac{5}{12}$

Solution

$1\dfrac{2}{9}$

Example 1.7.3

Use the associative property to simplify $6(3x)$.

Solution

Step 1: **Change the grouping.**

$$(6\times 3)x$$

Step 2: **Multiply in the parentheses.**

$$18x$$

Notice that we can multiply 6×3 but we could not multiply $3x$ without having a value for x.

Try It

5) Use the associative property to simplify $8(4x)$

Solution

$32x$

6) Use the associative property to simplify $-9(7y)$

Solution

$-63y$

Use the Identity and Inverse Properties of Addition and Multiplication

What happens when we add 0 to any number? Adding 0 doesn't change the value. For this reason, we call 0 the **additive identity**.

For example,

$13+0$	$-14+0$	$0+(-8)$
13	-14	-8

These examples illustrate the *Identity Property of Addition* that states that for any real number, a, $a+0=a$ and $0+a=a$.

What happens when we multiply any number by one? Multiplying by 1 doesn't change the value. So we call 1 the **multiplicative identity**.

For example,

43×1	-27×1	$1\times\frac{3}{5}$
43	-27	$\frac{3}{5}$

These examples illustrate the *Identity Property of Multiplication* that states that for any real number, a, $a\cdot 1=a$ and $1\cdot a=a$.

We summarize the Identity Properties below.

Identity Property

of addition for any real number a: 0 is the additive identity	$a+0=a$	$0+a=a$
of multiplication for any real number a: 1 is the additive identity	$a\times 1=a$	$1\times a=a$

What number added to 5 gives the additive identity, 0?

$5+\_\_\_=0$ We know that $5+(-5)=0$

What number added to -6 gives the additive identity, 0?

$-6+\_\_\_=0$ We know $(-6)+6=0$

Notice that in each case, the missing number was the opposite of the number!

We call $-a$ the **additive inverse** of a. *The opposite of a number is its additive inverse.* A number and its opposite add to zero, which is the additive identity. This leads to the *Inverse Property of Addition* that states for any real number, a, $a+(-a)=0$. Remember, a number and its opposite add to zero.

What number multiplied by $\frac{2}{3}$ gives the multiplicative identity, 1? In other words, $\frac{2}{3}$ times what results in 1?

$\frac{2}{3}\cdot\_\_\_=1$ We know $\frac{2}{3}\cdot\frac{3}{2}=1$

What number multiplied by 2 gives the multiplicative identity, 1? In other words 2 times what results in 1?

$2\cdot\_\_\_=1$ We know $2\cdot\frac{1}{2}=1$

Notice that in each case, the missing number was the reciprocal of the number!

We call $\frac{1}{a}$ the **multiplicative inverse** of a. *The reciprocal of a number is its multiplicative inverse.* A number and its reciprocal multiply to one, which is the multiplicative identity. This leads to the *Inverse Property of Multiplication* that states that for any real number, a, $a\neq 0$, $a\times\frac{1}{a}=1$.

We'll formally state the inverse properties here:

Inverse Property

of addition	For any real number a, $-a$ is the additive inverse of a. A number and its opposite add to zero.	$a + (-a) = 0$
of multiplication	For any real number a, $\frac{1}{a}$ is the multiplicative inverse of a. A number and its reciprocal multiply to one.	$a \times \frac{1}{a} = 1$

Example 1.7.4

Find the additive inverse of

a. $\frac{5}{8}$

b. 0.6

c. -8

d. $\frac{4}{3}$

Solution

To find the additive inverse, we find the opposite.

a. The additive inverse of $\frac{5}{8}$ is the opposite of $\frac{5}{8}$ The additive inverse of $\frac{5}{8}$ is $-\frac{5}{8}$

---

b. The additive inverse of 0.6 is the opposite of 0.6. The additive inverse of 0.6 is -0.6.

---

c. The additive inverse of -8 is the opposite of -8. We write the opposite of -8 as $-(-8)$, and then simplify it to 8. Therefore, the additive inverse of -8 is 8.

---

d. The additive inverse of $-\frac{4}{3}$ is the opposite of $-\frac{4}{3}$. We write this as $-(-\frac{4}{3})$ and then simplify to $\frac{4}{3}$. Thus, the additive inverse of $-\frac{4}{3}$ is $\frac{4}{3}$.

Try It

7) Find the additive inverse of:

a. $\frac{7}{9}$
b. 1.2
c. -14
d. $-\frac{9}{4}$

Solution
a. $-\frac{7}{9}$
b. -1.2
c. 14
d. $\frac{9}{4}$

8) Find the additive inverse of:

a. $\frac{7}{13}$
b. 8.4

c. -46
d. $-\frac{5}{2}$

Solution
a. $-\frac{7}{13}$
b. -8.4
c. 46
d. $\frac{5}{2}$

Example 1.7.5

Find the multiplicative inverse of
a. 9
b. $-\frac{1}{9}$
c. 0.9

Solution
To find the multiplicative inverse, we find the reciprocal.

a. The multiplicative inverse of 9 is the reciprocal of 9, which is $\frac{1}{9}$. Therefore, the multiplicative inverse of 9 is $\frac{1}{9}$.

---

b. The multiplicative inverse of $-\frac{1}{9}$ is the reciprocal of $-\frac{1}{9}$ which is -9. Thus, the multiplicative inverse of $-\frac{1}{9}$ is -9.

c.
***Step 1:* To find the multiplicative inverse of 0.9, we first convert 0.9 to a fraction**

$$\frac{9}{10}$$

***Step 2:* Then we find the reciprocal of the fraction.**

$$\text{The reciprocal of } \frac{9}{10} \text{ is } \frac{10}{9}.$$

$$\text{So the multiplicative inverse of } 0.9 \text{ is } \frac{10}{9}.$$

Try It

9) Find the multiplicative inverse of

a. 4
b. $-\frac{1}{7}$
c. 0.3

Solution
a. $\frac{1}{4}$
b. -7
c. $\frac{10}{3}$

10) Find the multiplicative inverse of

a. 18

b. $-\frac{4}{5}$
c. 0.6

Solution

a. $\frac{1}{18}$
b. $-\frac{5}{4}$
c. $\frac{5}{3}$

Use the Properties of Zero

The identity property of addition says that when we add 0 to any number, the result is that same number. What happens when we multiply a number by 0? Multiplying by 0 makes the product equal zero.

Multiplication by Zero

For any real number a,

$$0 \times \ a = 0 \qquad a \times 0 = 0$$

The product of any real number and 0 is 0.

What about division involving zero? What is $0 \div 3$? Think about a real example: If there are no cookies in the cookie jar and 3 people are to share them, how many cookies does each person get? There are no cookies to share, so each person gets 0 cookies. So, $0 \div 3 = 0$

We can check division with the related multiplication fact.

$12 \div 6 = 2$ because $2 \times 6 = 12$

So we know $0 \div 3 = 0$ because $0 \times 3 = 0$.

Division of Zero

For any real number, a, except 0,

$\frac{0}{a} = 0$ and $0 \div a = 0$.

Zero is divided by any real number except zero is zero.

Now think about dividing *by* zero. What is the result of dividing 4 by 0? Think about the related multiplication fact: $4 \div 0 =?$ means $? \times 0 = 4$ Is there a number that multiplied by 0 gives 4? Since any real number multiplied by 0 gives 0, there is no real number that can be multiplied by 0 to obtain 4.

We conclude that there is no answer to $4 \div 0$ and so we say that division by 0 is undefined.

Division by Zero

For any real number a, except 0,

$\frac{a}{0}$ and $a \div 0$ are undefined.

Division by zero is undefined.

We summarize the properties of zero below.

Properties of Zero

Multiplication by Zero: For any real number a,

$a \times 0 = 0 \qquad 0 \times a = 0$	The product of any number and 0 is 0.

Division by Zero: For any real number a, where $a \neq 0$

$\frac{0}{a} = 0$	Zero divided by any real number except itself is zero.
$\frac{a}{0}$ is undefined	Division by zero is undefined.

Example 1.7.6

Simplify:

a. -8×0
b. $\frac{0}{-2}$
c. $\frac{-32}{0}$

Solution

a.

Step 1: **The product of any real number and 0 is 0.**

$$-8 \times 0$$
$$0$$

---

b.

Step 1: **The product of any real number and 0 is 0.**

$$\frac{0}{-2}$$
$$0$$

---

c.

Step 1: **Division by 0 is undefined.**

$$\frac{-32}{0}$$
$$\text{Undefined}$$

Try It

11) Simplify:

a. $-14 \cdot 0$

b. $\frac{0}{-6}$

c. $\frac{-2}{0}$

Solution

a. 0

b. 0

c. undefined

12) Simplify:

a. $0(-17)$

b. $\frac{0}{-10}$

c. $\frac{-5}{0}$

Solution

a. 0
b. 0
c. undefined

We will now practice using the properties of identities, inverses, and zero to simplify expressions.

Example 1.7.7

Simplify:

a. $\dfrac{0}{n+5}$, where $n \neq -5$
b. $\dfrac{10-3p}{0}$, where $10-3p \neq 0$

Solution

a.

Step 1: **Zero divided by any real number except itself is 0.**

$$\frac{0}{n+5}$$
$$0$$

---

b.

Step 1: **Division by 0 is undefined.**

$$\frac{10-3p}{0}$$
$$\text{Undefined}$$

Example 1.7.8

Simplify: $-84n + (-73n) + 84n$

Solution

Step 1: **Notice that the first and third terms are opposites; use the commutative property of addition to re-order the terms.**

$$-84n + 84n + (-73n)$$

Step 2: **Add left to right.**

$$0 + (-73)$$

Step 3: **Add.**

$$-73n$$

Try It

13) Simplify: $-27a + (-48) + 27a$

Solution

$-48a$

14) Simplify: $39x + (-92x) + (-39x)$

Solution

$-92x$

Now we will see how recognizing reciprocals is helpful. Before multiplying left to right, look for reciprocals—their product is 1.

Example 1.7.9

Simplify: $\frac{7}{15} \times \frac{8}{23} \times \frac{15}{7}$

Solution

Step 1: **Notice that the first and third terms are reciprocals, so use the commutative property of multiplication to re-order the factors.**

$$\frac{7}{15} \times \frac{15}{7} \times \frac{8}{23}$$

Step 2: **Multiply left to right.**

$$\frac{8}{23}$$

Try It

15) Simplify: $\frac{9}{16} \times \frac{5}{49} \times \frac{16}{9}$

Solution

$\frac{5}{49}$

16) Simplify: $\frac{6}{17} \times \frac{11}{25} \times \frac{17}{6}$

Solution

$\frac{11}{25}$

Try It

17) Simplify:

a. $\frac{0}{m+7}$, where $m \neq -7$

b. $\frac{18-6c}{0}$, where $18-6c \neq 0$

Solution

a. 0

b. undefined

18) Simplify:

a. $\frac{0}{d-4}$, where $d \neq 4$

b. $\frac{15-4q}{0}$, $15-4q \neq 0$

Solution

a. 0

b. undefined

Example 1.7.10

Simplify: $\frac{3}{4} \times \frac{4}{3}(6x+12)$

Solution

Step 1: **There is nothing to do in the parentheses, so multiply the two fractions first—notice, they are reciprocals.**

$$1(6x+12)$$

Step 2: **Simplify by recognizing the multiplicative identity.**

$$(6x+12)$$

Try It

19) Simplify: $\frac{2}{5} \times \frac{5}{2}(20y+50)$

Solution

$20y+50$

20) Simplify: $\frac{3}{8} \times \frac{8}{3}(12z+16)$

Solution

$12z+16$

Simplify Expressions Using the Distributive Property

Suppose that three friends are going to the movies. They each need $\$9.25$—that's 9 dollars and 1 quarter—to pay for their tickets. How much money do they need all together?

You can think about the dollars separately from the quarters. They need 3 times $\$9$ so $\$27$, and 3 times 1 quarter, so 75 cents. In total, they need $\$27.75$. If you think about doing the math in this way, you are using the *distributive property*.

Distributive Property

a(b+c)=ab+ac

If a, b, c are real numbers, then	$(b+c)a = ba + ca$
Also,	$a(b-c) = ab - ac$ $(b-c)a = ba - ca$

Back to our friends at the movies, we could find the total amount of money they need like this:

$$\begin{array}{l} 3(9.25) \\ 3(9+0.25) \\ 3(9)+3(0.25) \\ 27+0.75 \\ 27.75 \end{array}$$

In algebra, we use the distributive property to remove parentheses as we simplify expressions.

For example, if we are asked to simplify the expression $3(x+4)$ the order of operations says to work in the parentheses first. But we cannot add x and 4, since they are not like terms. So we use the distributive property, as shown in Example 1.7.11.

Some students find it helpful to draw in arrows to remind them how to use the distributive property. Then the first step in Example 1.7.11 would look like this:

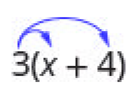


Figure 1.7.1

Example 1.7.11

Simplify: $3(x+4)$

Solution

***Step 1:* Distribute**

$$3 \times x + 3 \times 4$$

***Step 2:* Multiply.**

$$3x + 12$$

Try It

21) Simplify: $4(x+2)$

Solution

$4x + 8$

22) Simplify: $6(x+7)$

Solution

$6x + 42$

Example 1.7.12

Simplify: $8(\frac{3}{8}x+\frac{1}{4})$

Solution

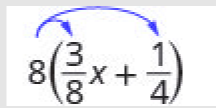


Figure 1.7.2

Step 1: **Distribute.**

$$8\cdot\frac{3}{8}x+8\cdot\frac{1}{4}$$

Step 2: **Multiply.**

$$3x+2$$

Try It

23) Simplify: $6(\frac{5}{6}y+\frac{1}{2})$

Solution

$5y+3$

24) Simplify: $12(\frac{1}{3}n+\frac{3}{4})$

Solution

$4n+9$

Using the distributive property as shown in Example 1.7.13 will be very useful when we solve money applications in later chapters.

Example 1.7.13

Simplify: $100(0.3+0.25q)$

Solution

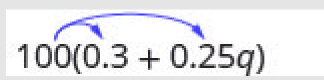


Figure 1.7.3

Step 1: **Distribute.**

$$100\left(0.3\right)+100\left(0.25q\right)$$

Step 2: **Multiply.**

$$30+25q$$

Try It

25) Simplify: $100(0.7+0.15p)$

Solution

$70 + 15p$

26) Simplify: $100(0.04 + 0.35d)$

Solution

$4 + 35d$

When we distribute a negative number, we need to be extra careful to get the signs correct!

Example 1.7.14

Simplify: $-2(4y + 1)$

Solution

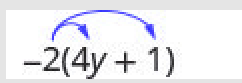


Figure 1.7.4

Step 1: **Distribute.**

$$-2 \cdot 4y + (-2) \cdot 1$$

Step 2: **Multiply.**

$$-8y - 2$$

Try It

27) Simplify: $-3(6m+5)$

Solution

$-18m-15$

28) Simplify: $-6(8n+11)$

Solution

$-48n-66$

Example 1.7.15

Simplify: $-11(4-3a)$

Solution

Step 1: **Distribute.**

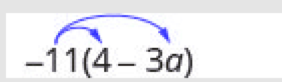


Figure 1.7.5

Step 2: **Multiply.**

$$-11\cdot4-(-11)\cdot13a$$
$$-44-(-33a)$$

Step 3: **Simplify.**

$$-44+33a$$

Notice that you could also write the result as $33a - 44$. Do you know why?

Try It

29) Simplify: $-5(2 - 3a)$

Solution

$-10 + 15a$

30) Simplify: $-7(8 - 15y)$

Solution

$-56 + 105y$

Example 1.7.16 will show how to use the distributive property to find the opposite of an expression.

Example 1.7.16

Simplify: $-(y + 5)$

Solution

Step 1: **Multiplying by -1 results in the opposite.**

$$-1(y + 5)$$

Step 2: **Distribute.**

$$-1 \times y + -1 \times 5$$

Step 3: **Simplify.**

$$-y + (-5)$$
$$-y - 5$$

Try It

31) Simplify: $-(z-11)$

Solution

$-z+11$

32) Simplify: $-(x-4)$

Solution

$-x+4$

There will be times when we'll need to use the distributive property as part of the order of operations. Start by looking at the parentheses. If the expression inside the parentheses cannot be simplified, the next step would be multiply using the distributive property, which removes the parentheses. The next two examples will illustrate this.

Example 1.7.17

Simplify: $8-2(x+3)$

Be sure to follow the order of operations. Multiplication comes before subtraction, so we will distribute the 2 first and then subtract.

Solution

***Step 1:* Distribute.**

$$8-2\times x-2\times 3$$

***Step 2:* Multiply.**

$$8-2x-6$$

***Step 3:* Combine like terms.**

$$-2x+2$$

Try It

33) Simplify: $9-3(x+2)$

Solution

$3-3x$

34) Simplify: $7x-5(x+4)$

Solution

$2x-20$

Example 1.7.18

Simplify: $4(x-8)-(x+3)$

Solution

Step 1: **Distribute.**

$$4x-32-x-3$$

Step 2: **Combine like terms.**

$$3x-35$$

Try It

35) Simplify: $6(x-9)-(x+12)$

Solution

$5x-66$

36) Simplify: $8(x-1)-(x+5)$

Solution

$7x-13$

All the properties of real numbers we have used in this chapter are summarized in the table below.

Commutative Property	
of addition: If a, b are real numbers, then	$a+b=b+a$
of multiplication: If a, b are real numbers, then	$a\times b=b\times a$
Associative Property	
of addition If a, b, c are real numbers, then	$(a+b)+c=a+(b+c)$
of multiplication If a, b, c are real numbers, then	$(a\times b)\times c=a\times(b\times c)$
Distributive Property	
If a, b, c are real numbers, then	$a(b+c)=ab+ac$
Identity Property	
of addition For any real number a: 0 is the additive identity	$a+0=a$ $0+a=a$
of multiplication For any real number a: 1 is the multiplicative identity	$1\times a=a$ $a\times 1=a$
Inverse Property	
of addition For any real number a, $-a$ is the additive inverse of a	$a+(-a)=0$

of multiplication For any real number a, $a \neq 0$ $\frac{1}{a}$ is the multiplicative inverse of a	$a \times \frac{1}{a} = 1$
Properties of Zero	
For any real number a, $a \times 0 = 0$	$a \times 0 = 0$
For any real number a,$a \neq 0$	$\frac{0}{a} = 0$ $\frac{a}{0}$ is undefined

Key Concepts

- **Commutative Property of**
  - *Addition:* If a, b, are real numbers, then $a + b = b + a$.
  - *Multiplication:* If a, b, are real numbers, then $a \cdot b = b \cdot a$. When adding or multiplying, changing the *order* gives the same result.
- **Associative Property of**
  - *Addition:* If a, b, c are real numbers, then $(a + b) + c = a + (b + c)$
  - *Multiplication:* If a, b, c are real numbers, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ When adding or multiplying, changing the *grouping* gives the same result.
- **Distributive Property:** If a, b, c are real numbers, then
  - $a(b + c) = ab + ac$
  - $(b + c)a = ba + ca$
  - $a(b - c) = ab - ac$

- $(b - c)a = ba - ca$

- **Identity Property**
  - *of Addition:* For any real number $a + 0 = a$, $0 + a = a$
    0 is the *additive identity*
  - *of Multiplication:* For any real number $a \cdot 1 = a$, $1 \cdot a = a$
    1 is the *multiplicative identity*

- **Inverse Property**
  - *of Addition:* For any real number a, $a + (-a) = 0$. A number and its *opposite* add to zero. $-a$ is the *additive inverse* of a.
  - *of Multiplication:* For any real number a, $a \neq 0$ $\frac{1}{a} = 1$. A number and its *reciprocal* multiply to one. $\frac{1}{a}$ is the *multiplicative inverse* of a.

- **Properties of Zero**
  - For any real number a,
    $a \cdot 0 = 0$, $0 \cdot a = 0$ – The product of any real number and 0 is 0.
  - $\frac{0}{a} = 0$ for $a \neq 0$ – Zero divided by any real number except zero is zero.
  - $\frac{a}{0}$ is undefined – Division by zero is undefined.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=363#h5p-7

b. After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

additive identity
The additive identity is the number 0; adding 0 to any number does not change its value.

additive inverse
The opposite of a number is its additive inverse. A number and it additive inverse add to 0.

multiplicative identity
The multiplicative identity is the number 1; multiplying 1 by any number does not change the value of the number.

multiplicative inverse
The reciprocal of a number is its multiplicative inverse. A number and its multiplicative inverse multiply to one.

1.8 UNIT SOURCES

Unit 1 Sources

1.0 "Foundations – Introduction" from Elementary Algebra 2e. by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

1.1 "Foundations – Introduction to Whole Numbers" from Elementary Algebra 2e. by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

1.2 "Foundations – use the Language of Algebra" from Elementary Algebra 2e. by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

1.3 "Foundations Add and Subtract Integers" and "Foundations Multiple and Divide Integers " from Elementary Algebra 2e. by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

1.4 "Foundations – Visualize Fractions" and "Foundations – Add and Subtract Fractions" from Elementary Algebra 2e. by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

1.5 "Foundations – Decimals" from Elementary Algebra 2e. by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

1.6 "Foundations – The Real Numbers" from Elementary Algebra 2e. by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

1.7 "Foundations – The Properties of Real Numbers" from Elementary Algebra 2e. by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

UNIT 2: MEASUREMENT

Chapter Outline

2.0 INTRODUCTION

Figure 2.0.1. The distance from Earth to the Moon may seem immense, but it is just a tiny fraction of the distance from Earth to other celestial bodies. "Crescent Moon, Earth's Atmosphere (NASA, International Space Station Science, 09/04/10)" by NASA's Marshall Space Flight Center is licensed under CC BY-NC 2.0.

The range of objects and phenomena studied in physics is immense. From the incredibly short lifetime of a nucleus to the age of the Earth, from the tiny sizes of sub-nuclear particles to the vast distance to the edges of the known universe, from the force exerted by a jumping flea to the force between Earth and the Sun, there are enough factors of 10 to challenge the imagination of even the most experienced scientist. Giving numerical values for physical quantities and equations for physical principles allows us to understand nature much more deeply than does qualitative description alone. To comprehend these vast ranges, we must also have accepted units in which to express them. And we shall find that (even in the potentially mundane discussion of meters, kilograms, and seconds) a profound simplicity of nature appears—all physical quantities can be expressed as combinations of only four fundamental physical quantities: length, mass, time, and electric current.

2.1 SYSTEMS OF MEASUREMENT

Learning Objectives

By the end of this section, you will be able to:

- Name the units in the Metric system (SI) and recognize their prefixes and their values.
- Perform metric-to-metric unit conversions using the decimal point method.
- Perform metric-to-metric unit conversions using dimensional analysis.
- Add and subtract SI units.
- Know and use the relationship between mL, g, and, cm^3.
- Recognize the U.S. system of units.
- Perform U.S. system-to-U.S. system conversions using dimensional analysis.
- Perform unit conversions (from any system) using dimensional analysis.
- Convert between Fahrenheit and Celsius temperatures.

Measurement Systems

The Metric System

Metric system (SI – international system of units): the most widely used system of measurement in the world. It is based on the basic units of meter, kilogram, second, etc.

In the metric system, units are related by powers of 10. The roots words of their names reflect this relation. For example, the basic unit for measuring length is a meter. One kilometer is 1,000 meters; the prefix *kilo* means *thousand*. One centimeter is $\frac{1}{100}$ of a meter, just like one cent is $\frac{1}{100}$ of one dollar.

SI common units:

Table 2.1.1

Quantity	Unit	Unit Symbol
Length	meter	m
Mass (or weight)	gram	kg
Volume	litre	L
Time	second	s
Temperature	degree (Celsius)	°C

Metric prefixes (SI prefixes): large and small numbers are made by adding SI prefixes, which is based on multiples of 10.

Metric conversion table:

Table 2.1.2

Prefix	Symbol (abbreviation)	Power of 10	Multiple value	
giga	G	10^{9}	1,000,000,000	1 Gm = 1,000,
mega	M	10^{6}	1,000,000	1 Mm = 1,000,
kilo-	k	10^{3}	1,000	1 km = 1,000 r
hecto-	h	10^{2}	100	1 hm = 100 m
deka-	da	10^{1}	10	1 dam = 10 m
meter/gram/litre		1 (100)		
deci-	d	10^{-1}	0.1	1 m = 10 dm
centi-	c	10^{-2}	0.01	1 m = 100 cm
milli-	m	10^{-3}	0.001	1 m = 1,000 m
micro	µ or mc	10^{-6}	0.000 001	1 m = 1,000,00
nano	n	10^{-9}	0.000 000 001	1 m = 1,000,00
pico	p	10^{-12}	0.000 000 000 001	1 m = 1,000,00

A good way to remember the order of the metric prefixes is by using a mnemonic device such as "Great Mighty King Henry died by drinking chocolate malted milk not poison". Notice that the first letter of each word reminds you of the metric prefix, and the word "by" represents the base units. Feel free to use this particular mnemonic device, or come up with your own!

Metric prefix for length, weight and volume:

Table 2.1.3

Prefix	Length (m – meter)		Weight (g – gram)		Liquid volume (L – lit	
giga (G)	Gm	(Gigameter)	Gg	(Gigagram)	GL	(Gigalitre)
mega (M)	Mm	(Megameter)	Mg	(Megagram)	ML	(Megalitre)
kilo (k)	km	(Kilometer)	kg	(Kilogram)	kL	(Kilolitre)
hecto (h)	hm	(hectometer)	hg	(hectogram)	hL	(hectolitre)
deka (da)	dam	(dekameter)	dag	(dekagram)	daL	(dekalitre)
meter/ gram/litre	m	(meter)	g	(gram)	L	(litre)
deci (d)	dm	(decimeter)	dg	(decigram)	dL	(decilitre)
centi (c)	cm	(centimeter)	cg	(centigram)	cL	(centilitre)
milli (m)	mm	(millimeter)	mg	(milligram)	mL	(millilitre)
micro (µ or mc)	µm or mcm	(micrometer)	µg or mcg	(microgram)	µL or mcL	(microlitre)
nano (n)	nm	(nanometer)	ng	(nanogram)	nL	(nanolitre)
pico (p)	pm	(picometer)	pg	(picogram)	pL	(picolitre)

The more commonly used equivalencies of measurements in the metric system are shown in Table 2.1.4. The common abbreviations for each measurement are given in parentheses. Please note, that you will need to be able to convert the units outside of this table as well.

Metric System of Measurement

Table 2.1.4

Length	Mass	Capacity
1 kilometer (km) = 1,000 m	1 kilogram (kg) = 1,000 g	1 kiloliter (kL) = 1,000 L
1 hectometer (hm) = 100 m	1 hectogram (hg) = 100 g	1 hectoliter (hL) = 100 L
1 dekameter (dam) = 10 m	1 dekagram (dag) = 10 g	1 dekaliter (daL) = 10 L
1 meter (m) = 1 m	1 gram (g) = 1 g	1 liter (L) = 1 L
1 decimeter (dm) = 0.1 m	1 decigram (dg) = 0.1 g	1 deciliter (dL) = 0.1 L
1 centimeter (cm) = 0.01 m	1 centigram (cg) = 0.01 g	1 centiliter (cL) = 0.01 L
1 millimeter (mm) = 0.001 m	1 milligram (mg) = 0.001 g	1 milliliter (mL) = 0.001 L
1 meter = 100 centimeters	1 gram = 100 centigrams	1 liter = 100 centiliters
1 meter = 1,000 millimeters	1 gram = 1,000 milligrams	1 liter = 1,000 milliliters

Performing Metric to Metric Conversions

One of the most convenient things about the metric system is that we can use its decimal nature to convert from one unit to the other simply by moving the decimal point to the left or to the right.

Steps for metric conversion:

- Identify the number of places to move the decimal point.

  – Convert a ***smaller*** unit ***to*** a ***larger*** unit: move the decimal point to the ***left***.
  – Convert a ***larger*** unit ***to*** a ***smaller*** unit: move the decimal point to the ***right***.

Example 2.1.1

326 mm = (?) m

Solution

***Step 1:* Identify mm (millimeters) and m (meters) on the conversion table.**

***Step 2:* Count places from mm to m:**

3 places left

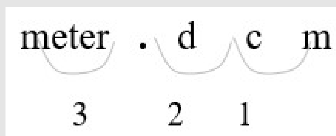


Figure 2.1.1

***Step 3:* Move 3 decimal places.**

Convert a smaller unit (mm) to a larger (m) unit: move the decimal point to the left.

$$(1m = 1000mm)$$

***Step 4:* Move the decimal point three places to the left.**

$$326.mm = 0.326m$$

Example 2.1.2

4.675 hg = (?) g

Solution

Step 1: **Identify hg (hectograms) and g (grams) on the conversion table.**
Step 2: **Count places from hg to g:**

2 places right

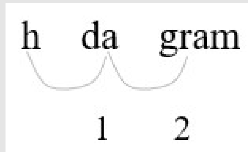


Figure 2.1.2

Step 3: **Move 2 decimal places.**

$$(1hg = 100g)$$

Step 4: **Convert a larger unit (hg) to a smaller (g) unit.**

Move the decimal point to the right.

Step 5: **Move the decimal point two places to the right.**

$$4.765hg = 476.5g$$

Try It

1) Convert 0.2744kg to micrograms.

Solution

274,400,000 mcg (or 274,400,00 µg)

2) Convert 12,940,000 nL to decilitres.

Solution

0.1294 dL

Being able to convert units by shifting the decimal point left or right is convenient and does work for a lot of our metric to metric conversions. However, as we will see later in the section, converting more complex units may be confusing if we are using the decimal point method. Thus, it is important to have an understanding of the technique of Dimensional Analysis (or the Unit Factor Method).

Dimensional Analysis (or the Unit Factor Method)

Convert units using the dimensional analysis or (the Unit Factor Method)

Step 1: **Write the original term as a fraction (over 1).**

Example: 10g can be written as $\frac{10g}{1}$

Step 2: **Write the conversion formula as a fraction, \frac{1}{( )} or \frac{( )}{1}**

Example: 1m = 100 cm can be written as $\frac{1m}{(100cm)}$ or $\frac{(100cm)}{1m}$

Step 3: **Put the desired or unknown unit on the top.**

Step 4: **Multiply the original term by \frac{1}{( )} or \frac{( )}{1} (Cancel out the same units)**

Example 2.1.3

1200 g = (?) kg

Solution

Step 1: **Write the original term (the left side) as a fraction.**

$$1200g = \frac{1200g}{1}$$

Step 2: **Write the conversion formula as a fraction.**

"kg" is the desired unit.

$$1kg = 1000g : \frac{1kg}{(1000g)}$$

Step 3: **Multiply.**

The units "g" cancel out.

$$\begin{aligned} 1200g &= \frac{1200\cancel{g}}{1} \times \frac{1kg}{(1000\cancel{g})} \\ &= \frac{1200kg}{1000} \\ &= 1.2kg \end{aligned}$$

Example 2.1.4

30 cm = (?) mm

Solution

Step 1: **Write the original term (the left side) as a fraction.**

$$30cm = \frac{30cm}{1}$$

Step 2: **Write the conversion formula as a fraction.**

"mm" is the desired unit.

$$1cm = 10mm : \frac{(10mm)}{1cm}$$

Step 3: **Multiply.**

The units "cm" cancel out.

$$30cm = \frac{30\cancel{cm}}{(1mm)} \times \frac{(10mm)}{1\cancel{cm}}$$
$$= \frac{(30)(10)mm}{1}$$
$$= 300mm$$

Try It

Use dimensional analysis to convert the following units:

3) Convert 28.4 dag to g.

Solution
284 g

4) Convert 0.00485kL to dL.

Solution
48.5 dL

Example 2.1.5

Have you ever run a 5K or 10K race? The length of those races are measured in kilometers. The metric system is commonly used in the United States when talking about the length of a race.

Nick ran a 10K race. How many meters did he run?

Solution

We will convert kilometers to meters using the identity property of multiplication.

Step 1: **Multiply the measurement to be converted by 1.**

$$10 \text{ kilometers } \times 1$$

Step 2: **Write 1 as a fraction relating kilometers and meters.**

$$10 \text{ kilometers} \cdot \frac{1,000 \text{ meters}}{1 \text{ kilometers}}$$

Step 3: **Simplify.**

$$\frac{10 \ \cancel{\text{kilometers}} \ \cdot 1,000 \ m}{1 \ \cancel{\text{kilometer}}}$$

Step 4: **Multiply.**

$10,000$ meters. Nick ran $10,000$ meters.

Try It

5) Sandy completed her first 5K race! How many meters did she run?

Solution

5,000 meters

6) Herman bought a rug 2.5 meters in length. How many centimeters is the length?

Solution

250 centimeters

Example 2.1.6

Eleanor's newborn baby weighed 3,200 grams. How many kilograms did the baby weigh?

Solution

We will convert grams into kilograms.

Step 1: **Multiply the measurement to be converted by 1.**

$$3,200 \text{ grams } \times \text{ } 1$$

Step 2: **Write 1 as a function relating kilograms and grams.**

$$3,200 \text{ grams} \cdot \frac{1kg}{1,000 \text{ grams}}$$

Step 3: **Simplify.**

$$3,200 \text{ } \cancel{\text{grams}} \cdot \frac{1kg}{1,000 \text{ } \cancel{\text{grams}}}$$

Step 4: **Multiply.**

$$\frac{3,200 \text{ kilograms}}{1,000}$$

Step 5: **Divide.**

3.2 kilograms. The baby weighed 3.2 kilograms.

Try It

7) Kari's newborn baby weighed 2,800 grams. How many kilograms did the baby weigh?

Solution

2.8 kilograms

8) Anderson received a package that was marked 4,500 grams. How many kilograms did this package weigh?

Solution
4.5 kilograms

Example 2.1.7

Samadia took 800mg of Ibuprofen for her inflammation. How many grams of Ibuprofen did she take?

Solution
We will convert milligrams to grams using the identity property of multiplication.

Step 1: **Multiply the measurement to be converted by 1.**

$$800 \text{ milligrams} \times 1$$

Step 2: **Write 1 as a fraction relating kilometres and metres.**

$$800 \text{ milligrams} \times \frac{1 \text{ gram}}{1000 \text{ milligrams}}$$

Step 3: **Simplify.**

$$800 \text{ milligrams} \times \frac{1 \text{ gram}}{1000 \text{ milligrams}}$$

Step 4: **Multiply.**

$$0.8 \text{ grams}$$

Samadia took 0.8 grams of Ibuprofen.

Example 2.1.8

Dena's recipe for lentil soup calls for 150 milliliters of olive oil. Dena wants to triple the recipe. How many liters of olive oil will she need?

Solution

We will find the amount of olive oil in millileters then convert to liters.

Step 1: **Translate to algebra.**

$$3 \times 150$$

Step 2: **Multiply.**

$$450mL$$

Step 3: **Convert to liters.**

$$450mL \times \frac{0.001L}{1mL}$$

Step 4: **Simplify.**

$$0.45L$$

Dena needs 0.45 liters of olive oil.

Try It

9) Klaudia took 0.125 grams of Ibuprofen for his headache. How many milligrams of the medication did she take?

Solution

125 milligrams

10) A recipe for Alfredo sauce calls for 250 milliliters of milk. Renata is making pasta with Alfredo sauce for a big party and needs to multiply the recipe amounts by 8. How many liters of milk will she need?

Solution
2 liters

11) To make one pan of baklava, Dorothea needs 400 grams of filo pastry. If Dorothea plans to make 6 pans of baklava, how many kilograms of filo pastry will she need?

Solution
2.4 kilograms

Example 2.1.9

The volume of blood coursing throughout an adult human body is about 5 litres. Convert it to millilitres.

Solution
We will convert litres to millilitres. In the Metric System of Measurement table, we see that 1 litre = 1,000 millilitres.

Step 1: **Multiply by 1, writing 1 as a fraction relating litres to millilitres.**

$$5L \times \frac{1000mL}{1L}$$

Step 2: **Simplify.**

$$5\,\cancel{L} \times \frac{1000mL}{1\,\cancel{L}} = 5 \times 1000mL$$

Step 3: **Multiply.**

$$5000mL$$

As we saw before, when we are converting metric to metric units, you may see a pattern. Since the system is based on multiples of ten, the calculations involve multiplying by multiples of ten. We have learned how to simplify these calculations by just moving the decimal.

Remember that to multiply by 10, 100, or 1,000, we move the decimal to the right one, two, or three places, respectively. To multiply by 0.1, 0.01, or 0.001, we move the decimal to the left one, two, or three places, respectively.

We can apply this pattern when we make measurement conversions in the metric system. In Figure 2.1.1 , we changed 3,200 grams to kilograms by multiplying by $\frac{1}{1000}$ (or 0.001). This is the same as moving the decimal three places to the left.

Figure 2.1.3

Example 2.1.10

Convert:

a. 350 L to kiloliters
b. 4.1 L to milliliters.

Solution

a. We will convert liters to kiloliters. In Table 2.1.4, we see that 1 kiloliter= 1,000 liters.

Step 1: **Multiply by 1, writing 1 as a fraction relating liters to kiloliters.**

$$350L \cdot \frac{1kL}{1,000L}$$

Step 2: **Simplify.**

$$350 \cancel{L} \cdot \frac{1kL}{1,000 \cancel{L}}$$

Step 3: **Move the decimal 3 units to the left.**

$$0.35kL$$

---

b. We will convert liters to milliliters. From Table 2.1.4 we see that 1 liter=1,000 milliliters.

Step 1: **Multiply by 1, writing 1 as a fraction relating liters to milliliters.**

$$4.1L \cdot \frac{1,000mL}{1L}$$

Step 2: **Simplify.**

$$4.1 \not{L} \cdot \frac{1,000mL}{1,000 \not{L}}$$

Step 3: **Move the decimal 3 units to the right.**

$$4rgb]1.0,0.0,0.0\overset{\rightarrow}{.}1_{rgb]1.0,0.0,0.01}0_{rgb]1.0,0.0,0.02}0_{rgb]1.0,0.0,0.03} = 4,100rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.0\ rgb]0.1,0.1,0.1mrgb]0.1,0.1,0.1L$$

Try It

12) Convert:

a. 725 L to kiloliters
b. 6.3 L to milliliters

Solution
a 7,250 kiloliters
b 6,300 milliliters

13) Convert:

a. 350 hL to liters
b. 4.1 L to centiliters

Solution
a 35,000 liters
b 410 centiliters

As we see, even when doing dimensional analysis, we can use the pattern of multiplying by powers of ten and shift our decimal point to the left or right accordingly to find our answers and make our calculations more simple. However, what might we do if we needed to convert from milligrams per decilitre to grams per litre. When we use these types of units, it can make it more difficult to simply move the decimal point to the left and right. In the following example, we see how dimensional analysis can help us stay organized and convert these types of units.

Example 2.1.11

100 m/s = (?) km/h

Solution

Step 1: **Write the original term (the left side) as a fraction.**

$$100m/s = \frac{100m}{1s}$$

Step 2: **Write the conversion formulas required as fractions.**

"km/h" is the desired unit

$$1000m = 1km \text{ and } 1h = 3600s$$
$$\frac{1km}{1000m} \text{ and } \frac{3600s}{1h}$$

Step 3: **Multiply.**

The units "m" and "s" cancel out.

$$\begin{aligned} 100m/s &= \frac{100\,\cancel{m}}{1\,\cancel{s}} \times \frac{1km}{1000\,\cancel{m}} \times \frac{3600\,\cancel{s}}{1h} \\ &= \frac{100 \times 3600km}{1 \times 1000h} \\ &= 360km/h \end{aligned}$$

Try It

14) Convert 0.000005kg/L to micrograms per decilitre.

Solution

500 mcg/dL or 500 mu g/dL.

Adding and subtracting SI measurements:

Example 2.1.12

Combine after converting to the same unit.

a. $\begin{array}{r} 3m \\ -2000mm \\ \hline \end{array}$

b. $\begin{array}{r} 25kg \\ 4g \\ \hline \end{array}$

Solution

a.

Step 1: **Convert to the same unit.**

$$1m = 1,000mm$$

Step 2: **Subtract.**

$$\begin{array}{r} 3000mm \\ -2000mm \\ \hline 1000mm \end{array}$$

---

b.

Step 1: **Convert to the same unit.**

$$1kg = 1000g$$

Step 2: **Add.**

$$\begin{array}{r} 25000g \\ 4g \\ \hline 25004g \end{array}$$

The Relationship between mL, g, and cm^3

How are mL, g, and cm^3 related?

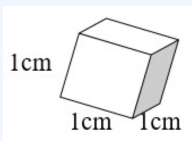


Figure 2.1.4

- A cube takes up 1 cm^3 of space (1 cm × 1 cm × 1 cm = 1cm^3).
- A cube holds 1 mL of water and has a mass of 1 gram at 4°C.
- 1 cm^3 = 1 mL = 1 g

Example 2.1.13

Convert.

a. $16cm^3$ = (?) g
b. 9 L = (?) cm^3
c. 35 cm^3 = (?) cL
d. 450 kg = (?) L

Solution
a. $16cm^3$ = (?) g

***Step 1:* Covert cm^3 to g.**

$$\begin{aligned} 1\ cm^3 &= 1g \\ 16\ cm^3 &= 16g \end{aligned}$$

---

b. 9 L = (?) cm^3

***Step 1:* Convert L to mL.**

$$\begin{aligned} 1\ L &= 1,000\ mL \\ 9\ L &= 9,000\ mL \end{aligned}$$

***Step 2:* Convert mL to cm^3**

$$\begin{aligned} 1\ mL &= 1\ cm^3 \\ &= 9000\ cm^3 \end{aligned}$$

---

c. 35 cm^3 = (?) cL

***Step 1:* Convert cm^3 to mL.**

$$\begin{aligned} 1\ cm^3 &= 1\ mL \\ 35\ cm^3 &= 35\ mL \end{aligned}$$

Step 2: **Move 1 decimal place left.**

$$= 3.5cL$$

---

d. 450 kg = (?) L

Step 1: **Convert kg to g.**

$$\begin{aligned} 1\ kg &= 1,000\ g \\ 450\ kg &= 450,000\ g \end{aligned}$$

Step 2: **Convert g to mL.**

$$\begin{aligned} 1\ g &= 1\ mL \\ &= 450,000\ mL \end{aligned}$$

Step 3: **Covert mL to L.**

$$\begin{aligned} 1\ L &= 1,000\ mL \\ &= 450\ L \end{aligned}$$

Example 2.1.14

A swimming pool measures 10 m by 8 m by 2 m. How many kilolitres of water will it hold?

Solution

Step 1: **Find the volume in m^3.**

$$160\ m^3 = (\ ?\ )\ kL$$

$$V = w \times l \times h = (8m)\ (10m)\ (2m) = 160\ m^3$$

Step 2: **Convert to cm^3**

$1m = 100cm$, $3 \times 2 = 6$, move 6 places right for volume.

$$160m^3 = 160,000,000\ cm^3$$

Step 3: **Convert to mL**

$$\begin{aligned} 1\ mL &= 1\ cm^3 \\ 160,000,000\ cm^3 &= 160,000,000\ mL \end{aligned}$$

Step 4: **Convert to kL.**

$$\begin{aligned} 160,000,000\ mL &= 160\ kL \\ 1\ kL &= 1,000,000\ mL \\ 160\ m^3 &= 160\ kL \end{aligned}$$

The swimming pool will hold 160 kL of water.

Use Mixed Units of Measurement in the Metric System

Performing arithmetic operations on measurements with mixed units of measures in the metric system requires care. Make sure to add or subtract like units.

Example 2.1.15

Ryland is 1.6 meters tall. His younger brother is 85 centimeters tall. How much taller is Ryland than his younger brother?

Solution
We can convert both measurements to either centimeters or meters. Since meters is the larger unit, we will subtract the lengths in meters. We convert 85 centimeters to meters by moving the decimal 2 places to the left.

Step 1: **Write the 85 centimeters as meters.**

85cm is 0.85m.

Step 2: **Subtract.**

$$\begin{array}{r} 1.6m \\ -0.85m \\ \hline 0.75m \end{array}$$

Ryland is 0.75 m taller than his brother.

Try It

15) Mariella is 1.58 meters tall. Her daughter is 75 centimeters tall. How much taller is Mariella than her daughter? Write the answer in centimeters.

Solution

83 centimeters

16) The fence around Hank's yard is 2 meters high. Hank is 96 centimeters tall. How much shorter than the fence is Hank? Write the answer in meters.

Solution

1.04 meters

Make Unit Conversions in the U.S. System

There are two systems of measurement commonly used around the world. Most countries use the metric system. The U.S. uses a different system of measurement, usually called the U.S. system. We will look at the U.S. system now.

The U.S. system of measurement uses units of inch, foot, yard, and mile to measure length and pound and ton to measure weight. For capacity, the units used are cup, pint, quart, and gallons. Both the U.S. system and the metric system measure time in seconds, minutes, and hours.

The equivalencies of measurements are following, also shows, in parentheses, the common abbreviations for each measurement.

Table 2.1.5

Length	1 foot (ft.)	= 12 inches (in.)	Volume	3 teaspoons (t)	= 1 tablespoon (T)
	1 yard (yd.)	= 3 feet (ft.)		16 tablespoons(T)	= 1 cup (C)
	1 mile (mi.)	= 5,280 feet (ft.)		1 cup (C)	= 8 fluid ounces (fl. oz.)
				1 pint (pt.)	= 2 cups (C)
				1 quart (qt)	= 2 pints (pt.)
				1 gallon (gal)	= 4 quarts (qt.)
Weight	1 pound (lb.)	= 16 ounces (oz.)	Time	1 minute	= 60 seconds (sec)
	1 ton	= 2000		1 hour (hr)	= 60 minutes (min)
				1 day	= 24 hours (hr)
				1 week (wk)	= 7 days
				1 year (yr)	= 365 days

In many real-life applications, we need to convert between units of measurement, such as feet and yards, minutes and seconds, quarts and gallons, etc. We will use the identity property of multiplication to do these conversions. We'll restate the identity property of multiplication here for easy reference.

Identity Property of Multiplication

For any real number a: $a \cdot 1 = a$ $1 \cdot a = a$

1 is the multiplicative identity

As we saw earlier in the section, dimensional analysis can be used to convert units. In the U.S System, since it is not a decimal system, it is best that we always use dimensional analysis to convert our units. Here, we elaborate on that concept.

To use the identity property of multiplication, we write 1 in a form that will help us convert the units. For example, suppose we want to change inches to feet. We know that 1 foot is equal to 12 inches, so we will write 1 as the fraction $\frac{1foot}{12inch}$ When we multiply by this fraction we do not change the value, but just change the units.

But $\frac{1 foot}{12 inch}$ also equals 1. How do we decide whether to multiply by $\frac{1 foot}{12 inch}$ or $\frac{1 foot}{12 inch}$? We choose the fraction that will make the units we want to convert *from* divide out. Treat the unit words like factors and "divide out" common units like we do common factors. If we want to convert 66 inches to feet, which multiplication will eliminate the inches?

$$66\ inches \cdot \frac{1\ foot}{12\ inches} \quad \text{or} \quad \cancel{66\ inches \frac{12\ inches}{1\ foot}}$$

The first form works since $66\ \cancel{inches} \cdot \frac{1\ foot}{12\ \cancel{inches}}$

The inches divide out and leave only feet. The second form does not have any units that will divide out and so will not help us.

Example 2.1.16

MaryAnne is 66 inches tall. Convert her height into feet.

Solution

***Step 1:* Multiply the measurement to be converted by 1; write 1 as a fraction relating the units given and the units needed.**

Multiply 66 inches by 1, writing 1 as a fraction relating inches and feet. We need inches in the denominator so that the inches will divide out!

$$66 \text{ inches} \times 1 = 66 \text{ inches } \times \frac{1 \text{ foot}}{12 \text{ inches}}$$

***Step 2:* Multiply.**

Think of 66 inches as $\frac{66 \text{ inches}}{1}$

$$\frac{66 \text{ inches } \times 1 \text{ foot}}{12 \text{ inches}}$$

***Step 3:* Simplify the fraction.**

Notice: inches divide out.

$$66\ \cancel{\text{inches}} \times \frac{1 \text{ foot}}{12\ \cancel{\text{inches}}} = \frac{66 \text{ inches}}{12}$$

***Step 4:* Simplify.**

Divide 66 by 12.

$$5.5feet$$

Try It

17) Lexie is 30 inches tall. Convert her height to feet.

Solution
2.5 feet

18) Rene bought a hose that is 18 yards long. Convert the length to feet.

Solution
54 feet

How To

Make Unit Conversions.

1. Multiply the measurement to be converted by 1; write 1 as a fraction relating the units given and the units needed.
2. Multiply.
3. Simplify the fraction.
4. Simplify.

When we use the identity property of multiplication to convert units, we need to make sure the units we want to change from will divide out. Usually this means we want the conversion fraction to have those units in the denominator.

Example 2.1.17

Eli's six months son is 102.4 ounces. Convert his weight to pounds.

Solution

To convert ounces into pounds we will multiply by conversion factors of 1.

Step 1: **Write 1 as** $\frac{1 \text{ pound}}{16 \text{ ounces}}$.

$$102.4 \text{ ounces} \times \frac{1 \text{ pound}}{16 \text{ ounces}}$$

Step 2: **Divide out the common units.**

$$102.4 \cancel{\text{ ounces}} \times \frac{1 \text{ pound}}{16 \cancel{\text{ ounces}}}$$

Step 3: **Simplify the fraction.**

$$\frac{102.4 \text{ ounces}}{16 \text{ ounces}}$$

Step 4: **Simplify.**

$$6.4 \text{ pounds}$$

Eli's six months son weights 6.4 pounds.

Example 2.1.18

Ndula, an elephant at the San Diego Safari Park, weighs almost 3.2 tons. Convert her weight to pounds.

Solution

We will convert 3.2 tons into pounds. We will use the identity property of multiplication, writing 1 as the fraction:

$$\frac{2000 \text{ pounds}}{1 \text{ ton}}$$

Step 1: **Multiply the measurement to be converted, by 1.**

$$3.2 \text{ tons} \times 1$$

Step 2: **Write 1 as a fraction relating tons and pounds.**

$$3.2 \text{ tons} \times \frac{2,000 \text{ pounds}}{1 \text{ ton}}$$

Step 3: **Simplify.**

$$\frac{3.2 \ \cancel{\text{tons}} \ \times \ 2,000 \text{ pounds}}{1 \ \cancel{\text{ton}}}$$

Step 4: **Multiply.**

$$6,400 \text{ pounds}$$

Try It

19) Arnold's SUV weighs about 4.3 tons. Convert the weight to pounds.

Solution

8,600 pounds

20) The Carnival *Destiny* cruise ship weighs 51,000 tons. Convert the weight to pounds.

Solution

102,000,000 pounds

21) One year old girl weights 11 pounds. Convert her weight to ounces.

Solution

176 ounces.

As was the case with the metric system, sometimes, to convert from one unit to another, we may need to use several other units in between, so we will need to multiply several fractions.

Example 2.1.19

Juliet is going with her family to their summer home. She will be away from her boyfriend for 9 weeks. Convert the time to minutes.

Solution

To convert weeks into minutes we will convert weeks into days, days into hours, and then hours into minutes. To do this we will multiply by conversion factors of 1.

***Step 1:* Write 1 as** $\frac{7 \text{ days}}{1 \text{ week}}$ **,** $\frac{24 \text{ hours}}{1 \text{ day}}$**, and** $\frac{60 \text{ minutes}}{1 \text{ hour}}$**.**

$$\frac{9wk}{1} \times \frac{7days}{1wk} \times \frac{24hr}{1day} \times \frac{60min}{1hr}$$

***Step 2:* Divide out the common units.**

$$\frac{9 \cancel{wk}}{1} \cdot \frac{7\ rgb]0.0,0.0,1.0 \cancel{days}}{1 \cancel{wk}} \cdot \frac{24\ rgb]1.0,0.0,0.0 \cancel{hr}}{1\ rgb]0.0,0.0,1.0 \cancel{day}} \cdot \frac{60\ min}{1\ rgb]1.0,0.0,0.0 \cancel{hr}}$$

***Step 3:* Multiply.**

$$\frac{9 \times 7 \times 24 \times 60 \text{ min}}{1 \times 1 \times 1 \times 1} = 90,720 \text{ minutes}$$

***Step 4:* Multiply.**

Juliet and her boyfriend will be apart for 90,720 minutes (although it may seem like an eternity!).

Try It

22) The distance between the earth and the moon is about 250,000 miles. Convert this length to yards.

Solution
440,000,000 yards

23) The astronauts of Expedition 28 on the International Space Station spend 15 weeks in space. Convert the time to minutes.

Solution
151,200 minutes

Example 2.1.20

How many ounces are in 1 gallon?

Solution
We will convert gallons to ounces by multiplying by several conversion factors. Refer to Table 2.1.5.

Step 1: **Multiply the measurement to be converted by 1.**

$$\frac{1 \text{ gallon}}{1} \times \frac{4 \text{ quarts}}{1 \text{ gallon}} \times \frac{2 \text{ pints}}{1 \text{ quart}} \times \frac{2 \text{ cups}}{1 \text{ pint}} \times \frac{8 \text{ ounces}}{1 \text{ cup}}$$

Step 2: **Use conversion factors to get to the right unit.**

Simplify.

$$\frac{1 \cancel{\text{ gallon}}}{1} \times \frac{4 \cancel{\textit{quarts}}}{1 \cancel{\text{ gallon}}} \times \frac{2 \cancel{\text{ pints}}}{1 \cancel{\text{ quarts}}} \times \frac{2 \cancel{\text{ cups}}}{1 \cancel{\text{ pint}}} \times \frac{8 \text{ ounces}}{1 \cancel{\text{ cup}}}$$

***Step 3:* Multiply.**

$$\frac{1 \times 4 \times 2 \times 2 \times 8 \text{ ounces}}{1 \times 1 \times 1 \times 1 \times 1}$$

***Step 4:* Simplify.**

$$128 \text{ ounces}$$

There are 128 ounces in a gallon.

Try It

24) How many cups are in 1 gallon?

Solution

16 cups

25) How many teaspoons are in 1 cup?

Solution

48 teaspoons

Use Mixed Units of Measurement in the U.S. System

We often use mixed units of measurement in everyday situations. Suppose Joe is 5 feet 10 inches tall, stays at work for 7 hours and 45 minutes, and then eats a 1 pound 2 ounce steak for dinner—all these measurements have mixed units.

Performing arithmetic operations on measurements with mixed units of measures requires care. Be sure to add or subtract like units!

Example 2.1.21

Seymour bought three steaks for a barbecue. Their weights were 14 ounces, 1 pound 2 ounces and 1 pound 6 ounces. How many total pounds of steak did he buy?

Solution

We will add the weights of the steaks to find the total weight of the steaks.

Step 1: **Add the ounces. Then add the pounds.**

$$\begin{aligned} & + \text{ 14 ounces} \\ \text{1 pound} & + \text{ 2 ounces} \\ \text{1 pound} & + \text{ 6 ounces} \\ = \text{ 2 pounds} & + \text{ 22 ounces} \end{aligned}$$

Step 2: **Convert 22 ounces to pounds and ounces.**

2 pounds 1 pound, 6 ounces.

Step 3: **Add the pounds.**

3 pounds, 6 ounces.

Seymour bought 3 pounds 6 ounces of steak.

Try It

26) Laura gave birth to triplets weighing 3 pounds 3 ounces, 3 pounds 3 ounces, and 2 pounds 9 ounces. What was the total birth weight of the three babies?

Solution

9 lbs. 8 oz

27) Stan cut two pieces of crown moulding for his family room that were 8 feet 7 inches and 12 feet 11 inches. What was the total length of the moulding?

Solution

21 ft. 6 in.

Example 2.1.22

Anthony bought four planks of wood that were each 6 feet 4 inches long. What is the total length of the wood he purchased?

Solution

We will multiply the length of one plank to find the total length.

Step 1: **Multiply the inches and then the feet.**

$$\begin{array}{rrr} & 6\ feet & 4\ inches \\ \times & & 4 \\ \hline & 24\ feet & 16\ inches \end{array}$$

Step 2: **Convert the 16 inches to feet.**

$$\text{16 inches} - \text{12 inches} = rgb]1.0,0.0,0.01rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0\text{foot}rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0{+}rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.04rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0\text{inches}rgb]1.0,0.0,0.0$$
$$rgb]1.0,0.0,0.01rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0\text{foot}rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0{+}rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.04rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0\text{inches} + \text{24 feet} = \text{25 feet} + \text{4 inches}$$

Step 3: **Add the feet.**

Anthony bought 25 feet and 4 inches of wood.

Try It

28) Henri wants to triple his spaghetti sauce recipe that uses 1 pound 8 ounces of ground turkey. How many pounds of ground turkey will he need?

Solution

4 lbs. 8 oz.

29) Joellen wants to double a solution of 5 gallons 3 quarts. How many gallons of solution will she have in all?

Solution

11 gallons 2 qt.

Convert Between the U.S. and the Metric Systems of Measurement

Many measurements in the United States are made in metric units. Our soda may come in 2-liter bottles, our calcium may come in 500-mg capsules, and we may run a 5K race. To work easily in both systems, we need to be able to convert between the two systems.

The table below shows some of the most common conversions.

Conversion Factors Between U.S. and Metric Systems

Table 2.1.6

Length	Mass	Capacity
1 in. = 2.54 cm	1 lb. = 0.45 kg	1 qt. = 0.95 L
1 ft. = 0.305 m	1 oz. = 28 g	1 fl. oz. = 30 mL
1 yd. = 0.914 m	1 kg = 2.2 lb.	1 L = 1.06 qt.
1 mi. = 1.61 km		
1 m = 3.28 ft.		

Figure 2.1.2 shows how inches and centimeters are related on a ruler.

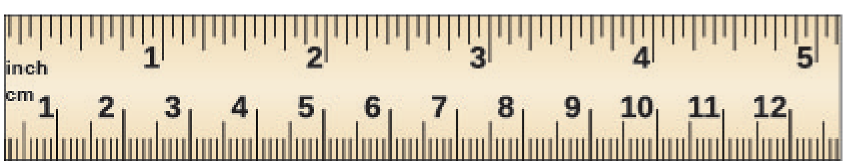


Figure 2.1.5: This ruler shows inches and centimeters.

Figure 2.1.3 shows the ounce and milliliter markings on a measuring cup.

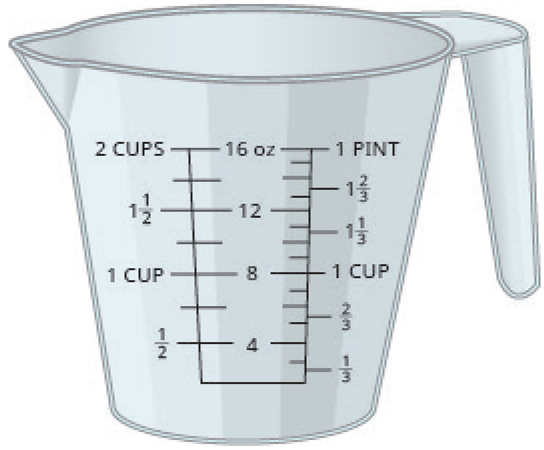


Figure 2.1.6: This measuring cup shows ounces and milliliters.

Figure 2.1.4 shows how pounds and kilograms marked on a bathroom scale.

Figure 2.1.7: This scale shows pounds and kilograms.

We make conversions between the systems just as we do within the systems—by multiplying by unit conversion factors.

Example 2.1.23

The plastic bag used for transfusion holds 500 mL of packed red cells. How many ounces are in the bag? Round to the nearest tenth of an ounce.

Solution

Step 1: **Multiply by a unit conversion factor relating mL and ounces.**

$$500 \text{ millilitres} \cdot \frac{1 \text{ ounce}}{30 \text{ millilitres}}$$

Step 2: **Simplify.**

$$\frac{50 \text{ ounces}}{30}$$

Step 3: **Divide.**

$$16.7 \text{ ounces}$$

The plastic bag has 16.7 ounces of packed red cells.

Try It

30) Adam donated 450 ml of blood. How many ounces is that?

Solution

15 ounces.

31) How many quarts of soda are in a 2-L bottle?

Solution

2.12 quarts

32)How many liters are in 4 quarts of milk?

Solution

3.8 liters

Example 2.1.24

Soleil was on a road trip and saw a sign that said the next rest stop was in 100 kilometers. How many miles until the next rest stop?

Solution

***Step 1:* Multiply by a unit conversion factor relating km and mi.**

$$100 \text{ kilometers} \cdot \frac{1 \text{ mile}}{1.61 \text{ kilometer}}$$

***Step 2:* Simplify.**

$$\frac{100 \text{ miles}}{1.61}$$

***Step 3:* Divide.**

$$62 \text{ miles}$$

Soleil will travel 62 miles.

Example 2.1.25

A human brain weights about 3 pounds. How many kilograms is that? Round to the nearest tenth of a kilogram.

Solution

***Step 1:* Multiply by a unit conversion factor relating km and mi.**

$$3 \text{ pounds} \times \frac{1 \text{ kilogram}}{2.2 \text{ pounds}}$$

***Step 2:* Simplify.**

$$\frac{3 \text{ kilograms}}{2.2}$$

Step 3: **Divide.**

$$1.4 \text{ kilograms}$$

A human brain weights around 1.4 kilograms.

Try It

33) A human liver normally weights approximately 1.5 kilograms. Convert it to pounds.

Solution

3.3 pounds

34) The height of Mount Kilimanjaro is 5,895 meters. Convert the height to feet.

Solution

19,335.6 feet

35) The flight distance from New York City to London is 5,586 kilometers. Convert the distance to miles.

Solution

8,993.46 km

Convert between Fahrenheit and Celsius Temperatures

Have you ever been in a foreign country and heard the weather forecast? If the forecast is for 22°C what does that mean?

The U.S. and metric systems use different scales to measure temperature. The U.S. system uses degrees Fahrenheit, written °F The metric system uses degrees Celsius, written °C. Figure 2.1.5 shows the relationship between the two systems. The diagram shows normal body temperature, along with the freezing and boiling temperatures of water in degrees Fahrenheit and degrees Celsius.

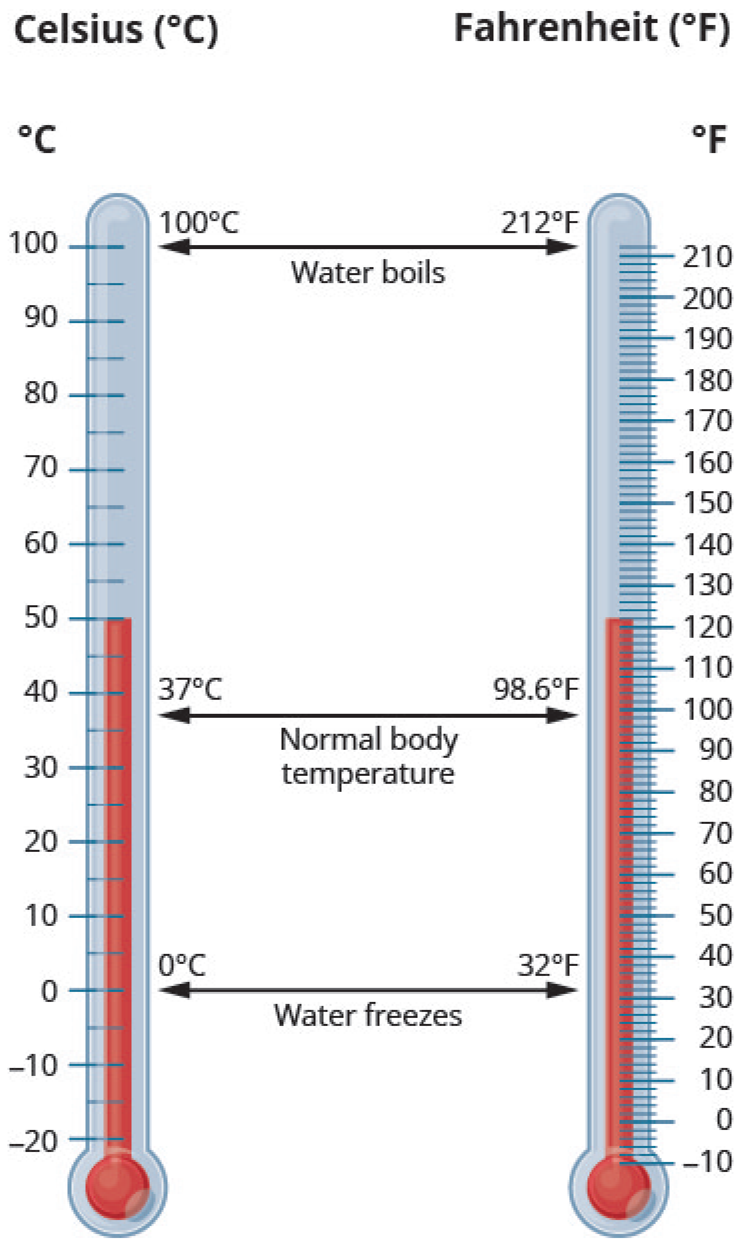


Figure 2.1.8

Temperature Conversion

To convert from Fahrenheit temperature, F, to Celsius temperature, C, use the formula:

$$C = \frac{5}{9}(F - 32)$$

To convert from Celsius temperature, C, to Fahrenheit temperature, F, use the formula:

$$F = \frac{9}{5}C32$$

Example 2.1.26

Convert 50° Fahrenheit into degrees Celsius.

Solution
We will substitute 50°F into the formula to find C.

Step 1: **Substitute 50 for F.**

$$C = \frac{5}{9}(rgb]1.0, 0.0, 0.050 - 32)$$

Step 2: **Simplify in parentheses.**

$$C = \frac{5}{9}(18)$$

Step 3: **Multiply.**

$$C = 10$$

So we found that 50°F is equivalent to 10°C.

Example 2.1.27

Before mixing, the Pfizer-BioNTech COVID-19 vaccine may be stored in an ultra-cold freezer between -112°F and -76°F. Convert the temperatures into degrees Celsius.

Solution

We will substitute a) -112°F and b) -76°F into the formula to find C.

a.

Step 1: **Substitute -112 for F.**

$$C = \frac{5}{9}(-112 - 32)$$

Step 2: **Simplify in parentheses.**

$$C = \frac{5}{9}(-144)$$

Step 3: **Multiply.**

$$C = -80$$

So we found that -112°F is equivalent to -80°C

---

b.

Step 1: **Substitute -76 for F.**

$$C = \frac{5}{9}(-76 - 32)$$

Step 2: **Simplify.**

$$C = \frac{5}{9}(-108)$$
$$C = -60$$

So we found that -76°F is equivalent to – 60°C.

Try It

36) Convert the Fahrenheit temperature to degrees Celsius: 59° Fahrenheit.

Solution

15°C

37) Convert the Fahrenheit temperature to degrees Celsius: 41° Fahrenheit.

Solution

5°C

Example 2.1.28

While visiting Paris, Woody saw the temperature was 20° Celsius. Convert the temperature into degrees Fahrenheit.

Solution

We will substitute 20°C into the formula to find F.

***Step 1:* Substitute 20 for C.**

$$F = \frac{9}{5}(rgb]1.0, 0.0, 0.020)\ 32$$

***Step 2:* Multiply.**

$$F = 3632$$

***Step 3:* Add.**

$$F = 68$$

So we found that 20°C is equivalent to 68°F.

Example 2.1.29

Once mixed, the Pfizer-BioNTech COVID-19 vaccine can be left at room temperature 2°C to 25°C. Convert the temperatures into degrees Fahrenheit.

Solution

We will substitute a) 2°C and b) 25°C into the formula to find F.

a.

Step 1: **Substitute 2 for C.**

$$F = \frac{9}{5} \times 232$$

Step 2: **Simplify.**

$$F = 35.6$$

So we found that 2°C is equivalent to 35.6°F.

---

b.

Step 1: **Substitute 25 for C.**

$$F = \frac{9}{5} \times 2532$$

Step 2: **Simplify.**

$$F = 77$$

So we found that 25°C is equivalent to 77°F.

Try It

38) Convert the Celsius temperature to degrees Fahrenheit: the temperature in Helsinki, Finland, was 15° Celsius.

Solution

59°F

39) Convert the Celsius temperature to degrees Fahrenheit: the temperature in Sydney, Australia, was 10° Celsius.

Solution

50°F

Key Concepts

- **Metric System of Measurement**

Metric System of Measurement

Length	Mass	Capacity
1 kilometer (km) = 1,000 m	1 kilogram (kg) = 1,000 g	1 kiloliter (kL) = 1,000 L
1 hectometer (hm) = 100 m	1 hectogram (hg) = 100 g	1 hectoliter (hL) = 100 L
1 dekameter (dam) = 10 m	1 dekagram (dag) = 10 g	1 dekaliter (daL) = 10 L
1 meter (m) = 1 m	1 gram (g) = 1 g	1 liter (L) = 1 L
1 decimeter (dm) = 0.1 m	1 decigram (dg) = 0.1 g	1 deciliter (dL) = 0.1 L
1 centimeter (cm) = 0.01 m	1 centigram (cg) = 0.01 g	1 centiliter (cL) = 0.01 L
1 millimeter (mm) = 0.001 m	1 milligram (mg) = 0.001 g	1 milliliter (mL) = 0.001 L
1 meter = 100 centimeters	1 gram = 100 centigrams	1 liter = 100 centiliters
1 meter = 1,000 millimeters	1 gram = 1,000 milligrams	1 liter = 1,000 milliliters

- **Temperature Conversion**
  - To convert from Fahrenheit temperature, F, to Celsius temperature, C, use the formula $C = \frac{5}{9}(F - 32)$
  - To convert from Celsius temperature, C, to Fahrenheit temperature, F, use the formula $F = \frac{9}{5}C32$

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it

online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=6020#h5p-34

b. Overall, after looking at the checklist, do you think you are well-prepared for the next Chapter? Why or why not?

2.2 ACCURACY, PRECISION, AND ROUNDING RULES

Learning Objectives

By the end of this section, you will be able to:

- Identify exact and inexact numbers.
- Recognize the number of significant figures in a given quantity.
- Identify the number of decimal places.
- Understand the difference between accuracy and precision.
- Calculate the percent uncertainty of a measurement.
- Apply the concept of significant figures to limit mathematical results to the proper number of digits.

Exact and Inexact Numbers

When considering measurement, we must consider the nature of the numbers that we are using in our calculations.

A number is exact if it is known with complete certainty. An exact number is a number that you can get by counting. Exact numbers have infinitely many significant figures and decimal places. For example, there are exactly 100 centimeters in one meter and exactly 4 quarts in a gallon.

A number is inexact if it has **uncertainty** associated with it. This uncertainty can arise due to measurement or rounding. Some examples would be a person's height or weight.

Accuracy and Precision of a Measurement

Science is based on observation and experiment—that is, on measurements. **Accuracy** is how close a measurement is to the correct value for that measurement. For example, let us say that you are measuring the length of standard computer paper. The packaging in which you purchased the paper states that it is 11.0 inches long. You measure the length of the paper three times and obtain the following measurements: 11.1 in., 11.2 in., and 10.9 in. These measurements are quite accurate because they are very close to the correct value of 11.0 inches. In contrast, if you had obtained a measurement of 12 inches, your measurement would not be very accurate.

The **precision** of a measurement system is refers to how close the agreement is between repeated measurements (which are repeated under the same conditions). Consider the example of the paper measurements. The precision of the measurements refers to the spread of the measured values. One way to analyze the precision of the measurements would be to determine the range, or difference, between the lowest and the highest measured values. In that case, the lowest value was 10.9 in. and the highest value was 11.2 in. Thus, the measured values deviated from each other by at most 0.3 in. These measurements were relatively precise because they did not vary too much in value. However, if the measured values had been 10.9, 11.1, and 11.9, then the measurements would not be very precise because there would be significant variation from one measurement to another.

The measurements in the paper example are both accurate and precise, but in some cases, measurements are accurate but not precise, or they are precise but not accurate. Let us consider an example of a GPS system that is attempting to locate the position of a restaurant in a city. Think of the restaurant location as existing at the centre of a bull's-eye target, and think of each GPS attempt to locate the restaurant as a black dot. In Figure 2.2.1 you can see that the GPS measurements are spread out far apart from each other, but they are all relatively close to the actual location of the restaurant at the centre of the target. This indicates a low precision, high accuracy measuring system. However, in Figure 2.2.2 the GPS measurements are concentrated quite closely to one another, but they are far away from the target location. This indicates a high precision, low accuracy measuring system.

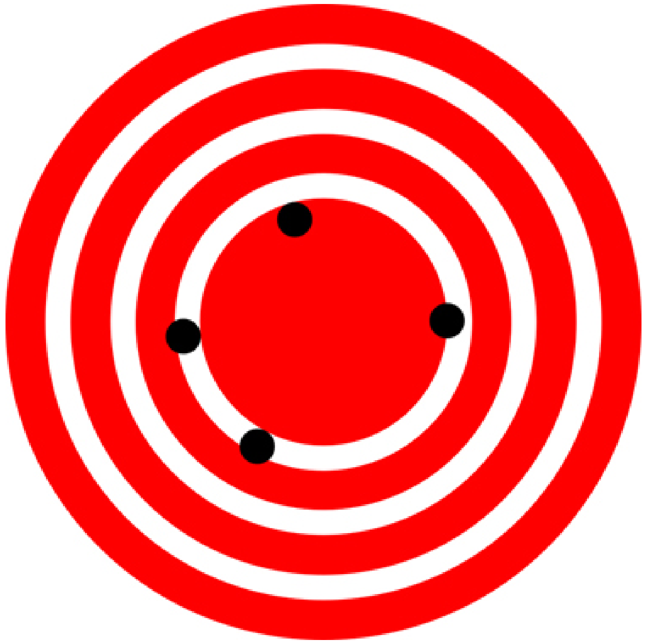


Figure 2.2.1**.** A GPS system attempts to locate a restaurant at the center of the bull's-eye. The black dots represent each attempt to pinpoint the location of the restaurant. The dots are spread out quite far apart from one another, indicating low precision, but they are each rather close to the actual location of the restaurant, indicating high accuracy. (credit: Dark Evil).

Figure 2.2.2. In this figure, the dots are concentrated rather closely to one another, indicating high precision, but they are rather far away from the actual location of the restaurant, indicating low accuracy. (credit: Dark Evil).

Accuracy, Precision, and Uncertainty

The degree of accuracy and precision of a measuring system are related to the **uncertainty** in the measurements. Uncertainty is a quantitative measure of how much your measured values deviate from a standard or expected value. If your measurements are not very accurate or precise, then the uncertainty of your values will be very high. In more general terms, uncertainty can be thought of as a disclaimer for your measured values. For example, if someone asked you to provide the mileage on your car, you might say that it is 45,000 miles, plus or minus 500 miles. The plus or minus amount is the uncertainty in your value. That is, you are indicating that the actual mileage of your car might be as low as 44,500 miles or as high as 45,500 miles, or anywhere in between. All measurements contain some amount of uncertainty. In our example of measuring the length of the paper, we might say that the length of the paper is 11 in., plus or minus 0.2 in. The uncertainty in a measurement, A, is often denoted as δA ("delta A"), so the measurement result would be recorded as $A \pm \delta A$. In our paper example, the length of the paper could be expressed as $11 in. \pm 0.2$.

The factors contributing to uncertainty in a measurement include:

1. Limitations of the measuring device,
2. The skill of the person making the measurement,
3. Irregularities in the object being measured,
4. Any other factors that affect the outcome (highly dependent on the situation).

In our example, such factors contributing to the uncertainty could be the following: the smallest division on the ruler is 0.1 in., the person using the ruler has bad eyesight, or one side of the paper is slightly longer than the other. At any rate, the uncertainty in a measurement must be based on a careful consideration of all the factors that might contribute and their possible effects.

Making Connections: Real World Connections – Fever or Chills?

Uncertainty is a critical piece of information, both in physics and in many other real-world applications. Imagine you are caring for a sick child. You suspect the child has a fever, so you check his or her temperature with a thermometer. What if the uncertainty of the thermometer were 3.0 °C? If the child's temperature reading was 37.0 °C (which is normal body temperature), the "true" temperature could be anywhere from a hypothermic 34.0 °C to a dangerously high 40.0 °C. A thermometer with an uncertainty of 3.0 °C would be useless.

Percent Uncertainty

One method of expressing uncertainty is as a percent of the measured value. If a measurement A is expressed with $uncertainty$, δA, the **percent uncertainty** (%unc) is defined to be:

$$\mathbf{\%\ unc} = \frac{\boldsymbol{\delta}\mathbf{A}}{\mathbf{A}} \times \mathbf{100\%}$$

Example 2.2.1

A grocery store sells 5-lb bags of apples. You purchase four bags over the course of a month and weigh the apples each time. You obtain the following measurements:

- Week 1 weight: **4.8 lb**
- Week 2 weight: **5.3 lb**
- Week 3 weight: **4.9 lb**
- Week 4 weight: **5.4 lb**

You determine that the weight of the 5-lb bag has an uncertainty of ±0.4 lb. What is the percent uncertainty of the bag's weight?

Strategy

First, observe that the expected value of the bag's weight, A, is 5 lb. The uncertainty in this value, δA, is 0.4 lb. We can use the following equation to determine the percent uncertainty of the weight:

$$\mathbf{\%\ unc = \frac{\delta A}{A} \times 100\%}$$

Solution

Plug the known values into the equation:

$$\begin{aligned} \%\ \text{unc} &= \frac{0.4\ \text{lb}}{5\ \text{lb}} \times\ 100\% \\ \%\ \text{unc} &= 8\% \end{aligned}$$

Discussion

We can conclude that the weight of the apple bag is 5 lb ±8%. Consider how this percent uncertainty would change if the bag of apples were half as heavy, but the uncertainty in the weight remained the same. Hint for future calculations: when calculating percent uncertainty, always remember that you must multiply the fraction by 100%. If you do not do this, you will have a decimal quantity, not a percent value.

Uncertainties in Calculations

There is an uncertainty in anything calculated from measured quantities. For example, the area of a floor calculated from measurements of its length and width has an uncertainty because the length and width have uncertainties. How big is the uncertainty in something you calculate by multiplication or division? If the measurements going into the calculation have small uncertainties (a few percent or less), then the **method of adding percents** can be used for multiplication or division. This method says that *the percent uncertainty in a quantity calculated by multiplication or division is the sum of the percent uncertainties in the items used to make the calculation*. For example, if a floor has a length of 4.00 m and a width of 3.00 m, with uncertainties of 2% and 1%, respectively, then the area of the floor is 12.0 m and has an uncertainty of 3%. (Expressed as an area this is 0.36 m^2, which we round to 0.4 m^2 since the area of the floor is given to a tenth of a square meter.)

Try It

1) A high school track coach has just purchased a new stopwatch. The stopwatch manual states that the stopwatch has an uncertainty of ±0.05 s. Runners on the track coach's team regularly clock 100-m sprints of 11.49 s to 15.01 s. At the school's last track meet, the first-place sprinter came in at 12.04 s and the second-place sprinter came in at 12.07 s. Will the coach's new stopwatch be helpful in timing the sprint team? Why or why not?

Solution

No, the uncertainty in the stopwatch is too great to effectively differentiate between the sprint times.

Precision of Measuring Tools and Significant Figures

An important factor in the accuracy and precision of measurements involves the precision of the measuring tool. In general, a precise measuring tool is one that can measure values in very small increments. For example, a standard ruler can measure length to the nearest millimeter, while a caliper can measure length to the nearest 0.01 millimeter. The caliper is a more precise measuring tool because it can measure extremely small differences in length. The more precise the measuring tool, the more precise and accurate the measurements can be.

When we express measured values, we can only list as many digits as we initially measured with our measuring tool. For example, if you use a standard ruler to measure the length of a stick, you may measure it to be 36.7 cm. You could not express this value as 36.71 cm because your measuring tool was not precise enough to measure a hundredth of a centimeter. It should be noted that the last digit in a measured value has been estimated in some way by the person performing the measurement. For example, the person measuring the length of a stick with a ruler notices that the stick length seems to be somewhere in between 36.6 cm and 36.7 cm, and he or she must estimate the value of the last digit. Using the method of significant figures, the rule is that *the last digit written down in a measurement is the first digit with some uncertainty*. In order to determine the number of significant digits in a value, start with the first measured value at the left and count the number of digits through the last digit written on the right. For example, the measured value 36.7 cm has three digits, or significant figures. Significant figures indicate the precision of a measuring tool that was used to measure a value.

Significant Figures: Zeros

Special consideration is given to zeros when counting significant figures. The zeros in 0.053 are not significant, because they are only placekeepers that locate the decimal point. There are two significant figures in 0.053. The zeros in 10.053 are not placekeepers but are significant—this number has five significant figures. The zeros in 1300 may or may not be significant depending on the style of writing numbers. They could mean the number is known to the last digit, or they could be placekeepers. So 1300 could have two, three, or four significant figures. (To avoid this ambiguity, write 1300 in scientific notation.) *Zeros are significant except when they serve only as placekeepers*.

Try It

2) Determine the number of significant figures in the following measurements:

a) 0.0009
b) 15,450.0
c) 6×10^3
d) 87.990
e) 30.42

Solution

a) 1
b) 6
c) 1
d) 5
e) 4

As you have probably realized by now, the biggest issue in determining the number of significant figures in a value is the zero. Is the zero significant or not? One way to unambiguously determine whether a zero is significant or not is to write a number in scientific notation. Scientific notation will include zeros in the coefficient of the number *only if they are significant*. Thus, the number 8.666×10^6 has four significant figures. However, the number 8.6660×10^6 has five significant figures. That last zero is significant; if it were not, it would not be written in the coefficient. So when in doubt about expressing the number of significant figures in a quantity, use scientific notation and include the number of zeros that are truly significant. We will learn more about scientific notation in a later section in the text.

Significant Figures

If you use a calculator to evaluate the expression $\frac{337}{217}$, you will get the following:

$$\frac{337}{217} = 1.55299539171...$$

and so on for many more digits. Although this answer is correct, it is somewhat presumptuous. You start with two values that each have three digits, and the answer has *twelve* digits? That does not make much sense from a strict numerical point of view.

Consider using a ruler to measure the width of an object, as shown in Figure 2.2.3 “Expressing Width”. The object is definitely more than 1 cm long, so we know that the first digit in our measurement is 1. We see by counting the tick marks on the ruler that the object is at least three ticks after the 1. If each tick represents 0.1 cm, then we know the object is at least 1.3 cm wide. But our ruler does not have any more ticks between the 0.3 and the 0.4 marks, so we can’t know exactly how much the next decimal place is. But with a practised eye we can estimate it. Let us estimate it as about six-tenths of the way between the third and fourth tick marks, which estimates our hundredths place as 6, so we identify a measurement of 1.36 cm for the width of the object.

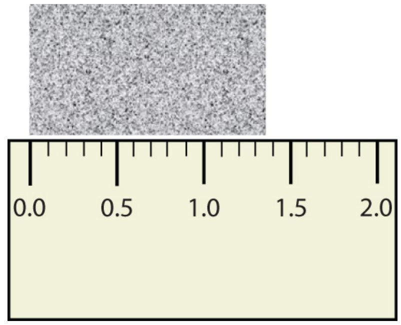


Figure 2.2.3. Expressing Width

What is the proper way to express the width of this object?

Does it make any sense to try to report a thousandths place for the measurement? No, it doesn't; we are not exactly sure of the hundredths place (after all, it was an estimate only), so it would be fruitless to estimate a thousandths place. Our best measurement, then, stops at the hundredths place, and we report 1.36 cm as proper measurement.

This concept of reporting the proper number of digits in a measurement or a calculation is called **significant figures**. Significant figures (sometimes called significant digits) represent the limits of what values of a measurement or a calculation we are sure of. The convention for a measurement is that the quantity reported should be all known values and the first estimated value. The conventions for calculations are discussed as follows.

Example 2.2.2

Use each diagram to report a measurement to the proper number of significant figures.

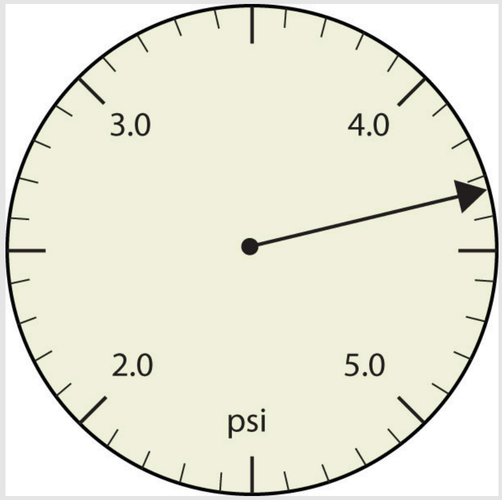
a. 

Figure 2.2.4. Pressure gauge in units of pounds per square inch

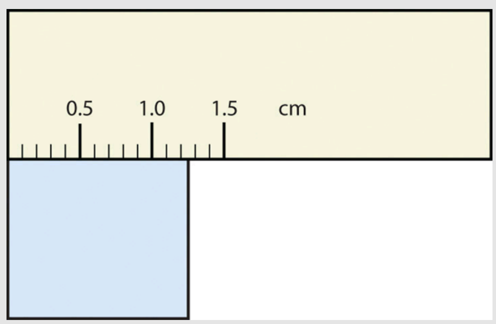
b. 

Figure 2.2.5. A measuring ruler

Solution

a. The arrow is between 4.0 and 5.0, so the measurement is at least 4.0. The arrow is between the third and fourth small tick marks, so it's at least 0.3. We will have to estimate the last place. It looks like about one-third of the way across the space, so let us estimate the hundredths place as 3. Combining the digits, we have a measurement of 4.33 psi (psi stands for "pounds per square inch" and is a unit of pressure, like air in a tire). We say that the measurement is reported to three significant figures.
b. The rectangle is at least 1.0 cm wide but certainly not 2.0 cm wide, so the first significant digit is 1. The rectangle's width is past the second tick mark but not the third; if each tick mark represents 0.1, then the rectangle is at least 0.2 in the next significant digit. We have to

estimate the next place because there are no markings to guide us. It appears to be about halfway between 0.2 and 0.3, so we will estimate the next place to be a 5. Thus, the measured width of the rectangle is 1.25 cm. Again, the measurement is reported to three significant figures

Try It

3) What would be the reported width of this rectangle?

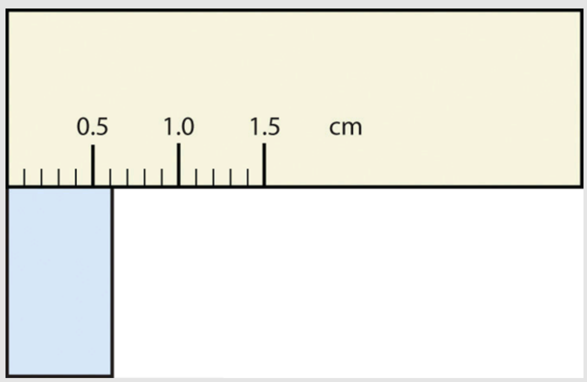


Figure 2.2.6. A measuring ruler

Solution

0.63cm

In many cases, you will be given a measurement. How can you tell by looking what digits are significant? For example, the reported population of the United States is 306,000,000. Does that mean that it is *exactly* three hundred six million or is some estimation occurring?

The following conventions dictate which numbers in a reported measurement are significant and which are not significant:

1. Any nonzero digit is significant.
2. Any zeros between nonzero digits (i.e., embedded zeros) are significant.
3. Zeros at the end of a number without a decimal point (i.e., trailing zeros) are not significant; they serve only to put the significant digits in the correct positions. However, zeros at the end of any number with a decimal point are significant.
4. Zeros at the beginning of a decimal number (i.e., leading zeros) are not significant; again, they serve only to put the significant digits in the correct positions.

So, by these rules, the population figure of the United States has only three significant figures: the 3, the 6, and the zero between them. The remaining six zeros simply put the 306 in the millions position.

Example 2.2.3

Give the number of significant figures in each measurement.

a. 36.7 m
b. 0.006606 s
c. 2,002 kg
d. 306,490,000 people

Solution

a. By rule 1, all nonzero digits are significant, so this measurement has three significant figures.
b. By rule 4, the first three zeros are not significant, but by rule 2 the zero between the sixes is; therefore, this number has four significant figures.
c. By rule 2, the two zeros between the twos are significant, so this measurement has four significant figures.

d. The four trailing zeros in the number are not significant, but the other five numbers are, so this number has five significant figures.

Try It

Give the number of significant figures in each measurement.

4) 0.000601 m

Solution

3 significant figures

5) 65.080 kg

Solution

5 significant figures

Significant Figures in Calculations

How are significant figures handled in calculations? It depends on what type of calculation is being performed. If the calculation is an addition or a subtraction, the rule is as follows: limit the reported answer to the rightmost column that all numbers have significant figures in common. For example, if you were to add 1.2 and 4.71, we note that the first number stops its significant figures in the tenths column, while the second number stops its significant figures in the hundredths column. We therefore limit our answer to the tenths column.

1.2
4.41
5.61
↑ limit final answer to the tenths column: 5.6

Figure 2.2.7.

We drop the last digit—the 1—because it is not significant to the final answer.

The dropping of positions in sums and differences brings up the topic of rounding. Although there are several conventions, in this text we will adopt the following rule: the final answer should be rounded up if the first dropped digit is 5 or greater and rounded down if the first dropped digit is less than 5.

77.2
<u>10.46</u>
87.66
↑ limit final answer to the tenths column and round up: 87.7

Figure 2.2.8.

Example 2.2.4

Express the final answer to the proper number of significant figures.

a. $101.2 + 18.702 = ?$
b. $202.88 - 1.013 = ?$

Solution

a. If we use a calculator to add these two numbers, we would get 119.902. However, most calculators do not understand significant figures, and we need to limit the final answer to the tenths place. Thus, we drop the 02 and report a final answer of 119.9 (rounding down).
b. A calculator would answer 201.867. However, we have to limit our final answer to the hundredths place. Because the first number being dropped is 7, which is greater than 7, we round up and report a final answer of 201.87.

Try It

6) Express the answer for $3.445 + 90.83 - 72.4$ to the proper number of significant figures.

Solution

21.9

If the operations being performed are multiplication or division, the rule is as follows: limit the answer to the number of significant figures that the data value with the *least* number of significant figures has. So if we are dividing 23 by 448, which have two and three significant figures each, we should limit the final reported answer to two significant figures (the lesser of two and three significant figures):

$$\frac{23}{448} = 0.051339286... = 0.051$$

The same rounding rules apply in multiplication and division as they do in addition and subtraction.

Example 2.2.5

Express the final answer to the proper number of significant figures.

a. $76.4 \times 180.4 =?$
b. $934.9 \div 0.00455 =?$

Solution

a. The first number has three significant figures, while the second number has four significant figures. Therefore, we limit our final answer to three significant figures:

$$76.4 \times 180.4 = 13,782.56 = 13,800.$$

---

b. The first number has four significant figures, while the second number has three significant figures. Therefore we limit our final answer to three significant figures:

$$934.9 \div 0.00455 = 205,472.5275\ldots = 205,000.$$

Try It

Express the final answer to the proper number of significant figures.

7) $22.4 \times 8.314 =$?

Solution

186

8) $1.381 \div 6.02 =$?

Solution

0.229

If we have to evaluate a problem with mixed operations (addition/subtraction and multiplication/division), we need to round according to the rounding rules above as we go through the operations in the order of operations.

Example 2.2.6

Express the final answer to the proper number of significant figures.

a. $(93.29 - 19.2) \times 1.2 =?$

b. $810.0 \div 2.43 + 2.4 = ?$

Solution

a. By the order of operations, we need to perform the subtraction first. Calculating this result would give 74.09 but we can only give our answer to the nearest tenth, so we need to round to 74.1. Then, we need to multiply 74.1 by 1.2 which gives 88.92 and we need to report this answer with two significant figures. Our answer is 89.
b. By the order of operations, we need to perform the division first. This gives us $333.\bar{3}$. Since 2.43 only has 3 significant figures, we need to limit our result to 3 significant figures, which gives us 333. Then, we need to add 2.4 to get 335.4. Finally, we need to give our answer to the nearest unit, and so our answer is 335.

Summary

When combining measurements with different degrees of accuracy and precision, *the number of significant digits in the final answer can be no greater than the number of significant digits in the least precise measured value*. There are two different rules, one for multiplication and division and the other for addition and subtraction, as discussed below.

1. For multiplication and division: *The result should have the same number of significant figures as the quantity having the least significant figures entering into the calculation*. For example, the area of a circle can be calculated from its radius using $A = \pi \mathrm{r}^2$. Let us see how many significant figures the area has if the radius has only two—say, $r = 1.2m$. Then,

$$A = \pi \mathrm{r}^2 = 3.1415927 \times (1.2 \text{ m})^2 = 4.5238934 \text{ m}^2$$

is what you would get using a calculator that has an eight-digit output. But because the radius has only two significant figures, it limits the calculated quantity to two significant figures or

$$A = \pi \mathrm{r}^2 = 3.1415927 \times (1.2 \text{ m})^2 = 4.52 \text{ m}^2$$

even though π is good to at least eight digits.

2. For addition and subtraction: *The answer can contain no more decimal places than the least precise measurement*. Suppose that you buy 7.56-kg of potatoes in a grocery store as measured with a scale with precision 0.01 kg. Then you drop off 6.052-kg of potatoes at your laboratory as measured by a scale with precision 0.001 kg. Finally, you go home and add 13.7 kg of potatoes as measured by a bathroom scale with precision 0.1 kg. How many kilograms of potatoes do you now have, and how many significant figures are appropriate in the answer? The mass is found by simple addition and subtraction:

$$\begin{array}{r} 7.56kg \\ -6.052kg \\ +\ \ 13.7kg \\ \hline 15.208kg = 15.2kg \end{array}$$

Next, we identify the least precise measurement: 13.7 kg. This measurement is expressed to the 0.1 decimal place, so our final answer must also be expressed to the 0.1 decimal place. Thus, the answer is rounded to the tenths place, giving us 15.2 kg.

3. For mixed operations: Round according to the rules above while following the order of operations for the calculations required. For example, perform the following operations and round to the appropriate number of significant figures.

$$(3.495 + 12.45) \div 2.4$$

We must add the values within the parentheses first, giving us 15.945. However, 12.45 is only precise to the nearest hundredth and so we need to round this result to the nearest hundredth, giving us 15.95. Then, we need to divide that by 2.4, which yields $6.6458\overline{3}$. Since 15.95 has fours significant figures, and 2.4 only has two significant figures, we must round the answer to two significant figures, 6.6.

Rounding in Practical Situations

It is important to note that the rounding rules above are the theoretical rules used in chemical and physical laboratory papers and results. In various health care professions, the precision of the measurement tool being used will often dictate how you would round your medication dosages. In this sense, it is important to understand these theoretical rules and be aware that in practical situations, you may be taught best practices to coincide with the measurement tools available to you.

Key Concepts:

- Accuracy of a measured value refers to how close a measurement is to the correct value. The uncertainty in a measurement is an estimate of the amount by which the measurement result may differ from this value.
- Precision of measured values refers to how close the agreement is between repeated measurements.

- The precision of a *measuring tool* is related to the size of its measurement increments. The smaller the measurement increment, the more precise the tool.
- Significant figures express the precision of a measuring tool.
- Significant figures in a quantity indicate the number of known values plus one place that is estimated.

- The following conventions dictate which numbers in a reported measurement are significant and which are not significant:

  1. Any nonzero digit is significant.
  2. Any zeros between nonzero digits (i.e., embedded zeros) are significant.
  3. Zeros at the end of a number without a decimal point (i.e., trailing zeros) are not significant; they serve only to put the significant digits in the correct positions. However, zeros at the end of any number with a decimal point are significant.
  4. Zeros at the beginning of a decimal number (i.e., leading zeros) are not significant; again, they serve only to put the significant digits in the correct positions.

- When multiplying or dividing measured values, the final answer can contain only as many significant figures as the least precise value.
- When adding or subtracting measured values, the final answer cannot contain more decimal places than the least precise value.
- When performing mixed operations, round according to the rules for addition/subtraction and multiplication/division while using the order of operations.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it

online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=6816#h5p-35

b. Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Glossary

accuracy
the degree to which a measure value agrees with the correct value for that measurement

method of adding percents
the percent uncertainty in a quantity calculated by multiplication or division is the sum of the percent uncertainties in the items used to make the calculation

percent uncertainty
the ratio of the uncertainty of a measurement to the measure value, express as a percentage

precision
the degree to which repeated measurements agree with each other

significant figures
express the precision of a measuring tool used to measure a value

uncertainty
a quantitative measure of how much your measured values deviate from a standard or expected value

2.3 SCIENTIFIC NOTATION

Learning Objectives

By the end of this section, you will be able to:

- Recall the Laws of Exponents
- Convert from decimal notation to scientific notation
- Convert scientific notation to decimal form
- Multiply and divide using scientific notation

Try It

Before you get started, take this readiness quiz:

1) What is the place value of the 6 in the number $64,891$?
2) Name the decimal: 0.0012.
3) Subtract: $5-(-3)$.

Review of the Exponent Properties

If a and b are real numbers, and m and n are integers, then:

Product Property	$\frac{a^m}{a^n} = a^{m+n}$
Power Property	$(a^m)^n = a^{mn}$
Product to a Power	$(ab)^m = a^m b^m$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}$ if $m > n$ and $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$, if $n > m, a \neq 0$
Zero Exponent Property	$a^0 = 1, a \neq 0$
Quotient to a Power Property	$a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n, a \neq 0$
Quotient to a Negative Exponent	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

Scientific Notation

Sometimes, in real-life scenarios, we may need to deal with numbers that are very large, or numbers that are very small. Thus, when dealing with numbers like $1,280,000,000$ or 0.00000000274, it may be beneficial to represent these numbers in a different way. We will use scientific notation to help us with this.

Scientific Notation and Significant Figures

Before we work on converting between forms of numbers, let's consider the concept of significant figures and scientific notation. We can determine the number of significant figures in a number in scientific notation in the same way we would a number in decimal notation. Let's remind ourselves of the rules for significant figures that were covered in the previous section:

The following conventions dictate which numbers in a reported measurement are significant and which are not significant:

1. Any nonzero digit is significant.
2. Any zeros between nonzero digits (i.e., embedded zeros) are significant.

3. Zeros at the end of a number without a decimal point (i.e., trailing zeros) are not significant; they serve only to put the significant digits in the correct positions. However, zeros at the end of any number with a decimal point are significant.
4. Zeros at the beginning of a decimal number (i.e., leading zeros) are not significant; again, they serve only to put the significant digits in the correct positions.

Of course, in scientific notation, we no longer have to worry about leading zeros. This is one of the benefits of using scientific notation, as the digits present are always significant.

Example 2.3.1

Determine the number of significant figures in the following numbers:

a. 1.41×10^3
b. 1.034×10^{-5}
c. 4.0000×10^{-2}

Solution

a. All non-zero digits are significant so there are 3 significant figures.
b. All confined zeros are significant, as well as all non-zero digits, so there are 4 significant figures.
c. All zeros at the end of the number after a decimal point are significant, so there are 5 significant figures.

Try It

4) Determine the number of significant figures in the following numbers:

a. 1.9×10^3
b. 4.00×10^{-5}
c. 5.001×10^{-2}

Solution

a. 2 significant figures
b. 3 significant figures
c. 4 significant figures

Convert from Decimal Notation to Scientific Notation

Remember working with place value for whole numbers and decimals? Our number system is based on powers of 10. We use tens, hundreds, thousands, and so on. Our decimal numbers are also based on powers of tens—tenths, hundredths, thousandths, and so on. Consider the numbers 4,000 and 0.004. We know that 4,000 means 4×1000 and 0.004 means $4 \times \frac{1}{1,000}$

If we write the 1000 as a power of ten in exponential form, we can rewrite these numbers in this way:

4000	0.004
4×1000	$4 \times \frac{1}{1000}$
4×10^3	$4 \times \frac{1}{10^3} = 4 \times 10^{-3}$

When a number is written as a product of two numbers, where the first factor is a number greater than or equal to one but less than 10, and the second factor is a power of 10 written in exponential form, it is said to be in **scientific notation**.

Scientific Notation

A number is in scientific notation if it is in the form $M \times 10^n$ where $1 \leq M < 10$.

It is customary in scientific notation to use as the $\times$ multiplication sign, even though we avoid using this sign elsewhere in algebra.

If we look at what happened to the decimal point, we can see a method to easily convert from decimal notation to scientific notation.

4\;\times\;10^3\;=\;\text{Decimal moved 3 to the right.}\;\;4{\color[rgb]{1.0, 0.0, 0.0}\overrightarrow.}0_{\color[rgb]{1.0, 0.0, 0.0}1}0_{\color[rgb]{1.0, 0.0, 0.0}2}0_{\color[rgb]{1.0, 0.0, 0.0}3}\;=\;4000{\color[rgb]{1.0, 0.0, 0.0}.}

4\;\times\;10^{-3}\;=\;\text{Decimal moved 3 to the left.}\;\;0_{\color[rgb]{1.0, 0.0, 0.0}3}0_{\color[rgb]{1.0, 0.0, 0.0}2}0_{\color[rgb]{1.0, 0.0, 0.0}1}4{\color[rgb]{1.0, 0.0, 0.0}\overleftarrow.}\;=\;0{\color[rgb]{1.0, 0.0, 0.0}.}{\color[rgb]{0.1, 0.1, 0.1}004}

In both cases, the decimal was moved 3 places to get the first factor between 1 and 10.

The power of 10 is positive when the number is larger than 1: $4,000 = 4 \times 10^3$

The power of 10 is negative when the number is between 0 and 1: $0.004 = 4 \times 10^{-3}$

Example 2.3.2

Write in scientific notation: 37,000.

Solution

Step 1: **Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.**

Remember, there is a decimal at the end of 37,000.

Move the decimal after the 3. 3.700 is between 1 and 10.

Step 2: **Count the number of decimal places, n, that the decimal point was moved.**

The decimal place was moved 4 places to the left.

Decimal moved 4 to the left. $3_{rgb]1.0,0.0,0.04}7_{rgb]1.0,0.0,0.03}0_{rgb]1.0,0.0,0.02}0_{rgb]1.0,0.0,0.01}0 rgb]1.0, 0.0, 0.0\overleftarrow{.} rgb]1.0, 0.0, 0.0\ rgb]0.1, 0.1, 0.1 = rgb]1.0, 0.0, 0.0\ rgb]0.1, 0.1, 0.13 rgb]1.0, 0.0, 0.0. rgb]0.1, 0.1, 0.17$

Step 3: **Write the number as a product with a power of 10.**

If the original number is:

Greater than 1, the power of 10 will be 10^a.
Between 0 and 1, the power of 10 will be 10^{-a}.

37,000 is greater than 1 so the power of 10 will have exponent 4.

$$3.7 \times 10^4$$

Step 4: **Check to see if your answer makes sense.**

10^4 is 10,000 and 10,000 times 3.7 will be 37,000.

$$3.7 \times 10^4 = 37,000$$

Try It

5) Write in scientific notation: $96,000$.

Solution

$9.6 \times\ 10^4$

6) Write in scientific notation: $48,300$.

Solution

$4.83 \times\ 10^4$

7) Write in scientific notation: $54,\tilde{0}00$

Solution

$5.40 \times\ 10^4$

HOW TO

Convert from decimal notation to scientific notation

1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.

2. Count the number of decimal places, n, that the decimal point was moved.
3. Write the number as a product with a power of 10.
   If the original number is:
   - greater than 1, the power of 10 will be 10^n.
   - between 0 and 1, the power of 10 will be 10^{-n}.
4. Check.

Example 2.3.3

Write in scientific notation: 0.0052

Solution

The original number, 0.0052, is between 0 and 1 so we will have a negative power of 10.

Step 1: **Move the decimal point to get 5.2, a number between 1 and 10.**

$$0rgb]1.0, 0.0, 0.0.0_{rgb]1.0,0.0,0.0\uparrow}0_{rgb]1.0,0.0,0.0\uparrow}5_{rgb]1.0,0.0,0.0\uparrow}2 \;=\; 5rgb]1.0, 0.0, 0.0.2$$

Step 2: **Count the number of decimal places the point was moved.**

0{\color[rgb]{1.0, 0.0, 0.0}.}0_{\color[rgb]{1.0, 0.0, 0.0}1}0_{\color[rgb]{1.0, 0.0, 0.0}2}5_{\color[rgb]{1.0, 0.0, 0.0}3}2\;=\;\text{The decimal was moved }{\color[rgb]{1.0, 0.0, 0.0}\text{3}}\text{ places.}

Step 3: **Write as a product with a power of 10.**

$$5.2 \times \ 10^{-3}$$

Step 4: **Check.**

$$5.2\times\ 10^{-3}$$
$$5.2\times\ \frac{1}{10^3}$$
$$5.2\times\ \frac{1}{1000}$$
$$5.2\times 0.001\ =\ 0.0052$$
$$0.0052 = 5.2\times\ 10^{-3}$$

Try It

8) Write in scientific notation: 0.0078

Solution

$7.8\times\ 10^{-3}$

9)Write in scientific notation: 0.0129

Solution

$1.29\times\ 10^{-2}$

Preserving Significant Zeros Using Scientific Notation

As we saw in the previous section, when considering a number with trailing zeros, it is not easy to determine whether or not they are significant. For example, the number 10,000 could have one significant figure only, but is is possible that some of the zeros are significant as well. One way to communicate a significant zero, when there is no decimal present, is to use a tilde to indicate its significance. For instance, $10,0\tilde{0}0$ indicates that the third zero is significant, which makes the zeros in between significant as well. In this sense, this number $10,0\tilde{0}0$ has 4 significant figures. If we wanted to put this number into scientific notation, we would need to include those significant zeros and it would be represented by 1.000×10^4.

Try It

10) Convert the following numbers to scientific notation. Be sure to preserve the number of significant figures.

a. 0.002840
b. 129.00
c. $1800\tilde{0}$

Solution

a. 2.840×10^{-3}
b. 1.2900×10^{2}
c. 1.8000×10^{4}

Convert Scientific Notation to Decimal Form

How can we convert from scientific notation to decimal form? Let's look at two numbers written in scientific notation and see.

$$[latex] \quad \begin{array}{rl} 9.12 \times & 10^4 \\ 9.12 \times & 10^{-4} \\ 9.12 \times & 10,000 \\ 9.12 \times & 0.0001 \\ 91,200 \times & 0.000912 \end{array} \quad \text{[/latex]}$$

If we look at the location of the decimal point, we can see an easy method to convert a number from scientific notation to decimal form.

9.12\times\;10^4=\text{Move the decimal 4 to the right.}\;9{\color[rgb]{1.0, 0.0, 0.0}\overrightarrow.}1_{\color[rgb]{1.0, 0.0, 0.0}1}2_{\color[rgb]{1.0, 0.0, 0.0}2}0_{\color[rgb]{1.0, 0.0, 0.0}3}0_{\color[rgb]{1.0, 0.0, 0.0}4}\;=\;91,200{\color[rgb]{1.0, 0.0, 0.0}.}

9.12\times\;10^{-4}=\text{Move the decimal 4 to the left.}\;=\;0_{\color[rgb]{1.0, 0.0, 0.0}4}0_{\color[rgb]{1.0, 0.0, 0.0}3}0_{\color[rgb]{1.0, 0.0, 0.0}2}0_{\color[rgb]{1.0, 0.0, 0.0}1}9{\color[rgb]{1.0, 0.0, 0.0}\overleftarrow.}12\;=\;0{\color[rgb]{1.0, 0.0, 0.0}.}000912

In both cases the decimal point moved 4 places. When the exponent was positive, the decimal moved to the right. When the exponent was negative, the decimal point moved to the left.

Example 2.3.4

Convert to decimal form: $6.2 \times \ 10^3$

Solution

***Step 1:* Determine the exponent, n, on the factor 10.**

The exponent is 3.

$$6.2 \times 10^3$$

***Step 2:* Move the decimal n spaces, adding zeros if needed.**

If the exponent is positive, move the decimal point n places to the right.
If the exponent is negative, move the decimal point $|n|$ places to the left.

$$\text{Exponent is positive so we move the decimal 3 spaces to the right, adding 2 zeros as placeholders.}$$
$$6rgb]1.0,0.0,0.0\overrightarrow{.}2_{rgb]1.0,0.0,0.01}0_{rgb]1.0,0.0,0.02}0_{rgb]1.0,0.0,0.03} = 6,200rgb]1.0,0.0,0.0.$$

***Step 3:* Check to see if your answer makes sense.**

10^3 is 1000 and 1000 times 6.2 is $6,200$.

Try It

11) Convert to decimal form: 1.3×10^3

Solution

$1,300$

12) Convert to decimal form: 9.25×10^4

Solution

$92,500$

13) Convert to decimal form: 3.900×10^5

Solution

$390,\tilde{0}00$

HOW TO

Convert scientific notation to decimal form.

The steps are summarized below.

To convert scientific notation to decimal form:

1. Determine the exponent, n, on the factor 10.
2. Move the decimal n places, adding zeros if needed.
    - If the exponent is positive, move the decimal point n places to the right.
    - If the exponent is negative, move the decimal point $|n|$ places to the left.
3. Check.

Example 2.3.5

Convert to decimal form: 8.9×10^{-2}

Solution

Step 1: **Determine the exponent, n, on the factor 10.**

The exponent is -2.

Step 2: **Since the exponent is negative, move the decimal point 2 places to the left.**

$$8.9 \ \times \ 10^{-2} \ = \ 0_{rgb]1.0,0.0,0.02}0_{rgb]1.0,0.0,0.0,0.01}8rgb]1.0,0.0,0.0\overleftarrow{.}9$$

Step 3: **Add zeros as needed for placeholders.**

$$8.9 \ \times \ 10^{-2} \ = \ 0.089$$

Try It

14) Convert to decimal form: 1.2×10^{-4}

Solution

0.00012

15) Convert to decimal form: 7.5×10^{-2}

Solution

0.075

Multiply and Divide Using Scientific Notation

Astronomers use very large numbers to describe distances in the universe and ages of stars and planets. Chemists use very small numbers to describe the size of an atom or the charge on an electron. When scientists perform calculations with very large or very small numbers, they use scientific notation. Scientific notation provides a way for the calculations to be done without writing a lot of zeros. We will see how the Properties of Exponents are used to multiply and divide numbers in scientific notation.

Example 2.3.6

Multiply. Write answers in decimal form: $(4 \times 10^5)(2 \times 10^{-7})$

Solution

Step 1: **Use the Commutative Property to rearrange the factors.**

$$4 \times 2 \times 10^5 \times 10^{-7}$$

Step 2: **Multiply.**

$$8 \times 10^{-2}$$

Step 3: **Change to decimal form by moving the decimal two places left.**

$$0.08$$

Try It

16) Multiply $(310^6)(2 \times 10^{-8})$. Write answers in decimal form.

Solution

0.06

17) Multiply $(3 \times 10^{-2})(3 \times 10^{-1})$. Write answers in decimal form.

Solution

0.009

Example 2.3.7

Divide. Write answers in decimal form: $\dfrac{9\times 10^{3}}{3\times 10^{-2}}$

Solution

Step 1: **Separate the factors, rewriting as the product of two fractions.**

$$\frac{9\times 10^{3}}{3\times 10^{-2}}$$

Step 2: **Divide.**

$$3\times\ 10^{5}$$

Step 3: **Change to decimal form by moving the decimal five places right.**

$$300,000$$

Try It

18) Divide $\dfrac{8\times 10^{4}}{2\times 10^{-1}}$ Write answers in decimal form.

Solution

$400,000$

19) Divide $\dfrac{8\times 10^{2}}{4\times 10^{-2}}$. Write answers in decimal form.

Solution

$20,000$

Access these online resources for additional instruction and practice with integer exponents and scientific notation:

- Negative Exponents
- Scientific Notation
- Scientific Notation 2

Key Concepts

- **To convert a decimal to scientific notation:**
  1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.
  2. Count the number of decimal places, n, that the decimal point was moved.
  3. Write the number as a product with a power of 10. If the original number is:
     - greater than 1, the power of 10 will be 10^n
     - between 0 and 1, the power of 10 will be 10^{-n}
  4. Check.
- **To convert scientific notation to decimal form:**
  1. Determine the exponent, n, on the factor 10.
  2. Move the decimal n places, adding zeros if needed.
     - If the exponent is positive, move the decimal point n places to the right.

    - If the exponent is negative, move the decimal point $|n|$ places to the left.
3. Check.

- Use a tilde to indicate a significant zero when necessary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=4449#h5p-35

b. Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Glossary

scientific notation

A number is expressed in scientific notation when it is of the form $M \times 10^n$ where $1 \leq M < 10$ and n is an integer.

2.4 MORE UNIT CONVERSIONS AND ROUNDING RULES

Learning Objectives

By the end of this section, you will be able to:

- Perform unit conversions while respecting the appropriate rounding rules for accuracy and precision.
- Perform operations with scientific notation while respecting the appropriate rounding rules for accuracy and precision.

Consider a simple example: how many feet are there in 4 yards? Most people will almost automatically answer that there are 12 feet in 4 yards. How did you make this determination? Well, if there are 3 feet in 1 yard and there are 4 yards, then there are 4 × 3 = 12 feet in 4 yards.

This is correct, of course, but it is informal. Let us formalize it in a way that can be applied more generally. We know that 1 yard (yd) equals 3 feet (ft):

1 yd = 3 ft

In math, this expression is called an *equality*. The rules of algebra say that you can change (i.e., multiply or divide or add or subtract) the equality (as long as you don't divide by zero) and the new expression will still be an equality. For example, if we divide both sides by 2, we get

$$\frac{1}{2}yd = \frac{3}{2}ft$$

We see that one-half of a yard equals 3/2, or one and a half, feet—something we also know to be true, so the above equation is still an equality. Going back to the original equality, suppose we divide both sides of the equation by 1 yard (number *and* unit):

$$\frac{1yd}{1yd} = \frac{3ft}{1yd}$$

The expression is still an equality, by the rules of algebra. The left fraction equals 1. It has the same quantity in the numerator and the denominator, so it must equal 1. The quantities in the numerator and denominator cancel, both the number *and* the unit:

$$\frac{\cancel{1}\ \cancel{yd}}{\cancel{1}\ \cancel{yd}} = \frac{3ft}{1yd}$$

When everything cancels in a fraction, the fraction reduces to 1:

$$1 = \frac{3ft}{1yd}$$

We have an expression, 3 ft1 yd, that equals 1. This is a strange way to write 1, but it makes sense: 3 ft equal 1 yd, so the quantities in the numerator and denominator are the same quantity, just expressed with different units. The expression 3 ft 1 yd is called a *conversion factor*, and it is used to formally change the unit of a quantity into another unit. (The process of converting units in such a formal fashion is sometimes called *dimensional analysis* or the *factor label method*.)

To see how this happens, let us start with the original quantity:

4 yd

Now let us multiply this quantity by 1. When you multiply anything by 1, you don't change the value of the quantity. Rather than multiplying by just 1, let us write 1 as 3 ft1 yd:

$$4yd \times \frac{3ft}{1yd}$$

The 4 yd term can be thought of as $\frac{4yd}{1}$; that is, it can be thought of as a fraction with 1 in the denominator. We are essentially multiplying fractions. If the same thing appears in the numerator and denominator of a fraction, they cancel. In this case, what cancels is the unit *yard*:

$$4\ \cancel{yd} \times \frac{3ft}{1\ \cancel{yd}}$$

$$\frac{4 \times 3ft}{1} = \frac{12ft}{1} = 12ft$$

That is all that we can cancel. Now, multiply and divide all the numbers to get the final answer:

Again, we get an answer of 12 ft, just as we did originally. But in this case, we used a more formal procedure that is applicable to a variety of problems.

How many millimeters are in 14.66 m? To answer this, we need to construct a conversion factor between millimeters and meters and apply it correctly to the original quantity. We start with the definition of a millimeter, which is

$$1mm = \frac{1}{1,000m}$$

The 1/1,000 is what the prefix *milli-* means. Most people are more comfortable working without fractions, so we will rewrite this equation by bringing the 1,000 into the numerator of the other side of the equation:

1,000 mm = 1 m

Now we construct a conversion factor by dividing one quantity into both sides. But now a question arises: which quantity do we divide by? It turns out that we have two choices, and the two choices will give us different conversion factors, both of which equal 1:

$$\frac{1000mm}{1000mm} = \frac{1m}{1000mm}\ or\ \frac{1000mm}{1m} = \frac{1m}{1m}$$

$$1 = \frac{1m}{1000mm}\ or\ \frac{1000mm}{1m} = 1$$

Which conversion factor do we use? The answer is based on *what unit you want to get rid of in your initial quantity*. The original unit of our quantity is meters, which we want to convert to millimeters. Because the original unit is assumed to be in the numerator, to get rid of it, we want the meter unit in the *denominator*; then they will cancel. Therefore, we will use the second conversion factor. Cancelling units and performing the mathematics, we get

$$14.66\,\cancel{m} \times \frac{1000mm}{1\,\cancel{m}} = 14660mm$$

Note how m cancels, leaving mm, which is the unit of interest.

The ability to construct and apply proper conversion factors is a very powerful mathematical technique in chemistry. You need to master this technique if you are going to be successful in this and future courses.

Example 2.4.1

a. Convert 35.9 kL to liters.

b. Convert 555 nm to meters.

Solution

a.

Step 1: **We will use the fact that $1kL = 1,000L$.**

Of the two conversion factors that can be defined, the one that will work is $\frac{1,000L}{1kL}$.

Step 2: **Applying this conversion factor, we get:**

$$35.9\,\cancel{kL} \times \frac{1000L}{1\,\cancel{kL}} = 35900L$$

---

b.

Step 1: **We will use the fact that** $1nm = \frac{1}{1,000,000,000m}$**, which we will rewrite as 1,000,000,000 nm = 1 m, or 10 nm = 1 m.**

Step 2: **Of the two possible conversion factors, the appropriate one has the nm unit in the denominator.**

$$\frac{1m}{10^9 nm}$$

Step 3: **Applying this conversion factor, we get:**

$$555nm \ \times \ \frac{1m}{10^9 nm} = 0.000000555m$$

Step 4: **In the final step, we expressed the answer in scientific notation.**

$$= 5.55 \times 10^{-7} m$$

Try It

1) Convert $67.08 \mu L$ to liters. Give your answer in scientific notation.

2) Convert $56.8m$ to kilometers. Give your answer in scientific notation.

Solution

1. 6.708×10^{-5} L
2. 5.68×10^{-2} km

What if we have a derived unit that is the product of more than one unit, such as m^2? Suppose we want to convert square meters to square centimeters? The key is to remember that m^2 means m × m, which means we have *two* meter units in our derived unit. That means we have to include *two* conversion factors, one for each unit. For example, to convert 17.6 m^2 to square centimeters, we perform the conversion as follows:

$$17.6m^2 = 17.6\,(\cancel{m} \times \cancel{m}) \times \frac{100cm}{1\,\cancel{m}} \times \frac{100cm}{1\,\cancel{m}} = 17000cm \times cm = 1.76 \times 10^5 cm^2$$

Example 2.4.2

How many cubic centimeters are in 0.883 m^3?

Solution

With an exponent of 3, we have three length units, so by extension we need to use three conversion factors between meters and centimeters. Thus, we have

$$0.833\,\cancel{m^3} \times \frac{100cm}{1\,\cancel{m}} \times \frac{100cm}{1\,\cancel{m}} \times \frac{100cm}{1\,\cancel{m}} = 883000cm^3 = 8.83 \times 10^5 cm^3$$

You should demonstrate to yourself that the three meter units do indeed cancel.

Try It

3) How many cubic millimeters are present in 0.0923 m^3? Give your answer in scientific notation.

Solution

9.23×10^7 mm^3

Suppose the unit you want to convert is in the denominator of a derived unit; what then? Then, in the conversion factor, the unit you want to remove must be in the *numerator*. This will cancel with the original unit in the denominator and introduce a new unit in the denominator. The following example illustrates this situation.

Example 2.4.3

Convert 88.4 m/min to meters/second.

Solution

Step 1: **We want to change the unit in the denominator from minutes to seconds.**

Because there are 60 seconds in 1 minute (60 s = 1 min), we construct a conversion factor so that the unit we want to remove, minutes, is in the numerator: $1min/60$s.

Step 2: **Apply and perform the math.**

$$\frac{88.4m}{min} \times \frac{1min}{60s} = 1.47m/s$$

Notice how the 88.4 automatically goes in the numerator. That's because any number can be thought of as being in the numerator of a fraction divided by 1.

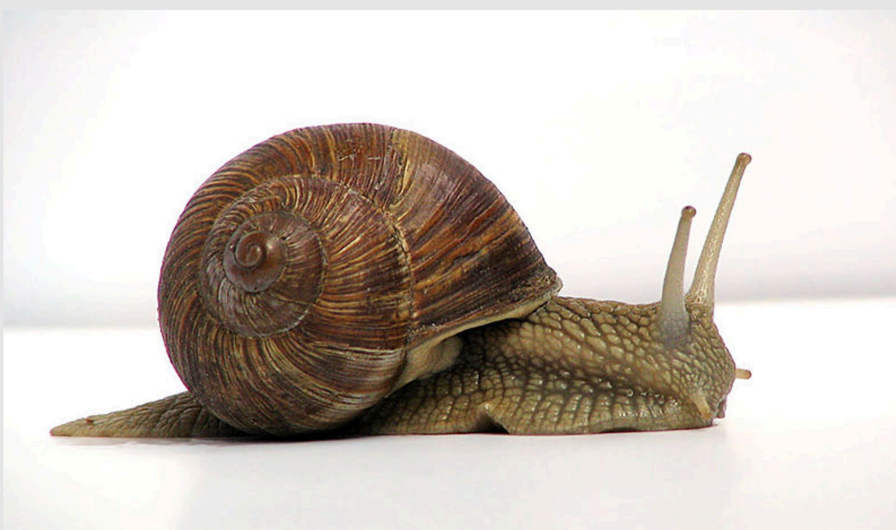


Figure 2.4.1. How Fast Is Fast? A common garden snail moves at a rate of about 0.2 m/min, which is about 0.003 m/s, which is 3 mm/s!
Source: "Grapevine snail"by Jürgen Schoneris licensed under the Creative Commons Attribution-Share Alike 3.0 Unported license.

Try It

4) Convert 0.203 m/min to meters/second. Give your answer in scientific notation.

Solution
0.00338 m/s or 3.38×10^{-3} m/s

Sometimes there will be a need to convert from one unit with one numerical prefix to another unit with a different numerical prefix. How do we handle those conversions? Well, you could memorize the conversion factors that interrelate all numerical prefixes. Or you can go the easier route: first convert the quantity to the base unit, the unit with no numerical prefix, using the definition of the original prefix. Then convert the quantity in the base unit to the desired unit using the definition of the second prefix. You can do the conversion in two separate steps or as one long algebraic step. For example, to convert 2.77 kg to milligrams:

$$2.77\ \cancel{kg} \times \frac{1000g}{1\ \cancel{kg}} = 2770g\ \ (convert\ to\ the\ base\ unit\ of\ grams)$$

$$2770\ \cancel{g} \times \frac{1000mg}{1\ \cancel{g}} = 2770000mg = 2.77 \times 10^6\ mg\ \ (convert\ to\ the\ desired\ unit)$$

Alternatively, it can be done in a single multistep process:

$$2.77\ \cancel{kg} \times \frac{1000\ \cancel{g}}{1\ \cancel{kg}} \times \frac{1000mg}{1\ \cancel{g}} = 2770000m = 2.77 \times 10^6 mg$$

You get the same answer either way.

Example 2.4.4

How many nanoseconds are in 368.09 μs?

Solution
You can either do this as a one-step conversion from microseconds to nanoseconds or convert to the base unit first and then to the final desired unit. We will use the second method here, showing the two steps in a single line. Using the definitions of the prefixes *micro-* and *nano-*,

$$368.09\ \cancel{\mu s} \times \frac{1\ \cancel{s}}{10^6\ \cancel{\mu s}} \times \frac{10^9 ns}{1\ \cancel{s}} = 368090ns = 3.6809 \times 10^5 ns$$

Try It

5) How many milliliters are in 607.8 kL? Give your answer in scientific notation.

Solution
6.078×10^8 mL

When considering the significant figures of a final numerical answer in a conversion, there is one important case where a number does not impact the number of significant figures in a final answer—the so-called exact number. An exact number is a number from a defined relationship, not a measured one. For example, the prefix *kilo-* means 1,000—*exactly* 1,000, no more or no less. Thus, in constructing the conversion factor

$$\frac{1000g}{1kg}$$

neither the 1,000 nor the 1 enter into our consideration of significant figures. The numbers in the numerator and denominator are defined exactly by what the prefix *kilo-* means. Another way of thinking about it is that these numbers can be thought of as having an infinite number of significant figures, such as

$$\frac{1000.0000000000...g}{1.0000000000...kg}$$

The other numbers in the calculation will determine the number of significant figures in the final answer.

Example 2.4.5

A rectangular plot in a garden has the dimensions 36.7 cm by 128.8 cm. What is the area of the garden plot in square meters? Express your answer in the proper number of significant figures.

Solution

Area is defined as the product of the two dimensions, which we then have to convert to square meters and express our final answer to the correct number of significant figures, which in this case will be three.

$$36.7\,\cancel{cm} \times 128.8\,\cancel{cm} \times \frac{1m}{100\,\cancel{cm}} \times \frac{1m}{100\,\cancel{cm}} = 0.472696m^2 = 0.473m^2$$

The 1 and 100 in the conversion factors do not affect the determination of significant figures because they are exact numbers, defined by the centi- prefix.

Try It

6) What is the volume of a block in cubic meters whose dimensions are 2.1 cm × 34.0 cm × 118 cm?

Solution

0.0084 m^3

In the examples above, the answers provided are all consistent with the rounding rules from the previous section, but we did not emphasize what choices were being made while reporting our answers. In the next sections, we will go through a series of examples to combine what we've learned over the last few sections. Throughout this section (and for the rest of the course), be sure to always pay attention to the instructions so that you are always rounding appropriately. Sometimes, you will be required to follow the rounding rules for accuracy and precision, and other times, you will be asked to round to a specific place value. It is always important to pay attention to detail and to answer questions with care.

Rounding Rules When Converting Within the Same System of Measurement

When considering metric-to-metric or U.S. System-to-U.S. System conversions, we must first realize that our conversion equations within each system are exact. For example, $100cm = 1m$ means that there are exactly 100 centimeters in 1 meter and $1ft = 12in$ means that there are exactly 12 inches in 1 foot. Since these conversion equations are exact, this means that the numbers involved have infinitely many significant figures and infinitely many decimal places. As such, when we are performing metric-to-metric or U.S. System-to-U.S. System conversions, we will use the measurement that we are converting to determine the appropriate number of significant figures to include in our answer. Please note that metric-to-metric conversion equations must be recreated by memory, whereas units from the U.S. system will always be provided in our course.

Example 2.4.6

Convert 31.5 meters per minute to meters per second. Give your answer with the appropriate number of significant figures.

Solution

Step 1: **Write down any relevant unit conversion equations.**

$$1min = 60s$$

Step 2: **Convert the units using dimensional analysis.**

$$= \frac{31.5m}{1min} \times \frac{1min}{60s}$$
$$= \frac{31.5m}{60s}$$
$$= 0.525\ m/s$$

Step 3: **Be sure that the answer is given to the proper number of significant figures.**

Since the operations we performed were all multiplications and divisions, we need to round to the least number of significant figures. The unit conversion equation used, $1min = 60s$, is exact so there are infinitely many significant figures in those values. In this case, we need to look at the measurement given. Since $31.5m/min$ has 3 significant figures, our answer should have 3 significant figures as well. So our answer of $0.525m/s$ is appropriate.

Example 2.4.7

Convert $18\tilde{0}mg/dL$ to g/L. Give your answer with the appropriate number of significant figures.

Solution

Step 1: **Write down any relevant unit conversion equations.**

$1000mg = 1g$ and $10dL = 1L$

For the purpose of the work, we will make note that there is a tilde over the zero and so $18\tilde{0}$ has 3 significant figures.

Step 2: **Convert the units using dimensional analysis.**

$$= \frac{180mg}{1dL} \times \frac{10dL}{1L} \times \frac{1g}{1000mg}$$
$$= \frac{180\ \times 10g}{1\ times\ 1000L}$$
$$= 1.8\ g/L$$

Step 3: **Be sure that the answer is given to the proper number of significant figures.**

The operations we used in this question were multiplications and divisions, and so we need to round to the least number of significant figures in our values. Our calculator gives us $1.8g/L$, but we need to take into account the fact that our measurement had 3 significant figures. Thus, our answer in this case should have 3 significant figures.

Our answer in this case is that $18\tilde{0}mg/dL = 1.80g/L$.

Example 2.4.8

Convert 32.8 ounces to cups. Give your answer with the appropriate number of significant figures.

Solutions

The following unit conversion equations may be useful for this example: $1cup = 8oz$.

Step 1: **Write down any relevant unit conversion equations.**

Note, that the U.S. system conversion equation is given.

Step 2: **Convert the units using dimensional analysis.**

$$= \frac{32.8oz}{1} \times \frac{1cup}{8oz}$$
$$= \frac{32.8\ cup}{1 \cdot 8}$$
$$= 4.1\ cups$$

Step 3: **Be sure that the answer is given to the proper number of significant figures.**

The operations we used in this question were multiplications and divisions, and so we need to round to the least number of significant figures in our values. Our calculator does not take into account significant figures. Since our unit conversion equation is exact (i.e. there are exactly 8 ounces in 1 cup), those numbers have infinitely many significant figures. Thus, we need to look to our measurement to decide how many significant figures our answer needs. 32.8 ounces has 3 significant figures, so our answer must also have 3 significant figures.

Therefore, $32.8oz = 4.10cups$.

Example 2.4.9

Convert 161.8 miles per hour to feet per second. Give your answer with the appropriate number of significant figures.

Solution

The following unit conversion equations may be useful for this example: $1mi = 5280ft$.

Step 1: **Write down any relevant unit conversion equations.**

Note, that the U.S. system conversion equation is given. To convert from hours to seconds, we need to know: $1h = 3600s$.

Step 2: **Convert the units using dimensional analysis.**

$$= \frac{161.8mi}{1h} \times \frac{5280ft}{1mi} \times \frac{1h}{3600s}$$
$$= \frac{161.8 \cdot 5280\ ft}{1 \cdot 3600s}$$
$$= 237.30\overline{6}ft/s$$

Step 3: **Be sure that the answer is given to the proper number of significant figures.**

The operations we used in this question were multiplications and divisions, and so we need to round to the least number of significant figures in our values. Our calculator does not take into account significant figures. Since our unit conversion equation is exact, the numbers involved in the conversion equations have infinitely many significant figures. Thus, we need to look to our measurement to decide how many significant figures our answer needs. 161.8 mi/h has 4 significant figures, so our answer must also have 4 significant figures.

Therefore, $161.8mi/h = 237.2ft/s$.

As we have seen in the examples above, when we are converting metric-to-metric or U.S. System-to-U.S. System units, we round according to the measurement that we are converting.

Rounding Rules When Converting Between Two Different Systems of Measurement

When considering the unit conversion equations that establish relationships between two units of different measurement systems, it is important to note the number of significant figures present in the equations. This is due to the fact that when we convert from the metric system to the U.S. system of measurement, the relationships between the units are not exact, but are rather approximations. For example, if we wanted to convert between grams (metric) and pounds (U.S. system), we may be given one of the following unit conversion equations.

Unit Conversion Equations for Converting Between Grams and Pounds

$1.00lb = 454g$

$1.000lb = 453.6g$

$1.0000lb = 453.59g$

$1.00000lb = 453.592g$

The equations above show that the precision of the unit conversion equation depends on the precision of the measurement tool that was used. Therefore, these numbers in these unit conversion equations are approximate and will be taken into account while performing our unit conversions.

Example 2.4.10

Convert 8.4 miles to kilometers. Give your answer with the appropriate number of significant figures.

Solution

The following unit conversion equations may be useful for this example:

$1.0000mi = 1.6093km$.

***Step 1:* Write down any relevant unit conversion equations.**

Note, that the U.S. system to metric conversion equation is provided.

***Step 2:* Convert the units using dimensional analysis.**

$$= \frac{8.4mi}{1} \times \frac{1.6093km}{1.0000mi}$$
$$= \frac{8.4 \cdot 1.6093km}{1}$$
$$= 13.51812km$$

***Step 3:* Be sure that the answer is given to the proper number of significant figures.**

All the operations in our conversion above were multiplications and divisions, so we need to round to the least number of significant figures. Don't forget, our calculator does not take into account significant figures. Since our unit conversion is 1.0000 mi = 1.6093 km, each of those numbers has 5 significant figures, whereas our measurement of 8.4mi has only 2 significant figures. Thus, we need to round our final answer to 2 significant figures.

Therefore, $8.4mi = 14km$.

Example 2.4.11

Convert 20.18 kilogram per liter to pound per gallon. Give your answer with the appropriate number of significant figures.

Solution

The following unit conversion equations may be useful for this example: $1.00kg = 2.20lbs$ and $1.00L = 0.264gal(U.S)$.

Step 1: **Write down any relevant unit conversion equations.**

Note, that the U.S. system to metric conversion equations are provided.

Step 2: **Convert the units using dimensional analysis.**

$$= \frac{20.18kg}{1L} \times \frac{2.20lbs}{1.00kg} \frac{1.00L}{0.264gal}$$
$$= \frac{20.18 \times\ 2.20 \times\ 1.00lbs}{1 \times\ 1.00 \times\ 0.264gal}$$
$$= 168.1\overline{6}lbs/gal$$

Step 3: **Be sure that the answer is given to the proper number of significant figures.**

All the operations in our conversion above were multiplications and divisions, so we need to round to the least number of significant figures. Don't forget, our calculator does not take into account significant figures. Since our unit conversions are $1.00kg = 2.20lbs$ and $1.00L = 0.264U.S.gal$, each of those numbers have 3 significant figures, whereas our measurement of 20.18 kg/L has 4 significant figures. Thus, we need to round our final answer to 3 significant figures.

Therefore, $20.18kg/L = 168lbs/gal$.

Rounding Rules With Scientific Notation

Now that we know how to use scientific notation, we can combine this with our knowledge of our rounding rules to round appropriately. In the following examples, we will perform the indicated operations and round according to the instructions provided.

Example 2.4.12

Perform the indicated operations. Give your answer with the appropriate number of significant figures.

a. $3.92 \times 10^4 + 2.3 \times 10^4$
b. $1.9 \times 10^{-2} - 1.49 \times 10^{-1}$
c. $(5.8 \times 10^5) \times (2.84 \times 10^{-7})$
d. $\dfrac{(3.8 \times 10^2)(2.93 \times 10^3)}{(8 \times 10^{-3})(8.935 \times 10^3)}$

Solution

a.

Step 1: **Since the powers of ten are the same, we can simply add the values.**

$$3.92 \times 10^4 + 2.3 \times 10^4 = 6.22 \times 10^4$$

Step 2: **As we were adding, we need to round to the correct precision. This means, we need to round according to the** 2.3×10^4 **which has 2 significant figures.**

Thus, the answer is: 6.2×10^4

---

b.

Step 1: **First, we need to write the numbers with the same power of ten.**

$$\begin{aligned} &= 1.9 \times 10^{-2} - 1.49 \times 10^{-1} \\ &= 0.19 \times 10^{-1} - 1.49 \times 10^{-1} \\ &= -1.3 \times 10^{-1} \end{aligned}$$

Step 2: **Our answer already has the correct precision, so no further rounding is necessary.**

---

c.

Step 1: **To multiply scientific notation, multiply the coefficients and add the exponents on the tens.**

$$= (5.8 \times 10^5) \times (2.84 \times 10^{-7})10^{-1}$$
$$= (5.8 \cdot 2.84) \times 10^{5+(-7)}$$
$$= 16.472 \times 10^{-2}$$

Step 2: **Before we round, we need to write this number in correct scientific notation.**

$$= 1.6472 \times 10^{-1}$$

Step 3: **Now, since we multiplied to get this result, we need to round to the least number of significant figures, which in this case is 2.**

$$= 1.6 \times 10^{-1}$$

---

d.

Step 1: **To answer this question, let's first perform the multiplications in the numerator and denominator, and then we will do the division.**

$$\frac{(3.8 \times 10^2)(2.93 \times 10^3)}{(8 \times 10^{-3})(8.935 \times 10^3)}$$

=\frac{3.8\times \ 2.93\times 10^{2+3}}{8\times \ 8.935 \times 10^{-3+3}}

$$= \frac{11.134 \times 10^5}{71.48 \times 10^0}$$

Step 2: **Now, we can perform the division of the coefficients and subtract the exponents on the tens.**

$$= 0.15576385 \times 10^5$$

Step 3: **Finally, let's put this into correct scientific notation before we round appropriately.**

$$= 1.5576385 \times 10^4$$

Step 4: **Since the 8 in the initial problem only has 1 significant figure, our answer should only have 1 significant figure.**

$$= 2 \times 10^4$$

Important Note

Instructions on rounding expectations will always be given on evaluations. Be sure to read instructions carefully to know if you are expected to use the rounding rules covered in this section, or if there are any other expectations. If questions are vague and do not specify rounding instructions, do not round your answer.

Key Concepts

- Units can be converted to other units using the proper conversion factors.
- Conversion factors are constructed from equalities that relate two different units.
- Conversions can be a single step or multistep.
- Unit conversion is a powerful mathematical technique in chemistry, physics, and mathematics, that must be mastered.
- Exact numbers do not affect the determination of significant figures.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=6819#h5p-35

b. Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

2.5 UNIT SOURCES

Unit 2 Sources

2.1 Up to & Including Example 2.1.4. taken from "Measurement Systems" from Mathematics for Public and Occupational Health Professionals is licensed under a Creative Commons Attribution-ShareAlike 4.0 International License, except where otherwise noted.

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UNIT 3: SOLVING LINEAR EQUATIONS, GRAPHS OF LINEAR EQUATIONS, AND APPLICATIONS OF LINEAR FUNCTIONS

Chapter Outline

3.0 INTRODUCTION

If we carefully placed more rocks of equal weight on both sides of this formation, it would still balance. Similarly, the expressions in an equation remain balanced when we add the same quantity to both sides of the equation. In this chapter, we will solve equations, remembering that what we do to one side of the equation, we must also do to the other side.

Figure 3.0.1 : The rocks in this formation must remain perfectly balanced around the center for the formation to hold its shape Photo by Robert Anasch Unsplash License

Mathematical formulas model phenomena in every facet of our lives. They are used to explain events and predict outcomes in fields such as transportation, business, economics, medicine, chemistry, engineering, and many more. In this chapter, we will apply our skills in solving equations to solve problems in a variety of situations.

Figure 3.0.2 :Sophisticated mathematical models are used to predict traffic patterns on our nation's highways. [Photo](#) by [Derek Lee](#) [Unsplash License](#)

Graphs are found in all areas of our lives—from commercials showing you which cell phone carrier provides the best coverage, to bank statements and news articles, to the boardroom of major corporations. In this chapter, we will study the rectangular coordinate system, which is the basis for most consumer graphs. We will look at linear graphs, slopes of lines, equations of lines, and linear inequalities.

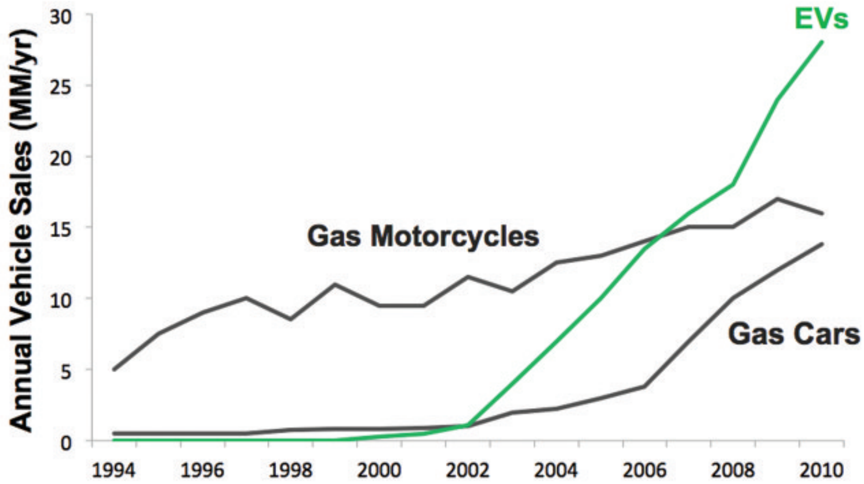


Figure 3.0.3: This graph illustrates the annual vehicle sales of gas motorcycles, gas cars, and electric vehicles from 1994 to 2010. It is a line graph with x- and y-axes, one of the most common types of graphs. Image by Steve Jurvetson

3.1 SOLVE EQUATIONS USING THE SUBTRACTION AND ADDITION PROPERTIES OF EQUALITY

Learning Objectives

By the end of this section, you will be able to:

- Verify a solution of an equation
- Solve equations using the Subtraction and Addition Properties of Equality
- Solve equations that require simplification
- Translate to an equation and solve
- Translate and solve applications

Try It

Before you get started, take this readiness quiz:

1) Evaluate $x+4$ when $x=-3$.
2) Evaluate $15-y$ when $y=-5$.
3) Simplify $4(4n+1)-15n$.
4) Translate into algebra "5 is less than x".

Verify a Solution of an Equation

Solving an equation is like discovering the answer to a puzzle. The purpose in solving an equation is to find the value or values of the variable that make each side of the equation the same – so that we end up with a true statement. Any value of the variable that makes the equation true is called a solution to the equation. It is the answer to the puzzle!

Solution of an equation

A **solution of an equation** is a value of a variable that makes a true statement when substituted into the equation.

HOW TO

To determine whether a number is a solution to an equation.

1. Substitute the number in for the variable in the equation.
2. Simplify the expressions on both sides of the equation.
3. Determine whether the resulting equation is true (the left side is equal to the right side)
    - If it is true, the number is a solution.
    - If it is not true, the number is not a solution.

Example 3.1.1

Determine whether $x = \dfrac{3}{2}$ is a solution of $4x - 2 = 2x + 1$.

Solution

Since a solution to an equation is a value of the variable that makes the equation true, begin by substituting the value of the solution for the variable.

Step 1: **Substitute $\frac{3}{2}$ for x.**

$$4\left(rgb]1.0,0.0,0.0\frac{3}{2}\right)-2\stackrel{?}{=}2\left(rgb]1.0,0.0,0.0\frac{3}{2}\right)+1$$

Step 2: **Multiply.**

$$6-2\stackrel{?}{=}3+1$$

Step 3: **Subtract.**

$$4=4\checkmark$$

Since $x=\frac{3}{2}$ results in a true equation (4 is in fact equal to 4), $\frac{3}{2}$ is a solution to the equation $4x-2=2x+1$

Try It

5) Is $y=\frac{4}{3}$ a solution of $9y+2=6y+3?$

Solution

no

6) Is $y=\frac{7}{5}$ a solution of $5y+3=10y-4$?

Solution

yes

Solve Equations Using the Subtraction and Addition

Properties of Equality

We are going to use a model to clarify the process of solving an equation. An envelope represents the variable – since its contents are unknown – and each counter represents one. We will set out one envelope and some counters on our workspace, as shown in Figure 3.1.1. Both sides of the workspace have the same number of counters, but some counters are "hidden" in the envelope. Can you tell how many counters are in the envelope?

The illustration shows a model of an equation with one variable. On the left side of the workspace is an unknown (envelope) and three counters, while on the right side of the workspace are eight counters.

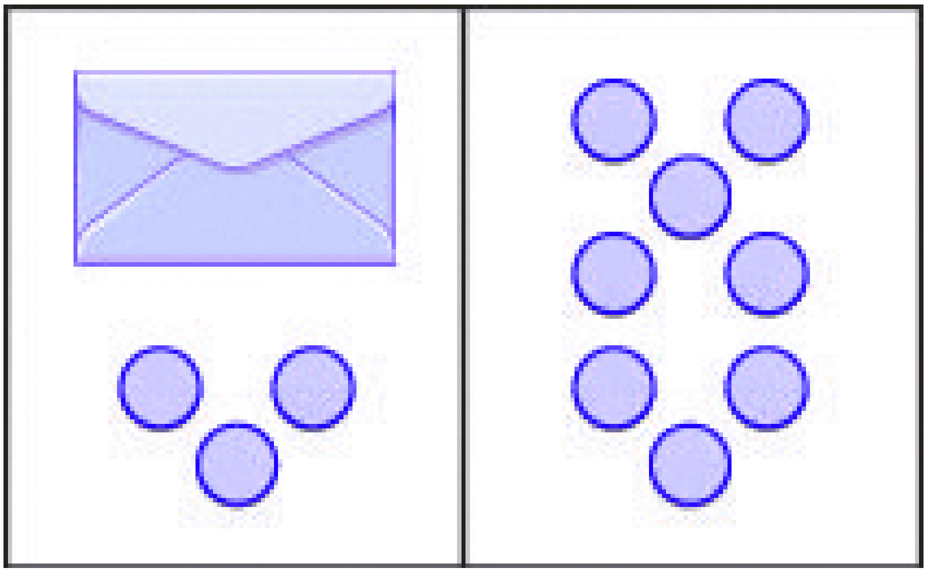


Figure 3.1.1 The illustration shows a model of an equation with one variable. On the left side of the workspace is an unknown (envelope) and three counters, while on the right side of the workspace are eight counters.

What are you thinking? What steps are you taking in your mind to figure out how many counters are in the envelope?

Perhaps you are thinking: "I need to remove the 3 counters at the bottom left to get the envelope by itself. The 3 counters on the left can be matched with 3 on the right and so I can take them away from both sides. That leaves five on the right—so there must be 5 counters in the envelope." See Figure 3.1.2. for an illustration of this process.

The illustration shows a model for solving an equation with one variable. On both sides of the workspace remove three counters, leaving only the unknown (envelope) and five counters on the right side. The unknown is equal to five counters.

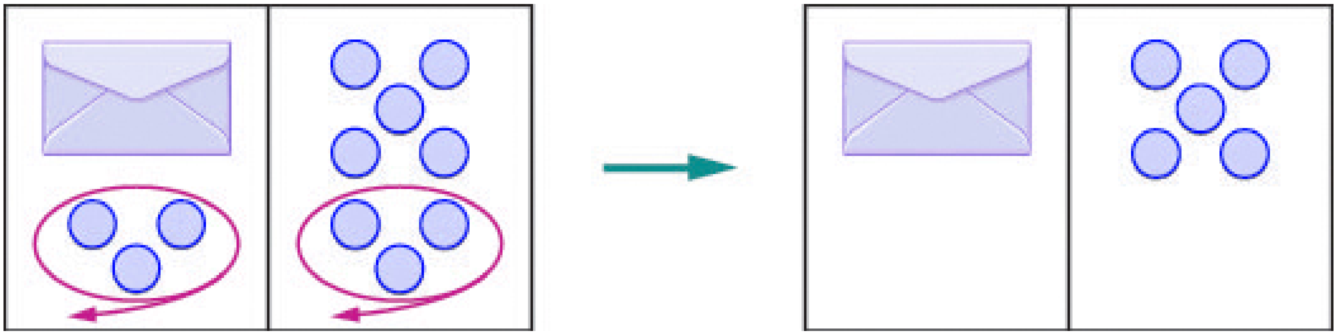
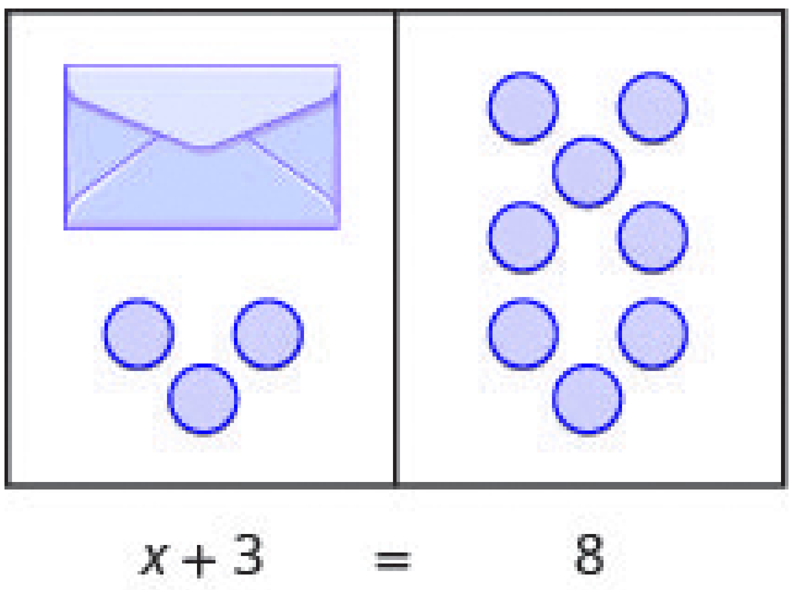


Figure 3.1.2-The illustration shows a model for solving an equation with one variable. On both sides of the workspace remove three counters, leaving only the unknown (envelope) and five counters on the right side. The unknown is equal to five counters.

What algebraic equation would match this situation? In Figure 3.1.3 each side of the workspace represents an expression and the centre line takes the place of the equal sign. We will call the contents of the envelope x.

The illustration shows a model for the equation $x + 3 = 8$.



3.1.3 – The illustration shows a model for the equation ◇+3=8.

Let's write algebraically the steps we took to discover how many counters were in the envelope:

	$x+3=8$
First, we took away three from each side.	$x+3rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.03=8rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.03$
Then we were left with five.	$x=5$
Check:	Five in the envelope plus three more does equal eight!
	$5+3=8$

Our model has given us an idea of what we need to do to solve one kind of equation. The goal is to isolate the variable by itself on one side of the equation. To solve equations such as these mathematically, we use the *Subtraction Property of Equality*.

Subtraction Property of Equality

For any numbers a, b, and c,

if $\quad a=b$,
then $\quad a-c=b-c$

When you subtract the same quantity from both sides of an equation, you still have equality.

Let's see how to use this property to solve an equation. Remember, the goal is to isolate the variable on one side of the equation. And we check our solutions by substituting the value into the equation to make sure we have a true statement.

Example 3.1.2

Solve: $y + 37 = -13$.

Solution

To get y by itself, we will undo the addition of 37 by using the Subtraction Property of Equality.

Step 1: **Subtract 37 from each side to 'undo' the addition.**

$$y + 37 rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.0 37 = -13 rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.0 37$$

Step 2: **Simplify.**

$$y = -50$$

Step 3: **Check:**

$$y + 37 = -13$$

Step 4: **Substitute** $y = -50$

$$rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.0 50 + 37 = -13$$

$$-13 \stackrel{?}{=} -13 \checkmark$$

Since $y = -50$ makes $y + 37 = -13$ a true statement, we have the solution to this equation.

Try It

7) Solve: $x + 19 = -27$.

Solution

$x = -46$

8) Solve: $x+16=-34$.

Solution

$x=-50$

What happens when an equation has a number subtracted from the variable, as in the equation $x-5=8$? We use another property of equations to solve equations where a number is subtracted from the variable. We want to isolate the variable, so to 'undo' the subtraction we will add the number to both sides. We use the Addition Property of Equality.

Addition Property of Equality

For any numbers a, b, and c,

if $a=b$,
then $a+c=b+c$
When you add the same quantity to both sides of an equation, you still have equality.

In Example 3.1.2, 37 was added to the y and so we subtracted 37 to 'undo' the addition. In 3.1.3, we will need to 'undo' subtraction by using the Addition Property of Equality.

Example 3.1.3

Solve: $a-28=-37$.

Solution
Step 1: **Add 28 to each side to 'undo' the subtraction.**

$a-28rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.028=-37rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.028$

Step 2: **Simplify.**

$$a = -9$$

Step 3: **Check:**

$$a - 28 = -37$$

Step 4: **Substitute** $a = -9$

$$rgb]1.0, 0.0, 0.0{-}rgb]1.0, 0.0, 0.09 - 28 = 37$$

$$-37 \stackrel{?}{=} -37\checkmark$$

The solution to $a - 28 = -37$ is $a = -9$.

Try It

9) Solve: $n - 61 = -75$.

Solution

$n = -14$

10) Solve: $p - 41 = -73$.

Solution

$p = -32$

Example 3.1.4

Solve: $x-\frac{5}{8}=\frac{3}{4}$

Solution

Step 1: **Use the Addition Property of Equality.**

$$x-\frac{5}{8}rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.0\frac{5}{8}=\frac{6}{8}rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.0\frac{5}{8}$$

Step 2: **Find the LCD to add the fractions on the right.**

$$x-\frac{5}{8}+\frac{5}{8}=\frac{6}{8}+\frac{5}{8}$$

$$\text{Simplify} \qquad x=\frac{11}{8}$$

Step 3: **Check:**

$$x-\frac{5}{8}=\frac{3}{4}$$

Step 4: **Substitute** $x=\frac{11}{8}$**.**

$$rgb]1.0,0.0,0.0\frac{11}{8}-\frac{5}{8}\stackrel{?}{=}\frac{3}{4}$$

$$\text{Subtract.} \qquad \frac{6}{8}\stackrel{?}{=}\frac{3}{4}$$

$$\text{Simplify.} \qquad \frac{3}{4}=\frac{3}{4}\checkmark$$

The solution to $x-\frac{5}{8}=\frac{3}{4}$ is $x=\frac{11}{8}$

Try It

11) Solve: $p-\frac{2}{3}=\frac{5}{6}$

Solution

$p=\frac{3}{2}$

12) Solve: $q-\frac{1}{2}=\frac{5}{6}$

Solution

$q=\frac{4}{3}$

The next example will be an equation with decimals.

Example 3.1.5

Solve: $n-0.63=-4.2$.

Solution

Step 1: **Use the Addition Property of Equality.**

$$\begin{array}{ll} & n-0.63 rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.00rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.063=-4.2 rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.00rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.063 \\ \text{Add.} & n=-3.57 \end{array}$$

Step 2: **Check:**

$$n=-3.57$$

Step 3: **Let** $n=-3.57$**.**

$$rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.03.57-0.63\stackrel{?}{=}rgb]0.1,0.1,0.1-rgb]0.1,0.1,0.14rgb]0.1,0.1,0.1.rgb]0.1,0.1,0.12$$
$$-4.2-4.2\checkmark$$

Try It

13) Solve: $b-0.47=-2.1$.

Solution

$b=-1.63$

14) Solve: $c-0.93=-4.6$.

Solution

$c=-3.67$

Solve Equations That Require Simplification

In the previous examples, we were able to isolate the variable with just one operation. Most of the equations we encounter in algebra will take more steps to solve. Usually, we will need to simplify one or both sides of an equation before using the *Subtraction or Addition Properties of Equality*.

You should always simplify as much as possible before you try to isolate the variable. Remember that to simplify an expression means to do all the operations in the expression. Simplify one side of the equation at a time. Note that simplification is different from the process used to solve an equation in which we apply an operation to both sides.

Example 3.1.6

Solve: $9x-5-8x-6=7$.

Solution

Step 1: **Simplify the expressions on each side as much as possible.**

Rearrange the terms, using the Commutative Property of Addition.

$$\text{Combine like terms.} \qquad \begin{aligned} 9x-5-8x-6&=7 \\ 9x-8x-5-6&=7 \\ x-11&=7 \end{aligned}$$

Notice that each side is now simplified as much as possible.

Step 2: **Isolate the variable.**

Now isolate x.

Undo subtraction by adding 11 to both sides. $x-11rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.011=7rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.011$

Step 3: **Simplify the expressions on both sides of the equation.**

$$x=18$$

Step 4: **Check the solution.**

Substitute $x=18$

$$\begin{aligned} 9x-5-8x-6&=7 \\ 9rgb]1.0,0.0,0.0\,(18)-5-8rgb]1.0,0.0,0.0\,(8)-6&\overset{?}{=}7 \\ 162-5-144-6&\overset{?}{=}7 \\ 157-144&\overset{?}{=}7 \\ 13-6&\overset{?}{=}7 \\ 7&=7\checkmark \end{aligned}$$

The solution to $9x-5-8x-6=7$ is $x=18$

Try It

15) Solve: $8y - 4 - 7y - 7 = 4$.

Solution

$y = 15$

16) Solve: $6z + 5 - 5z - 4 = 3$.

Solution

$z = 2$

Example 3.1.7

Solve: $5(n - 4) - 4n = -8$.

Solution

We simplify both sides of the equation as much as possible before we try to isolate the variable.

Step 1: **Distribute on the left.**

$$5n - 20 - 4n = -8$$

Step 2: **Use the Commutative Property to rearrange terms.**

$$5n - 4n - 20 = -8$$

Combine like terms. $\quad n - 20 = -8$

Step 3: **Each side is as simplified as possible. Next, isolate n.**

Step 4: **Undo subtraction by using the Addition Property of Equality.**

$$n - 20 rgb]1.0, 0.0, 0.0 + rgb]1.0, 0.0, 0.020 = -8 rgb]1.0, 0.0, 0.0 + rgb]1.0, 0.0, 0.020$$

Add. $\quad n = 12$

***Step 5:* Check.**

Substitute $n = 12$.

$$\begin{aligned} 5(n-4)-4n &= -8 \\ 5(rgb]1.0,0.0,0.012-4)-4(rgb]1.0,0.0,0.012) &\stackrel{?}{=} -8 \\ 5(8)-48 &\stackrel{?}{=} -8 \\ 40-48 &\stackrel{?}{=} -8 \\ -8 &= -8\checkmark \end{aligned}$$

The solution to $5(n-4)-4n=-8$ is $n = 12$.

Try It

17) Solve: $5(p-3)-4p=-10$.

Solution

$p = 5$

18) Solve: $4(q+2)-3q=-8$.

Solution

$q = -16$

Example 3.1.8

Solve: $3(2y-1)-5y=2(y+1)-2(y+3)$.

Solution

We simplify both sides of the equation before we isolate the variable.

Step 1: **Distribute on both sides.**

$$6y-3-5y=2y+2-2y-6$$

Step 2: **Use the Commutative Property of Addition.**

$$6y-5y-3=2y-2y+2-6$$

$$\text{Combine like terms.}\qquad y-3=-4$$

Step 3: **Each side is as simplified as possible. Next, isolate** y**.**

Step 4: **Undo subtraction by using the Addition Property of Equality.**

$$y-3rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.03=-4rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.03$$

$$\text{Add.}\qquad y=-1$$

Step 5: **Check.**

Let $y=-1$.

$$3(2y-1)-5y=2(y+1)-2(y+3)$$

$$3(2(rgb]1.0,0.0,0.012-1)-1)-5(rgb]1.0,0.0,0.0-1)\stackrel{?}{=}2(-1+1)-2(-1+3)$$

$$3(-2-1)+5\stackrel{?}{=}2(0)-2(2)$$

$$3(-3)+5\stackrel{?}{=}-4$$

$$-9+5=-4$$

$$-4=-4\checkmark$$

The solution to $3(2y-1)-5y=2(y+1)-2(y+3)$ is $y=-1$.

Try It

19) Solve: $4(2h-3)-7h=6(h-2)-6(h-1)$

Solution

$h=6$

20) Solve: $2(5x+2)-9x=3(x-2)-3(x-4)$

Solution

$x=2$

Translate to an Equation and Solve

To solve applications algebraically, we will begin by translating from English sentences into equations. Our first step is to look for the word (or words) that would translate to the equals sign. Below table shows us some of the words that are commonly used.

Equals =	is is equal to is the same as the result is gives was will be

The steps we use to translate a sentence into an equation are listed below.

HOW TO

Translate an English sentence to an algebraic equation.

1. Locate the "equals" word(s). Translate to an equals sign (=).
2. Translate the words to the left of the "equals" word(s) into an algebraic expression.
3. Translate the words to the right of the "equals" word(s) into an algebraic expression.

Example 3.1.9

Translate and solve: Eleven more than x is equal to 54.

Solution

Step 1: **Translate.**

$$\underbrace{Eleven\ more\ than\ x}_{x+11}\ \underbrace{is\ equal\ to}_{=}\ \underbrace{54}_{54}$$

Step 2: **Subtract 11 from both sides.**

$$x+11rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.011=54rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.011$$

Step 3: **Simplify.**

$$x=43$$

Step 4: **Check**

$$\begin{aligned} 43+11 &\stackrel{?}{=} 54 \\ 54 &= 54 \end{aligned}$$

Try It

21) Translate and solve: Ten more than x is equal to 41.

Solution

$x+10=41; \quad x=31$

22) Translate and solve: Twelve less than x is equal to 51.

Solution

$y-12=51; \quad y=63$

Example 3.1.10

Translate and solve: The difference of $12t$ and $11t$ is -14.

Solution

Step 1: Translate.

$$\underbrace{The\ difference\ of\ 12t\ and\ 11t}_{12t-11t}\ \underbrace{is}_{=}\ \underbrace{-14}_{-14}$$

***Step 2:* Simplify.**

$$t=-14$$

***Step 3:* Check:**

$$\begin{aligned} 12(-14)-11(-14) &\stackrel{?}{=} -14 \\ (-168+154) &\stackrel{?}{=} -14 \\ -14 &= -14 \end{aligned}$$

Try It

23) Translate and solve: The difference of $4x$ and $3x$ is 14.

Solution

$$\begin{aligned} 4x - 3x &= 14 \\ x &= 14 \end{aligned}$$

24) Translate and solve: The difference of $7a$ and $6a$ is -8.

Solution

$$\begin{aligned} 7a - 6a &= -8 \\ a &= -8 \end{aligned}$$

Translate and Solve Applications

Most of the time a question that requires an algebraic solution comes out of a real-life question. To begin with, that question is asked in English (or the language of the person asking) and not in math symbols. Because of this, it is an important skill to be able to translate an everyday situation into algebraic language.

We will start by restating the problem in just one sentence, assigning a variable, and then translating the sentence into an equation to solve. When assigning a variable, choose a letter that reminds you of what you are looking for. For example, you might use q for the number of quarters if you were solving a problem about coins.

Example 3.1.11

The MacIntyre family recycled newspapers for two months. The two months of newspapers weighed a total of 57 pounds. The second month, the newspapers weighed 28 pounds. How much did the newspapers weigh the first month?

Solution

Step 1: **Read the problem.**

Make sure all the words and ideas are understood.

The problem is about the weight of newspapers.

Step 2: **Identify what we are asked to find.**

What are we asked to find?

"How much did the newspapers wight the 2nd month?"

Step 3: **Name what we are looking for.**

Choose a variable to represent that quantity.

Let w = weight of the newspapers the 1st month

Step 4: **Translate into an equation.**

It may be helpful to restate the problem in one sentence with the important information.

Restate the problem.

Weight of newspapers the 1st month plus the weight of the newspapers the 2nd month equals 57 pounds.

We know the weight of the newspapers the second month is 28 pounds.

Weight from 1st month plus 2nd month equals 57 pounds.

Translate into an equation, using the variable w.

$$w + 28 = 57$$

Step 5: **Solve the equation using good algebra techniques.**

$$w + 28rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.028 = 57rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.028$$
$$w = 29$$

Step 6: **Check the answer in the problem and make sure it makes sense.**

Does 1st month's weight plus 2nd month's weight equal 57 pounds?

$$29 + 28 \stackrel{?}{=} 57$$
$$57 = 57\checkmark$$

Step 7: **Answer the question with a complete sentence.**

Write a sentence to answer "How much did the newspapers weigh the 2nd month?"

The 2nd month the newspapers weighed 29 pounds.

Translate into an algebraic equation and solve:

HOW TO

Solve an application.

1. Read the problem. Make sure all the words and ideas are understood.
2. Identify what we are looking for.
3. Name what we are looking for. Choose a variable to represent that quantity.
4. Translate into an equation. It may be helpful to restate the problem in one sentence with the important information.
5. Solve the equation using good algebra techniques.
6. Check the answer in the problem and make sure it makes sense.
7. Answer the question with a complete sentence.

Example 3.1.12

Randell paid $\$28,675$ for his new car. This was $\$875$ less than the sticker price. What was the sticker price of the car?

Solution

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

"What was the sticker price of the car?"

Step 3: **Name what we are looking for.**

Choose a variable to represent that quantity.

Let s = the sticker price of the car.

Step 4: **Translate into an equation. Restate the problem in one sentence.**

$\$28,675$ is $\$875$ less than the sticker price

$\$28,675$ is $\$875$ less than s

Step 5: **Solve the equation.**

$$\begin{aligned} 28675 &= s - 875 \\ 28675 + 875 &= s - 875 + 875 \\ 29550 &= s \end{aligned}$$

Step 6: **Check the answer**.

Is $\$875$ less than $\$29,550$? Equal to $\$28,675$?

$$\begin{aligned} 29,550 - 875 &\stackrel{?}{=} 28,675 \\ 28,675 &= 28,675 \end{aligned}$$

Step 7: **Answer the question with a complete sentence.**

The sticker price of the car was $\$29,550$.

Try It

25) Translate into an algebraic equation and solve:

The Pappas family has two cats, Zeus and Athena. Together, they weigh 23 pounds. Zeus weighs 16 pounds. How much does Athena weigh?

Solution

7 pounds

26) Translate into an algebraic equation and solve:

Sam and Henry are roommates. Together, they have 68 books. Sam has 26 books. How many books does Henry have?

Solution
42 books

27) Translate into an algebraic equation and solve:

Eddie paid $19,875 for his new car. This was $1,025 less than the sticker price. What was the sticker price of the car?

Solution
$20,900

28) Translate into an algebraic equation and solve:

The admission price for the movies during the day is $7.75. This is $3.25 less the price at night. How much does the movie cost at night?

Solution
$11.00

Key Concepts

- **To Determine Whether a Number is a Solution to an Equation**
    1. **Substitute the number in for the variable in the equation.**
    2. **Simplify the expressions on both sides of the equation.**
    3. **Determine whether the resulting statement is true.**
        - If it is true, the number is a solution.
        - If it is not true, the number is not a solution.

- **Addition Property of Equality**
    - For any numbers a, b, and c, if $a = b$, then $a + c = b + c$.
- **Subtraction Property of Equality**
    - For any numbers
    - a, b, and c, if $a = b$, then $a - c = b - c$.
- **To Translate a Sentence to an Equation**
    1. Locate the "equals" word(s). Translate to an equal sign ($=$).
    2. Translate the words to the left of the "equals" word(s) into an algebraic expression.
    3. Translate the words to the right of the "equals" word(s) into an algebraic expression.
- **To Solve an Application**
    1. Read the problem. Make sure all the words and ideas are understood.
    2. Identify what we are looking for.
    3. Name what we are looking for. Choose a variable to represent that quantity.
    4. Translate into an equation. It may be helpful to restate the problem in one sentence with the important information.
    5. Solve the equation using good algebra techniques.
    6. Check the answer in the problem and make sure it makes sense.
    7. Answer the question with a complete sentence.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=556#h5p-8

Glossary

solution of an equation
solution of an equation is a value of a variable that makes a true statement when substituted into the equation.

3.2 SOLVE EQUATIONS USING THE DIVISION AND MULTIPLICATION PROPERTIES OF EQUALITY

Learning Objectives

By the end of this section, you will be able to:

- Solve equations using the Division and Multiplication Properties of Equality
- Solve equations that require simplification
- Translate to an equation and solve
- Translate and solve applications

Try It

Before you get started, take this readiness quiz:

1) Simplify: $-7 \cdot \frac{1}{-7}$.

2) Evaluate $9x + 2$ when $x = -3$.

Solve Equations Using the Division and Multiplication

Properties of Equality

You may have noticed that all of the equations we have solved so far have been of the form $x + a = b$ or $x - a = b$. We were able to isolate the variable by adding or subtracting the constant term on the side of the equation with the variable. Now we will see how to solve equations that have a variable multiplied by a constant and so will require the division to isolate the variable.

Let's look at our puzzle again with the envelopes and counters in Figure 3.2.1.

The illustration shows a model of an equation with one variable multiplied by a constant. On the left side of the workspace are two instances of the unknown (envelope), while on the right side of the workspace are six counters.

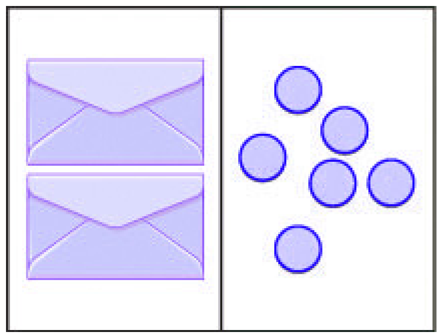


Figure 3.2.1 – The illustration shows a model of an equation with one variable multiplied by a constant. On the left side of the workspace are two instances of the unknown (envelope), while on the right side of the workspace are six counters.

In the illustration, there are two identical envelopes that contain the same number of counters. Remember, the left side of the workspace must equal the right side, but the counters on the left side are "hidden" in the envelopes. So how many counters are in each envelope?

How do we determine the number? We have to separate the counters on the right side into two groups of the same size to correspond with the two envelopes on the left side. The 6 counters divided into 2 equal groups gives 3 counters in each group (since $6 \div 2 = 3$).

What equation models the situation shown in Figure 3.2.2? There are two envelopes, and each contains x counters. Together, the two envelopes must contain a total of 6 counters.

The illustration shows a model of the equation $2x = 6$.

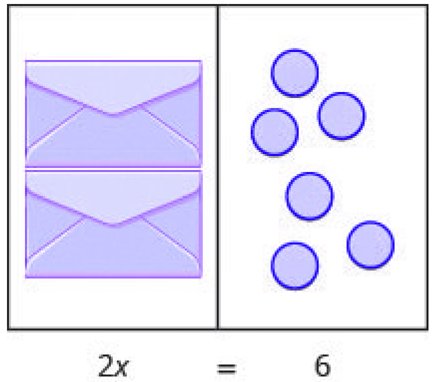


Figure 3.2.2 – The illustration shows a model of the equation [latex]2�=6[/latex].

	$2x = 6$
If we divide both sides of the equation by 2, as we did with the envelopes and counters,	$\frac{2x}{2} = \frac{6}{2}$
we get:	$x = 3$

We found that each envelope contains 3 counters. Does this check? We know $2 \cdot 3 = 6$, so it works! Three counters in each of two envelopes do equal six!

This example leads to the Division Property of Equality.

The Division Property of Equality

For any numbers a, b, and c, if $a = b$, and $c \neq 0$,

If $a = b$,
then $\frac{a}{c} = \frac{b}{c}$
When you divide both sides of an equation by any non-zero number, you still have equality.

The goal in solving an equation is to 'undo' the operation on the variable. In the next example, the variable is multiplied by 5, so we will divide both sides by 5 to 'undo' the multiplication.

Example 3.2.1

Solve: $5x = -27$.

Solution

Step 1: **To isolate x, "undo" the multiplication by 5.**

$$5x = -27$$

Step 2: **Divide to 'undo' the multiplication.**

$$\frac{5x}{rgb]1.0, 0.0, 0.05} = -\frac{27}{rgb]1.0, 0.0, 0.05}$$

$$\text{Simplify.} \qquad x = -\frac{27}{5}$$

Step 3: **Check:**

$$5x = -27$$

Step 5: **Substitute** $-\frac{27}{5}$ **for** x**.**

\begin{eqnarray*}5\left({\color[rgb]{1.0, 0.0, 0.0}-}{\color[rgb]{1.0, 0.0, 0.0}\frac{27}5}\right)&\overset?=-27\\-27&=-27\checkmark\end{eqnarray*}

Since this is a true statement, $x = -\frac{27}{5}$ is the solution to $5x = -27$.

Try It

3) Solve: $3y = -41$.

Solution

$$y = \frac{-41}{3}$$

4) Solve: $4z = -55$.

Solution

$$z = -\frac{55}{4}$$

Consider the equation $\frac{x}{4} = 3$. We want to know what number divided by 4 gives 3. So to "undo" the division, we will need to multiply by 4. The Multiplication Property of Equality will allow us to do this. This property says that if we start with two equal quantities and multiply both by the same number, the results are equal.

The Multiplication Property of Equality

For any numbers a, b, and c,

If $\quad a = b$,
Then $ac = bc$
If you multiply both sides of an equation by the same number, you still have equality.

Example 3.2.2

Solve: $\frac{y}{-7} = -14$.

Solution

Here y is divided by -7. We must multiply by -7 to isolate y.

Step 1: **Multiply both sides by -7.**

$$\frac{5x}{rgb]1.0,0.0,0.05} = -\frac{27}{rgb]1.0,0.0,0.05}$$

Simplify. $$x = -\frac{27}{5}$$

Step 2: **Check:** $\frac{y}{-7} = -14$

Step 3: **Substitute** $y = 98$.

$$\frac{rgb]1.0,0.0,0.098}{-7} \stackrel{?}{=} -14$$

$$-14 = -14✓$$

Try It

5) Solve: $\frac{a}{-7} = -42$.

Solution

$a = 294$

6) Solve: $\frac{b}{-6} = -24$.

Solution

$b = 144$

Example 3.2.3

Solve: $-n = 9$.

Solution

***Step 1:* Remember $-n$ is equivalent to $-1n$.**

$$-1n = 9$$

***Step 2:* Divide both sides by -1.**

$$\text{Divide} \quad \frac{-1n}{rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.01} = \frac{9}{rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.01} \quad n = -9$$

Notice that there are two other ways to solve $-n = 9$. We can also solve this equation by multiplying both sides by -1 and also by taking the opposite of both sides.

***Step 3:* Check:**

$$-n = 9$$

***Step 4:* Substitute $n = -9$.**

$$-(-rgb]1.0,0.0,0.09) \stackrel{?}{=} 9$$

$$\text{Simplify} \quad 9 = 9\checkmark$$

Try It

7) Solve: $-k = 8$.

Solution

$k = -8$

8) Solve: $-g = 3$.

Solution

$g = -3$

Example 3.2.4

Solve: $\frac{3}{4}x = 12$.

Solution

Since the product of a number and its reciprocal is 1, our strategy will be to isolate x by multiplying by the reciprocal of $\frac{3}{4}$.

Step 1: **Multiply by the reciprocal of $\frac{3}{4}$.**

$$rgb]1.0, 0.0, 0.0\frac{4}{3} \cdot \frac{3}{4}x = rgb]1.0, 0.0, 0.0\frac{4}{3} \cdot 12$$

$$\text{Reciprocals multiply to 1} \qquad 1x = \frac{4}{3} \cdot \frac{12}{1}$$

$$\text{Mulitply.} \qquad x = 16$$

Notice that we could have divided both sides of the equation $\frac{3}{4}x = 12$ by $\frac{3}{4}$ to isolate x. While this would work, most people would find multiplying by the reciprocal easier.

Step 2: **Check:**

Step 5: **Substitute $x = 16$.**

$$\frac{3}{4}x = 12$$
$$\frac{3}{4} \cdot rgb]1.0,0.0,0.016 \stackrel{?}{=} 12$$
$$12 = 12\checkmark$$

Try It

9) Solve: $\frac{2}{5}n = 14$.

Solution

$n = 35$

10) Solve: $\frac{5}{6}y = 15$.

Solution

$y = 18$

In the next example, all the variable terms are on the right side of the equation. As always, our goal in solving the equation is to isolate the variable.

Example 3.2.5

Solve: $\frac{8}{15} = -\frac{4}{5}x$.

Solution

***Step 1:* Multiply by the reciprocal of $-\frac{4}{5}$.**

	$\left(rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\frac{5}{4}\right)\left(\frac{8}{15}\right) = \left(rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\frac{5}{4}\right)\left(-\frac{4}{5}x\right)$
Reciprocals multiply to 1	$-\frac{\cancel{5}\cdot\cancel{4}\cdot 2}{\cancel{4}\cdot 3\cdot\cancel{5}} = 1x$
Mulitply.	$-\frac{2}{3} = x$

***Step 2:* Let $x = -\frac{2}{3}$.**

$$\frac{8}{15} = -\frac{4}{5}x$$
$$\frac{8}{15} = -\frac{4}{5}\left(rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\frac{2}{3}\right)$$
$$\frac{8}{15} = \frac{8}{15}\checkmark$$

Try It

11) Solve: $\frac{9}{25} = -\frac{4}{5}z$.

Solution

$z = -\frac{9}{5}$

12) Solve: $\frac{5}{6} = -\frac{8}{3}r$

Solution

$r = -\frac{5}{16}$

Solve Equations That Require Simplification

Many equations start out more complicated than the ones we have been working with.

With these more complicated equations, the first step is to simplify both sides of the equation as much as possible. This usually involves combining like terms or using the distributive property.

Example 3.2.6

Solve: $14 - 23 = 12y - 4y - 5y$.

Solution

Begin by simplifying each side of the equation.

Step 1: **Simplify each side.**

$$-9 = 3y$$

Step 2: **Divide both sides by 3 to isolate y.**

$$\frac{-9}{3} = \frac{3y}{3}$$

$$\text{Divide.} \qquad -3 = y$$

Step 3: **Check:**

$$14 - 23 \stackrel{?}{=} 12y - 4y - 5y$$

***Step 4:* Substitute $y = -3$.**

$$14 - 23 \stackrel{?}{=} 12\,(rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.03) - 4\,(rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.03) - 5\,(rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.03)$$
$$14 - 23 \stackrel{?}{=} -36 + 12 + 15$$
$$-9 = -9\checkmark$$

Try It

13) Solve: $18 - 27 = 15c - 9c - 3c$.

Solution

$c = -3$

14) Solve: $18 - 22 = 12x - x - 4x$.

Solution

$x = -\dfrac{4}{7}$

Example 3.2.7

Solve: $-4(a - 3) - 7 = 25$.

Solution

Here we will simplify each side of the equation by using the distributive property first.

***Step 1:* Distribute.**

$$-4a + 12 - 7 = 25$$

Simplify. $-4a + 5 = 25$

Simplify. $-4a = 20$

Step 2: **Divide both sides by -4 to isolate a.**

$$\left(\frac{-4a}{rgb]1.0, 0.0, 0.0{-}rgb]1.0, 0.0, 0.04}\right) = \left(\frac{20}{rgb]1.0, 0.0, 0.0{-}rgb]1.0, 0.0, 0.04}\right)$$

Divide. $a = -5$

Step 3: **Check:**

$$-4(a - 3) - 7 = 25$$

Step 4: **Substitute $a = -5$.**

$$-4\left(rgb]1.0, 0.0, 0.0{-}rgb]1.0, 0.0, 0.05 - 3\right) - 7 \stackrel{?}{=} 25$$

$$-4\left(-8\right) - 7 \stackrel{?}{=} 25$$

$$32 - 7 \stackrel{?}{=} 25$$

$$25 = 25\checkmark$$

Try It

15) Solve: $-4(q - 2) - 8 = 24$.

Solution

$q = -6$

16) Solve: $-6(r - 2) - 12 = 30$.

Solution

$r = -5$

Now we have covered all four properties of equality—subtraction, addition, division, and multiplication. We'll list them all together here for easy reference.

Properties of Equality

Subtraction Property of Equality for any real numbers a, b and c, If $a = b$, then $a - c = b - c$.	**Addition Property of Equality** for any real numbers a, b and c, If $a = b$, then $a + c = b + c$.
Division Property of Equality for any real numbers a, b and c, and $c \neq 0$ If $a = b$, then $\frac{a}{c} = \frac{b}{c}$.	**Multiplication Property of Equality** for any real numbers a, b and c, If $a = b$, then $a \times c = b \times c$.

When you add, subtract, multiply, or divide the same quantity from both sides of an equation, you still have equality.

Translate to an Equation and Solve

In the next few examples, we will translate sentences into equations and then solve the equations. You might want to review the translation table in the previous chapter.

Example 3.2.8

Translate and solve: The number 143 is the product of -11 and y.

Solution

Begin by translating the sentence into an equation.

Step 1: **Translate.**

$$\underbrace{The\ number\ 143}_{143}\ \underbrace{is}_{=}\ \underbrace{the\ product\ of\ -11\ and\ y.}_{-11y}$$

Step 2: **Divide by -11.**

$$\frac{143}{rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.011} = \frac{-11y}{rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.011}$$

Step 3: **Simplify.**

$$-13 = y$$

Step 4: **Check:**

$$\begin{aligned} 143 &= -11y \\ 143 &\stackrel{?}{=} -11(-13) \\ 143 &= 143\checkmark \end{aligned}$$

Try It

17) Translate and solve: The number 132 is the product of -12 and y.

Solution

$132 = -12y; \quad y = -11$

18) Translate and solve: The number 117 is the product of -13 and z.

Solution

$117 = -13z; \quad z = -9$

Example 3.2.9

Translate and solve: n divided by 8 is -32.

Solution

Step 1: **Begin by translating the sentence into an equation.**

$$\underbrace{n\ divided\ by\ 8}_{\frac{n}{8}}\ \underbrace{is}_{=}\ \underbrace{-32.}_{-32}$$

Step 2: **Multiple both sides by 8.**

$$rgb]1.0, 0.0, 0.08 \cdot \frac{n}{8} = rgb]1.0, 0.0, 0.08\,(-32)$$

$$\text{Simplify.} \qquad n = -256$$

Step 3: **Check:**

Is n divided by 8 equal to -32?

Step 4: **Let $n = -256$.**

$$\frac{-256}{8} \stackrel{?}{=} -32$$

$$-31 = -32\checkmark$$

Try It

19) Translate and solve: n divided by 7 is equal to -21.

Solution

$$\frac{n}{7}=-21; \qquad n=-147$$

20) Translate and solve: n divided by 8 is equal to -56.

Solution

$$\frac{n}{8}=-56; \qquad n=-448$$

Example 3.2.10

Translate and solve: The quotient of y and -4 is 68.

Solution

Begin by translating the sentence into an equation.

Step 1: **Translate.**

$$\underbrace{The\ quotient\ of\ y\ and\ -4}_{\frac{y}{-4}}\ \underbrace{is}_{=}\ \underbrace{68.}_{68}$$

Step 2: **Multiply both sides by -4.**

$$rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.04\left(\frac{y}{-4}\right)=rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.04\,(68)$$

$$\text{Simplify.} \qquad y=-272$$

Step 3: **Check:**

Is the quotient of y and -4 equal to 68?

Step 4: **Let** $y = -272$**.**

$$\frac{-272}{-4} \stackrel{?}{=} 68$$
$$68 = 68\checkmark$$

Try It

21) Translate and solve: The quotient of q and -8 is 72.

Solution

$\dfrac{q}{-8} = 72;\qquad q = -576$

22) Translate and solve: The quotient of p and -9 is 81.

Solution

$\dfrac{p}{-9} = 81;\qquad p = -729$

Example 3.2.11

Translate and solve: Three-fourths of p is 18.

Solution

Begin by translating the sentence into an equation. Remember, “of” translates into multiplication.

Step 1: **Translate.**

$$\underbrace{Three - fourths\ of\ p}_{\frac{3}{4}p}\ \underbrace{is}_{=}\ \underbrace{18.}_{18}$$

Step 2: **Multiply both sides by $\frac{4}{3}$.**

$$rgb]1.0,0.0,0.0\frac{4}{3}\cdot\frac{3}{4}p = rgb]1.0,0.0,0.0\frac{4}{3}\cdot 18$$

$$\text{Simplify.} \qquad p = 24$$

Step 3: **Check:**

Is three-fourths of p equal to 18?

Step 5: **Let $p = 24$.**

Is three-fourths of 24 equal to 18?

$$\frac{3}{4}\times 24 \stackrel{?}{=} 18$$

$$18 = 18\checkmark$$

Try It

23) Translate and solve: Two-fifths of f is 16.

Solution

$\frac{2}{5} = 16;\ f = 40$

24) Translate and solve: Three-fourths of f is 21.

Solution

$\frac{3}{4} = 21; f = 28$

Example 3.2.12

Translate and solve: The sum of three-eighths and x is one-half.

Solution

Begin by translating the sentence into an equation.

***Step 1:* Translate.**

$$\underbrace{The\ sum\ of\ three-eighths\ and\ x}_{\frac{3}{8}+x}\ \underbrace{is}_{=}\ \underbrace{\frac{1}{2}}_{\frac{1}{2}}$$

***Step 2:* Subtract $\frac{3}{8}$ from each side.**

$$\frac{3}{8} - rgb]1.0,0.0,0.0\frac{3}{8} + x = \frac{1}{2} - rgb]1.0,0.0,0.0\frac{3}{8}$$

Simplify and rewrite fractions with common denominators. $\quad x = \frac{4}{8} - \frac{3}{8}$

Simplify. $\quad x = \frac{1}{8}$

***Step 3:* Check:**

Is the sum of three-eighths and x equal to one-half?

***Step 4:* Let $x = \frac{1}{8}$**

Is the sum of three-eighths and one-eighth equal to one-half?

$$\frac{3}{8} + \frac{1}{8} \stackrel{?}{=} \frac{1}{2}$$

Simplify. $\quad \frac{4}{8} \stackrel{?}{=} \frac{1}{2}$

Simplify. $\quad \frac{1}{2} = \frac{1}{2}\checkmark$

Try It

25) Translate and solve: The sum of five-eighths and x is one-fourth.

Solution

$\frac{5}{8}+x=\frac{1}{4}; \qquad x=-\frac{3}{8}$

26) Translate and solve: The sum of three-fourths and x is five-sixths.

Solution

$\frac{3}{4}+x=\frac{5}{6}; \qquad x=\frac{1}{12}$

Translate and Solve Applications

To solve applications using the Division and Multiplication Properties of Equality, we will follow the same steps we used in the last section. We will restate the problem in just one sentence, assign a variable, and then translate the sentence into an equation to solve.

Example 3.2.13

Denae bought 6 pounds of grapes for $\$10.74$. What was the cost of one pound of grapes?

Solution

Step 1: **What are you asked to find?**

The cost of 1 pound of grapes

Step 2: **Assign a variable.**

Let c = the cost of one pound.

Step 3: **Write a sentence that gives the information to find it.**

The cost of 6 pounds is $\$10.74$.

Step 4: **Translate into an equation.**

$$6c = 10.74$$

Step 5: **Solve.**

$$\frac{4}{3}(12,000) = \frac{4}{3} \times \frac{3}{4}p$$
$$16,000 = p$$

The grapes cost $\$1.79$ per pound.

Step 6: **Check:**

If one pound costs $\$1.79$, do 6 pounds cost $\$10.74$?

$$6\,(1.79) \stackrel{?}{=} 10.74$$
$$10.74 = 10.74\checkmark$$

Try It

27) Translate and solve:

Arianna bought a 24-pack of water bottles for $\$9.36$. What was the cost of one water bottle?

Solution

$\$0.39$

28) Translate and solve:

At JB's Bowling Alley, 6 people can play on one lane for $\$34.98$. What is the cost for each person?

Solution

$\$5.83$

Example 3.2.14

Andreas bought a used car for $\$12,000$. Because the car was 4-years old, its price was $\frac{3}{4}$ of the original price, when the car was new. What was the original price of the car?

Solution

Step 1: **What are you asked to find?**

The original price of the car

Step 2: **Assign a variable.**

p = the original price.

Step 3: **Write a sentence that gives the information to find it.**

$\$12,000$ is $\frac{3}{4}$ of the original price.

Step 4: **Translate into an equation.**

$$12,000 = \frac{3}{4}p$$

Step 5: **Solve.**

$$\frac{4}{3} \times (12,000) = \frac{4}{3} \times \frac{3}{4}p$$
$$16,000 = p$$

The original cost of the car was $\$16,000$.

Step 6: **Check:**

Is $\frac{3}{4}$ of $\$16,000$ equal to $\$12,000$?

$$\frac{3}{4} \times (16,000) \stackrel{?}{=} 12,000$$
$$12,000 = 12,000 \checkmark$$

Try It

29) Translate and solve:

The annual property tax on the Mehta's house is $\$1,800$, calculated as $\frac{15}{1,000}$ of the assessed value of the house. What is the assessed value of the Mehta's house?

Solution

$\$120,000$

30) Translate and solve:

Stella planted 14 flats of flowers in $\frac{2}{3}$ of her garden. How many flats of flowers would she need to fill the whole garden?

Solution

21 flats

Key Concepts

- **The Division Property of Equality**—For any numbers a, b, and c, and $c \neq 0$, if $a = b$, then $\frac{a}{c} = \frac{b}{c}$.
  When you divide both sides of an equation by any non-zero number, you still have equality.
- **The Multiplication Property of Equality**—For any numbers a, b, and c, if $a = b$, then $ac = bc$.
  If you multiply both sides of an equation by the same number, you still have equality.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=631#h5p-16

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

3.3 SOLVE EQUATIONS WITH VARIABLES AND CONSTANTS ON BOTH SIDES

Learning Objectives

By the end of this section, you will be able to:

- Solve an equation with constants on both sides
- Solve an equation with variables on both sides
- Solve an equation with variables and constants on both sides

Try It

Before you get started, take this readiness quiz:

1) Simplify: $4y-9+9$.

Solve Equations with Constants on Both Sides

In all the equations we have solved so far, all the variable terms were on only one side of the equation with the constants on the other side. This does not happen all the time—so now we will learn to solve equations in which the variable terms, or constant terms, or both are on both sides of the equation.

Our strategy will involve choosing one side of the equation to be the "variable side", and the other side of the equation to be the "constant side." Then, we will use the Subtraction and Addition Properties of Equality to get all the variable terms together on one side of the equation and the constant terms together on the other side.

By doing this, we will transform the equation that began with variables and constants on both sides into the form $ax = b$. We already know how to solve equations of this form by using the Division or Multiplication Properties of Equality.

Example 3.3.1

Solve: $7x + 8 = -13$.

Solution

In this equation, the variable is found only on the left side. It makes sense to call the left side the "variable" side. Therefore, the right side will be the "constant" side. We will write the labels above the equation to help us remember what goes where.

$$\underset{7x+8}{rgb]1.0,0.0,0.0v rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0r rgb]1.0,0.0,0.0i rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0b rgb]1.0,0.0,0.0l rgb]1.0,0.0,0.0e} = \underset{-13}{rgb]1.0,0.0,0.0c rgb]1.0,0.0,0.0o rgb]1.0,0.0,0.0n rgb]1.0,0.0,0.0s rgb]1.0,0.0,0.0t rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0n rgb]1.0,0.0,0.0}$$

Since the left side is the "x", or variable side, the 8 is out of place. We must "undo" adding 8 by subtracting 8, and to keep the equality we must subtract 8 from both sides.

$$\underset{7x+8}{rgb]1.0,0.0,0.0v rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0r rgb]1.0,0.0,0.0i rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0b rgb]1.0,0.0,0.0l rgb]1.0,0.0,0.0e} = \underset{-13}{rgb]1.0,0.0,0.0c rgb]1.0,0.0,0.0o rgb]1.0,0.0,0.0n rgb]1.0,0.0,0.0s rgb]1.0,0.0,0.0t rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0n rgb]1.0,0.0,0.0}$$

Step 1: **Use the Subtraction Property of Equality.**

$$\begin{array}{lrcl} & 7x + 8\ rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.08 & = & -13\ rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.08 \\ \text{Simplify} & 7x & = & -21 \end{array}$$

Now all the variables are on the left and the constant on the right.
The equation looks like those you learned to solve earlier.

Step 2: **Use the Division Property of Equality.**

$$\begin{array}{lc} & \dfrac{7x}{rgb]1.0,0.0,0.07} = \dfrac{-21}{rgb]1.0,0.0,0.07} \\ \text{Simplify} & x = -3 \end{array}$$

Step 3: **Check:**

$$7x + 8 = -13$$

***Step 6:* Let $x = -3$.**

$$7\,(rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.03) + 8 \stackrel{?}{=} -13$$
$$-21 + 8 \stackrel{?}{=} -13$$
$$-13 = -13\checkmark$$

Try It

2) Solve: $3x + 4 = -8$.

Solution

$x = -4$

3) Solve: $5a + 3 = -37$.

Solution

$a = -8$

Example 3.3.2

Solve: $8y - 9 = 31$.

Solution

Notice, the variable is only on the left side of the equation, so we will call this side the "variable" side, and the right side will be the "constant" side. Since the left side is the "variable" side, the 9 is out of place. It is subtracted from the $8y$, so to "undo" subtraction, add 9 to both sides. Remember, whatever you do to the left, you must do to the right.

$rgb]1.0,0.0,0.0v rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0r rgb]1.0,0.0,0.0i rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0b rgb]1.0,0.0,0.0l rgb]1.0,0.0,0.0e \quad = \quad rgb]1.0,0.0,0.0c rgb]1.0,0.0,0.0o rgb]1.0,0.0,0.0n rgb]1.0,0.0,0.0s rgb]1.0,0.0,0.0t rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0n rgb]1.0,0.0,0.0t$

$8y - 9 \quad = \quad 31$

Step 1: **Add 9 to both sides.**

$$8y - 9rgb]1.0, 0.0, 0.0{+}rgb]1.0, 0.0, 0.09 = 31rgb]1.0, 0.0, 0.0{+}rgb]1.0, 0.0, 0.09$$

$$\text{Simplify} \qquad 8y = 40$$

The variables are now on one side and the constants on the other.

We continue from here as we did earlier.

Step 2: **Divide both sides by 8.**

$$\frac{8y}{rgb]1.0, 0.0, 0.08} = \frac{40}{rgb]1.0, 0.0, 0.08}$$

$$\text{Simplify} \qquad y = 5$$

Step 3: **Check:**

$$8y - 9 = 31$$

Step 4: **Let $y = 5$.**

$$8 \cdot rgb]1.0, 0.0, 0.05 - 9 \stackrel{?}{=} 31$$

$$40 - 9 \stackrel{?}{=} 31$$

$$31 = 31\checkmark$$

Try It

4) Solve: $5y - 9 = 16$.

Solution

$y = 5$

5) Solve: $3m - 8 = 19$.

Solution

$m = 9$

Solve Equations with Variables on Both Sides

What if there are variables on both sides of the equation? For equations like this, begin as we did above—choose a "variable" side and a "constant" side, and then use the subtraction and addition properties of equality to collect all variables on one side and all constants on the other side.

Example 3.3.3

Solve: $9x = 8x - 6$.

Solution

Here the variable is on both sides, but the constants only appear on the right side, so let's make the right side the "constant" side. Then the left side will be the "variable" side.

$$\underset{9x}{rgb]1.0,0.0,0.0v rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0r rgb]1.0,0.0,0.0i rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0b rgb]1.0,0.0,0.0l rgb]1.0,0.0,0.0e} = \underset{8x-6}{rgb]1.0,0.0,0.0c rgb]1.0,0.0,0.0o rgb]1.0,0.0,0.0n rgb]1.0,0.0,0.0s rgb]1.0,0.0,0.0t rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0n rgb]1.0,0.0,0.0t}$$

Step 1: **We don't want any x's on the right, so subtract the $8x$ from both sides.**

$$9x rgb]1.0,0.0,0.0- rgb]1.0,0.0,0.08 rgb]1.0,0.0,0.0x = 8x rgb]1.0,0.0,0.0- rgb]1.0,0.0,0.08 rgb]1.0,0.0,0.0x - 6$$

$$\text{Simplify} \qquad x = 6$$

We succeeded in getting the variables on one side and the constants on the other, and have obtained the solution.

Step 3: **Check:**

$$9x = 8x - 6$$

Step 4: **Let $x = -6$.**

$$9x\,(rgb]1.0,0.0,0.0- rgb]1.0,0.0,0.06) \overset{?}{=} 8x\,(rgb]1.0,0.0,0.0- rgb]1.0,0.0,0.06) - 6$$

$$-54 \overset{?}{=} -48 - 6$$

$$-54 = -54\checkmark$$

Try It

6) Solve: $6n = 5n - 10$.

Solution

$n = -10$

7) Solve: $-6c = -7c - 1$.

Solution

$c = -1$

Example 3.3.4

Solve: $5y - 9 = 8y$.

Solution

The only constant is on the left and the y's are on both sides. Let's leave the constant on the left and get the variables to the right.

$$\underset{rgb]1.0,0.0,0.0c rgb]1.0,0.0,0.0o rgb]1.0,0.0,0.0n rgb]1.0,0.0,0.0s rgb]1.0,0.0,0.0t rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0n rgb]1.0,0.0,0.0t}{5y - 9} = \underset{rgb]1.0,0.0,0.0v rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0r rgb]1.0,0.0,0.0i rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0b rgb]1.0,0.0,0.0l rgb]1.0,0.0,0.0e}{8y}$$

Step 1: **Subtract $5y$ from both sides.**

$$5y rgb]1.0,0.0,0.0- rgb]1.0,0.0,0.05 rgb]1.0,0.0,0.0y - 9 = 8y rgb]1.0,0.0,0.0- rgb]1.0,0.0,0.05 rgb]1.0,0.0,0.0y$$

Simplify $-9 = 3y$

Step 2: **We have the y's on the right and the constants on the left. Divide both sides by 3.**

$$\frac{-9}{rgb]1.0,0.0,0.03} = \frac{3y}{rgb]1.0,0.0,0.03}$$

Simplify $-3 = y$

***Step 3:* Check:**

$$5y - 9 = 8y$$

***Step 4:* Let $y = -3$.**

$$5\,(rgb]1.0, 0.0, 0.0{-}rgb]1.0, 0.0, 0.03) - 9 \stackrel{?}{=} 8\,(rgb]1.0, 0.0, 0.0{-}rgb]1.0, 0.0, 0.03)$$
$$-15 - 9 \stackrel{?}{=} -24$$
$$-24 = -24\checkmark$$

Try It

8) Solve: $3p - 14 = 5p$.

Solution

$p = -7$

9) Solve: $8m + 9 = 5m$.

Solution

$m = -3$

Example 3.3.5

Solve: $12x = -x + 26$.

Solution

The only constant is on the right, so let the left side be the “variable” side.

$$\underset{12x}{rgb]1.0,0.0,0.0v rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0r rgb]1.0,0.0,0.0i rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0b rgb]1.0,0.0,0.0l rgb]1.0,0.0,0.0e} = \underset{-x+26}{rgb]1.0,0.0,0.0c rgb]1.0,0.0,0.0o rgb]1.0,0.0,0.0n rgb]1.0,0.0,0.0s rgb]1.0,0.0,0.0t rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0n rgb]1.0,0.0,0.0t}$$

Step 1: **Remove the $-x$ from the right side by adding x to both sides.**

$$12x rgb]1.0,0.0,0.0{+} rgb]1.0,0.0,0.0x = -x rgb]1.0,0.0,0.0{+} rgb]1.0,0.0,0.0x + 26$$

$$\text{Simplify} \qquad 13x = 26$$

Step 2: **All the x's are on the left and the constants are on the right. Divide both sides by 13.**

$$\frac{13x}{rgb]1.0,0.0,0.013} = \frac{26}{rgb]1.0,0.0,0.013}$$

$$\text{Simplify} \qquad 1x = 2$$

Try It

10) Solve: $12j = -4j + 32$.

Solution

$j = 2$

11) Solve: $8h = -4h + 12$.

Solution

$h = 1$

Solve Equations with Variables and Constants on Both Sides

The next example will be the first to have variables and constants on both sides of the equation. It may take several steps to solve this equation, so we need a clear and organized strategy.

Example 3.3.6

Solve: $7x + 5 = 6x + 2$.

Solution

Step 1: **Choose which side will be the "variable" side – the other side will be the "constant" side.**

The variable terms are $7x$ and $6x$.

Since 7 is greater than 6, we will make the left side the "x" side.

The right side will be the "constant" side.

$$\underset{7x+5}{\text{variable}} = \underset{6x+2}{\text{constant}}$$

Step 2: **Collect the variable terms to the "variable" side of the equation, using the addition or subtraction property of equality.**

With the right side as the "constant" side, the $6x$ is out of place, so subtract $6x$ from both sides.

$$7x - 6x + 5 = 6x - 6x + 2$$

Combine like terms. $x + 5 = 2$

Now, the variable is only on the left side!

Step 3: **Collect all the constants to the other side of the equation, using the addition or subtraction property of equality.**

The right side is the "constant" side, so the 5 is out of place.

Subtract from both sides. $x + 5 - 5 = 2 - 5$

Simplify. $x = -3$

Step 4: **Make the coefficient of the variable equal 1, using the multiplication or division property of equality.**

The coefficient of x is one.

The equation is solved.

Step 5: **Check.**

Let x=-3.	$7x+6=6x+2$
Simplify.	$(rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.03)+5=6\,(rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.03)+2$
Add.	$-21+5=-18+2$
	$-16=-16\checkmark$

Try It

12) Solve: $12x+8=6x+2$.

Solution

$x=-1$

13) Solve: $9y+4=7y+12$.

Solution

$y=4$

We'll list the steps below so you can easily refer to them. But we'll call this the 'Beginning Strategy' because we'll be adding some steps later in this chapter.

HOW TO

Beginning Strategy for Solving Equations with Variables and Constants on Both Sides of the Equation.

1. Choose which side will be the "variable" side—the other side will be the "constant" side.
2. Collect the variable terms to the "variable" side of the equation, using the Addition or Subtraction Property of Equality.
3. Collect all the constants to the other side of the equation, using the Addition or

Subtraction Property of Equality.
4. Make the coefficient of the variable equal 1, using the Multiplication or Division Property of Equality.
5. Check the solution by substituting it into the original equation.

In Step 1, a helpful approach is to make the "variable" side the side that has the variable with the larger coefficient. This usually makes the arithmetic easier.

Example 3.3.7

Solve: $8n - 4 = -2n + 6$.

Solution

In the first step, choose the variable side by comparing the coefficients of the variables on each side.

Step 1: **Since $8 > -2$, make the left side the "variable" side.**

$$\underset{8n-4}{rgb]1.0,0.0,0.0v rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0r rgb]1.0,0.0,0.0i rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0b rgb]1.0,0.0,0.0l rgb]1.0,0.0,0.0e} = \underset{-2n+6}{rgb]1.0,0.0,0.0c rgb]1.0,0.0,0.0o rgb]1.0,0.0,0.0n rgb]1.0,0.0,0.0s rgb]1.0,0.0,0.0t rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0n rgb]1.0,0.0,0.0t}$$

Step 2: **We don't want variable terms on the right side—add $2n$ to both sides to leave only constants on the right.**

$$8n rgb]1.0,0.0,0.0 + rgb]1.0,0.0,0.02 rgb]1.0,0.0,0.0n - 4 = -2n rgb]1.0,0.0,0.0 + rgb]1.0,0.0,0.02 rgb]1.0,0.0,0.0n + 6$$

Combine like terms. $\quad 10n - 4 = 6$

Step 3: **We don't want any constants on the left side, so add 4 to both sides.**

$$10n - 4 rgb]1.0,0.0,0.0 + rgb]1.0,0.0,0.04 = 6 rgb]1.0,0.0,0.0 + rgb]1.0,0.0,0.04$$

Simplify $\quad 10n = 10$

Step 4: **The variable term is on the left and the constant term is on the right. To get the coefficient of n to be one, divide both sides by 10.**

$$\frac{10n}{rgb]1.0,0.0,0.010} = \frac{10}{rgb]1.0,0.0,0.010}$$

Simplify $\quad n = 1$

Step 5: **Check:**

$$8n - 4 = -2n + 6$$

Step 6: **Let** $n = 1$.

$$8 \cdot rgb]1.0, 0.0, 0.01 - 4 \stackrel{?}{=} 2 \cdot rgb]1.0, 0.0, 0.01 + 6$$
$$8 - 4 \stackrel{?}{=} -2 + 6$$
$$4 = 4\checkmark$$

Try It

14) Solve: $8q - 5 = -4q + 7$.

Solution

$q = 1$

15) Solve: $7n - 3 = n + 3$.

Solution

$n = 1$

Example 3.3.8

Solve: $7a - 3 = 13a + 7$.

Solution

In the first step, choose the variable side by comparing the coefficients of the variables on each side.

Since $13 > 7$, make the right side the "variable" side and the left side the "constant" side.

$$rgb]1.0,0.0,0.0crgb]1.0,0.0,0.0orgb]1.0,0.0,0.0nrgb]1.0,0.0,0.0srgb]1.0,0.0,0.0trgb]1.0,0.0,0.0argb]1.0,0.0,0.0nrgb]1.0,0.0,0.0t \quad 7a-3 = 13a+7 \quad rgb]1.0,0.0,0.0vrgb]1.0,0.0,0.0argb]1.0,0.0,0.0rrgb]1.0,0.0,0.0irgb]1.0,0.0,0.0argb]1.0,0.0,0.0brgb]1.0,0.0,0.0lrgb]1.0,0.0,0.0e$$

Step 1: **Subtract $7a$ from both sides to remove the variable term from the left.**

$$7argb]1.0,0.0,0.0-rgb]1.0,0.0,0.07rgb]1.0,0.0,0.0a-3 = 13argb]1.0,0.0,0.0-rgb]1.0,0.0,0.07rgb]1.0,0.0,0.0a+7$$

Combine like terms. $-3 = 6a+7$

Step 2: **Subtract 7 from both sides to remove the constant from the right.**

$$-3rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.07 = 6a+7rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.07$$

Simplify. $-10 = 6a$

Step 3: **Divide both sides by 6 to make 1 the coefficient of a.**

$$\frac{-10}{rgb]1.0,0.0,0.06} = \frac{6a}{rgb]1.0,0.0,0.06}$$

Simplify. $-\frac{5}{3} = a$

Step 4: **Check:**

$$7a-3 = 13a+7$$

Step 5: **Let $a = -\frac{5}{3}$.**

$$8\left(rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\frac{5}{3}\right) - 3 \stackrel{?}{=} 13\left(rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\frac{5}{3}\right)+7$$

$$-\frac{35}{3} - \frac{9}{3} \stackrel{?}{=} -\frac{65}{3} + \frac{21}{3}$$

$$-\frac{54}{3} = -\frac{54}{3} \checkmark$$

Try It

16) Solve: $2a-2 = 6a+18$.

Solution

$a = -5$

17) Solve: $4k - 1 = 7k + 17$.

Solution

$k = -6$

In the last example, we could have made the left side the "variable" side, but it would have led to a negative coefficient on the variable term. (Try it!) While we could work with the negative, there is less chance of errors when working with the positives. The strategy outlined above helps avoid the negatives!

To solve an equation with fractions, we just follow the steps of our strategy to get the solution!

Example 3.3.9

Solve: $\frac{5}{4}x + 6 = \frac{1}{4}x - 2$.

Solution

Since $\frac{5}{4} > \frac{1}{4}$, make the left side the "variable" side and the right side the "constant" side.

rgb]1.0,0.0,0.0vrgb]1.0,0.0,0.0argb]1.0,0.0,0.0rrgb]1.0,0.0,0.0irgb]1.0,0.0,0.0argb]1.0,0.0,0.0brgb]1.0,0.0,0.0lrgb]1.0,0.0,0.0e		rgb]1.0,0.0,0.0crgb]1.0,0.0,0.0orgb]1.0,0.0,0.0nrgb]1.0,0.0,0.0srgb]1.0,0.0,0.0trgb]1.0,0.0,0.0argb]1.0,0.0,0.0nrgb]1.0,0.0,0.0t
$\frac{5}{4}x + 6$	$=$	$\frac{1}{4}x - 2$

Step 1: **Subtract $\frac{1}{4}x$ from both sides.**

$$\frac{5}{4}x rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\frac{1}{4}rgb]1.0,0.0,0.0x + 6 = \frac{1}{4}x rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\frac{1}{4}rgb]1.0,0.0,0.0x - 2$$

Combine like terms. $x + 6 = -2$

Step 2: **Subtract 6 from both sides.**

$$x + 6rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.06 = -2rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.06$$

Step 3: **Simplify.**

$$x = -8$$

Step 4: **Check: Let $x = -8$**

$$\frac{5}{4}x+6=\frac{1}{4}x-2$$
$$\frac{5}{4}(-8)+6\stackrel{?}{=}\frac{1}{4}(-8)-2$$
$$-10+6\stackrel{?}{=}-2-2$$
$$-4=-4\checkmark$$

Try It

18) Solve: $\frac{7}{8}x-12=-\frac{1}{8}x-2$.

Solution

$x=10$

19) Solve: $\frac{7}{6}y+11=\frac{1}{6}y+8$.

Solution

$y=-3$

We will use the same strategy to find the solution for an equation with decimals.

Example 3.3.10

Solve: $7.8x+4=5.4x-8$.

Solution

Since $7.8 > 5.4$, make the left side the "variable" side and the right side the "constant" side.

$$\underbrace{7.8x+4}_{\text{variable side}} = \underbrace{5.4x-8}_{\text{constant side}}$$

Step 1: **Subtract $5.4x$ from both sides.**

$$7.8x \; rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.05rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.04rgb]1.0,0.0,0.0x + 4 = 5.4x \; rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.05rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.04rgb]1.0,0.0,0.0x - 8$$

Combine like terms. $2.4x + 4 = -8$

Step 2: **Subtract 4 from both sides.**

$$2.4x + 4 \; rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.04 = -8 \; rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.04$$

Simplify. $2.4x = -12$

Step 3: **Use the Division Property of Equality.**

$$\frac{2.4x}{rgb]1.0,0.0,0.02rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.04} = \frac{-12}{rgb]1.0,0.0,0.02rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.04}$$

Simplify. $x = -5$

Step 4: **Check:**

$$7.8x + 4 = 5.4x - 5$$

Step 5: **Let $x = -5$.**

$$7.8 \, rgb]1.0,0.0,0.0\,(-5) + 4 = 5.4 \, rgb]1.0,0.0,0.0\,(-5) - 8$$

$$-39 + 4 \stackrel{?}{=} -27 - 8$$

$$-35 = -35 \checkmark$$

Try It

20) Solve: $2.8x + 12 = -1.4x - 9$.

Solution

$x = -5$

21) Solve: $3.6y + 8 = 1.2y - 4$.

Solution

$y = -5$

Key Concepts

Beginning Strategy for Solving an Equation with Variables and Constants on Both Sides of the Equation

1. Choose which side will be the "variable" side—the other side will be the "constant" side.
2. Collect the variable terms to the "variable" side of the equation, using the Addition or Subtraction Property of Equality.
3. Collect all the constants to the other side of the equation, using the Addition or Subtraction Property of Equality.
4. Make the coefficient of the variable equal 1, using the Multiplication or Division Property of Equality.
5. Check the solution by substituting it into the original equation.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it

online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=716#h5p-17

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

3.4 USE A GENERAL STRATEGY TO SOLVE LINEAR EQUATIONS

Learning Objectives

By the end of this section, you will be able to:

- Solve equations using a general strategy
- Classify equations

Try It

Before you get started, take this readiness quiz:

1) Simplify: $-(a-4)$
2) Multiply: $\dfrac{3}{2}(12x+20)$.
3) Simplify: $5-2(n+1)$.
4) Multiply: $3(7y+9)$.
5) Multiply: $(2.5)(6.4)$.

Solve Equations Using the General Strategy

Until now we have dealt with solving one specific form of a linear equation. It is time now to lay out one overall strategy that can be used to solve any linear equation. Some equations we solve will not require all these steps to solve, but many will.

Beginning by simplifying each side of the equation makes the remaining steps easier.

Example 3.4.1

Solve: $-6(x+3)=24$.

Solution

***Step 1:* Simplify each side of the equation as much as possible.**

Use the Distributive Property.

$$-6\left(x+3\right)=24$$
$$-6x-18=24$$

Notice that each side of the equation is simplified as much as possible.

***Step 2:* Collect all variable terms on one side of the equation.**

Nothing to to – all x's are on the left side.

***Step 3:* Collect constant terms on the other side of the equation.**

To get constants only on the right, add 18 to each side.

$$-6x-18rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.018=24rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.018$$

Simplify. $-6x=42$

***Step 4:* Make the coefficient of the variable term to equal 1.**

Divide each side by -6. $\dfrac{-6x}{rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.06}=\dfrac{42}{rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.06}$

Simplify. $x=-7$

***Step 5:* Check the solution.**

Let $x=-7$

$$-6\,(x+3) = 24$$
$$-6\,(rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.07+3) \stackrel{?}{=} 24$$
$$\text{Simplify.} -6\,(-4) \stackrel{?}{=} 24$$
$$\text{Multiply.}\ 24 = 24$$

Try It

6) Solve: $5(x+3)=35$.

Solution

$x=4$

7) Solve: $6(y-4)=-18$.

Solution

$y=1$

HOW TO

General strategy for solving linear equations.

1. Simplify each side of the equation as much as possible.
   Use the Distributive Property to remove any parentheses.
   Combine like terms.
2. Collect all the variable terms on one side of the equation.
   Use the Addition or Subtraction Property of Equality.
3. Collect all the constant terms on the other side of the equation.

Use the Addition or Subtraction Property of Equality.
4. Make the coefficient of the variable term to equal to 1.
Use the Multiplication or Division Property of Equality.
State the solution to the equation.
5. Check the solution.
Substitute the solution into the original equation to make sure the result is a true statement.

Example 3.4.2

Solve: $-(y+9)=8$.

Solution
Step 1: **Simplify each side of the equation as much as possible by distributing.**

$$-y-9=8$$

Step 2: **The only y term is on the left side, so all variable terms are on the left side of the equation.**

Step 3: **Add 9 to both sides to get all constant terms on the right side of the equation.**

$$-y-9rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.09=8rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.09$$
$$\text{Simplify.} \qquad -y=17$$

Step 4: **Rewrite $-y$ as $-1y$.**

$$-1y=17$$

Step 5: **Make the coefficient of the variable term to equal to 1 by dividing both sides by -1.**

$$\frac{-1y}{-1}=\frac{17}{-1}$$
$$\text{Simplify.} \qquad y=-17$$

Step 6: **Check:**

$$-(y+9)=8$$

***Step 7:* Let $y=-17$.**

$$-\left(rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.017+9\right)\stackrel{?}{=}8$$
$$-(-8)\stackrel{?}{=}8$$
$$8=8\checkmark$$

Try It

8) Solve: $-(y+8)=-2$.

Solution
$y=-6$

9) Solve: $-(z+4)=-12$

Solution
$z=8$

Example 3.4.3

Solve: $5(a-3)+5=-10$.

Solution
***Step 1:* Simplify each side of the equation as much as possible.**

$$\begin{array}{ll} \text{Distribute.} & 5a - 15 + 5 = -10 \\ \text{Combine like terms.} & 5a - 10 = -10 \end{array}$$

Step 2: **The only a term is on the left side, so all variable terms are on one side of the equation.**

Step 3: **Add 10 to both sides to get all constant terms on the other side of the equation.**

$$5a - 10rgb]1.0,0.0,0.0{+}rgb]1.0,0.0,0.010 = -10rgb]1.0,0.0,0.0{+}rgb]1.0,0.0,0.010$$

$$\text{Simplify.} \qquad 5a = 0$$

Step 4: **Make the coefficient of the variable term to equal to *1* by dividing both sides by 5.**

$$\frac{5a}{rgb]1.0,0.0,0.05} = \frac{0}{rgb]1.0,0.0,0.05}$$

$$\text{Simplify.} \qquad a = 0$$

Step 5: **Check:**

$$5(a-3)+5=-10$$

Step 6: **Let** $a=0$.

$$\begin{array}{r} 5\left(rgb]1.0,0.0,0.00-3\right)+5 \stackrel{?}{=} -10 \\ 5\left(-3\right)+5 \stackrel{?}{=} -10 \\ -15+5 \stackrel{?}{=} -10 \\ -10 = -10\checkmark \end{array}$$

Try It

10) Solve: $2(m-4)+3=-1$.

Solution

$m=2$

11) Solve: $7(n-3)-8=-15$.

Solution

$n = 2$

Example 3.4.4

Solve: $\frac{2}{3}(6m - 3) = 8 - m$.

Solution

Step 1: **Distribute.**

$$4m - 2 = 8 - m$$

Step 2: **Add m to get the variables only to the left.**

$$4m rgb]1.0, 0.0, 0.0{+} rgb]1.0, 0.0, 0.0m - 2 = 8 - m rgb]1.0, 0.0, 0.0{+} rgb]1.0, 0.0, 0.0m$$

$$\text{Simplify.} \qquad 5m - 2 = 8$$

Step 3: **Add 2 to get constants only on the right.**

$$5m - 2 rgb]1.0, 0.0, 0.0{+} rgb]1.0, 0.0, 0.02 = 8 rgb]1.0, 0.0, 0.0{+} rgb]1.0, 0.0, 0.02$$

$$\text{Simplify.} \qquad 5m = 10$$

Step 4: **Divide by 5.**

$$\frac{5m}{rgb]1.0, 0.0, 0.05} = \frac{10}{rgb]1.0, 0.0, 0.05}$$

$$\text{Simplify.} \qquad m = 2$$

Step 5: **Check:**

$$\frac{2}{3}(6m - 3) = 8 - m$$

Step 6: **Let $m = 2$.**

$$\frac{2}{3}(6 \cdot rgb]1.0, 0.0, 0.02 \cdot -3) \stackrel{?}{=} 8 - rgb]1.0, 0.0, 0.02$$
$$\frac{2}{3}(12 - 3) \stackrel{?}{=} 6$$
$$\frac{2}{3}(9) \stackrel{?}{=} 6$$
$$6 = 6\checkmark$$

Try It

12) Solve: $\frac{1}{3}(6u + 3) = 7 - u$.

Solution

$u = 2$

13) Solve: $\frac{2}{3}(9x - 12) = 8 + 2x$.

Solution

$x = 4$

Example 3.4.5

Solve: $8 - 2(3y + 5) = 0$.

Solution

Step 1: **Simplify—use the Distributive Property.**

$$8-6y-10=0$$

$$\text{Combine like terms.} \qquad -6y-2=0$$

Step 2: **Add 2 to both sides to collect constants on the right.**

$$-6y-2rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.02=0rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.02$$

$$\text{Simplify.} \qquad -6y=2$$

Step 3: **Divide both sides by -6.**

$$\frac{-6y}{rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.06}=\frac{2}{rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.06}$$

$$\text{Simplify.} \qquad y=-\frac{1}{3}$$

Step 4: **Check:**

Let $y=-\frac{1}{3}$

$$8-2\left(3y+5\right)=0$$

$$8-2\left[3\left(rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\frac{1}{3}\right)+5\right]=0$$

$$8-2\left(-1+5\right)\overset{?}{=}0$$

$$8-2\left(4\right)\overset{?}{=}0$$

$$8-8\overset{?}{=}0$$

$$0=0\checkmark$$

Try It

14) Solve: $12-3(4j+3)=-17$

Solution

$j = \frac{5}{3}$

15) Solve: $-6 - 8(k - 2) = -10$.

Solution

$k = \frac{5}{2}$

Example 3.4.6

Solve: $4(x - 1) - 2 = 5(2x + 3) + 6$.

Solution

Step 1: **Distribute.**

$$\begin{array}{rr} & 4x - 4 - 2 = 10x + 15 + 6 \\ \text{Combine like terms.} & 4x - 6 = 10x + 21 \end{array}$$

Step 2: **Subtract $4x$ to get the variables only on the right side since $10 > 4$.**

$$\begin{array}{lr} & 4x rgb]1.0,0.0,0.0 - rgb]1.0,0.0,0.0 4 rgb]1.0,0.0,0.0 x - 6 = 10 rgb]1.0,0.0,0.0 - rgb]1.0,0.0,0.0 4 rgb]1.0,0.0,0.0 x + 21 \\ \text{Simplify.} & -6 = 6x + 21 \end{array}$$

Step 3: **Subtract 21 to get the constants on left.**

$$\begin{array}{lr} & 6 rgb]1.0,0.0,0.0 - rgb]1.0,0.0,0.0 21 = 6x + 21 rgb]1.0,0.0,0.0 - rgb]1.0,0.0,0.0 21 \\ \text{Simplify.} & -27 = 6x \end{array}$$

Step 4: **Divide by 6.**

$$\begin{array}{lr} & \frac{-27}{rgb]1.0,0.0,0.06} = \frac{6x}{rgb]1.0,0.0,0.06} \\ \text{Simplify.} & -\frac{9}{2} = x \end{array}$$

Step 5: **Check:**

$$4(x - 1) - 2 = 5(2x + 3) + 6$$

Step 6: **Let** $x = -\frac{9}{2}$.

$$4\left(rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\frac{9}{2}\right) - 2 \stackrel{?}{=} 5\left[2\left(rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\frac{9}{2}\right) + 3\right] + 6$$

$$4\left(-\frac{11}{2}\right) - 2 \stackrel{?}{=} 5\left(-9+3\right) + 6$$

$$-22 - 2 \stackrel{?}{=} 5\left(-6\right) + 6$$

$$-24 \stackrel{?}{=} -30 + 6$$

$$-24 = -24\checkmark$$

Try It

16) Solve: $6(p-3) - 7 = 5(4p+3) - 12$.

Solution

$p = -2$

17) Solve: $8(q+1) - 5 = 3(2q-4) - 1$.

Solution

$q = -8$

Example 3.4.7

Solve: $10[3 - 8(2s-5)] = 15(40-5s)$.

Solution

Step 1: **Simplify from the innermost parentheses first.**

$$10[3 - 16s + 40] = 15(40 - 5s)$$

Step 2: **Combine like terms in the brackets.**

$$10[43 - 16s] = 15(40 - 5s)$$

Step 3: **Distribute.**

$$430 - 160s = 600 - 75s$$

Step 4: **Add $160s$ to get the s's to the right.**

$$430 - 160rgb]1.0, 0.0, 0.0{+}rgb]1.0, 0.0, 0.0160rgb]1.0, 0.0, 0.0s = 600 - 85srgb]1.0, 0.0, 0.0{+}rgb]1.0, 0.0, 0.0160rgb]1.0, 0.0, 0.0s$$

Simplify. $430 = 600 + 85s$

Step 5: **Subtract 600 to get the constants to the left.**

$$430 - rgb]1.0, 0.0, 0.0600 = 85s + 600rgb]1.0, 0.0, 0.0{-}rgb]1.0, 0.0, 0.0600$$

Simplify. $-170 = 85s$

Step 6: **Divide.**

$$\frac{-170}{rgb]1.0, 0.0, 0.085} = \frac{85s}{rgb]1.0, 0.0, 0.085}$$

Simplify. $-2 = s$

Step 7: **Check:**

$$10[3 - 8(2s - 5)] = 15(40 - 5s)$$

Step 8: **Substitute $s = -2$.**

$$10\left[3 - 8\left(2rgb]0.1, 0.1, 0.1\left(rgb]1.0, 0.0, 0.0{-}rgb]1.0, 0.0, 0.02\right) - 5\right)\right] \stackrel{?}{=} 15\left(40 - 5\left(rgb]1.0, 0.0, 0.0{-}rgb]1.0, 0.0, 0.02\right)\right)$$
$$10\left[3 - 8\left(-4 - 5\right)\right] \stackrel{?}{=} 15\left(40 + 10\right)$$
$$10\left[3 - 8\left(-9\right)\right] \stackrel{?}{=} 15\left(50\right)$$
$$10\left[3 + 72\right] \stackrel{?}{=} 750$$
$$10\left[75\right] \stackrel{?}{=} 750$$
$$750 = 750 \checkmark$$

Try It

18) Solve: $6[4-2(7y-1)]=8(13-8y)$.

Solution

$y=-\frac{17}{5}$

19) Solve: $12[1-5(4z-1)]=3(24+11z)$.

Solution

$z=0$

Example 3.4.8

Solve: $0.36(100n+5)=0.6(30n+15)$.

Solution

Step 1: **Distribute.**

$$36n+1.8=18n+9$$

Step 2: **Subtract $18n$ to get the variables to the left.**

$$36nrgb]1.0,0.0,0.0-rgb]1.0,0.0,0.018rgb]1.0,0.0,0.0n+1.8=18nrgb]1.0,0.0,0.0-rgb]1.0,0.0,0.018rgb]1.0,0.0,0.0n+9$$

Simplify. $18n+1.8=9$

Step 3: **Subtract 1.8 to get the constants to the right.**

$$18n+1.8rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.01rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.08=9rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.01rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.08$$

Simplify. $18n=7.2$

Step 4: **Divide.**

$$\frac{18n}{rgb]1.0, 0.0, 0.018} = \frac{7.2}{rgb]1.0, 0.0, 0.018}$$

Simplify. $n = 0.4$

Step 5: **Check:**

$$0.35(100n + 5) = 0.6(30n + 15)$$

Step 6: **Let** $n = 0.4$.

$$0.36\,(100\,(rgb]1.0, 0.0, 0.00rgb]1.0, 0.0, 0.0.rgb]1.0, 0.0, 0.04) + 5) \stackrel{?}{=} 0.6\,(30\,(rgb]1.0, 0.0, 0.00rgb]1.0, 0.0, 0.0.rgb]1.0, 0.0, 0.04) + 15)$$
$$0.36\,(40 + 5) \stackrel{?}{=} 0.6\,(12 + 15)$$
$$0.36\,(45) \stackrel{?}{=} 0.6\,(27)$$
$$16.2 = 16.2\checkmark$$

Try It

20) Solve: $0.55(100n + 8) = 0.6(85n + 14)$.

Solution

$n = 1$

21) Solve: $0.15(40m - 120) = 0.5(60m + 12)$.

Solution

$m = -1$

Classify Equations

Consider the equation we solved at the start of the last section, $7x + 8 = -13$. The solution we found was $x = -3$. This means the equation $7x + 8 = -13$ is true when we replace the variable, x, with the value -3. We showed this when we checked the solution $x = -3$ and evaluated $7x + 8 = -13$ for $x = -3$.

$$7\,(rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.03) + 8 \stackrel{?}{=} -13$$
$$-21 + 8 \stackrel{?}{=} -13$$
$$-13 = -13✓$$

If we evaluate $7x + 8$ for a different value of x, the left side will not be -13.

The equation $7x + 8 = -13$ is true when we replace the variable, x, with the value -3, but not true when we replace x with any other value. Whether or not the equation $7x + 8 = -13$ is true depends on the value of the variable. Equations like this are called conditional equations.

All the equations we have solved so far are **conditional equations**.

Conditional equation

An equation that is true for one or more values of the variable and false for all other values of the variable is a conditional equation.

Now let's consider the equation $2y + 6 = 2(y + 3)$. Do you recognize that the left side and the right side are equivalent? Let's see what happens when we solve for y.

	$2y + 6 = 2(y + 3)$
Distribute	$-2y + 6 = 2y + 6$
Subtract $2y$ to get the $y's$ to one side.	$2y rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.02 rgb]1.0,0.0,0.0y + 6 = 2 rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.02 rgb]1.0,0.0,0.0y + 6$
Simplify-the $y's$ are gone!	$6 = 6$

But $6 = 6$ is true.

This means that the equation $2y + 6 = 2(y + 3)$ is true for any value of y. We say the solution to the equation is all of the real numbers. An equation that is true for any value of a variable like this is called an **identity**.

Identity

An equation that is true for any value of the variable is called an identity.

The solution of an identity is all real numbers.

What happens when we solve the equation $5z = 5z - 1$?

$$\begin{array}{ll} & 5z = 5z - 1 \\ \text{Subtract } 5z \text{ to get the constant alone on the right.} & 5z rgb]1.0,0.0,0.0 - rgb]1.0,0.0,0.05 rgb]1.0,0.0,0.0 z = 5z rgb]1.0,0.0,0.0 - rgb]1.0,0.0,0.05 rgb]1.0,0.0,0.0 z - 1 \\ \text{Simplify-the } z's \text{ are gone!} & 0 \neq -1 \end{array}$$

But $0 \neq -1$

Solving the equation $5z = 5z - 1$ led to the false statement $0 = -1$. The equation $5z = 5z - 1$ will not be true for any value of z. It has no solution. An equation that has no solution, or that is false for all values of the variable, is called a **contradiction**.

Contradiction

An equation that is false for all values of the variable is called a contradiction.

A contradiction has no solution.

Example 3.4.9

Classify the equation as a conditional equation, an identity, or a contradiction. Then state the solution.

$6(2n - 1) + 3 = 2n - 8 + 5(2n + 1)$

Solution

Step 1: **Distribute.**

$$\begin{array}{ll} & 12n - 6 + 3 = 2n - 8 + 10n + 5 \\ \text{Combine like terms.} & 12n - 3 = 12n - 3 \end{array}$$

Step 2: **Subtract $12n$ to get the n's to one side.**

$$\begin{array}{ll} & 12n rgb]1.0,0.0,0.0 - rgb]1.0,0.0,0.012 rgb]1.0,0.0,0.0 n - 3 = 12n rgb]1.0,0.0,0.0 - rgb]1.0,0.0,0.012 rgb]1.0,0.0,0.0 n - 3 \\ \text{Simplify.} & -3 = -3 \end{array}$$

This is a true statement.

The equation is an identity.
The solution is all real numbers.

Try It

22) Classify the equation as a conditional equation, an identity, or a contradiction, and then state the solution:

$$4+9(3x-7)=-42x-13+23(3x-2)$$

Solution
identity; all real numbers

23) Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$$8(1-3x)+15(2x+7)=2(x+50)+4(x+3)+1$$

Solution
identity; all real numbers

Example 3.4.10

Classify as a conditional equation, an identity, or a contradiction. Then state the solution.

$10+4(p-5)=0$

Solution
Step 1: **Distribute.**

$$\begin{array}{rr} & 10+4p-20=0 \\ \text{Combine like terms.} & 4p-10=0 \end{array}$$

Step 2: **Add 10 to both sides.**

$$4p - 10 rgb]1.0,0.0,0.0{+} rgb]1.0,0.0,0.0{10} = 0 rgb]1.0,0.0,0.0{+} rgb]1.0,0.0,0.0{10}$$

$$\text{Simplify.} \qquad 4p = 10$$

Step 3: **Divide.**

$$\frac{4p}{rgb]1.0,0.0,0.0{4}} = \frac{10}{rgb]1.0,0.0,0.0{4}}$$

$$\text{Simplify.} \qquad p = \frac{5}{2}$$

The equation is true when $p = \frac{5}{2}$.

This is a conditional equation.

The solution is $p = \frac{5}{2}$.

Try It

24) Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $11(q+3) - 5 = 19$

Solution

conditional equation; $q = \frac{9}{11}$

25) Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $6 + 14(k-8) = 95$

Solution

conditional equation; $k = \frac{193}{14}$

Example 3.4.11

Classify the equation as a conditional equation, an identity, or a contradiction. Then state the solution.

$5m + 3(9 + 3m) = 2(7m - 11)$

Solution

Step 1: **Distribute.**

$$5m + 27 + 9m = 14m - 22$$

$$\text{Combine like terms.} \qquad 14m + 27 = 14m - 22$$

Step 2: **Subtract $14m$ from both sides.**

$14m + 27rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.014rgb]1.0, 0.0, 0.0m = 14m - 22rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.014rgb]1.0, 0.0, 0.0m$

$\text{Simplify.} \qquad 27 \neq -22$

But $27 \neq -22.$

The equation is a contradiction.
It has no solution.

Try It

26) Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$$5m + 3(9 + 3m) = 2(7m - 11)$$

Solution

contradiction; no solution

27) Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution:

$$4(7d+18)=13(3d-2)-11d$$

Solution

contradiction; no solution

Type of equation	What happens when you solve it?	Solution
Conditional Equation	True for one or more values of the variables and false for all other values	One or more values
Identity	**True** for any value of the variable	All real numbers
Contradiction	**False** for all values of the variable	No solution

Key Concepts

- **General Strategy for Solving Linear Equations**
  1. Simplify each side of the equation as much as possible.
     Use the Distributive Property to remove any parentheses.
     Combine like terms.
  2. Collect all the variable terms on one side of the equation.
     Use the Addition or Subtraction Property of Equality.
  3. Collect all the constant terms on the other side of the equation.
     Use the Addition or Subtraction Property of Equality.
  4. Make the coefficient of the variable term to equal to 1.

Use the Multiplication or Division Property of Equality.
State the solution to the equation.

5. Check the solution.
   Substitute the solution into the original equation.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objective of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=838#h5p-18

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

conditional equation
An equation that is true for one or more values of the variable and false for all other values of the variable is a conditional equation.

contradiction
An equation that is false for all values of the variable is called a contradiction. A contradiction

has no solution.

identity

An equation that is true for any value of the variable is called an identity. The solution of an identity is all real numbers.

3.5 SOLVE EQUATIONS WITH FRACTIONS OR DECIMALS

Learning Objectives

By the end of this section, you will be able to:

- Solve equations with fraction coefficients
- Solve equations with decimal coefficients

Before you get started, take this readiness quiz:

Try It

1) Multiply: $8 \times \dfrac{3}{8}$
2) Find the LCD of $\dfrac{5}{6}$ and $\dfrac{1}{4}$
3) Multiply 4.78 by 100

Solve Equations with Fraction Coefficients

Let's use the general strategy for solving linear equations introduced earlier to solve the equation, $\dfrac{1}{8}x + \dfrac{1}{2} = \dfrac{1}{4}$.

	$\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$
To isolate the x term, subtract $\frac{1}{2}$ from both sides.	$\frac{1}{8}x + \frac{1}{2} rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.0\frac{1}{2} = \frac{1}{4} rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.0\frac{1}{2}$
Simplify the left side.	$\frac{1}{8}x = \frac{1}{4} - \frac{1}{2}$
Change the constants to equivalent fractions with the LCD.	$\frac{1}{8}x = \frac{1}{4} - \frac{2}{4}$
Subtract.	$\frac{1}{8}x = -\frac{1}{4}$
Multiply both sides by the reciprocal of $\frac{1}{8}$	$rgb]1.0, 0.0, 0.0\frac{8}{1} \cdot \frac{1}{8}x = rgb]1.0, 0.0, 0.0\frac{8}{1}\left(-\frac{1}{4}\right)$
Simplify.	$x = -2$

This method worked fine, but many students do not feel very confident when they see all those fractions. So, we are going to show an alternate method to solve equations with fractions. This alternate method eliminates the fractions.

We will apply the *Multiplication Property of Equality* and multiply both sides of an equation by the least common denominator of all the fractions in the equation. The result of this operation will be a new equation, equivalent to the first, but without fractions. This process is called "clearing" the equation of fractions.

Let's solve a similar equation, but this time use the method that eliminates the fractions.

Example 3.5.1

Solve: $\frac{1}{6}y - \frac{1}{3} = \frac{5}{6}$

Solution

Step 1: **Find the least common denominator of *all* the fractions in the equation.**

What is the LCD of $\frac{1}{6}$, $\frac{1}{3}$, and $\frac{5}{6}$?

$$\frac{1}{6}y - \frac{1}{3} = \frac{5}{6} \quad LCD = 6$$

Step 2: **Multiply both sides of the equation by that LCD. This clears the fractions.**

Multiply both sides of the equation by the LCD 6.	$rgb]1.0,0.0,0.06rgb]1.0,0.0,0.0\left(rgb]0.1,0.1,0.1\frac{1}{6}rgb]0.1,0.1,0.1yrgb]0.1,0.1,0.1-rgb]0.1,0.1,0.1\frac{1}{3}\right)rgb]0.1,0.1,0.1=rgb]1.0,0.0,0.06rgb]1.0,0.0,0.0\left(rgb]0.1,0.1,0.1\frac{5}{6}\right)$
Use the Distributive Property.	$rgb]0.1,0.1,0.16rgb]0.1,0.1,0.1\cdot rgb]0.1,0.1,0.1\frac{1}{6}rgb]0.1,0.1,0.1yrgb]0.1,0.1,0.1-rgb]0.1,0.1,0.16rgb]0.1,0.1,0.1\cdot rgb]0.1,0.1,0.1\frac{1}{3}rgb]0.1,0.1,0.1=rgb]0.1,0.1,0.16rgb]0.1,0.1,0.1\cdot rgb]0.1,0.1,0.1\frac{5}{6}$
Simplify - and notice, no more fractions!	$rgb]0.1,0.1,0.1yrgb]0.1,0.1,0.1-rgb]0.1,0.1,0.12rgb]0.1,0.1,0.1=rgb]0.1,0.1,0.15$

Step 3: **Sole using the General Strategy for Solving Linear Equations.**

To isolate the "y" term, add 2.	$y-2rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.02=5rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.02$
Simplify.	$y=7$

Try It

4) Solve: $\frac{1}{4}x + \frac{1}{2} = \frac{5}{8}$

Solution

$x = \frac{1}{2}$

5) Solve: $\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$

Solution

$x = -2$

Notice in Example 3.5.1, once we cleared the equation of fractions, the equation was like those we solved earlier in this chapter. We changed the problem to one we already knew how to solve! We then used the *General Strategy for Solving Linear Equations*.

HOW TO

Strategy to solve equations with fraction coefficients.

1. Find the least common denominator of *all* the fractions in the equation.
2. Multiply both sides of the equation by that LCD. This clears the fractions.
3. Solve using the General Strategy for Solving Linear Equations.

Example 3.5.2

Solve: $6 = \frac{1}{2}v + \frac{2}{5}v - \frac{3}{4}v$

Solution

We want to clear the fractions by multiplying both sides of the equation by the LCD of all the fractions in the equation.

***Step 1:* Find the LCD of all fractions in the equation.**

$$6 = \frac{1}{2}v + \frac{2}{5}v - \frac{3}{4}v$$

Step 2: **The LCD is 20.**

Step 3: **Multiply both sides of the equation by 20.**

$$rgb]1.0, 0.0, 0.020\,(6) = rgb]1.0, 0.0, 0.020 \cdot \left(\frac{1}{2}v + \frac{2}{5}v - \frac{3}{4}v\right)$$

Step 4: **Distribute.**

$$20(6) = 20 \cdot \frac{1}{2}v + 20 \cdot \frac{2}{5}v - 20 \cdot \frac{3}{4}v$$

Simplify - notice, no more fractions!. $\quad 120 = 10v + 8v - 15v$

Step 5: **Combine like terms.**

$$120 = 3v$$

Step 6: **Divide by 3.**

$$\frac{120}{rgb]1.0, 0.0, 0.03} = \frac{3v}{rgb]1.0, 0.0, 0.03}$$

Simplify. $\quad 40 = v$

Step 7: **Check:**

$$6 = \frac{1}{2}v + \frac{2}{5}v - \frac{3}{4}v$$

Step 8: **Let $v = 40$.**

$$6 \stackrel{?}{=} \frac{1}{2}(rgb]1.0, 0.0, 0.040) + \frac{2}{5}(rgb]1.0, 0.0, 0.040) - \frac{3}{4}(rgb]1.0, 0.0, 0.040)$$

$$6 \stackrel{?}{=} 20 + 16 - 30$$

$$6 = 6\checkmark$$

Try It

6) Solve: $7 = \frac{1}{2}x + \frac{3}{4}x - \frac{2}{3}x$

Solution

$x = 12$

7) Solve: $-1 = \frac{1}{2}u + \frac{1}{4}u - \frac{2}{3}u$

Solution

$u = -12$

In the next example, we again have variables on both sides of the equation.

Example 3.5.3

Solve: $a + \frac{3}{4} = \frac{3}{8}a - \frac{1}{2}$

Solution

Step 1: **Find the LCD of all fractions in the equation.**

The LCD is 8.

Step 2: **Multiply both sides by the LCD.**

$$rgb]1.0, 0.0, 0.08\left(a + \frac{3}{4}\right) = rgb]1.0, 0.0, 0.08\left(\frac{3}{8}a - \frac{1}{2}\right)$$

Step 3: **Distribute.**

\begin{align*}&\;&8\cdot a+8\cdot\frac34&=8\cdot\frac38a-8\cdot\frac12\\
&\text{Simplify - no more fractions.}\;&8a+6&=3a-4\end{align*}

Step 4: **Subtract $3a$ from both sides.**

$$8a rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.03rgb]1.0,0.0,0.0a+6=3a rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.03rgb]1.0,0.0,0.0a-4$$

$\text{Simplify.}\quad 5a+6=-4$

Step 5: **Subtract 6 from both sides.**

$$5a+6rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.06=-4rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.06$$

$\text{Simplify.}\quad 5a=-10$

Step 6: **Divide by 5.**

$$\frac{5a}{rgb]1.0,0.0,0.05}=\frac{-10}{rgb]1.0,0.0,0.05}$$

$\text{Simplify.}\quad a=-2$

Step 7: **Check:**

$$a+\frac{3}{4}=\frac{3}{8}a-\frac{1}{2}$$

Step 8: **Let $a=-2$.**

$$rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.02+\frac{3}{4}\overset{?}{=}\frac{3}{8}(rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.02)-\frac{1}{2}$$

$$-\frac{8}{4}+\frac{3}{4}\overset{?}{=}-\frac{16}{8}-\frac{4}{8}$$

$$-\frac{5}{4}=-\frac{10}{8}$$

$$-\frac{5}{4}=-\frac{5}{4}\checkmark$$

Try It

8) Solve: $x+\frac{1}{3}=\frac{1}{6}x-\frac{1}{2}$

Solution

$x=-1$

9) Solve: $c+\frac{3}{4}=\frac{1}{2}c-\frac{1}{4}$

Solution

$c=-2$

In the next example, we start by using the *Distributive Property*. This step clears the fractions right away.

Example 3.5.4

Solve: $-5=\frac{1}{4}(8x+4)$

Solution

Step 1: **Distribute.**

$$-5=\frac{1}{4}\cdot 8x+\frac{1}{4}\cdot 4$$

$$\text{Simplify.}\qquad -5=2x+1$$

Now there are no fractions.

Step 2: **Subtract 1 from both sides.**

$$-5rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.01=2+1rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.01$$

$$\text{Simplify.}\qquad -6=2x$$

Step 3: **Divide by 2.**

$$\frac{-6}{rgb]1.0,0.0,0.02}=\frac{2x}{rgb]1.0,0.0,0.02}$$

$$\text{Simplify.}\qquad -3=x$$

Step 4: **Check:**

$$-5 = \frac{1}{4}(8x + 4)$$

Step 5: **Let** $x = -3$.

$$-5 \stackrel{?}{=} \frac{1}{2}(4\,(rgb]1.0, 0.0, 0.0{-}rgb]1.0, 0.0, 0.03) + 2)$$
$$-5 \stackrel{?}{=} \frac{1}{2}(-12 + 2)$$
$$-5 \stackrel{?}{=} \frac{1}{2}(-10)$$
$$-5 = -5\checkmark$$

Try It

10) Solve: $-11 = \frac{1}{2}(6p + 2)$

Solution

$p = -4$

11) Solve: $8 = \frac{1}{3}(9q + 6)$

Solution

$q = 2$

In the next example, even after distributing, we still have fractions to clear.

Example 3.5.5

Solve: $\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$

Solution

Step 1: **Distribute.**

$$\frac{1}{2}\cdot y-\frac{1}{2}\cdot 5=\frac{1}{4}\cdot y-\frac{1}{4}\cdot 1$$

$$\text{Simplify.} \qquad \frac{1}{2}y-\frac{5}{2}=\frac{1}{4}y-\frac{1}{4}$$

Step 2: **Multiply by the LCD,** 4**.**

$$rgb]1.0,0.0,0.04\left(\frac{1}{2}y-\frac{5}{2}\right)=rgb]1.0,0.0,0.04\left(\frac{1}{4}y-\frac{1}{4}\right)$$

$$\text{Distribute.} \qquad 4\cdot\frac{1}{2}y-4\cdot\frac{5}{2}=4\cdot\frac{1}{4}y-4\cdot\frac{1}{4}$$

$$\text{Simplify.} \qquad 2y-10=y-1$$

Step 3: **Collect the variables to the left.**

$$2yrgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0y-10=yrgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0y-1$$

$$\text{Simplify.} \qquad y-10=-1$$

Step 4: **Collect the constants to the right.**

$$y-10rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.010=-1rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.010$$

$$\text{Simplify.} \qquad y=9$$

Step 5: **Check:**

$$\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$$

Step 6: **Let** $y=9$**.**

$$\frac{1}{2}(rgb]1.0,0.0,0.09-5)\stackrel{?}{=}\frac{1}{4}(rgb]1.0,0.0,0.09-1)$$

Finish the check on your own.

Try It

12) Solve: $\frac{1}{5}(n+3)=\frac{1}{4}(n+2)$

Solution

$n=2$

13) Solve: $\frac{1}{2}(m-3)=\frac{1}{4}(m-7)$

Solution

$m=-1$

Example 3.5.6

Solve: $\frac{5x-3}{4}=\frac{x}{2}$

Solution

Step 1: **Multiply by the LCD,** 4**.**

$$rgb]1.0,0.0,0.04\left(\frac{5x-3}{4}\right)=rgb]1.0,0.0,0.04\left(\frac{x}{2}\right)$$

$$\text{Simplify.} \qquad 5x-3=2x$$

Step 2: **Collect the variables to the right.**

$$5xrgb]1.0,0.0,0.0-rgb]1.0,0.0,0.05rgb]1.0,0.0,0.0x-3=2xrgb]1.0,0.0,0.0-rgb]1.0,0.0,0.05rgb]1.0,0.0,0.0x$$

$$\text{Simplify.} \qquad -3=-3x$$

Step 3: **Divide.**

$$\frac{-3}{rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.03} = \frac{-3x}{rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.03}$$

Simplify. $1 = x$

Step 4: **Check:**

$$\frac{5x-3}{4} = \frac{x}{2}$$

Step 5: **Let** $x = 1$.

$$\frac{5\left(rgb]1.0,0.0,0.01\right)-3}{4} \stackrel{?}{=} \frac{rgb]1.0,0.0,0.01}{2}$$

$$\frac{2}{4} \stackrel{?}{=} \frac{1}{2}$$

$$\frac{1}{2} = \frac{1}{2}\checkmark$$

Try It

14) Solve: $\dfrac{4y-7}{3} = \dfrac{y}{6}$

Solution

$y = 2$

15) Solve: $\dfrac{-2z-5}{4} = \dfrac{z}{8}$

Solution

$z = -2$

Example 3.5.7

Solve: $\frac{a}{6}+2=\frac{a}{4}+3$

Solution

Step 1: **Multiply by the LCD, 12.**

$$rgb]1.0,0.0,0.012\left(\frac{a}{6}+2\right)=rgb]1.0,0.0,0.012\left(\frac{a}{4}+3\right)$$

Step 2: **Distribute.**

$$12\cdot\frac{a}{6}+12\cdot 2=12\cdot\frac{a}{4}+12\cdot 3$$

$$\text{Simplify.}\qquad 2a+24=3a+36$$

Step 3: **Collect the variables to the right.**

$$2argb]1.0,0.0,0.0-rgb]1.0,0.0,0.02rgb]1.0,0.0,0.0a+24=3argb]1.0,0.0,0.0-rgb]1.0,0.0,0.02rgb]1.0,0.0,0.0a+36$$

$$\text{Simplify.}\qquad 24=a+36$$

Step 4: **Collect the constants to the left.**

$$24rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.036=a+36rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.036$$

$$\text{Simplify.}\qquad a=-12$$

Step 5: **Check:**

$$\frac{a}{6}+2=\frac{a}{4}+3$$

Step 6: **Let $a=-12$.**

$$\frac{rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.012}{6}+2\stackrel{?}{=}\frac{rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.012}{4}+3$$

$$-2+2\stackrel{?}{=}-3+3$$

$$0=0\checkmark$$

Try It

16) Solve: $\frac{b}{10} + 2 = \frac{b}{4} + 5$

Solution

$b = -20$

17) Solve: $\frac{c}{6} + 3 = \frac{c}{3} + 4$

Solution

$c = -6$

Example 3.5.8

Solve: $\frac{4q+3}{2} + 6 = \frac{3q+5}{4}$

Solution

Step 1: **Multiply by the LCD,** 4**.**

$$rgb]1.0, 0.0, 0.04 \left(\frac{4q+3}{2} + 6 \right) = rgb]1.0, 0.0, 0.04 \left(\frac{3q+5}{4} \right)$$

Step 2: **Distribute.**

$$4\left(\frac{4q+3}{2}\right)+4\cdot 6=4\cdot\left(\frac{3q+5}{4}\right)$$

Simplify. $$2(4q+3)+24=3q+5$$
$$8q+6+24=3q+5$$
$$8q+30=3q+5$$

Step 3: **Collect the variables to the left.**

$$8q rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.03 rgb]1.0,0.0,0.0q+30=3q rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.03 rgb]1.0,0.0,0.0q+5$$

Simplify. $$5q+30=5$$

Step 4: **Collect the constants to the right.**

$$5q+30 rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.030=5 rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.030$$

Simplify. $$5q=-25$$

Step 5: **Divide by 5.**

$$\frac{5q}{rgb]1.0,0.0,0.05}=\frac{-25}{rgb]1.0,0.0,0.05}$$

Simplify. $$q=-5$$

Step 6: **Check:**

$$\frac{4q+3}{2}+6=\frac{3q+5}{4}$$

Step 7: **Let $q=-5$.**

$$\frac{4\left(rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.05\right)+3}{2}+6\stackrel{?}{=}\frac{3\left(rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.05\right)+5}{4}$$

Finish the check on your own.

Try It

18) Solve: $\dfrac{3r+5}{6}+1=\dfrac{4r+3}{3}$

Solution

$r = 1$

19) Solve: $\frac{2s+3}{2} + 1 = \frac{3s+2}{4}$

Solution

$s = -8$

Solve Equations with Decimal Coefficients

Some equations have decimals in them. This kind of equation will occur when we solve problems dealing with money or percentages. But decimals can also be expressed as fractions. For example, $0.3 = \frac{3}{10}$ and $0.17 = \frac{17}{100}$. So, with an equation with decimals, we can use the same method we used to clear fractions—multiply both sides of the equation by the least common denominator.

Example 3.5.9

Solve: $0.06x + 0.02 = 0.25x - 1.5$

Solution

Look at the decimals and think of the equivalent fractions.

$$0.06 = \frac{6}{100}$$ 0.02=\frac{2}{100} $$0.25 = \frac{25}{100}$$ $$1.5 = 1\frac{5}{10}$$

Notice, the LCD is 100.

By multiplying by the LCD, we will clear the decimals from the equation.

$$0.06x + 0.02 = 0.25x - 1.5$$

Step 1: **Multiply both sides by 100.**

$$rgb]1.0, 0.0, 0.0100(0.06x + 0.02) = rgb]1.0, 0.0, 0.0100(0.25x - 1.5)$$

Step 2: **Distribute.**

$$100(0.06x) + 100(0.02) = 100(0.25x) - 100(1.5)$$

Step 3: **Multiply, and now we have no more decimals.**

$$6x + 2 = 25x - 150$$

Step 4: **Collect the variables to the right.**

$$6x rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.06 rgb]1.0, 0.0, 0.0x + 2 = 25x rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.06 rgb]1.0, 0.0, 0.0x - 150$$

$$\text{Simplify.} \qquad 2 = 19x - 150$$

Step 5: **Collect the constants to the left.**

$$2 rgb]1.0, 0.0, 0.0 + rgb]1.0, 0.0, 0.0150 = 19x - 150 rgb]1.0, 0.0, 0.0 + rgb]1.0, 0.0, 0.0150$$

$$\text{Simplify.} \qquad 152 = 19x$$

Step 6: **Divide by** 19**.**

$$\frac{152}{rgb]1.0, 0.0, 0.019} = \frac{19x}{rgb]1.0, 0.0, 0.019}$$

$$\text{Simplify.} \qquad 8 = x$$

Step 7: **Check: Let** $x = 8$**.**

$$0.06\,(rgb]1.0, 0.0, 0.08) + 0.02 \stackrel{?}{=} 0.25\,(rgb]1.0, 0.0, 0.08) - 1.5$$

$$0.48 + 0.02 \stackrel{?}{=} 2.00 - 1.5$$

$$0.50 = 0.50\checkmark$$

Try It

20) Solve: $0.14h + 0.12 = 0.35h - 2.4$

Solution

$h = 12$

21) Solve: $0.65k - 0.1 = 0.4k - 0.35$

Solution

$k = -1$

The next example uses an equation that is typical of the money applications in the next chapter. Notice that we distribute the decimal before we clear all the decimals.

Example 3.5.10

Solve: $0.25x + 0.05(x + 3) = 2.85$

Solution

Step 1: **Distribute first.**

$$0.25x + 0.05x + 0.15 = 2.85$$

Step 2: **Combine like terms.**

$$0.30x + 0.15 = 2.85$$

Step 3: **To clear decimals, multiply by 100.**

$$rgb]1.0, 0.0, 0.0100\,(0.30x + 0.15) = rgb]1.0, 0.0, 0.0100\,(2.8)$$

Step 4: **Distribute.**

$$30x + 15 = 285$$

Step 5: **Subtract 15 from both sides.**

$$30x + 15rgb]1.0, 0.0, 0.0{-}rgb]1.0, 0.0, 0.015 = 285rgb]1.0, 0.0, 0.0{-}rgb]1.0, 0.0, 0.015$$

$$\text{Simplify.} \qquad 30x = 270$$

Step 6: **Divide by 30.**

$$\frac{30x}{rgb]1.0, 0.0, 0.030} = \frac{270}{rgb]1.0, 0.0, 0.030}$$

$$\text{Simplify.} \qquad x = 9$$

Step 7: **Check it yourself by substituting $x = 9$ into the original equation.**

Try It

22) Solve: $0.25n + 0.05(n + 5) = 2.95$

Solution

$n = 9$

23) Solve: $0.10d + 0.05(d - 5) = 2.15$

Solution

$d = 16$

Key Concepts

- **Strategy to Solve an Equation with Fraction Coefficients**
    1. Find the least common denominator of all the fractions in the equation.
    2. Multiply both sides of the equation by that LCD. This clears the fractions.
    3. Solve using the General Strategy for Solving Linear Equations.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=958#h5p-19

b. Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

3.6 SOLVE A FORMULA FOR A SPECIFIC VARIABLE

Learning Objectives

By the end of this section, you will be able to:

- Use the Distance, Rate, and Time formula
- Solve a formula for a specific variable

Try It

Before you get started, take this readiness quiz:

1) Solve: $15t = 120$
2) Solve: $6x + 24 = 96$

Use the Distance, Rate, and Time Formula

One formula you will use often in algebra and in everyday life is the formula for distance travelled by an object moving at a constant rate. Rate is an equivalent word for "speed." The basic idea of rate may already familiar to you. Do you know what distance you travel if you drive at a steady rate of 60 miles per hour for 2 hours?

(This might happen if you use your car's cruise control while driving on the highway.) If you said 120 miles, you already know how to use this formula!

For an object moving at a uniform (constant) rate, the distance travelled, the elapsed time, and the rate are related by the formula:

$$\begin{aligned} d &= rt, \quad when \\ d &= distance \\ r &= rate \\ t &= time \end{aligned}$$

We will use the *Strategy for Solving Applications* that we used earlier in this chapter. When our problem requires a formula, we change Step 4: In place of writing a sentence, we write the appropriate formula. We write the revised steps here for reference.

HOW TO

Solve an application (with a formula).

1. Read the problem. Make sure all the words and ideas are understood.
2. Identify what we are looking for.
3. Name what we are looking for. Choose a variable to represent that quantity.
4. Translate into an equation. Write the appropriate formula for the situation. Substitute in the given information.
5. Solve the equation using good algebra techniques.
6. Check the answer in the problem and make sure it makes sense.
7. Answer the question with a complete sentence.

You may want to create a mini-chart to summarize the information in the problem. See the chart in this first example.

Example 3.6.1

Jamal rides his bike at a uniform rate of 12 miles per hour for $3\frac{1}{2}$ hours. What distance has he travelled?

Solution

Step 1: **Read the problem.**

Step 2**: Identify what you are looking for.**

distance travelled

Step 3**: Name. Choose a variable to represent it.**

Let d = distance.

Step 4: **Translate: Write the appropriate formula.**

$$d = rt$$
$$d = ?$$
$$r = 12\ mph$$
$$t = 3\frac{1}{2}\ hours$$

Step 5**: Substitute in the given information.**

$$d = 12 \cdot 3\frac{1}{2}$$

Step 6**: Solve the equation.**

$$d = 42\ miles$$

Step 7: **Check**

Does 42 miles make sense?

Jamal rides:

Figure 3.6.1

***Step 8*: Answer the question with a complete sentence.**

Jamal rode 42 miles.

Try It

3) Lindsay drove for $5\frac{1}{2}$ hours at 60 miles per hour. How much distance did she travel?

Solution

330 miles

4) Trinh walked for $2\frac{1}{3}$ hours at 3 miles per hour. How far did she walk?

Solution

7 miles

Example 3.6.2

Rey is planning to drive from his house in San Diego to visit his grandmother in Sacramento, a distance of 520 miles. If he can drive at a steady rate of 65 miles per hour, how many hours will the trip take?

Solution

Step 1: **Read the problem.**

Step 2: **Identify what you are looking for.**

How many hours (time)

Step 3: **Name.**

Choose a variable to represent it.

Let t = time.

$$d = 520\ miles$$
$$r = 65\ mph$$
$$t = ?\ hours$$

Step 4: **Translate.**

Write the appropriate formula. Substitute in the given information.

$$d = rt$$
$$520 = 65t$$
$$t = 8$$

Step 5: **Solve the equation.**

$$t = 8$$

Step 6: **Check.**

Substitute the numbers into the formula and make sure the result is a true statement.

$$d = rt$$
$$520 \stackrel{?}{=} 65 \times 8$$
$$520 = 520\checkmark$$

Step 7: **Answer the question with a complete sentence.**

Rey's trip will take 8 hours.

Try It

5) Lee wants to drive from Phoenix to his brother's apartment in San Francisco, a distance of 770 miles. If he drives at a steady rate of 70 miles per hour, how many hours will the trip take?

Solution

11 hours

6) Yesenia is 168 miles from Chicago. If she needs to be in Chicago in 3 hours, at what rate does she need to drive?

Solution

56 mph

Solve a Formula for a Specific Variable

You are probably familiar with some geometry formulas. A formula is a mathematical description of the relationship between variables. Formulas are also used in the sciences, such as chemistry, physics, and biology. In medicine, they are used for calculations for dispensing medicine or determining body mass index. Spreadsheet programs rely on formulas to make calculations. It is important to be familiar with formulas and be able to manipulate them easily.

In Example 3.6.1 and Example 3.6.2, we used the formula $d = rt$. This formula gives the value of d, distance, when you substitute in the values of r and t, the rate and time. But in Example 3.6.2 we had to find the value of t. We substituted in values of d and r and then used algebra to solve for t. If you had to do this often, you might wonder why there is not a formula that gives the value of t when you substitute in the values of d and r. We can make a formula like this by solving the formula $d = rt$ for t.

To solve a formula for a specific variable means to isolate that variable on one side of the equals sign with a coefficient of 1. All other variables and constants are on the other side of the equals sign. To see how to solve a formula for a specific variable, we will start with the distance, rate and time formula.

Example 3.6.3

Solve the formula $d = rt$ for t:

a. when $d = 520$ and $r = 65$
b. in general

Solution
We will write the solutions side-by-side to demonstrate that solving a formula in general uses the same steps as when we have numbers to substitute.

a. when $d = 520$ **and** $r = 65$		**b.**
Step 1: **Write the formula.**	$d = rt$	***Step 1:*** **Write the formula.**
Step 2: **Substitute.**	$520 = 65t$	
Step 3: **Divide, to isolate** t**.**	$\frac{520}{65} = \frac{65t}{65}$	***Step 2:*** **Divide, to isolate** t**.**
Step 4: **Simplify.**	$8 = t$	***Step 3:*** **Simplify.**

We say the formula $t = \frac{d}{r}$ is solved for t.

Try It

7) Solve the formula $d = rt$ for r

a. when $d = 180$ and $t = 4$
b. in general

Solution

a.$r = 45$
b.$r = \frac{d}{t}$

8) Solve the formula $d = rt$ for r

a. when $d = 780$ and $t = 12$
b. in general

Solution

a.$r = 65$
b. $r = \frac{d}{t}$

Example 3.6.4

Solve the formula $A = \frac{1}{2}bh$ for h:

a. when $A = 90$ and $b = 15$
b. in general

Solution

a. when $A = 90$ and $b = 15$		b. i	
Step 1: **Write the formula.**	$A = \frac{1}{2}bh$	***Step 1:*** **Write the formula.**	
Step 2: **Substitute.**	$90 = \frac{1}{2} \cdot rgb]1.0, 0.0, 0.015 \cdot h$		
Step 3: **Clear the fractions.**	$rgb]1.0, 0.0, 0.02 \cdot 90 = rgb]1.0, 0.0, 0.02 \cdot \frac{1}{2}15h$	***Step 2:*** **Clear the fractions.**	
Step 4: **Simplify.**	$180 = 15h$	***Step 3:*** **Simplify.**	
Step 5: **Solve for** h**.**	$12 = h$	***Step 4:*** **Solve for** h**.**	

We can now find the height of a triangle, if we know the area and the base, by using the formula $h = \frac{2A}{b}$

Try It

9) Use the formula $A = \frac{1}{2}bh$ to solve for h

a. when $A = 170$ and $b = 17$
b. in general

Solution

a. $h = 20$
b. $h = \frac{2A}{b}$

10) Use the formula $A = \frac{1}{2}bh$ to solve for b

a. when $A = 62$ and $h = 31$
b. in general

Solution

a.$b = 4$
b.$b = \frac{2A}{h}$

The formula $I = Prt$ is used to calculate simple interest, I, for a principal, P, invested at rate, r, for t years.

Example 3.6.5

Solve the formula $I = Prt$ to find the principal, P:

a. when $I = \$5,600$, $r = 4\%$, $t = 7\ years$
b. in general

Solution

a. $\$5,600, r = 4\%, t = 7years$			**b. in g**
Step 1: **Write the formula.**	$I = Prt$	***Step 1:*** **Write the formula.**	
Step 2: **Substitute.**	$5600 = P(0.04)(7)$		
Step 3: **Simplify.**	$5600 = P(0.28)$	***Step 2:*** **Simplify.**	
Step 4: **Divide, to isolate** P.	$\frac{560}{rgb]1.0,0.0,0.00rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.028} = \frac{P(0.28)}{rgb]1.0,0.0,0.00rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.028}$	***Step 3:*** **Divide, to isolate** P.	
Step 5: **Simplify.**	$20,000 = P$	***Step 4:*** **Simplify.**	
The principal is	$\$20,000$		

Try It

11) Use the formula $I = Prt$ to find the principal, P:

a. when $I = \$2,160$, $r = 6\%$, $t = 3years$
b. in general

Solution
a.$\$12,000$
b. $P = \frac{I}{rt}$

12) Use the formula $I = Prt$ to find the principal, P:

a. when $I = \$5,400$, $r = 12\%$, $t = 5\ years$
b. in general

Solution
a. $\$9,000$
b. $P = \frac{I}{rt}$

Later in this class, and in future algebra classes, you'll encounter equations that relate two variables, usually x and y. You might be given an equation that is solved for y and need to solve it for x, or vice versa. In the following example, we're given an equation with both x and y on the same side and we'll solve it for y.

Example 3.6.6

Solve the formula $3x + 2y = 18$ for y:

a. when $x = 4$
b. in general

Solution

a. when $x = 4$		**b. in general**	
Step 1: **Substitute.**	$3(4) + 2y = 18$		
Step 2: **Subtract to isolate the** y**-term.**	$12 rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.0 12 + 2y = 18 rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.0 12$	***Step 1:*** **Subtract to isolate the** y**-term.**	$3x rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.0 3 rgb]1.0, 0.0, 0.0 x$
Step 3: **Divide.**	$\frac{2y}{rgb]1.0, 0.0, 0.0 2} = \frac{6}{rgb]1.0, 0.0, 0.0 2}$	***Step 2:*** **Divide.**	$\frac{2y}{rgb]1.0, 0.0, 0.0 2} = \frac{}{rgb]1.0}$
Step 4: **Simplify.**	$y = 3$	***Step 3:*** **Simplify.**	$y = -\frac{3x}{2}$

Try It

13) Solve the formula $3x + 4y = 10$ for y:

a. when $x = \frac{14}{3}$
b. in general

Solution

a. $y = 1$
b. $y = \frac{10 - 3x}{4}$

14) Solve the formula $5x + 2y = 18$ for y:

a. when $x = 4$
b. in general

Solution

a. $y = -1$
b. $y = \dfrac{18 - 5x}{2}$

In Examples 3.6.3 through 3.6.6 we used the numbers in part a. as a guide to solving in general in part b. Now we will solve a formula in general without using numbers as a guide.

Example 3.6.7

Solve the formula $P = a + b + c$ for a

Solution

Step 1: **We will isolate a on one side of the equation.**

$$P = a + b + c$$

Step 2: **Both b and c are added to a, so we subtract them from both sides of the equation.**

$Prgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.0brgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.0crgb]1.0, 0.0, 0.0 = a + b + crgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.0brgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.0c$

Step 3: **Simplify.**

$$P - b - c = a$$
$$a = P - b - b$$

Try It

15) Solve the formula $P = a + b + c$ for b

Solution

$b = P - a - c$

16) Solve the formula $P = a + b + c$ for c

Solution

$c = P - a - b$

Example 3.6.8

Solve the formula $6x + 5y = 13$ for y

Solution

Step 1: **Subtract $6x$ from both sides to isolate the term with y**

$$6x rgb]1.0,0.0,0.0 - rgb]1.0,0.0,0.06 rgb]1.0,0.0,0.0 x + 5y = 13 rgb]1.0,0.0,0.0 - rgb]1.0,0.0,0.06 rgb]1.0,0.0,0.0 x$$

$$\text{Simplify.} \qquad 5y = 13 - 6x$$

Step 2: **Divide by 5 to make the coefficient 1.**

$$\frac{5y}{rgb]1.0,0.0,0.05} = \frac{13 - 6x}{rgb]1.0,0.0,0.05}$$

$$\text{Simplify.} \qquad y = \frac{13 - 6x}{5}$$

The fraction is simplified. We cannot divide $13 - 6x$ by 5.

Try It

17) Solve the formula $4x + 7y = 9$ for y.

Solution

$$y = \frac{9 - 4x}{7}$$

18) Solve the formula $5x + 8y = 1$ for y.

Solution

$$y = \frac{1 - 5x}{8}$$

In the next example, we will solve this formula for the height.

Example 3.6.9

Solve the formula $V = \frac{1}{3}\pi r^2 h$ for h.

Solution

Step 1: **Write the formula.**

$$V = \frac{1}{3}\pi r^2 h$$

Step 2: **Remove the fraction on the right.**

$$rgb]1.0, 0.0, 0.03rgb]1.0, 0.0, 0.0\cdot V = rgb]1.0, 0.0, 0.03rgb]1.0, 0.0, 0.0\cdot \frac{1}{3}\pi r^2 h$$

Simplify. $3V = \pi r^2 h$

***Step 4:* Divide both sides by πr^2**

$$\frac{3V}{\pi r^2} = h$$

We could now use this formula to find the height of a right circular cone when we know the volume and the radius of the base, by using the formula $h = \frac{3V}{\pi r^2}$.

Try It

19) Use the formula $A = \frac{1}{2}bh$ to solve for b.

Solution

$b = \frac{2A}{h}$

20) Use the formula $A = \frac{1}{2}bh$ to solve for h.

Solution

$h = \frac{2A}{b}$

In the sciences, we often need to change temperature from Fahrenheit to Celsius or vice versa. If you travel in a foreign country, you may want to change the Celsius temperature to the more familiar Fahrenheit temperature.

Example 3.6.10

Solve the formula $C = \frac{5}{9}(F - 32)$ for F.

Solution

Step 1: **Write the formula.**

$$C = \frac{5}{9}(F - 32)$$

Step 2: **Remove the fraction on the right.**

$$rgb]1.0,0.0,0.0\frac{9}{5}C = rgb]1.0,0.0,0.0\frac{9}{5} \cdot \frac{5}{9}(F - 32)$$

$$\text{Simplify.} \qquad \frac{9}{5}C = (F - 32)$$

Step 4: **Add 32 to both sides.**

$$\frac{9}{5}C + 32 = F$$

We can now use the formula $F = \frac{9}{5}C + 32$ to find the Fahrenheit temperature when we know the Celsius temperature.

Try It

21) Solve the formula $F = \frac{9}{5}C + 32$ for C.

Solution

$C = \frac{5}{9}(F - 32)$

22) Solve the formula $A = \frac{1}{2}h(b + B)$ for b.

Solution

$b = \frac{2A - Bh}{h}$

The next example uses the formula for the surface area of a right cylinder.

Example 3.6.11

Solve the formula $S = 2\pi r^2 + 2\pi rh$ for h.

Solution

Step 1: **Write the formula.**

$$S = 2\pi r^2 + 2\pi rh$$

Step 2: **Isolate the h term by subtracting $2\pi r^2$ from each side.**

$$Srgb]1.0,0.0,0.0-rgb]1.0,0.0,0.02rgb]1.0,0.0,0.0\pi rgb]1.0,0.0,0.0r^{rgb]1.0,0.0,0.02} = 2\pi r^2 rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.02rgb]1.0,0.0,0.0\pi rgb]1.0,0.0,0.0r^{rgb]1.0,0.0,0.02} + 2\pi rh$$

Simplify. $S - 2\pi r^2 = 2\pi rh$

Step 3: **Solve for h by dividing both sides by $2\pi r$**

$$\frac{S - 2\pi r^2}{2\pi r} = \frac{2\pi rh}{2\pi r}$$

Simplify. $$\frac{S - 2\pi r^2}{2\pi r} = h$$

Try It

23) Solve the formula $A = P + Prt$ for t.

Solution

$$t = \frac{A - P}{Pr}$$

24) Solve the formula $A = P + Prt$ for r.

Solution

$$r = \frac{A - P}{Pt}$$

Key Concepts

- **To Solve an Application (with a formula)**
  1. Read the problem. Make sure all the words and ideas are understood.
  2. Identify what we are looking for.
  3. Name what we are looking for. Choose a variable to represent that quantity.
  4. Translate into an equation. Write the appropriate formula for the situation. Substitute in the given information.
  5. Solve the equation using good algebra techniques.
  6. Check the answer in the problem and make sure it makes sense.
  7. Answer the question with a complete sentence.
- **Distance, Rate and Time**
  For an object moving at a uniform (constant) rate, the distance travelled, the elapsed time,

and the rate are related by the formula: $d = rt$ where $d = distance$, $r = rate$, $t = time$.

- **To solve a formula for a specific variable** means to get that variable by itself with a coefficient of 1 on one side of the equation and all other variables and constants on the other side.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=1002#h5p-20

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

3.7 USE A PROBLEM-SOLVING STRATEGY AND APPLICATIONS

Learning Objectives

By the end of this section, you will be able to:

- Approach word problems with a positive attitude
- Use a problem-solving strategy for word problems
- Solve number problems
- Translate and solve basic percent equations
- Solve percent applications
- Find percent increase and percent decrease
- Solve simple interest applications
- Solve applications with discount or mark-up

Try It

Before you get started, take this readiness quiz:

1) Translate “6 less than twice x” into an algebraic expression.

2) Solve: $\frac{2}{3}x = 24$

3) Solve: $3x+8=14$
4) Convert 4.5% to a decimal.
5) Convert 0.6 to a percent.
6) Round 0.875 to the nearest hundredth.
7) Multiply $(4.5)(2.38)$
8) Solve $3.5=0.7n$
9) Subtract $50-37.45$

Approach Word Problems with a Positive Attitude

"If you think you can… or think you can't… you're right."—Henry Ford

The world is full of word problems! Will my income qualify me to rent that apartment? How much punch do I need to make for the party? What size diamond can I afford to buy my girlfriend? Should I fly or drive to my family reunion?

How much money do I need to fill the car with gas? How much tip should I leave at a restaurant? How many socks should I pack for vacation? What size turkey do I need to buy for Thanksgiving dinner, and then what time do I need to put it in the oven? If my sister and I buy our mother a present, how much does each of us pay?

Now that we can solve equations, we are ready to apply our new skills to word problems. Do you know anyone who has had negative experiences in the past with word problems? Have you ever had thoughts like the student below?

Negative thoughts can be barriers to success.

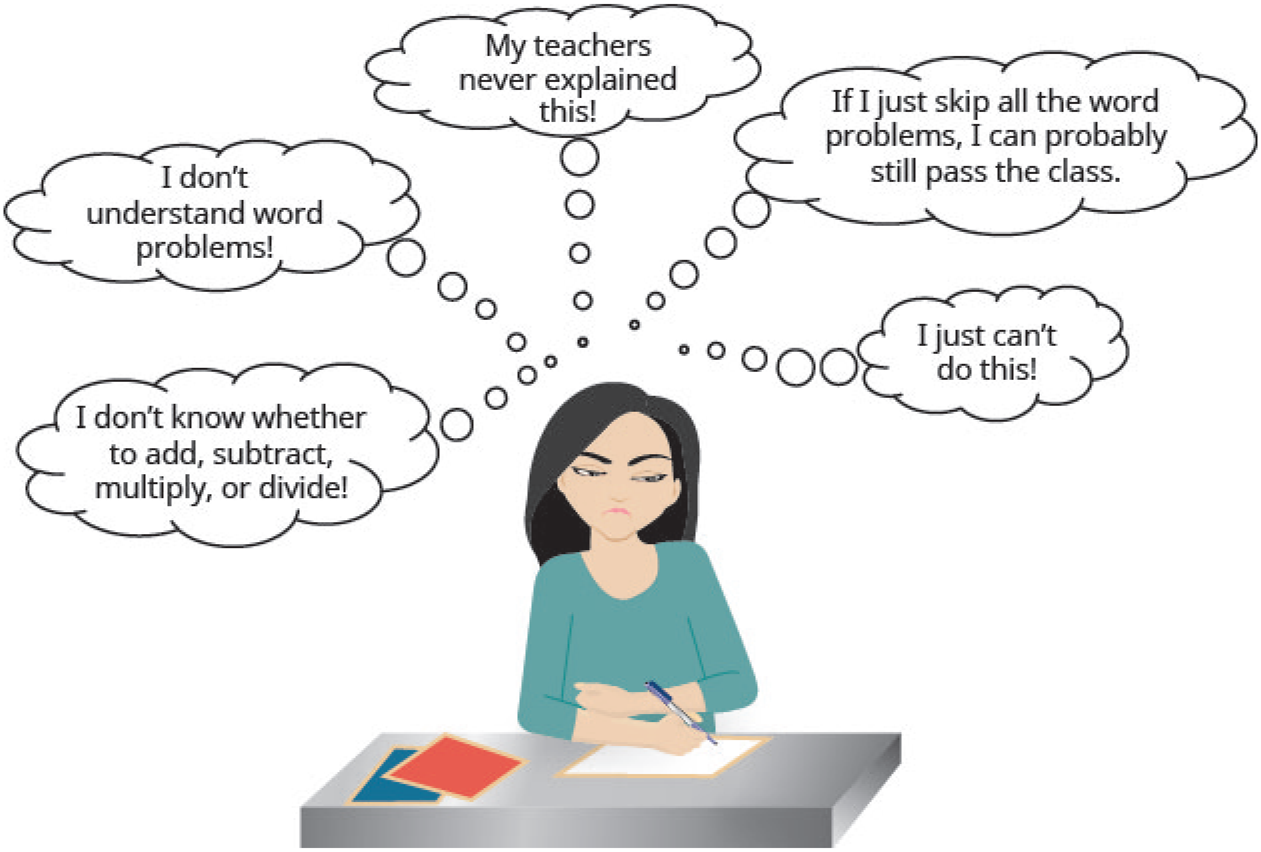


Figure 3.7.1 Negative thoughts can be barriers to success.

When we feel we have no control, and continue repeating negative thoughts, we set up barriers to success. We need to calm our fears and change our negative feelings.

Start with a fresh slate and begin to think positive thoughts. If we take control and believe we can be successful, we will be able to master word problems! Read the positive thoughts in Figure 3.7.2 and say them out loud.

Thinking positive thoughts is a first step towards success.

Figure 3.7.2 Thinking positive thoughts is a first step towards success.

Think of something, outside of school, that you can do now but couldn't do 3 years ago. Is it driving a car? Snowboarding? Cooking a gourmet meal? Speaking a new language? Your past experiences with word problems happened when you were younger—now you're older and ready to succeed!

Use a Problem-Solving Strategy for Word Problems

We have reviewed translating English phrases into algebraic expressions, using some basic mathematical vocabulary and symbols. We have also translated English sentences into algebraic equations and solved some word problems. The word problems applied math to everyday situations. We restated the situation in one sentence, assigned a variable, and then wrote an equation to solve the problem. This method works as long as the situation is familiar and the math is not too complicated.

Now, we'll expand our strategy so we can use it to successfully solve any word problem. We'll list the strategy here, and then we'll use it to solve some problems. We summarize below an effective strategy for problem-solving.

HOW TO

Use a Problem-Solving Strategy to Solve Word Problems.

1. Read the problem. Make sure all the words and ideas are understood.
2. Identify what we are looking for.
3. Name what we are looking for. Choose a variable to represent that quantity.
4. Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. Solve the equation using good algebra techniques.
6. Check the answer in the problem and make sure it makes sense.
7. Answer the question with a complete sentence.

Example 3.7.1

Pilar bought a purse on sale for $\$18$, which is one-half of the original price. What was the original price of the purse?

Solution

Step 1: **Read the problem. Read the problem two or more times if necessary. Look up any unfamiliar words in a dictionary or on the internet.**

In this problem, is it clear what is being discussed? Is every word familiar?

Step 2: **Identify what you are looking for.**

Did you ever go into your bedroom to get something and then forget what you were looking for? It's hard to find something if you are not sure what it is! Read the problem again and look for words that tell you what you are looking for!

In this problem, the words "what was the original price of the purse" tell us what we need to find.

Step 3: **Name what we are looking for.**

Choose a variable to represent that quantity. We can use any letter for the variable, but choose one that makes it easy to remember what it represents.

Let p = the original price of the purse.

Step 4: **Translate into an equation.**

It may be helpful to restate the problem in one sentence with all the important information. Translate the English sentence into an algebraic equation.

Reread the problem carefully to see how the given information is related. Often, there is one sentence that gives this information, or it may help to write one sentence with all the important information. Look for clue words to help translate the sentence into algebra. Translate the sentence into an equation.

Restate the problem in one sentence with all the important information.

$$\underbrace{18}\ \ \underbrace{\text{is}}\ \ \underbrace{\text{one-half the original price.}}$$

Translate into an equation.

$$18 \ = \ \frac{1}{2} \ \times \ p$$

Step 5: Solve the equation using good algebraic techniques.

Even if you know the solution right away, using good algebraic techniques here will better prepare you to solve problems that do not have obvious answers.

Solve the equation:

$$18 = \frac{1}{2}p$$

Multiply both sides by 2.	$rgb]1.0, 0.0, 0.02 \ \times \ 18 = rgb]1.0, 0.0, 0.02 \ \times \ \frac{1}{2}p$
Simplify	$36 = p$

Step 6: **Check the answer in the problem to make sure it makes sense.**

We solved the equation and found that $p = 36$, which means "the original price" was $\$36$.

Does $\$36$ make sense in the problem? Yes, because 18 is one-half of 36, and the purse was on sale at half the original price.

Step 7: **Answer the question with a complete sentence.** The problem asked "What was the original price of the purse?"

The answer to the question is: "The original price of the purse was $\$36$."

If this were a homework exercise, our work might look like this:

Pilar bought a purse on sale for $\$18$, which is one-half the original price. What was the original price of the purse?

Let $p =$ the original price.

18 is one-half the original price.

$$18 = \frac{1}{2}p$$

Multiply both sides by 2.	$rgb]1.0, 0.0, 0.02 \times 18 = rgb]1.0, 0.0, 0.02 \times \frac{1}{2}p$
Simplify	$36 = p$

Step 8: **Check. Is $\$36$ a reasonable price for a purse?**

Yes.

Is 18 one half of 36?

$$18 \stackrel{?}{=} \frac{1}{2} \times 36$$

$$18 = 18\checkmark$$

The original price of the purse was $\$36$.

Try It

10) Joaquin bought a bookcase on sale for $\$120$, which was two-thirds of the original price. What was the original price of the bookcase?

Solution

$\$180$

11) Two-fifths of the songs in Mariel's playlist are country. If there are 16 country songs, what is the total number of songs in the playlist?

Solution

40

Let's try this approach with another example.

Example 3.7.2

Ginny and her classmates formed a study group. The number of girls in the study group was three more than twice the number of boys. There were 11 girls in the study group. How many boys were in the study group?

Solution

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

How many boys were in the study group?

Step 3: **Name. Choose a variable to represent the number of boys.**

Let $n =$ the number of boys.

Step 4: **Translate. Restate the problem in one sentence with all the important information.**

$$\underbrace{\text{The number of girls } (11)}\ \underbrace{\text{was}}\ \underbrace{\text{three more than twice the number of boys.}}$$

Translate into an equation.

$$11 \;=\; 2b \;+3$$

Step 5: **Solve the equation.**

	$11 = 2b + 3$
Subtract 3 from each side	$11\ rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.03 = 2b + 3\ rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.03$
Simplify	$8 = 2b$
Divide each side by 2	$\dfrac{8}{rgb]1.0,0.0,0.02} = \dfrac{2b}{rgb]1.0,0.0,0.02}$
Simplify	$4 = b$

Step 6: **Check. First, is our answer reasonable?**

Yes, having 4 boys in a study group seems OK. The problem says the number of girls was 3

more than twice the number of boys. If there are four boys, does that make eleven girls? Twice 4 boys is 8. Three more than 8 is 11.

Step 7: **Answer the question.**

There were 4 boys in the study group.

Try It

12) Guillermo bought textbooks and notebooks at the bookstore. The number of textbooks was 3 more than twice the number of notebooks. He bought 7 textbooks. How many notebooks did he buy?

Solution

2

13) Gerry worked Sudoku puzzles and crossword puzzles this week. The number of Sudoku puzzles he completed is eight more than twice the number of crossword puzzles. He completed 22 Sudoku puzzles. How many crossword puzzles did he do?

Solution

7

Solve Number Problems

Now that we have a problem-solving strategy, we will use it on several different types of word problems. The first type we will work on is "number problems." Number problems give some clues about one or more numbers. We use these clues to write an equation. Number problems don't usually arise on an everyday basis, but they provide a good introduction to practicing the problem-solving strategy outlined above.

Example 3.7.3

The difference of a number and six is 13. Find the number.

Solution

Step 1: **Read the problem. Are all the words familiar?**

Step 2: **Identify what we are looking for.**

The number.

Step 3: **Name. Choose a variable to represent the number.**

Let n = the number.

Step 4: **Translate. Remember to look for clue words like "difference... of... and..."**

Restate the problem as one sentence.

$$\underbrace{\text{The difference of the number and 6}}\ \underbrace{\text{is}}\ \underbrace{13.}$$

Translate into an equation.

$$n - 6 = 13$$

Step 5: **Solve the equation.**

Simplify.

$$\begin{aligned} n - 6 &= 13 \\ n &= 19 \end{aligned}$$

Step 6: **Check.**

The difference of 19 and 6 is 13. It checks!

Step 7: **Answer the question.**

The number is 19.

Try It

14) The difference of a number and eight is 17. Find the number.

Solution

25

15) The difference of a number and eleven is -7. Find the number.

Solution

4

Example 3.7.4

The sum of twice a number and seven is 15. Find the number.

Solution

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

The number.

Step 3: **Name. Choose a variable to represent the number.**

Let n = the number.

Step 4: **Translate.**

Restate the problem as one sentence.

$$\underbrace{\text{The sum of twice a number and 7}}\ \underbrace{\text{is}}\ \underbrace{13.}$$

Translate into an equation.

$$2n + 7 = 15$$

Step 5: **Solve the equation.**

Subtract 7 from each side and simplify.	$2n + 7 = 15$
Divide each side by 2 and simplify.	$2n = 8$
	$n = 4$

Step 6: **Check.**

Is the sum of twice 4 and 7 equal to 15?

Step 7: **Answer the question.**

The number is 4.

Did you notice that we left out some of the steps as we solved this equation? If you're not yet ready to leave out these steps, write down as many as you need.

Try It

16) The sum of four times a number and two is 14. Find the number.

Solution

3

17) The sum of three times a number and seven is 25. Find the number.

Solution

6

Some number word problems ask us to find two or more numbers. It may be tempting to name them all with different variables, but so far we have only solved equations with one variable. In order to avoid using more than one variable, we will define the numbers in terms of the same variable. Be sure to read the problem carefully to discover how all the numbers relate to each other.

Example 3.7.5

One number is five more than another. The sum of the numbers is 21. Find the numbers.

Solution

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

We are looking for two numbers.

Step 3: **Name. We have two numbers to name and need a name for each.**

Choose a variable to represent the first number.

$$n = 1^{st}\ number$$

What do we know about the second number?

$$n + 5 = 2^{nd}\ number$$

Step 4: **Translate. Restate the problem as one sentence with all the important information.**

The sum of the 1st number and the 2nd number is 21.

Translate into an equation.

$$\underbrace{1^{st}\ \text{number}} + \underbrace{2^{nd}\ \text{number}}\ \underbrace{=}\ \underbrace{21}$$

Substitute the variable expressions.

$$n + n + 5 = 21$$

Step 5: **Solve the equation.**

	$n + n + 5 = 21$	
Combine like terms.	$2n + 5 = 21$	
Subtract 5 from both sides and simplify.	$2n = 16$	
Divide by 2 and simplify.	$n = 8$	$1^{st}\ number$
Find the second number, too	$rgb]1.0, 0.0, 0.0n + 5$	$2^{nd}\ number$
	$rgb]1.0, 0.0, 0.08 + 5 = 13$	

***Step 6:* Check.**

Do these numbers check in the problem?

Is one number 5 more than the other?	$13 \stackrel{?}{=} 8+5$
Is thirteen 5 more than 8? Yes.	$13 = 13\checkmark$
Is the sum of the two numbers 21?	$8+13 \stackrel{?}{=} 21$
	$21 = 21\checkmark$

***Step 7:* Answer the question.**

The numbers are 8 and 13.

Try It

18) One number is six more than another. The sum of the numbers is twenty-four. Find the numbers.

Solution

9, 15

19) The sum of two numbers is fifty-eight. One number is four more than the other. Find the numbers.

Solution

27, 31

Example 3.7.6

The sum of two numbers is negative fourteen. One number is four less than the other. Find the numbers.

Solution

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

We are looking for two numbers.

Step 3: **Name.**

Choose a variable.

$$n = 1^{st}\ number$$

One number is 4 less than the other.

$$n - 4 = 2^{nd}\ number$$

Step 4: **Translate.**

The sum of the 2 numbers is negative 14.

Write as one sentence.

$$\underbrace{1^{st}\text{ number}} + \underbrace{2^{nd}\text{ number}}\ \underbrace{\text{is}}\ \underbrace{\text{negative }14}$$

Translate into an equation.

$$n\ +\ n - 4\ =\ -14$$

Step 5: **Solve the equation.**

$$n + n - 4 = -14$$

Combine like terms. $\quad 2n - 4 = -14$

Add 4 to each side and simplify. $\quad 2n = -10$

$$n = -5 \qquad 1^{st}\ number$$

Simplify. $\quad rgb]1.0, 0.0, 0.0n - 4 \qquad 2^{nd}\ number$

$$rgb]1.0, 0.0, 0.0 - 5 - 4 = -9$$

Step 6: **Check.**

Is -9 four less than -5?	$-5 - 4 \stackrel{?}{=} -9$
	$-9 = -9\checkmark$
Is their sum -14?	$-5 + (-9) \stackrel{?}{=} -14$
	$-14 = -14\checkmark$

***Step 7:* Answer the question.**

The numbers are -5 and -9.

Try It

20) The sum of two numbers is negative twenty-three. One number is seven less than the other. Find the numbers.

Solution

-15, -8

21) The sum of two numbers is -18. One number is 40 more than the other. Find the numbers.

Solution

$-29, 11$

Example 3.7.7

One number is ten more than twice another. Their sum is one. Find the numbers.

Solution

Step 1: **Read the problem.**

Step 2: **Identify what you are looking for.**

We are looking for two numbers.

Step 3: **Name.**

Choose a variable.

$$x = 1^{st}\ number$$

One number is 10 more than twice another.

$$2x + 10 = 2^{nd}\ number$$

Step 4: **Translate.**

Their sum is one.

Restate as one sentence.

The sum of the two numbers is 1.

Translate into an equation.

$$x + 2x + 10 = 1$$

Step 5: **Solve the equation.**

	$x + 2x + 10 = 1$	
Combine like terms.	$3x + 10 = 1$	
Subtract 10 from each side.	$3x = -9$	
Divide each side by 3.	$x = -3$	$1^{st}\ number$
	$2rgb]1.0, 0.0, 0.0x + 10$	$2^{nd}\ number$
	$2(rgb]1.0, 0.0, 0.0 - 3) + 10 = 4$	

Step 6: **Check.**

Is ten more than twice -3 equal to 4?	$2(-3) + 10 \stackrel{?}{=} 4$
	$-6 + 10 \stackrel{?}{=} 4$
	$4 = 4\checkmark$
Is their sum 1?	$-3 + 4 \stackrel{?}{=} 1$
	$1 = 1$

Step 7: **Answer the question.**

The numbers are -3 and -4.

Try It

22) One number is eight more than twice another. Their sum is negative four. Find the numbers.

Solution

$-4, 0$

23) One number is three more than three times another. Their sum is -5. Find the numbers.

Solution

$-3, -2$

Some number problems involve consecutive integers. *Consecutive integers* are integers that immediately follow each other. Examples of consecutive integers are:

$$1, 2, 3, 4$$
$$-10, -9, -8, -7$$
$$150, 151, 152, 153$$

Notice that each number is one more than the number preceding it. So if we define the first integer as n, the next consecutive integer is $n+1$. The one after that is one more than $n+1$, so it is $n+1+1$, which is $n+2$.

n	1^{st} integer
$n+1$	2nd consecutive integer
$n+2$	3rd consecutive integer… etc.

Example 3.7.8

The sum of two consecutive integers is 47. Find the numbers.

Solution

***Step 1:* Read the problem.**

***Step 2:* Identify what you are looking for.**

Two consecutive integers.

***Step 3:* Name each number.**

$$\begin{aligned} n = &\ 1^{st}\ integer \\ n+1 = &\ 2^{nd}\ next\ consecutive\ integer \end{aligned}$$

***Step 4:* Translate.**

Restate as one sentence.

The sum of the integers is 47.

Translate into an equation.

$$n+n+1=47$$

***Step 5:* Solve the equation.**

$$\begin{array}{lrl} & n+n+1=47 & \\ \text{Combine like terms.} & 2x+1=47 & \\ \text{Subtract 1 from each side.} & 2n=46 & \\ \text{Divide each side by 2.} & n=23 & 1^{st}\ integer \\ & rgb]1.0,0.0,0.0n+1 & 2^{nd}\ integer \\ & rgb]1.0,0.0,0.023+1=24 & \end{array}$$

***Step 6:* Check.**

$$\begin{aligned} 23+24 &\stackrel{?}{=} 47 \\ 47 &= 47\checkmark \end{aligned}$$

***Step 7:* Answer the question.**

The two consecutive integers are 23 and 24.

Try It

24) The sum of two consecutive integers is 95. Find the numbers.

Solution

47, 48

25) The sum of two consecutive integers is -31. Find the numbers.

Solution

-16, -15

Example 3.7.9

Find three consecutive integers whose sum is -42.

Solution

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

three consecutive integers

Step 3: **Name each of the three numbers.**

$$\begin{aligned} n &= 1^{st}\ integer \\ n+1 &= 2^{nd}\ consecutive\ integer \\ n+2 &= 3^{rd}\ consecutive\ integer \end{aligned}$$

Step 4: **Translate.**

Restate as one sentence.

The sum of the three integers is -42.

Translate into an equation.

$$n+n+1+n+2=-42$$

Step 5: **Solve the equation.**

	$n+n+1+n+2=-42$	
Combine like terms.	$3n+3=-42$	
Subtract 3 from each side.	$3n=-45$	
Divide each side by 3.	$n=-15$	1^{st} *integer*
	$rgb]1.0,0.0,0.0n+1$	
	$rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.015+1=-14$	2^{nd} *integer*
	$rgb]1.0,0.0,0.0-15+2=-13$	3^{rd} *integer*

Step 6: **Check.**

$$-13+(-14)+(-15)\stackrel{?}{=}-42$$
$$-42=-42\checkmark$$

Step 7: **Answer the question.**

The three consecutive integers are -13, -14, and -15.

Try It

26) Find three consecutive integers whose sum is -96.

Solution

-33, -32, -31

27) Find three consecutive integers whose sum is -36.

Solution

-13, -12, -11

Now that we have worked with consecutive integers, we will expand our work to include consecutive even integers and consecutive odd integers. *Consecutive even integers* are even integers that immediately follow one another. Examples of consecutive even integers are:

$$18, 20, 22$$
$$64, 66, 68$$
$$-12, -10, -8$$

Notice each integer is 2 more than the number preceding it. If we call the first one n, then the next one is $n+2$. The next one would be $n+2+2$ or $n+4$.

n	1st even integer
$n+2$	2nd consecutive even integer
$n+4$	3rd consecutive even integer… etc.

Consecutive odd integers are odd integers that immediately follow one another. Consider the consecutive odd integers 77, 79, and 81.

$$77, 79, 81$$
$$n, n+2, n+4$$

n	1st odd integer
$n+2$	2nd consecutive odd integer
$n+4$	3rd consecutive odd integer… etc.

Does it seem strange to add 2 (an even number) to get from one odd integer to the next? Do you get an odd number or an even number when we add 2 to 3? to 11? to 47?

Whether the problem asks for consecutive even numbers or odd numbers, you don't have to do anything different. The pattern is still the same—to get from one odd or one even integer to the next, add 2 .

Example 3.7.10

Find three consecutive even integers whose sum is 84.

Solution

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

three consecutive even integers

Step 3: **Name the integers.**

$$n = 1^{st}\ integer$$

Let $n + 2 = 2^{nd}\ consecutive\ even\ integer$

$$n + 4 = 3^{rd}\ consecutive\ even\ integer$$

Step 4: **Translate.**

Restate as one sentence.

The sum of the three even integers is 84.

Translate into an equation.

$$n + n + 2 + n + 4 = 84$$

Step 5: **Solve the equation.**

	$n + n + 2 + n + 4 = 84$	
Combine like terms.	$3n + 6 = 84$	
Subtract 6 from each side.	$3n + 6 - 6 = 84 - 6$	
Divide each side by 3.	$3n = 78$	
	$n = 26$	$1^{st}\ integer$
	$rgb]1.0, 0.0, 0.0n + 2$	$2^{nd}\ integer$
	$rgb]1.0, 0.0, 0.026 + 2 = 28$	
	$rgb]1.0, 0.0, 0.0n + 4$	$3^{rd}\ integer$
	$rgb]1.0, 0.0, 0.026 + 4 = 30$	

Step 6: **Check.**

$$26 + 28 + 30 \stackrel{?}{=} 84$$

$$84 = 84 \checkmark$$

Step 7: **Answer the question.**

The three consecutive integers are 26, 28, and 30.

Try It

28) Find three consecutive even integers whose sum is 102.

Solution

32, 34, 36

29) Find three consecutive even integers whose sum is -24.

Solution

-10, -8, -6

Example 3.7.11

A married couple together earns $\$110,000$ a year. The wife earns $\$16,000$ less than twice what her husband earns. What does the husband earn?

Solution

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

How much does the husband earn?

Step 3: **Name.**

Choose a variable to represent the amount the husband earns.

Let $h =$ the amount the husband earns.

The wife earns $\$16,000$ less than twice that.

$2h - 16,000$ the amount the wife earns.

Together the husband and wife earn *$110,000*.

Step 4: **Translate.**

Restate the problem in one sentence with all the important information.

$$\underbrace{\text{The amount the husband earns}} + \underbrace{\text{the amount the wife earns}} \underbrace{\text{is}} \underbrace{\$110,000}$$

Translate into an equation.

$$h + 2h - 16,000 = 110,000$$

Step 5: **Solve the equation.**

	$h + 2h - 16,000 = 110,000$	
Combine like terms.	$3h - 16,000 = 110,000$	
Add 16,000 to both sides and simplify.	$3h = 126,000$	
Divide each side by 3.	$h = 42,000$	
	$\$42,000$	*the amount the husband earns*
	$2rgb]1.0, 0.0, 0.0h - 16$	*the amount the wife earns*
	$2(rgb]1.0, 0.0, 0.042,000) - 16,000$	
	$84,000 - 16,000 = 68,000$	

Step 6: **Check.**

If the wife earns $\$68,000$ and the husband earns $\$42,000$ is the total $\$110,000$? Yes!

Step 7: **Answer the question.**

The husband earns $\$42,000$ a year.

Try It

30) According to the National Automobile Dealers Association, the average cost of a car in 2014 was $\$28,500$. This was $\$1,500$ less than 6 times the cost in 1975. What was the average cost of a car in 1975?

Solution

$\$5,000$

31) U.S. Census data shows that the median price of new home in the United States in November

2014 was $\$280,900$. This was $\$10,700$ more than 14 times the price in November 1964. What was the median price of a new home in November 1964?

Solution

$\$19,300$

Translate and Solve Basic Percent Equations

We will solve percent equations using the methods we used to solve equations with fractions or decimals. Without the tools of algebra, the best method available to solve percent problems was by setting them up as proportions. Now as an algebra student, you can just translate English sentences into algebraic equations and then solve the equations.

We can use any letter you like as a variable, but it is a good idea to choose a letter that will remind us of what you are looking for. We must be sure to change the given percent to a decimal when we put it in the equation.

Example 3.7.12

Translate and solve: What number is 35% of 90?

Solution

$$\underbrace{\text{What number}}\ \underbrace{\text{is}}\ \underbrace{35\%}\ \underbrace{\text{of}}\ \underbrace{90?}$$

Step 1: **Translate into algebra. Let $n=$ the number.**

Remember "of" means multiply, "is" means equals.

$$n = 0.25 \cdot 90$$

Step 2: **Multiply.**

$$n = 31.5$$

31.5 is 35% of 90.

Try It

32) Translate and solve:

What number is 45% of 80?

Solution

36

33) Translate and solve:

What number is 55% of 60?

Solution

33

We must be very careful when we translate the words in the next example. The unknown quantity will not be isolated at first, like it was in Example 3.7.12. We will again use direct translation to write the equation.

Example 3.7.13

Translate and solve: 6.5% of what number is $\$1.17$?

Solution

$$\underbrace{6.5\%}\ \underbrace{\text{of}}\ \underbrace{\text{what number}}\ \underbrace{\text{is}}\ \underbrace{\$1.17?}$$

Step 1: **Translate. Let $n =$ the number.**

$$0.065 \cdot n = 1.17$$

Step 2: **Solve.**

$$0.065 \cdot n = 1.17$$

Multiply. $$0.065n = 1.17$$

Divide both sides by 0.065 and simplify. $$n = 18$$

6.5% of $\$18$ is $\$1.17$.

Try It

34) Translate and solve: 7.5% of what number is $\$1.95$?

Solution

$\$26$

35) Translate and solve: 8.5% of what number is $\$3.06$?

Solution

$\$36$

In the next example, we are looking for the percent.

Example 3.7.14

Translate and solve: 144 is what percent of 96?

Solution

$$\underbrace{144}\ \underbrace{\text{is}}\ \underbrace{\text{what percent}}\ \underbrace{\text{of}}\ \underbrace{96?}$$

Step 1: **Translate into algebra. Let $p =$ the percent.**

$$144 = p \cdot 96$$

Step 2: **Solve.**

$$\begin{array}{ll} & 144 = p \cdot 96 \\ \text{Multiply.} & 144 = 96p \\ \text{Divide both sides by 96 and simplify.} & 1.5 = p \\ \text{Convert to percent.} & 150\% = p \end{array}$$

144 is 150% of 96.

Note that we are asked to find percent, so we must have our final result in percent form.

Try It

36) Translate and solve:

110 is what percent of 88?

Solution

125%

37) Translate and solve:

126 is what percent of 72?

Solution

175%

Solve Applications of Percent

Many applications of percent—such as tips, sales tax, discounts, and **interest**—occur in our daily lives. To solve these applications we'll translate to a basic percent equation, just like those we solved in previous examples. Once we translate the sentence into a percent equation, we know how to solve it.

We will restate the problem solving strategy we used earlier for easy reference.

HOW TO

Use a Problem-Solving Strategy to Solve an Application.

1. Read the problem. Make sure all the words and ideas are understood.
2. Identify what we are looking for.
3. Name what we are looking for. Choose a variable to represent that quantity.
4. Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. Solve the equation using good algebra techniques.
6. Check the answer in the problem and make sure it makes sense.
7. Answer the question with a complete sentence.

Now that we have the strategy to refer to, and have practised solving basic percent equations, we are ready to solve percent applications. Be sure to ask yourself if your final answer makes sense—since many of the applications will involve everyday situations, you can rely on your own experience.

Example 3.7.15

Dezohn and his girlfriend enjoyed a nice dinner at a restaurant and his bill was $\$68.50$. He wants to leave an 18% tip. If the tip will be 18% of the total bill, how much tip should he leave?

Solution

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

the amount of tip should Dezohn leave

Step 3: **Name what we are looking for.**

Choose a variable to represent it.

Let t = amount of tip.

Step 4: **Translate into an equation.**

Write a sentence that gives the information to find it.

The tip is 18% of the total bill.

Translate the sentence into an equation.

$$\underbrace{\text{The tip}}\ \underbrace{\text{is}}\ \underbrace{18\%}\ \underbrace{\text{of}}\ \underbrace{\$68.50}$$

Step 5: **Solve the equation.**

$$t = 0.18 \cdot 68.50$$

$$\text{Multiply.} \qquad t = 12.33$$

Step 6: **Check. Does this make sense?**

Yes, 20% of $\$70$ is $\$14$.

Step 7: **Answer the question with a complete sentence.**

Dezohn should leave a tip of $\$12.33$.

Notice that we used t to represent the unknown tip.

Try It

38) Cierra and her sister enjoyed a dinner in a restaurant and the bill was $\$81.50$. If she wants to leave 18% of the total bill as her tip, how much should she leave?

Solution

$\$14.67$

39) Kimngoc had lunch at her favourite restaurant. She wants to leave 15% of the total bill as her tip. If her bill was $\$14.40$, how much will she leave for the tip?

Solution

$\$2.16$

Example 3.7.16

The label on Masao's breakfast cereal said that one serving of cereal provides 85 milligrams (mg) of potassium, which is 2% of the recommended daily amount. What is the total recommended daily amount of potassium?

Solution

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

the total amount of potassium that is recommended

Step 3: **Name what we are looking for**.

Choose a variable to represent it.

Let $a =$ total amount of potassium.

Step 4: **Translate.**

Write a sentence that gives the information to find it.

$$\underbrace{85mg}\ \underbrace{\text{is}}\ \underbrace{2\%}\ \underbrace{\text{of the}}\ \underbrace{\text{total amount}}$$

Translate into an equation.

$$85 = 0.02 \cdot a$$

Step 5: **Solve the equation.**

$$\begin{aligned} 85 &= 0.02 \cdot a \\ 4,250 &= a \end{aligned}$$

Step 6: **Check. Does this make sense?**

Yes, 2% is a small percent and 85 is a small part of $4,250$.

Step 7: **Answer the question with a complete sentence.**

The amount of potassium that is recommended is $4,250mg$.

Try It

40) One serving of wheat square cereal has seven grams of fibre, which is 28% of the recommended daily amount. What is the total recommended daily amount of fibre?

Solution

$25g$

41) One serving of rice cereal has $190mg$ of sodium, which is 8% of the recommended daily amount. What is the total recommended daily amount of sodium?

Solution

$2,375mg$

Example 3.7.17

Mitzi received some gourmet brownies as a gift. The wrapper said each brownie was 480 calories, and had 240 calories of fat. What percent of the total calories in each brownie comes from fat?

Solution

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

the percent of the total calories from fat

Step 3: **Name what we are looking for.**

Choose a variable to represent it.

Let $p=$ percent of fat.

Step 4: **Translate. Write a sentence that gives the information to find it.**

$$\underbrace{\text{What percent}}\ \underbrace{\text{of}}\ \underbrace{480}\ \underbrace{\text{is}}\ \underbrace{240?}$$

Translate into an equation.

$$p\cdot 480=240$$

Step 5: **Solve the equation.**

$$\begin{array}{ll} & 480p=240 \\ \text{Divide by 480} & p=0.5 \\ \text{Put in percent form} & p=50\% \end{array}$$

Step 6: **Check. Does this make sense?**

Yes, 240 is half of 480, so 50% makes sense.

Step 7: **Answer the question with a complete sentence.**

Of the total calories in each brownie, 50% is fat.

Try It

42) Solve. Round to the nearest whole percent.

Veronica is planning to make muffins from a mix. The package says each muffin will be 230 calories and 60 calories will be from fat. What percent of the total calories is from fat?

Solution

26%

43) Solve. Round to the nearest whole percent.

The mix Ricardo plans to use to make brownies says that each brownie will be 190 calories, and 76 calories are from fat. What percent of the total calories are from fat?

Solution

40%

Find Percent Increase and Percent Decrease

People in the media often talk about how much an amount has increased or decreased over a certain period of time. They usually express this increase or decrease as a percent.

To find the percent increase, first we find the amount of increase, the difference of the new amount and the original amount. Then we find what percent the amount of increase is of the original amount.

HOW TO

Find the Percent Increase.

1. Find the amount of increase.
   new amount − original amount = increase
2. Find the percent increase.
   The increase is what percent of the original amount?

Example 3.7.18

In 2011, the California governor proposed raising community college fees from $\$26$ a unit to $\$36$ a unit. Find the percent increase. (Round to the nearest tenth of a percent.)

Solution

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

the percent increase

Step 3: **Name what we are looking for.**

Choose a variable to represent it.

Let $p =$ the percent.

Step 4: **Translate**.

Write a sentence that gives the information to find it.

new amount − original amount = increase

First find the amount of increase.

$$36 - 26 = 10$$

Find the percent.

Increase is what percent of the original amount?

$$\underbrace{10}\ \underbrace{\text{is}}\ \underbrace{\text{what percent}}\ \underbrace{\text{of}}\ \underbrace{26?}$$

Translate into an equation.

$$10 = p \cdot 26$$

Step 5: **Solve the equation.**

$$\begin{array}{lr} & 10 = 26p \\ \text{Divide by } 26. & 0.384 = p \\ \text{Change to percent form; round to the nearest tenth.} & 38.4\% = p \end{array}$$

Step 6: **Check. Does this make sense?**

Yes, 38.4% is close to $\frac{1}{3}$ and 10 is close to $\frac{1}{3}$ of 26.

Step 7: **Answer the question with a complete sentence.**

The new fees represent a 38.4% increase over the old fees.

Notice that we rounded the division to the nearest thousandth in order to round the percent to the nearest tenth.

Try It

44) Find the percent increase. (Round to the nearest tenth of a percent.)

In 2011, the IRS increased the deductible mileage cost to 55.5 cents from 51 cents.

Solution

8.8%

45) Find the percent increase.

In 1995, the standard bus fare in Chicago was $\$1.50$. In 2008, the standard bus fare was $\$2.25$.

Solution

50%

Finding the percent decrease is very similar to finding the percent increase, but now the amount of decrease is the difference of the original amount and the new amount. Then we find what percent the amount of decrease is of the original amount.

HOW TO

Find the Percent Decrease.

1. Find the amount of decrease.
   original amount − new amount = decrease
2. Find the percent decrease.
   Decrease is what percent of the original amount?

Example 3.7.19

The average price of a gallon of gas in one city in June 2014 was $\$3.71$. The average price in that city in July was $\$3.64$. Find the percent decrease.

Solution

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

the percent decrease

Step 3: **Name what we are looking for.**

Choose a variable to represent that quantity.

Let p = the percent decrease.

Step 4: **Translate.**

Write a sentence that gives the information to find it.

First find the amount of decrease.

$$3.71 - 3.64 = 0.07$$

Find the percent.

Decrease is what percent of the original amount?

$$\underbrace{0.07}\ \underbrace{\text{is}}\ \underbrace{\text{what percent}}\ \underbrace{\text{of}}\ \underbrace{3.71?}$$

Translate into an equation.

$$0.07 = p \cdot 3.71$$

Step 5: **Solve the equation.**

$$\begin{array}{ll} & 0.07 = 3.71p \\ \text{Divide by } 3.71. & 0.019 = p \\ \text{Change to percent form; round to the nearest tenth.} & 1.9\% = p \end{array}$$

Step 6: **Check. Does this make sense?**

Yes, if the original price was $\$4$, a 2% decrease would be 8 cents.

Step 7: **Answer the question with a complete sentence.**

The price of gas decreased 1.9%.

Try It

46) Find the percent decrease. (Round to the nearest tenth of a percent.)

The population of North Dakota was about $672,000$ in 2010. The population is projected to be about $630,000$ in 2020.

Solution

6.3%

47) Find the percent decrease.

Last year, Sheila's salary was $\$42,000$. Because of furlough days, this year, her salary was $\$37,800$.

Solution

10%

Solve Simple Interest Applications

Do you know that banks pay you to keep your money? The money a customer puts in the bank is called the principal, P, and the money the bank pays the customer is called the **interest**. The interest is computed as a certain percent of the principal; called the **rate of interest**, r. We usually express rate of interest as a percent per year, and we calculate it by using the decimal equivalent of the percent. The variable t, (for *time*) represents the number of years the money is in the account.

To find the interest we use the **simple interest** formula, $I = Prt$.

Simple Interest

If an amount of money, P, called the principal, is invested for a period of t years at an annual interest rate r, the amount of interest, I, earned is

$I = Prt$	where	$I = interest$ $P = principal$ $r = rate$ $t = time$

Interest earned according to this formula is called simple interest.

Interest may also be calculated another way, called compound interest. This type of interest will be covered in later math classes.

The formula we use to calculate simple interest is $I = Prt$. To use the formula, we substitute in the values the problem gives us for the variables, and then solve for the unknown variable. It may be helpful to organize the information in a chart.

Example 3.7.20

Nathaly deposited $\$12,500$ in her bank account where it will earn 4% interest. How much interest will Nathaly earn in 5 years?

$$\begin{aligned} &I = ? \\ &P = \$12,500 \\ &r = 4\% \\ &t = 5\ years \end{aligned}$$

Solution

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

the amount of interest earned

Step 3: **Name what we are looking for.**

Choose a variable to represent that quantity.

Let $I =$ the amount of interest.

Step 4: **Translate into an equation.**

$$\begin{array}{ll} \text{Write the formula} & I = Prt \\ \text{Substitute in the given information} & I = (12,500)(0.04)(5) \end{array}$$

Step 5: **Solve the equation.**

$$I = 2,500$$

Step 6: **Check**: **Does this make sense?**

Is $\$2,500$ a reasonable interest on $\$12,500$? Yes.

Step 7: **Answer the question with a complete sentence.**

The interest is $\$2,500$.

Try It

48) Areli invested a principal of $\$950$ in her bank account with interest rate 3%. How much interest did she earn in 5 years?

Solution

$\$142.50$

49) Susana invested a principal of $\$36,000$ in her bank account with interest rate 6.5%. How much interest did she earn in 3 years?

Solution

$\$7,020$

There may be times when we know the amount of interest earned on a given principal over a certain length of time, but we don't know the rate. To find the rate, we use the simple interest formula, substitute in the given values for the principal and time, and then solve for the rate.

Example 3.7.21

Loren loaned his brother $\$3,000$ to help him buy a car. In 4 years his brother paid him back the $\$3,000$ plus $\$660$ in interest. What was the rate of interest?

$$I = \$660$$
$$P = \$3,000$$
$$r =$$
$$t = 4 \; years$$

Solution

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

the rate of interest

Step 3: **Name what we are looking for.**

Choose a variable to represent that quantity.

Let $r =$ the rate of interest.

Step 4: **Translate into an equation.**

$$\begin{array}{ll} \text{Write the formula} & I = Prt \\ \text{Substitute in the given information} & 660 = (3000)r(4) \end{array}$$

Step 5: **Solve the equation.**

$$\begin{array}{ll} & 660 = (3,000)r(4) \\ \text{Multiply.} & 660 = (12,000)r \\ \text{Divide.} & 0.055 = r \\ \text{Change to percent form.} & 5.5\% = r \end{array}$$

Step 6: **Check**: **Does this make sense?**

$$I = Prt$$
$$660 \stackrel{?}{=} (3000)(0.55)(4)$$
$$660 = 660\checkmark$$

Step 7: **Answer the question with a complete sentence.**

The rate of interest was 5.5%.

Notice that in this example, Loren's brother paid Loren interest, just like a bank would have paid interest if Loren invested his money there.

Try It

50) Jim loaned his sister $\$5,000$ to help her buy a house. In 3 years, she paid him the $\$5,000$, plus $\$900$ interest. What was the rate of interest?

Solution

6%

51) Hang borrowed $\$7,500$ from her parents to pay her tuition. In 5 years, she paid them $\$1,500$ interest in addition to the $\$7,500$ she borrowed. What was the rate of interest?

Solution

4%

Example 3.7.22

Eduardo noticed that his new car loan papers stated that with a 7.5% interest rate, he would pay $\$6,596.25$ in interest over 5 years. How much did he borrow to pay for his car?

Solution

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

the amount borrowed (the principal)

Step 3**: Name what we are looking for.**

Choose a variable to represent that quantity.

Let P = principal borrowed.

Step 4: **Translate into an equation.**

$$\begin{array}{lr} \text{Write the formula} & I = Prt \\ \text{Substitute in the given information} & 6,596.25 = P(0.075)(5) \end{array}$$

Step 5: **Solve the equation.**

$$\begin{array}{lr} & 6,596 = P(0.075)(5) \\ \text{Multiply.} & 6,596.25 = 0.375P \\ \text{Divide.} & 17,590 = P \end{array}$$

Step 6: **Check**: **Does this make sense?**

$$I = Prt$$
$$6.596.25 \stackrel{?}{=} (14,590)(0.075)(5)$$
$$6,596.25 = 6,596.25\checkmark$$

Step 7: **Answer the question with a complete sentence.**

The principal was $\$17,590$.

Try It

52) Sean's new car loan statement said he would pay $\$4,866.25$ in interest from an interest rate of 8.5% over 5 years. How much did he borrow to buy his new car?

Solution

$\$11,450$

53) In 5 years, Gloria's bank account earned $\$2,400$ interest at 5%. How much had she deposited in the account?

Solution

$\$9,600$

Solve Applications with Discount or Mark-up

Applications of discount are very common in retail settings. When you buy an item on sale, the original price has been discounted by some dollar amount. The **discount rate**, usually given as a percent, is used to determine the amount of the discount. To determine the **amount of discount**, we multiply the discount rate by the original price.

We summarize the discount model in the box below.

Discount

amount of discount = discount rate × original price

sale price = original price − amount of discount

Keep in mind that the sale price should always be less than the original price.

Example 3.7.23

Elise bought a dress that was discounted 35% off of the original price of $\$140$.

What was a. the amount of discount and b. the sale price of the dress?

Solution

a. Original Price = $\$140$ Discount rate = 35% Discount = ?

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

the amount of discount

Step 3: **Name what we are looking for.**

Choose a variable to represent that quantity.

Let $d =$ the amount of discount.

***Step 4:* Translate into an equation.**

Write a sentence that gives the information to find it.

The discount is 35% of $140.

Translate into an equation.

$$d = 0.35(140)$$

***Step 5:* Solve the equation.**

$$d = 0.35(140)$$
$$d = 49$$

***Step 6*: Check: Does this make sense?**

Is a $\$49$ discount reasonable for a $\$140$ dress? Yes.

***Step 7:* Write a complete sentence to answer the question.**

The amount of discount was $\$49$.

---

b. Read the problem again.

***Step 1:* Identify what we are looking for.**

the sale price of the dress

***Step 2:* Name what we are looking for.**

Choose a variable to represent that quantity.

Let s = the sale price.

***Step 3:* Translate into an equation.**

Write a sentence that gives the information to find it.

$$\underbrace{\text{The sale price}}\ \underbrace{\text{is}}\ \underbrace{\text{the \$140}}\ \underbrace{\text{minus}}\ \underbrace{\text{the \$49 discount}}$$

Translate into an equation.

$$s = 140 - 49$$

***Step 4:* Solve the equation.**

$$s = 140 - 49$$
$$s = 91$$

Step 5: **Check. Does this make sense?**

Is the sale price less than the original price?

Yes, $\$91$ is less than $\$140$.

Step 6: **Answer the question with a complete sentence.**

The sale price of the dress was $\$91$.

Try It

54) Find
a. the amount of discount and
b. the sale price:

Sergio bought a belt that was discounted 40% from an original price of $\$29$.

Solution

a. $\$11.60$
b. $\$17.40$

55) Find
a. the amount of discount and
b. the sale price:

Oscar bought a barbecue that was discounted 65% from an original price of $\$395$.

Solution

a. $\$256.75$
b. $\$138.25$

There may be times when we know the original price and the sale price, and we want to know the discount rate. To find the discount rate, first we will find the amount of discount and then use it to compute the rate as a percent of the original price. Example 3.7.24 will show this case.

Example 3.7.24

Jeannette bought a swimsuit at a sale price of $\$13.95$. The original price of the swimsuit was $\$31$. Find the:

a. amount of discount and
b. discount rate.

Solution

a.

Original price = $\$31$
Discount = ?
Sale price = $\$13.95$

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

the amount of discount

Step 3: **Name what we are looking for.**

Choose a variable to represent that quantity.

Let $d =$ the amount of discount.

Step 4: **Translate into an equation.**

Write a sentence that gives the information to find it.

The discount is the difference between the original price and the sale price.

Translate into an equation.

$$d = 31 - 13.95$$

Step 5: **Solve the equation.**

$$d = 31 - 13.95$$
$$d = 17.05$$

Step 6: **Check**: **Does this make sense?**

Is 17.05 less than 31? Yes.

Step 7: **Answer the question with a complete sentence.**

The amount of discount was $\$17.05$.

---

b. Read the problem again.
1. When we translate this into an equation, we obtain 17.05 equals r times 31. We are told to solve the equation 17.05 equals $31r$. We divide by 31 to obtain 0.55 equals r. We put this in percent form to obtain r equals 55. We are told to check: does this make sense? Is 7.05 equal to 55 of > 1? Below this, we have 17.05 equals with a question mark over it 0.55 times 31. Below this, we have 17.05 equals 17.05 with a checkmark next to it. Then we are told to answer the question with a complete sentence: The rate of discount was 55%.

Step 1: **Identify what we are looking for.**

the discount rate

Step 2: **Name what we are looking for.**

Choose a variable to represent it.

Let $r =$ the discount rate.

Step 3: **Translate into an equation.**

Write a sentence that gives the information to find it.

$$\underbrace{\text{The discount of \$17.05}}\ \underbrace{\text{is}}\ \underbrace{\text{what percent}}\ \underbrace{\text{of}}\ \underbrace{\text{\$31?}}$$

Translate into an equation.

$$17.05 = r \cdot 31$$

Step 4: **Solve the equation.**

$$\begin{array}{rr} & 17.05 = 31r \\ \text{Divide both sides by 31.} & 0.55 = r \\ \text{Change to percent form} & r = 55\% \end{array}$$

Step 5: **Check. Does this make sense?**

Is $\$17.05$ equal to 55% of $\$31$?

$$17.05 = 0.55 \times (31)$$
$$17.05 = 17.05 \checkmark$$

Step 6: **Answer the question with a complete sentence.**

The rate of discount was 55%.

Try It

56) Find
a. the amount of discount and
b. the discount rate.

Lena bought a kitchen table at the sale price of $\$375.20$. The original price of the table was $\$560$.

Solution
a. $\$184.80$
b. 33%

57) Find
a. the amount of discount and
b. the discount rate.

Nick bought a multi-room air conditioner at a sale price of $\$340$. The original price of the air conditioner was $\$400$.

Solution
a. $\$60$
b. 15%

Applications of **mark-up** are very common in retail settings. The price a retailer pays for an item is called the original cost. The retailer then adds a mark-up to the original cost to get the list price, the price he sells the item for. The mark-up is usually calculated as a percent of the original cost. To determine the amount of mark-up, multiply the mark-up rate by the original cost.

We summarize the mark-up model in the box below.

Mark-Up

amount of mark-up = mark-up rate × original cost

list price = original cost + amount of mark up

Keep in mind that the **list price** should always be more than the **original cost**.

Example 3.7.25

Adam's art gallery bought a photograph at original cost $\$250$. Adam marked the price up 40%. Find the:

a. amount of mark-up and

b. the list price of the photograph.

Solution

a.

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

the amount of mark-up

Step 3: **Name what we are looking for.**

Choose a variable to represent it.

Let $m =$ the amount of markup.

Step 4: **Translate into an equation.**

Write a sentence that gives the information to find it.

$$\underbrace{\text{The mark-up}}\ \underbrace{\text{is}}\ \underbrace{40\%}\ \underbrace{\text{of the \$250 original cost}}$$

Translate into an equation.

$$m = 0.40 \cdot 250$$

***Step 5:* Solve the equation.**

$$m = 0.40 \cdot 250)$$
$$m = 100$$

***Step 6:* Check. Does this make sense?**

Yes, 40% is less than one-half and 100 is less than half of 250.

***Step 7:* Answer the question with a complete sentence.**

The mark-up on the photograph was $\$100$.

---

b.

***Step 1:* Read the problem again.**

***Step 2:* Identify what we are looking for.**

the list price

***Step 3:* Name what we are looking for.**

Choose a variable to represent it.

Let $p =$ the list price.

***Step 4:* Translate into an equation.**

Write a sentence that gives the information to find it.

$$\underbrace{\text{The list price}} \ \underbrace{\text{is}} \ \underbrace{\text{original cost}} \ \underbrace{\text{plus}} \ \underbrace{\text{the mark-up}}$$

Translate into an equation.

$$p = 250 + 100$$

***Step 5:* Solve the equation.**

$$p = 250 + 100$$
$$p = 350$$

***Step 6:* Check. Does this make sense?**

Is the list price more than the net price?
Is $\$350$ more than $\$250$? Yes

***Step 7:* Answer the question with a complete sentence.**

The list price of the photograph was $\$350$.

Try It

58) Find
a. the amount of mark-up and
b. the list price.

Jim's music store bought a guitar at the original cost $\$1,200$. Jim marked the price up 50%.

Solution

a. $\$600$
b. $\$1,800$

59) Find
a. the amount of mark-up and
b. the list price.

The Auto Resale Store-bought Pablo's Toyota for $\$8,500$. They marked the price up 35%.

Solution

a. $\$2,975$
b. $\$11,475$

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it

online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=1269#h5p-21

Glossary

amount of discount
: The amount of discount is the amount resulting when a discount rate is multiplied by the original price of an item.

discount rate
: The discount rate is the percent used to determine the amount of a discount, common in retail settings.

interest
: Interest is the money that a bank pays its customers for keeping their money in the bank.

list price
: The list price is the price a retailer sells an item for.

mark-up
: A mark-up is a percentage of the original cost used to increase the price of an item.

original cost
: The original cost in a retail setting, is the price that a retailer pays for an item.

principal
: The principal is the original amount of money invested or borrowed for a period of time at a specific interest rate.

rate of interest
: The rate of interest is a percent of the principal, usually expressed as a percent per year.

simple interest
: Simple interest is the interest earned according to the formula $I = Prt$.

3.8 SOLVE MIXTURE AND UNIFORM MOTION APPLICATIONS

Learning Objectives

By the end of this section, you will be able to:

- Solve coin word problems
- Solve ticket and stamp word problems
- Solve mixture word problems
- Use the mixture model to solve investment problems using simple interest
- Solve uniform motion applications

Try It

Before you get started, take this readiness quiz:

1) Multiply: $14(0.25)$.
2) Solve: $(0.25x)+0.10(x+4)=2.5$
3) The number of dimes is three more than the number of quarters. Let q represent the number of quarters. Write an expression for the number of dimes.
4) Find the distance traveled by a car going 70 miles per hour for 3 hours.

5) Solve $x+1.2(x-10)=98$.

6) Convert 90 minutes to hours.

Solve Coin Word Problems

In **mixture problems**, we will have two or more items with different values to combine together. The mixture model is used by grocers and bartenders to make sure they set fair prices for the products they sell. Many other professionals, like chemists, investment bankers, and landscapers also use the mixture model. Doing the Manipulative Mathematics activity *Coin Lab* will help you develop a better understanding of mixture word problems.

We will start by looking at an application everyone is familiar with—money!

Imagine that we take a handful of coins from a pocket or purse and place them on a desk. How would we determine the value of that pile of coins? If we can form a step-by-step plan for finding the total value of the coins, it will help us as we begin solving coin word problems.

So what would we do? To get some order to the mess of coins, we could separate the coins into piles according to their value. Quarters would go with quarters, dimes with dimes, nickels with nickels, and so on. To get the total value of all the coins, we would add the total value of each pile.

Figure 3.8.1.

How would we determine the value of each pile? Think about the dime pile—how much is it worth? If we count the number of dimes, we'll know how many we have—the *number* of dimes.

But this does not tell us the *value* of all the dimes. Say we counted 17 dimes, how much are they worth? Each dime is worth $0.10—that is the *value* of one dime. To find the total value of the pile of 17 dimes, multiply 17 by $0.10 to get $1.70. This is the total value of all 17 dimes. This method leads to the following model.

HOW TO

Total Value of Coins

For the same type of coin, the total value of a number of coins is found by using the model

$$number \times value = total$$

Where

number is the number of coins

value is the value of each coin

total value is the total value of all the coins

The number of dimes times the value of each dime equals the total value of the dimes.

$$Number \times Value = Total\ Value$$

We could continue this process for each type of coin, and then we would know the total value of each type of coin. To get the total value of *all* the coins, add the total value of each type of coin.

Let's look at a specific case. Suppose there are 14 quarters, 17 dimes, 21 nickels, and 39 pennies.

Type	Number ·	Value($)	= Total Value ($)
Quarters	14	0.25	3.50
Dimes	17	0.10	1.70
Nickels	21	0.05	1.05
Pennies	39	0.01	0.39
			6.64

The total value of all the coins is $\$6.64$.

Notice how the chart helps organize all the information! Let's see how we use this method to solve a coin word problem.

Example 3.8.1

Adalberto has $\$2.25$ in dimes and nickels in his pocket. He has nine more nickels than dimes. How many of each type of coin does he have?

Solution

***Step 1:* Read the problem. Make sure all the words and ideas are understood.**

- **Determine** the types of coins involved.
  Think about the strategy we used to find the value of the handful of coins. The first thing we need is to notice what types of coins are involved. Adalberto has dimes and nickels.
- **Create a table** to organize the information. See chart below.
  - Label the columns "type," "number," "value," "total value."
  - List the types of coins.
  - Write in the value of each type of coin.
  - Write in the total value of all the coins.

Type	Number ·	Value ($)	= Total Value ($)
Dimes		0.10	
Nickels		0.05	
			2.25

We can work this problem all in cents or in dollars. Here we will do it in dollars and put in the dollar sign ($) in the table as a reminder.
The value of a dime is $\$0.10$ and the value of a nickel is $\$0.05$. The total value of all the coins is $\$2.25$. The table below shows this information.

***Step 2:* Identify what we are looking for.**

- We are asked to find the number of dimes and nickels Adalberto has.

***Step 3:* Name what we are looking for. Choose a variable to represent that quantity.**

- Use variable expressions to represent the number of each type of coin and write them in the table.
- Multiply the number times the value to get the total value of each type of coin.

Next we counted the number of each type of coin. In this problem we cannot count each type of coin—that is what you are looking for—but we have a clue. There are nine more nickels than dimes. The number of nickels is nine more than the number of dimes.

Let $d =$ number of dimes.
$d + 9 =$ number of nickles

Fill in the "number" column in the table to help get everything organized.

Type	Number ·	Value ($)	= Total Value ($)
Dimes	d	0.10	
Nickels	$d+9$	0.05	
			2.25

Now we have all the information we need from the problem!

We multiply the number times the value to get the total value of each type of coin. While we do not know the actual number, we do have an expression to represent it.

And so now multiply $number \times value = total\ \ value$ See how this is done in the table below.

Type	Number ·	Value ($)	= Total Value ($)
Dimes	d	0.10	$0.10d$
Nickels	$d+9$	0.05	$0.05(d+9)$
			2.25

Notice that we made the heading of the table show the model.

Step 4: **Translate into an equation. It may be helpful to restate the problem in one sentence. Translate the English sentence into an algebraic equation.**

Write the equation by adding the total values of all the types of coins.

	Value of dimes	+	Value of nickels	=	Total value of coins
Translate to an equation	$0.10d$	+	$0.50(d+9)$	=	2.25

Step 5: **Solve the equation using good algebra techniques.**

Now solve this equation	$0.10d + 0.05(d+9) = 2.25$
Distribute	$0.10d + 0.05d + 0.45 = 2.25$
Combine like terms	$0.15d + 0.45 = 2.25$
Subtract 0.45 from each side	$0.15d = 2.25$
Divide	$d = 12$
So there are 12 dimes	
The number of nickels is ({\color[rgb]{1.0, 0.0, 0.0}d}+9)	$rgb]1.0, 0.0, 0.0d + 9$
	$rgb]1.0, 0.0, 0.012 + 9 = 21$

Step 6: **Check the answer in the problem and make sure it makes sense.**

Does this check?

12 dimes	$12(0.10) = 1.20$
21 nickels	$21(0.05) = 2.25$√

Step 7: **Answer the question with a complete sentence.**

- Adalberto has twelve dimes and twenty-one nickels.

If this were a homework exercise, our work might look like the following.

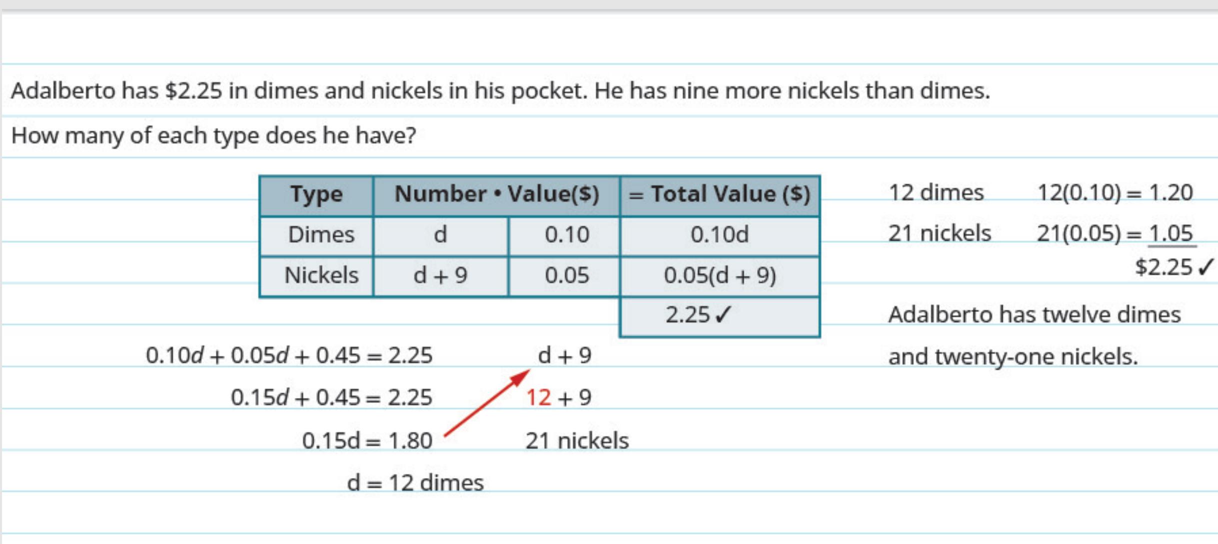


Figure 3.8.2

Try It

7) Michaela has $\$2.05$ in dimes and nickels in her change purse. She has seven more dimes than nickels. How many coins of each type does she have?

Solution

9 nickels, 16 dimes

8) Liliana has $\$2.10$ in nickels and quarters in her backpack. She has 12 more nickels than quarters. How many coins of each type does she have?

Solution

17 nickels, 5 quarters

HOW TO

Solve Coin Word Problems.

Step 1: **Read the problem. Make sure all the words and ideas are understood.**

- Determine the types of coins involved.
- Create a table to organize the information.
- Label the columns "type," "number," "value," "total value."
- List the types of coins.
- Write in the value of each type of coin.
- Write in the total value of all the coins.

Type	Number	• Value($)	= Total Value($)

Figure 3.8.3

Step 2: **Identify what we are looking for.**

Step 3: **Name what we are looking for. Choose a variable to represent that quantity.**

Use variable expressions to represent the number of each type of coin and write them in the table.

Multiply the number times the value to get the total value of each type of coin.

Step 4: **Translate into an equation.**

It may be helpful to restate the problem in one sentence with all the important information. Then, translate the sentence into an equation.

Write the equation by adding the total values of all the types of coins.

Step 5: **Solve the equation using good algebra techniques.**

Step 6: **Check the answer in the problem and make sure it makes sense.**

Step 7: **Answer the question with a complete sentence.**

Example 3.8.2

Maria has $\$2.43$ in quarters and pennies in her wallet. She has twice as many pennies as quarters. How many coins of each type does she have?

Solution

Step 1: **Read the problem.**

Determine the types of coins involved.

We know that Maria has quarters and pennies.
Create a table to organize the information.

- Label the columns "type," "number," "value," "total value."
- List the types of coins.
- Write in the value of each type of coin.
- Write in the total value of all the coins.

Type	Number ·	Value ($)	= Total Value ($)
Quarters		0.25	
Pennies		0.01	
			2.43

Step 2: **Identify what you are looking for.**

- We are looking for the number of quarters and pennies.

Step 3: **Name. Represent the number of quarters and pennies using variables.**

- We know Maria has twice as many pennies as quarters. The number of pennies is defined in terms of quarters.
- Let q represent the number of quarters.
- Then the number of pennies is $2q$.

Type	Number ·	Value ($)	= Total Value ($)
Quarters	q	0.25	
Pennies	$2q$	0.01	
			2.43

Multiply the 'number' and the 'value' to get the 'total value' of each type of coin.

Type	Number ·	Value ($)	= Total Value ($)
Quarters	q	0.25	$0.25q$
Pennies	$2q$	0.01	$0.01(2q)$
			2.43

Step 4: **Translate. Write the equation by adding the 'total value' of all the types of coins.**

Step 5: **Solve the equation.**

$$\begin{array}{ll} & 0.25q + 0.01(2q) = 2.43 \\ \text{Multiply} & 0.25q + 0.02q = 2.43 \\ \text{Combine like terms} & 0.27q = 2.43 \\ \text{Divide by 0.27} & d = 9 \\ \text{The number of pennies is 2q} & 2rgb]1.0, 0.0, 0.0q \\ & 2 \cdot rgb]1.0, 0.0, 0.09 = 18 \text{ pennies} \end{array}$$

Step 6: **Check the answer in the problem.**

Maria has 9 quarters and 18 pennies. Does this make $\$2.43$?

$$\begin{array}{ll} \text{9 quarters} & 9(0.25) = 2.25 \\ \text{18 pennies} & 18(0.01) = 0.18 \\ \text{Total} & \$2.43\checkmark \end{array}$$

Step 7: **Answer the question.**

Maria has nine quarters and eighteen pennies.

Try It

9) Sumanta has $\$4.20$ in nickels and dimes in her piggy bank. She has twice as many nickels as dimes. How many coins of each type does she have?

Solution

42 nickels, 21 dimes.

10) Alison has three times as many dimes as quarters in her purse. She has $\$9.35$ altogether. How many coins of each type does she have?

Solution
51 dimes, 17 quarters.

In the next example, we'll show only the completed table—remember the steps we take to fill in the table.

Example 3.8.3

Danny has $\$2.14$ worth of pennies and nickels in his piggy bank. The number of nickels is two more than ten times the number of pennies. How many nickels and how many pennies does Danny have?

Solution
Step 1: **Read the problem.**

Determine the types of coins involved.

pennies and nickels

Create a table. Write in the value of each type of coin.

Pennies are worth $\$0.01$.
Nickels are worth $\$0.05$.

Step 2: **Identify what we are looking for.**

the number of pennies and nickels

Step 3: **Name. Represent the number of each type of coin using variables.**

The number of nickels is defined in terms of the number of pennies, so start with pennies.

Let p = number of pennies

The number of nickels is two more than ten times the number of pennies.

Let $10p + 2$ = number of nickels

Multiply the number and the value to get the total value of each type of coin.

Type	Number ·	Value ($)	= Total Value ($)
Pennies	p	0.01	$0.01p$
Nickels	$10p+2$	0.05	$0.05(10p+2)$
			2.14

Step 4: **Translate. Write the equation by adding the total value of all the types of coins.**

$$0.01p \;+\; 0.05(10p \;+\; 2) \;=\; 2.14$$

Step 5: **Solve the equation.**

$$\begin{aligned} 0.01p \;+\; 0.50p \;+\; 0.10 &= 2.14 \\ 0.51p \;+\; 0.10 &= 2.14 \\ 0.51p &= 2.04 \\ p &= 4 \text{ pennies} \end{aligned}$$

How many nickels?

$$\begin{aligned} 10p \;+\; 2 \\ 10(rgb]1.0,0.0,0.04) \;+\; 2 \\ 42 \text{ nickles} \end{aligned}$$

Step 7: **Answer the question.**

Danny has four pennies and 42 nickels.

Try It

11) Jesse has $\$6.55$ worth of quarters and nickels in his pocket. The number of nickels is five more than two times the number of quarters. How many nickels and how many quarters does Jesse have?

Solution

41 nickels, 18 quarters

12) Elane has $\$7.00$ total in dimes and nickels in her coin jar. The number of dimes that Elane has is seven less than three times the number of nickels. How many of each coin does Elane have?

Solution

22 nickels, 59 dimes

Solve Ticket and Stamp Word Problems

Problems involving tickets or stamps are very much like coin problems. Each type of ticket and stamp has a value, just like each type of coin does. So to solve these problems, we will follow the same steps we used to solve coin problems.

Example 3.8.4

At a school concert, the total value of tickets sold was $\$1,506$. Student tickets sold for $\$6$ each and adult tickets sold for $\$9$ each. The number of adult tickets sold was five less than three times the number of student tickets sold. How many student tickets and how many adult tickets were sold?

Solution

***Step 1:* Read the problem.**

- Determine the types of tickets involved. There are student tickets and adult tickets.
- Create a table to organize the information.

Type	Number ·	Value ($)	= Total Value ($)
Student		6	
Adult		9	
			1506

Step 2: **Identify what we are looking for.**

- We are looking for the number of student and adult tickets.

Step 3: **Name. Represent the number of each type of ticket using variables.**

We know the number of adult tickets sold was five less than three times the number of student tickets sold.

- Let s be the number of student tickets.
- Then $3s-5$ is the number of adult tickets

Multiply the number times the value to get the total value of each type of ticket.

Type	Number ·	Value ($)	= Total Value ($)
Student	s	6	$6s$
Adult	$3s-5$	9	$9(3s-5)$
			1506

Step 4: **Translate. Write the equation by adding the total values of each type of ticket.**

$$6s+9(3s-5)=1506$$

Step 5: **Solve the equation.**

\begin{eqnarray*} \mbox{6s+27s-45} & = & 1506 \\ \\ \ 33s-45 & = & 1506 \\ \\ \ s & = & 47 \\ \\ \ & = & 3s-5 \\ \\ \ & = & 3(47)-5 \end{eqnarray*}

136 adult tickets.

Step 6: **Check the answer.**

There were 47 student tickets at $\$6$ each and 136 adult tickets at $\$9$ each. Is the total value $\$1,506$? We find the total value of each type of ticket by multiplying the number of tickets times its value then add to get the total value of all the tickets sold.

$$\begin{array}{rcrcl} 47 & \times & 6 & = & 282 \\ 136 & \times & 9 & = & 1,224 \\ & & & = & 1,506 \checkmark \end{array}$$

Step 7: Answer the question. They sold 47 student tickets and 136 adult tickets.

Try It

13) The first day of a water polo tournament the total value of tickets sold was $\$17,610$. One-day passes sold for $\$20$ and tournament passes sold for $\$30$. The number of tournament passes sold was 37 more than the number of day passes sold. How many day passes and how many tournament passes were sold?

Solution
330 day passes, 367 tournament passes

14) At the movie theatre, the total value of tickets sold was $\$2,612.50$. Adult tickets sold for $\$10$ each and senior/child tickets sold for $\$7.50$ each. The number of senior/child tickets sold was 25 less than twice the number of adult tickets sold. How many senior/child tickets and how many adult tickets were sold?

Solution
112 adult tickets, 199 senior/child tickets

We have learned how to find the total number of tickets when the number of one type of ticket is based on the number of the other type. Next, we'll look at an example where we know the total number of tickets and have to figure out how the two types of tickets relate.

Suppose Bianca sold a total of 100 tickets. Each ticket was either an adult ticket or a child ticket. If she sold 20 child tickets, how many adult tickets did she sell?

- *Did you say '80'? How did you figure that out? Did you subtract 20 from 100?*

If she sold 45 child tickets, how many adult tickets did she sell?

- *Did you say '55'? How did you find it? By subtracting 45 from 100?*

What if she sold 75 child tickets? How many adult tickets did she sell?

- *The number of adult tickets must be $100 - 75$. She sold 25 adult tickets.*

Now, suppose Bianca sold x child tickets. Then how many adult tickets did she sell? To find out, we would follow the same logic we used above. In each case, we subtracted the number of child tickets from 100 to get the number of adult tickets. We now do the same with x.

We have summarized this below.

Child tickets	Adult tickets
20	80
45	55
75	25
$rgb]1.0, 0.0, 0.0x$	$rgb]1.0, 0.0, 0.0100 - x$

We can apply these techniques to other examples

Example 3.8.5

Galen sold 810 tickets for his church's carnival for a total of $\$2,820$. Children's tickets cost $\$3$ each and adult tickets cost $\$5$ each. How many children's tickets and how many adult tickets did he sell?

Solution

***Step 1:* Read the problem.**

- Determine the types of tickets involved. There are children tickets and adult tickets.
- Create a table to organize the information.

Type	Number ·	Value ($)	= Total Value ($)
Children		3	
Adult		5	
			2820

***Step 2:* Identify what we are looking for.**

- We are looking for the number of children and adult tickets.

Step 3: **Name. Represent the number of each type of ticket using variables.**

- We know the total number of tickets sold was 810. This means the number of children's tickets plus the number of adult tickets must add up to 810.
- Let c be the number of children tickets.
- Then $810 - c$ is the number of adult tickets.
- Multiply the number times the value to get the total value of each type of ticket.

Type	Number ·	Value ($)	= Total Value ($)
Children	c	3	$3c$
Adult	$810 - c$	5	$5(810 - c)$
			2820

Step 4: **Translate.**

Write the equation by adding the total values of each type of ticket.

Step 5: **Solve the equation.**

\begin{eqnarray*} \mbox{3c + 5(810 - c)} & = & 2,820 \\ \\ \ 3c + 5(810 - c) & = & 2,820 \\ \\ \ -2c & = & -1,230 \\ \\ \ c & = & 615 \;\text{children tickets} \end{eqnarray*}

How many adults?

$$810 - c$$
$$810 - 615$$

195 adult tickets

Step 6: **Checkthe answer. There were 615 children's tickets at $\$3$ each and 195 adult tickets at $\$5$ each. Is the total value $\$2,820$?**

$$\begin{array}{rcrcl} 615 & \times & 3 & = & 1,845 \\ 195 & \times & 5 & = & 975 \\ & & & = & 2,820 \checkmark \end{array}$$

Step 7: **Answer the question. Galen sold 615 children's tickets and 195 adult tickets.**

Try It

15) During her shift at the museum ticket booth, Leah sold 115 tickets for a total of $\$1,163$. Adult tickets cost $\$12$ and student tickets cost $\$5$. How many adult tickets and how many student tickets did Leah sell?

Solution

84 adult tickets, 31 student tickets

16) A whale-watching ship had 40 paying passengers on board. The total collected from tickets was $\$1,196$. Full-fare passengers paid $\$32$ each and reduced-fare passengers paid $\$26$ each. How many full-fare passengers and how many reduced-fare passengers were on the ship?

Solution

26 full-fare, 14 reduced fare

Now, we'll do one where we fill in the table all at once.

Example 3.8.6

Monica paid $8.36 for stamps. The number of 41-cent stamps was four more than twice the number of two-cent stamps. How many 41-cent stamps and how many two-cent stamps did Monica buy?

Solution

The types of stamps are 41-cent stamps and two-cent stamps. Their names also give the value!

"The number of 41-cent stamps was four more than twice the number of two-cent stamps."

$$\text{Let } x = \text{number of 2 – cent stamps.}$$
$$2x + 4 = \text{number of 41 – cent stamps}$$

Type	Number ·	Value ($)	= Total Value ($)
41 cent stamps	$2x+4$	0.41	$0.41(2x+4)$
2 cent stamps	x	0.02	$0.02x$
			8.36

Step 1: **Write the equation from the total values.**

$$0.41(2x+4)+0.02x=8.36$$

Step 2: **Solve the equation.**

$$\begin{aligned} 0.82x+1.64+0.02x &= 8.36 \\ 0.84x+1.64 &= 8.36 \\ 0.84x &= 6.72 \\ x &= 8 \end{aligned}$$

Monica bought eight two-cent stamps.

$$2x+4 \text{ for } x=8$$

Step 3: **Find the number of 41-cent stamps she bought be evaluating.**

$$\begin{aligned} 2x+4 \\ 2(8)+4 \\ 20 \end{aligned}$$

Step 4: **Check.**

$$\begin{aligned} 8(0.02)+20(0.41) &\stackrel{?}{=} 8.36 \\ 0.16+8.20 &\stackrel{?}{=} 8.36 \\ 8.36 &= 8.36\checkmark \end{aligned}$$

Monica bought eight two-cent stamps and 20 41-cent stamps.

Try It

17) Eric paid $\$13.36$ for stamps. The number of 41-cent stamps was eight more than twice the number of two-cent stamps. How many 41-cent stamps and how many two-cent stamps did Eric buy?

Solution

32 at $\$0.41$, 12 at $\$0.02$

18) Kailee paid $\$12.66$ for stamps. The number of 41-cent stamps was four less than three times the number of 20-cent stamps. How many 41-cent stamps and how many 20-cent stamps did Kailee buy?

Solution

26 at $\$0.41$, 10 at $\$0.20$

Mixture Word Problems

Now we'll solve some more general applications of the mixture model. Grocers and bartenders use the mixture model to set a fair price for a product made from mixing two or more ingredients. Financial planners use the mixture model when they invest money in a variety of accounts and want to find the overall interest rate. Landscape designers use the mixture model when they have an assortment of plants and a fixed budget, and event coordinators do the same when choosing appetizers and entrees for a banquet.

Our first mixture word problem will be making trail mix from raisins and nuts.

Example 3.8.7

Henning is mixing raisins and nuts to make 10 pounds of trail mix. Raisins cost $\$2$ a pound and nuts cost $\$6$ a pound. If Henning wants his cost for the trail mix to be $\$5.20$ a pound, how many pounds of raisins and how many pounds of nuts should he use?

Solution

As before, we fill in a chart to organize our information.

The 10 pounds of trail mix will come from mixing raisins and nuts.

$$\text{x} = \text{number of pounds of raisins}$$

$$10 - x = \text{number of pounds of nuts}$$

We enter the price per pound for each item.

We multiply the number times the value to get the total value.

Type	Number of pounds ·	Price per pound ($)	= Total Value ($)
Raisins	x	2	$2x$
Nuts	$10 - x$	6	$6(10 - x)$
Trail mix	10	5.20	$10(5.20)$

Notice that the last line in the table gives the information for the total amount of the mixture.

We know the value of the raisins plus the value of the nuts will be the value of the trail mix.

Step 1: **Write the equation from the total values.**

$$2x + 6(10 + x) = 10(5.20)$$

Step 2: **Solve the equation.**

$$\begin{aligned} 2x + 60 - 6x &= 52 \\ -4x &= -8 \\ x &= 2 \text{ pounds of raisins} \end{aligned}$$

Step 3: **Find the number of pounds of nuts.**

$$10 \; - \; x$$
$$10 \; - \; rgb]1.0, 0.0, 0.02$$
$$rgb]0.1, 0.1, 0.18 \text{ pounds of nuts}$$

Step 4: **Check.**

$$2(\$2) + 8(\$6) \stackrel{?}{=} 10(\$5.20)$$
$$\$4 + 48 \stackrel{?}{=} \$52$$
$$\$52 = \$52 \checkmark$$

Henning mixed two pounds of raisins with eight pounds of nuts.

Try It

19) Orlando is mixing nuts and cereal squares to make a party mix. Nuts sell for $\$7$ a pound and cereal squares sell for $\$4$ a pound. Orlando wants to make 30 pounds of party mix at a cost of $\$6.50$ a pound, how many pounds of nuts and how many pounds of cereal squares should he use?

Solution

5 pounds cereal squares, 25 pounds nuts

20) Becca wants to mix fruit juice and soda to make a punch. She can buy fruit juice for $\$3$ a gallon and soda for $\$4$ a gallon. If she wants to make 28 gallons of punch at a cost of $\$3.25$ a gallon, how many gallons of fruit juice and how many gallons of soda should she buy?

Solution

21 gallons of fruit punch, 7 gallons of soda

We can also use the mixture model to solve investment problems using simple interest. We have used the simple interest formula, $I = Prt$, where t represented the number of years. When we just need to find the interest for one year, $t = 1$, so then $I = Pr$.

Example 3.8.8

Stacey has $\$20,000$ to invest in two different bank accounts. One account pays interest at 3 per year and the other account pays interest at 5 per year. How much should she invest in each account if she wants to earn 4.5 interest per year on the total amount?

Solution

We will fill in a chart to organize our information. We will use the simple interest formula to find the interest earned in the different accounts.

The interest on the mixed investment will come from adding the interest from the account earning 3 and the interest from the account earning 5 to get the total interest on the $\$20,000$.

\begin{eqnarray*} \mbox{Let x} & = & \text{amount invested at 3%} \\ \\ \ {20,000 - x} & = & \text{amount invested at 5%} \end{eqnarray*}

The amount invested is the *principal* for each account.

We enter the interest rate for each account.

We multiply the amount invested times the rate to get the interest.

Type	Amount Invested ·	Rate	= Interest
3%	x	0.03	$0.03x$
5%	$20,000-x$	0.05	$0.05(20,000-x)$
4.5%	$20,000$	0.045	$0.045(20,000)$

Notice that the total amount invested, $20,000$, is the sum of the amount invested at 3 and the amount invested at 5. And the total interest, $0.045(20,000)$ is the sum of the interest earned in the 3 account and the interest earned in the 5% account.

As with the other mixture applications, the last column in the table gives us the equation to solve.

Step 1: **Write the equation from the interest earned.**

$$0.03x+0.05(20,000-x)=0.045(20,000)$$

Step 2: **Solve the equation.**

$$0.03x + 0.05(20,000 - x) = 0.045(20,000)$$
$$0.03x + 1000 - 0.05x = 900$$
$$-0.02x + 1000 = 900$$
$$-0.02x - 100$$
$$x = 5,000 \text{ amount invested at } 3\%$$

Step 3: **Find the amount invested at 5%.**

$$20,000 - rgb]1.0, 0.0, 0.0x$$
$$20,000 - rgb]1.0, 0.0, 0.05rgb]1.0, 0.0, 0.0,rgb]1.0, 0.0, 0.0000$$
$$15,000 = \text{amount invested at } 5\%$$

Step 4: **Check.**

$$0.03x + 0.05(15,000 + x) \stackrel{?}{=} 0.045(20,000)$$
$$150 + 750 \stackrel{?}{=} -900$$
$$900 = 900\checkmark$$

Stacey should invest $5,000 in the account that earns 3% and $15,000 in the account that earns 5%.

Try It

21) Remy has $\$14,000$ to invest in two mutual funds. One fund pays interest at 4 per year and the other fund pays interest at 7 per year. How much should she invest in each fund if she wants to earn 6.1 interest on the total amount?

Solution

$\$4,200$ at 4, $\$9,800$ at 7

22) Marco has $\$8,000$ to save for his daughter's college education. He wants to divide it between one account that pays 3.2 interest per year and another account that pays 8 interest per year.

How much should he invest in each account if he wants the interest on the total investment to be 6.5?

Solution

$\$2,500$ at 3.2, $\$5,500$ at 8

Solve Uniform Motion Applications

When planning a road trip, it often helps to know how long it will take to reach the destination or how far to travel each day. We would use the distance, rate, and time formula, $D = rt$, which we have already seen.

In this section, we will use this formula in situations that require a little more algebra to solve than the ones we saw earlier. Generally, we will be looking at comparing two scenarios, such as two vehicles travelling at different rates or in opposite directions. When the speed of each vehicle is constant, we call applications like this *uniform motion problems*.

Our problem-solving strategies will still apply here, but we will add to the first step. The first step will include drawing a diagram that shows what is happening in the example. Drawing the diagram helps us understand what is happening so that we will write an appropriate equation. Then we will make a table to organize the information, like we did for the money applications.

The steps are listed here for easy reference:

HOW TO

Use a Problem-Solving Strategy in Distance, Rate, and Time Applications.

Step 1: **Read the problem. Make sure all the words and ideas are understood.**

- Draw a diagram to illustrate what it happening.
- Create a table to organize the information.
- Label the columns rate, time, distance.
- List the two scenarios.
- Write in the information you know.

	Rate •	Time =	Distance

Figure 3.8.4

Step 2: **Identify what we are looking for.**

Step 3: **Name what we are looking for. Choose a variable to represent that quantity.**

- Complete the chart.
- Use variable expressions to represent that quantity in each row.
- Multiply the rate times the time to get the distance.

Step 4: **Translate into an equation.**

- Restate the problem in one sentence with all the important information.
- Then, translate the sentence into an equation.

Step 5: **Solve the equation using good algebra techniques.**

Step 6: **Check the answer in the problem and make sure it makes sense.**

Step 7: **Answer the question with a complete sentence.**

Example 3.8.9

An express train and a local train leave Pittsburgh to travel to Washington, D.C. The express train can make the trip in 4 hours and the local train takes 5 hours for the trip. The speed of the express train is 12 miles per hour faster than the speed of the local train. Find the speed of both trains.

Solution

Step 1: **Read the problem. Make sure all the words and ideas are understood.**

- Draw a diagram to illustrate what it happening. Shown below is a sketch of what is happening in the example.

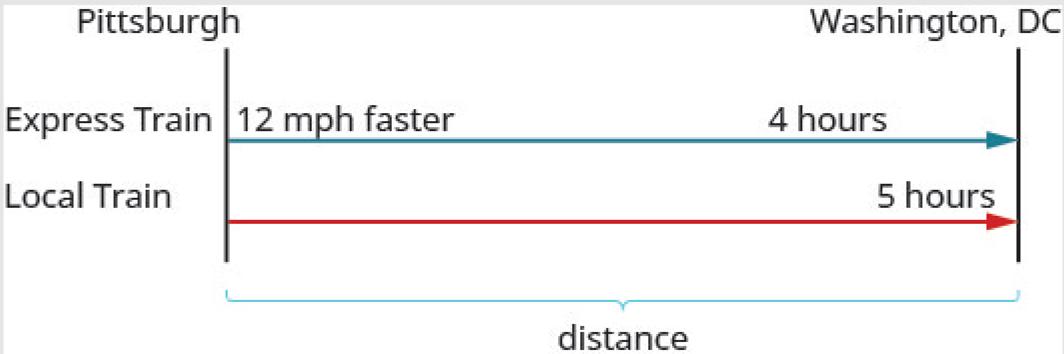


Figure 3.8.5

	Rate (mph) ·	Time (hrs)	= Distance (miles)
Express		$rgb]1.0, 0.0, 0.04$	
Local		$rgb]1.0, 0.0, 0.05$	

- Create a table to organize the information.
- Label the columns "Rate," "Time," and "Distance."
- List the two scenarios.
- Write in the information you know.

Step 2: **Identify what we are looking for.**

- We are asked to find the speed of both trains.
- Notice that the distance formula uses the word "rate," but it is more common to use "speed" when we talk about vehicles in everyday English.

Step 3: **Name what we are looking for. Choose a variable to represent that quantity.**

- Complete the chart
- Use variable expressions to represent that quantity in each row.

- We are looking for the speed of the trains. Let's let r represent the speed of the local train. Since the speed of the express train is 12 mph faster, we represent that as $r+12$.

  $r=$ speed of the local train

  $r+12=$ speed of the express train

Fill in the speeds into the chart.

	Rate (mph) ·	Time (hrs)	= Distance (miles)
Express	$rgb]1.0, 0.0, 0.0r+12$	$rgb]1.0, 0.0, 0.04$	
Local	$rgb]1.0, 0.0, 0.0r$	$rgb]1.0, 0.0, 0.05$	

Multiply the rate times the time to get the distance.

	Rate (mph) ·	Time (hrs)	= Distance (miles)
Express	$r+12$	4	$rgb]1.0, 0.0, 0.04(r+12)$
Local	r	5	$rgb]1.0, 0.0, 0.05r$

Step 4: **Translate into an equation.**

- Restate the problem in one sentence with all the important information.
- Then, translate the sentence into an equation.
- The equation to model this situation will come from the relation between the distances. Look at the diagram we drew above. How is the distance travelled by the express train related to the distance travelled by the local train?
- Since both trains leave from Pittsburgh and travel to Washington, D.C. they travel the same distance. So we write:

$$\begin{array}{lccc} & \text{distance travelled by express train} & = & \text{distance travelled by local train} \\ \text{Translate to an equation} & \underbrace{\qquad\qquad\qquad\qquad}_{4(r+12)} & = & \underbrace{\qquad\qquad\qquad\qquad}_{5r} \end{array}$$

Step 5: **Solve the equation using good algebra techniques.**

Now solve this equation.

$$\begin{aligned} 4(r+12) &= 5r \\ 4r+48 &= 5r \\ 48 &= r \end{aligned}$$

So the speed of the local train is 48 mph.

Find the speed of the express train.

$$\begin{aligned} &rgb]1.0,0.0,0.0r + 12 \\ &rgb]1.0,0.0,0.048 + 12 \\ &60 \end{aligned}$$

The speed of the express train is 60 mph.

Step 6: **Check the answer in the problem and make sure it makes sense.**

$$\begin{array}{ll} \text{express train} & 60\ mph(4\ hours) = 240\ miles \\ \text{local train} & 48\ mph(5\ hour) = 240\ miles \end{array}$$

Step 7: **Answer the question with a complete sentence.**

- The speed of the local train is 48 mph and the speed of the express train is 60 mph.

Try It

23) Wayne and Dennis like to ride the bike path from Riverside Park to the beach. Dennis's speed is seven miles per hour faster than Wayne's speed, so it takes Wayne 2 hours to ride to the beach while it takes Dennis 1.5 hours for the ride. Find the speed of both bikers.

Solution

Wayne 21 mph, Dennis 28 mph

24) Jeromy can drive from his house in Cleveland to his college in Chicago in 4.5 hours. It takes his mother 6 hours to make the same drive. Jeromy drives 20 miles per hour faster than his mother. Find Jeromy's speed and his mother's speed.

Solution

Jeromy 80 mph, mother 60 mph

In Example 3.8.9, the last example, we had two trains travelling the same distance. The diagram and the chart helped us write the equation we solved. Let's see how this works in another case.

Example 3.8.10

Christopher and his parents live 115 miles apart. They met at a restaurant between their homes to celebrate his mother's birthday. Christopher drove 1.5 hours while his parents drove 1 hour to get to the restaurant. Christopher's average speed was 10 miles per hour faster than his parents' average speed. What were the average speeds of Christopher and of his parents as they drove to the restaurant?

Solution
Step 1: **Read the problem. Make sure all the words and ideas are understood.**

- Draw a diagram to illustrate what it happening. Below shows a sketch of what is happening in the example.

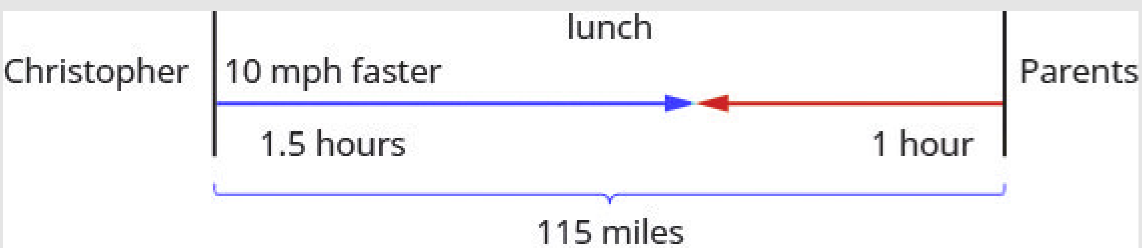


Figure 3.8.6

- Create a table to organize the information.
- Label the columns rate, time, distance.
- List the two scenarios.
- Write in the information you know.

	Rate (mph) ·	Time (hrs)	= Distance (miles)
Christopher		$rgb]1.0, 0.0, 0.01.5$	
Parents		$rgb]1.0, 0.0, 0.01$	
			$rgb]1.0, 0.0, 0.0115$

Step 2: **Identify what we are looking for.**

- We are asked to find the average speeds of Christopher and his parents.

Step 3: **Name what we are looking for. Choose a variable to represent that quantity.**

- Complete the chart.
- Use variable expressions to represent that quantity in each row.
- We are looking for their average speeds. Let's let r represent the average speed of the parents. Since the Christopher's speed is 10 mph faster, we represent that as $r + 10$.

Fill in the speeds into the chart.

	Rate (mph) ·	Time (hrs)	= Distance (miles)
Christopher	$r + 10$	1.5	$rgb]1.0, 0.0, 0.01.5(r + 10)$
Parents	r	1	$rgb]1.0, 0.0, 0.0r$
			$rgb]1.0, 0.0, 0.0115$

Multiply the rate times the time to get the distance.

Step 4: **Translate into an equation.**

- Restate the problem in one sentence with all the important information.
- Then, translate the sentence into an equation.
- Again, we need to identify a relationship between the distances in order to write an equation. Look at the diagram we created above and notice the relationship between the distance Christopher travelled and the distance his parents travelled.

The distance Christopher travelled plus the distance his parents travel must add up to 115 miles. So we write:

	distance travelled by Christopher	+	distance travelled by his parents	=	115
Translate to an equation	$1.5(r+10)$	+	r	=	115

Step 5: **Solve the equation using good algebra techniques.**

Now solve this equation.

$$\begin{aligned} 1.5(r+10)+r &= 115 \\ 1.5r+15+r &= 115 \\ 2.5r+15 &= 115 \\ 2.5r &= 100 \\ r &= 40 \end{aligned}$$

So the parents' speed was 40 mph.

Christopher's speed is $r+10$.

$$rgb]1.0, 0.0, 0.0r+10$$
$$rgb]1.0, 0.0, 0.040+10=50$$

Step 6: **Check the answer in the problem and make sure it makes sense.**

$$\begin{array}{lr} \text{Christopher drove} & 50\ mph(1.5\ hours) = 75\ miles \\ \text{His parents drove} & 40\ mph(1\ hour) = 40\ miles \\ & = 115\ miles \end{array}$$

Step 7: **Answer the question with a complete sentence.**

Christopher's speed was 50 mph.
His parents' speed was 40 mph.

Try It

25) Carina is driving from her home in Anaheim to Berkeley on the same day her brother is driving from Berkeley to Anaheim, so they decide to meet for lunch along the way in Buttonwillow. The distance from Anaheim to Berkeley is 410 miles. It takes Carina 3 hours to get to Buttonwillow,

while her brother drives 4 hours to get there. The average speed Carina's brother drove was 15 miles per hour faster than Carina's average speed. Find Carina's and her brother's average speeds.

Solution
Carina 50 mph, brother 65 mph

26) Ashley goes to college in Minneapolis, 234 miles from her home in Sioux Falls. She wants her parents to bring her more winter clothes, so they decide to meet at a restaurant on the road between Minneapolis and Sioux Falls. Ashley and her parents both drove 2 hours to the restaurant. Ashley's average speed was seven miles per hour faster than her parents' average speed. Find Ashley's and her parents' average speed.

Solution
Parents 55 mph, Ashley 62 mph

As you read the next example, think about the relationship of the distances travelled. Which of the previous two examples is more similar to this situation?

Example 3.8.11

Two truck drivers leave a rest area on the interstate at the same time. One truck travels east and the other one travels west. The truck travelling west travels at 70 mph and the truck travelling east has an average speed of 60 mph. How long will they travel before they are 325 miles apart?

Solution
***Step 1:* Read the problem. Make sure all the words and ideas are understood.**

- Draw a diagram to illustrate what it happening.

Figure 3.8.7

- Create a table to organize the information.

	Rate (mph) ·	Time (hrs)	= Distance (miles)
West		$rgb]1.0, 0.0, 0.070$	
East		$rgb]1.0, 0.0, 0.060$	
			$rgb]1.0, 0.0, 0.0325$

Step 2: **Identify what we are looking for.**

- We are asked to find the amount of time the trucks will travel until they are 325 miles apart.

Step 3: **Name what we are looking for. Choose a variable to represent that quantity.**

- We are looking for the time travelled. Both trucks will travel the same amount of time. Let's call the time t. Since their speeds are different, they will travel different distances.
- Complete the chart.

	Rate (mph) ·	Time (hrs)	= Distance (miles)
West	70	$rgb]1.0, 0.0, 0.0t$	$rgb]1.0, 0.0, 0.070t$
East	60	$rgb]1.0, 0.0, 0.0t$	$rgb]1.0, 0.0, 0.060t$
			325

Step 4: **Translate into an equation.**

- We need to find a relation between the distances in order to write an equation. Looking at the diagram, what is the relationship between the distance each of the trucks will travel?
- The distance traveled by the truck going west plus the distance travelled by the truck going

east must add up to 325 miles. So we write:

$$\underbrace{\text{distance travelled by westbound truck}}_{70t} \quad \underset{+}{+}\underbrace{\text{distance travelled by eastbound truck}}_{60t} \quad \underset{=32}{=32}$$

Step 5: **Solve the equation using good algebra techniques.**

$$\begin{aligned} 70t + 60t &= 325 \\ 130t &= 325 \\ t &= 2.5 \end{aligned}$$

So it will take the trucks 2.5 hours to be 325 miles apart.

Step 6: **Check the answer in the problem and make sure it makes sense.**

$$\begin{array}{ll} \text{Truck going West} & 70\ mph(2.5\ hours) = 175\ miles \\ \text{Truck going East} & 60\ mph(2.5\ hour) = 150\ miles \\ & = 325\ miles \end{array}$$

Step 7: **Answer the question with a complete sentence.**

It will take the trucks 2.5 hours to be 325 miles apart.

Try It

27) Pierre and Monique leave their home in Portland at the same time. Pierre drives north on the turnpike at a speed of 75 miles per hour while Monique drives south at a speed of 68 miles per hour. How long will it take them to be 429 miles apart?

Solution

3 hours

28) Thanh and Nhat leave their office in Sacramento at the same time. Thanh drives north on I-5 at a speed of 72 miles per hour. Nhat drives south on I-5 at a speed of 76 miles per hour. How long will it take them to be 330 miles apart?

Solution
2.2 hours

Matching Units in Problems

It is important to make sure the units match when we use the distance rate and time formula. For instance, if the rate is in miles per hour, then the time must be in hours.

Example 3.8.12

When Katie Mae walks to school, it takes her 30 minutes. If she rides her bike, it takes her 15 minutes. Her speed is three miles per hour faster when she rides her bike than when she walks. What are her walking speed and her speed riding her bike?

Solution
First, we draw a diagram that represents the situation to help us see what is happening.

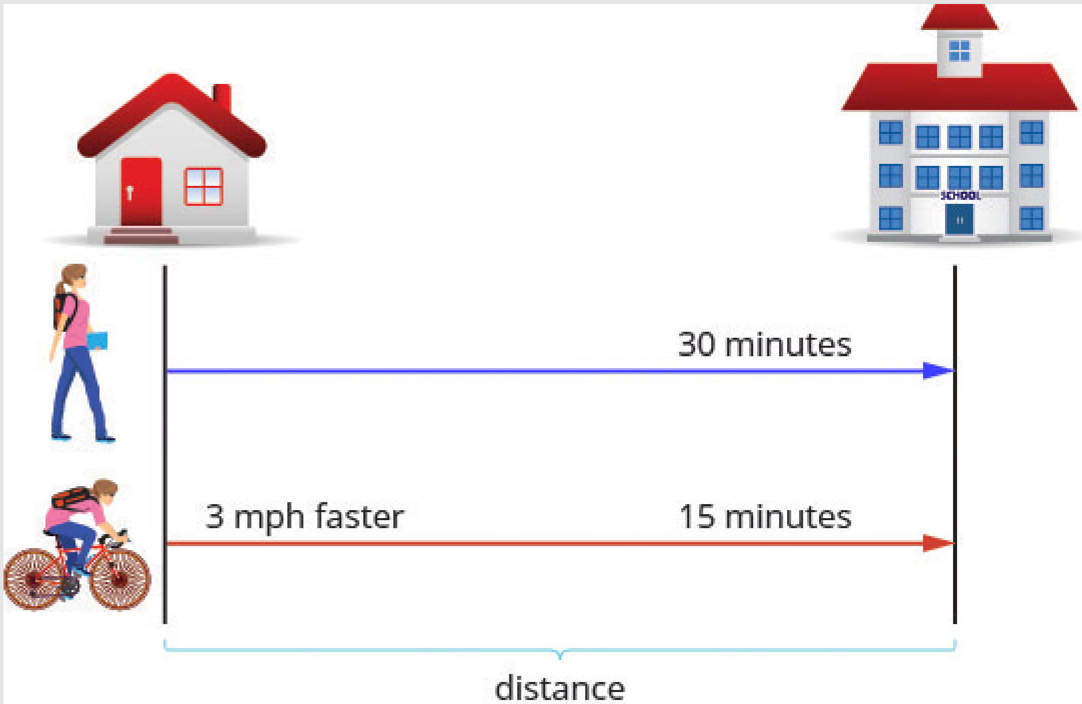


Figure 3.8.8

We are asked to find her speed walking and riding her bike. Let's call her walking speed r. Since her biking speed is three miles per hour faster, we will call that speed $r+3$. We write the speeds in the chart.

The speed is in miles per hour, so we need to express the times in hours, too, in order for the units to be the same. Remember, one hour is 60 minutes. So:

30 minutes is $\frac{30}{60}$ or $\frac{1}{2}$ hour

15 minutes is $\frac{15}{60}$ or $\frac{1}{4}$ hour

Next, we multiply rate times time to fill in the distance column.

	Rate (mph) ·	Time (hrs)	= Distance (miles)
Walk	r	$\frac{1}{2}$	$\frac{1}{2}r$
Bike	$r+3$	$\frac{1}{4}$	$\frac{1}{4}(r+3)$

The equation will come from the fact that the distance from Katie Mae's home to her school is the same whether she is walking or riding her bike.

So we say:

$$\underbrace{\text{distance walked}} = \underbrace{\text{distance covered by bike}}$$

Step 1: **Translate into an equation.**

$$\frac{1}{2}r = \frac{1}{4}(r + 3)$$

Step 2: **Solve this equation.**

$$\frac{1}{2}r = \frac{1}{4}(r + 3)$$

Step 3: **Clear the fractions by multiplying by the LCD of all the fractions in the equation.**

$$8 \times \frac{1}{2}r = 8 \times \frac{1}{4}(r + 3)$$

Step 4: **Simplify.**

$$\begin{aligned} 4r &= 2(r + 3) \\ 4r &= 2r + 6 \\ 2r &= 6 \\ r &= 3\ mph \end{aligned}$$

3 mph is Katie Mae's walking speed

$rgb]1.0, 0.0, 0.0r + 3$ biking speed

$rgb]1.0, 0.0, 0.03 + 3$

6 mph (Katie Mae's biking speed)

Step 5: **Let's check if this works.**

$$\begin{array}{ll} \text{Walk} & 3\ mph(0.5\ hours) = 1.5\ miles \\ \text{Bike} & 6\ mph(0.25\ hour) = 1.5\ miles \end{array}$$

Yes, either way Katie Mae travels 1.5 miles to school.

Katie Mae's walking speed is 3 mph.

Her speed riding her bike is 6 mph.

Try It

29) Suzy takes 50 minutes to hike uphill from the parking lot to the lookout tower. It takes her 30 minutes to hike back down to the parking lot. Her speed going downhill is 1.2 miles per hour faster than her speed going uphill. Find Suzy's uphill and downhill speeds.

Solution

Uphill 1.8 mph, downhill three mph

30) Llewyn takes 45 minutes to drive his boat upstream from the dock to his favorite fishing spot. It takes him 30 minutes to drive the boat back downstream to the dock. The boat's speed going downstream is four miles per hour faster than its speed going upstream. Find the boat's upstream and downstream speeds.

Solution

Upstream 8 mph, downstream 12 mph

In the distance, rate, and time formula, time represents the actual amount of elapsed time (in hours, minutes, etc.). If a problem gives us starting and ending times as clock times, we must find the elapsed time in order to use the formula.

Example 3.8.13

Hamilton loves to travel to Las Vegas, 255 miles from his home in Orange County. On his last trip, he left his house at 2:00 pm. The first part of his trip was on congested city freeways. At 4:00 pm, the traffic cleared and he was able to drive through the desert at a speed 1.75 times faster than when he drove in the congested area. He arrived in Las Vegas at 6:30 pm. How fast was he driving during each part of his trip?

Solution

A diagram will help us model this trip.

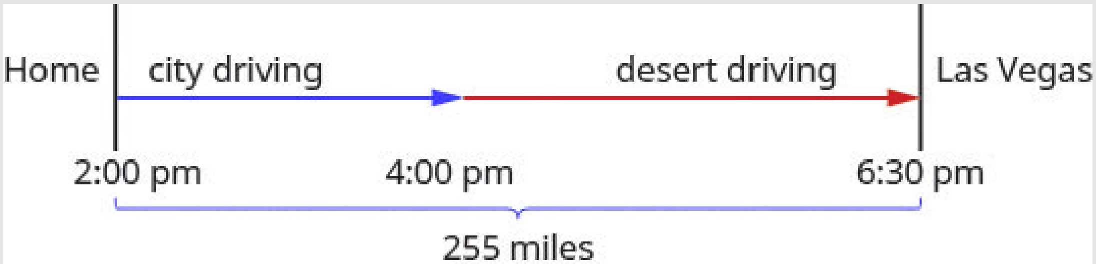


Figure 3.8.9

Next, we create a table to organize the information.

We know the total distance is 255 miles. We are looking for the rate of speed for each part of the trip. The rate in the desert is 1.75 times the rate in the city. If we let $r =$ the rate in the city, then the rate in the desert is $1.75r$.

The times here are given as clock times. Hamilton started from home at 2:00 pm and entered the desert at 4:30 pm. So he spent two hours driving the congested freeways in the city. Then he drove faster from 4:00 pm until 6:30 pm in the desert. So he drove 2.5 hours in the desert.

Now, we multiply the rates by the times.

	Rate (mph) ·	Time (hrs)	= Distance (miles)
City	r	2	$2r$
Desert	$1.75r$	2.5	$2.5(1.5r)$
			225

By looking at the diagram below, we can see that the sum of the distance driven in the city and the distance driven in the desert is 255 miles.

$$\underbrace{\text{distance driven in the city}} + \underbrace{\text{distance driven in desert}} = 255$$

Step 1: **Translate into an equation.**

$$2r \;+\; 2.5(1.75r) \;=\; 255$$

Step 2: **Solve this equation.**

$$\begin{aligned} 2r + 2.5(1.75r) &= 255 \\ 2r + 4.375r &= 255 \\ 6.375r &= 255 \\ r &= 40\ mph\ city \end{aligned}$$

$$1.75rgb]1.0, 0.0, 0.0r \text{ desert speed}$$
$$1.75(rgb]1.0, 0.0, 0.040)$$
$$70\ mph$$

Step 3: **Check.**

$$\text{City } 40\ mph\ (2 \text{ hours}) = 80\ miles$$
$$\text{Desert } 70\ mph\ (2.5 \text{ hours}) = \frac{175\ miles}{255\ miles}$$

Hamilton drove 40 mph in the city and 70 mph in the desert.

Try It

31) Cruz is training to compete in a triathlon. He left his house at 6:00 and ran until 7:30. Then he rode his bike until 9:45. He covered a total distance of 51 miles. His speed when biking was 1.6 times his speed when running. Find Cruz's biking and running speeds.

Solution

Biking 16 mph, running 10 mph

32) Phuong left home on his bicycle at 10:00. He rode on the flat street until 11:15, then rode uphill until 11:45. He rode a total of 31 miles. His speed riding uphill was 0.6 times his speed on the flat street. Find his speed biking uphill and on the flat street.

Solution

Uphill 12 mph, flat street 20 mph

Key Concepts

- **Total Value of Coins** For the same type of coin, the total value of a number of coins is found by using the model.
  - number × value=total value where *number* is the number of coins and *value* is the value of each coin; *total value* is the total value of all the coins
- **Problem-Solving Strategy—Coin Word Problems**
  1. **Read the problem. Make all the words and ideas are understood. Determine the types of coins involved.**
     - Create a table to organize the information.
     - Label the columns type, number, value, total value.
     - List the types of coins.
     - Write in the value of each type of coin.
     - Write in the total value of all the coins.
  2. **Identify what we are looking for.**
  3. **Name what we are looking for. Choose a variable to represent that quantity.**
     Use variable expressions to represent the number of each type of coin and write them in the table.
     Multiply the number times the value to get the total value of each type of coin.
  4. **Translate into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the sentence into an equation.**
     Write the equation by adding the total values of all the types of coins.
  5. **Solve the equation using good algebra techniques.**
  6. **Check the answer in the problem and make sure it makes sense.**
  7. **Answer the question with a complete sentence.**
- **Distance, Rate, and Time**
  - $D = rt$ where $D =$ distance, $r =$ rate, $t =$ time

- **Problem-Solving Strategy—Distance, Rate, and Time Applications**

  1. **Read the problem. Make sure all the words and ideas are understood.**
     Draw a diagram to illustrate what it happening.
     Create a table to organize the information: Label the columns rate, time, distance. List the two scenarios. Write in the information you know.
  2. **Identify what we are looking for.**
  3. **Name what we are looking for. Choose a variable to represent that quantity.**
     Complete the chart.
     Use variable expressions to represent that quantity in each row.
     Multiply the rate times the time to get the distance.
  4. **Translate into an equation.**
     Restate the problem in one sentence with all the important information.
     Then, translate the sentence into an equation.
  5. **Solve the equation using good algebra techniques.**
  6. **Check the answer in the problem and make sure it makes sense.**
  7. **Answer the question with a complete sentence.**

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=1362#h5p-22

b. After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

mixture problems
Mixture problems combine two or more items with different values together.

3.9 GRAPH LINEAR EQUATIONS IN TWO VARIABLES

Learning Objectives

By the end of this section, you will be able to:

- Plot points in a rectangular coordinate system
- Verify solutions to an equation in two variables
- Complete a table of solutions to a linear equation
- Find solutions to a linear equation in two variables
- Recognize the relationship between the solutions of an equation and its graph.
- Graph a linear equation by plotting points.
- Graph vertical and horizontal lines.
- Identify the x-intercept and y-intercept on a graph
- Find the x-intercept and y-intercept from an equation of a line
- Graph a line using the intercepts

Try It

Before you get started, take this readiness quiz:

1) Evaluate $x+3$ when $x=-1$.

2) Evaluate $2x - 5y$ when $x = 3$ and $y = -2$.
3) Solve for y: $40 - 4y = 20$.
4) Evaluate $3x + 2$ when $x = -1$.
5) Solve $3x + 2y = 12$ for y in general.
6) Solve: $3 \cdot 0 + 4y = -2$.

Plot Points on a Rectangular Coordinate System

Just like maps use a grid system to identify locations, a grid system is used in algebra to show a relationship between two variables in a **rectangular coordinate system**. The rectangular coordinate system is also called the xy-plane or the 'coordinate plane'.

The horizontal number line is called the x-*axis*. The vertical number line is called the y-*axis.* The x-axis and the y-axis together form the rectangular coordinate system. These axes divide a plane into four regions, called **quadrant**s. The quadrants are identified by Roman numerals, beginning on the upper right and proceeding counterclockwise. See Figure 3.9.1.

'Quadrant' has the root 'quad,' which means 'four.'

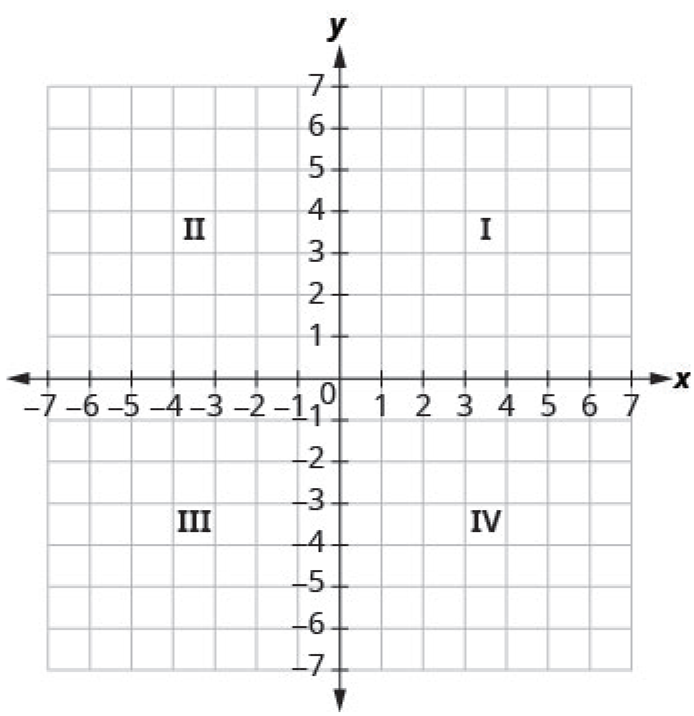


Figure 3.9.1-'Quadrant' has the root 'quad,' which means 'four.'

In the rectangular coordinate system, every point is represented by an **ordered pair**. The first number in the ordered pair is the **[latex]x[/latex]-coordinate** of the point, and the second number is the **[latex]y[/latex]-coordinate** of the point.

Ordered Pair

An ordered pair, (x, y), gives the coordinates of a point in a rectangular coordinate system.

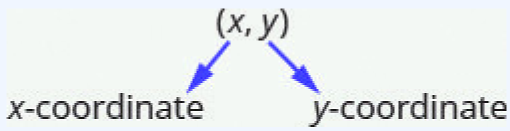


Figure 3.9.2

The first number is the x-coordinate.

The second number is the y-coordinate.

The phrase 'ordered pair' means the order is important. What is the ordered pair of the point where the axes cross? At that point both coordinates are zero, so its ordered pair is $(0, 0)$. The point $(0, 0)$ has a special name. It is called the **origin**.

The Origin

The point $(0, 0)$ is called the origin. It is the point where the x-axis and y-axis intersect.

We use the coordinates to locate a point on the xy-plane. Let's plot the point $(1, 3)$ as an example. First, locate 1 on the x-axis and lightly sketch a **vertical line** through $x = 1$. Then, locate 3 on the y-axis and sketch a **horizontal line** through $y = 3$. Now, find the point where these two lines meet—that is the point with coordinates $(1, 3)$.

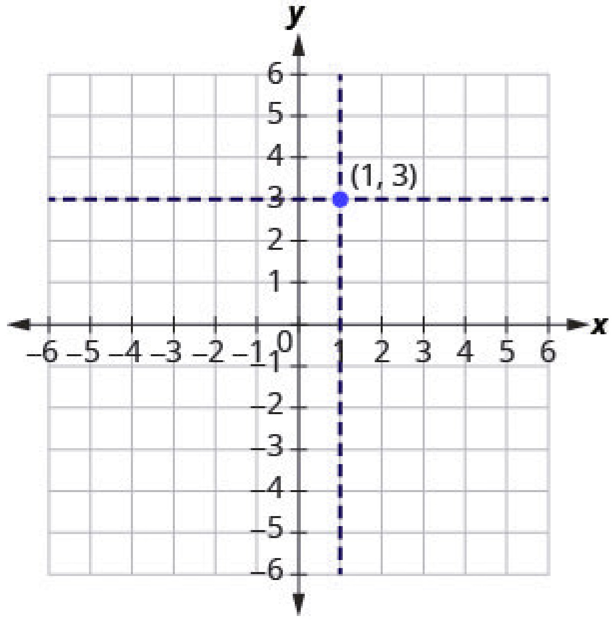


Figure 3.9.3

Notice that the vertical line through $x = 1$ and the horizontal line through $y = 3$ are not part of the graph. We just used them to help us locate the point $(1, 3)$.

Example 3.9.1

Plot each point in the rectangular coordinate system and identify the quadrant in which the point is located:

a. $(-5, 4)$
b. $(-3, -4)$
c. $(2, -3)$
d. $(-2, 3)$
e. $(3, \frac{5}{2})$

Solution
The first number of the coordinate pair is the x-coordinate, and the second number is the y-coordinate.

a. Since $x = -5$, the point is to the left of the y-axis. Also, since $y = 4$, the point is above the x-axis. The point $(-5, 4)$ is in Quadrant II.

b. Since $x = -3$, the point is to the left of the y-axis. Also, since $y = -4$, the point is below the x-axis. The point $(-3, -4)$ is in Quadrant III.

c. Since $x = 2$, the point is to the right of the y-axis. Since $y = -3$, the point is below the x-axis. The point $(2, -3)$ is in Quadrant IV.

d. Since $x = -2$, the point is to the left of the y-axis. Since $y = 3$, the point is above the x-axis. The point $(-2, 3)$ is in Quadrant II.

e. Since $x = 3$, the point is to the right of the y-axis. Since $y = \frac{5}{2}$, the point is above the x-axis. (It may be helpful to write $\frac{5}{2}$) as a mixed number or decimal.) The point $(3, \frac{5}{2})$ is in Quadrant I.

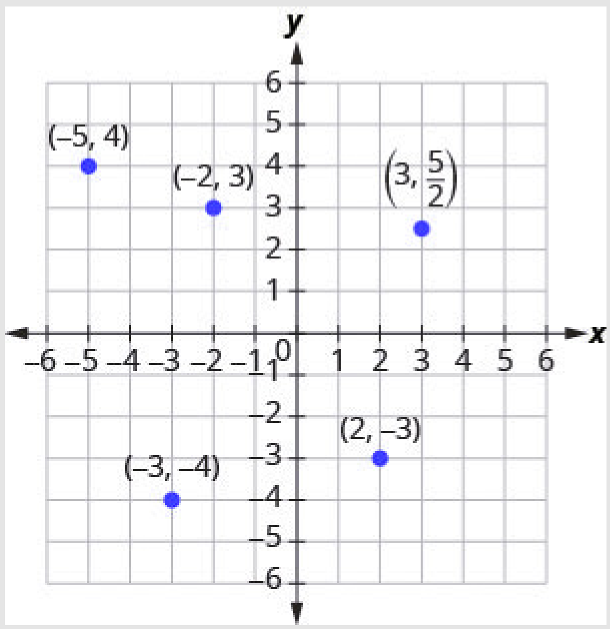


Figure 3.9.4

Try It

7) Plot each point in a rectangular coordinate system and identify the quadrant in which the point is located:

a. $(-2, 1)$
b. $(-3, -1)$
c. $(4, -4)$
d. $(-4, 4)$
e. $(-4, \frac{3}{2})$

Solution

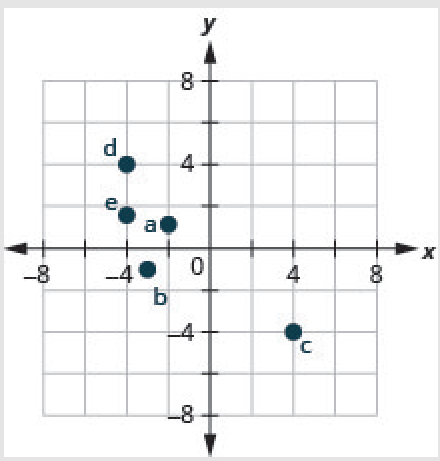


Figure 3.9.5

8) Plot each point in a rectangular coordinate system and identify the quadrant in which the point is located:

a. $(-4, 1)$
b. $(-2, 3)$
c. $(2, -5)$

d. $(-2, 5)$
e. $(-3, \frac{5}{2})$

Solution

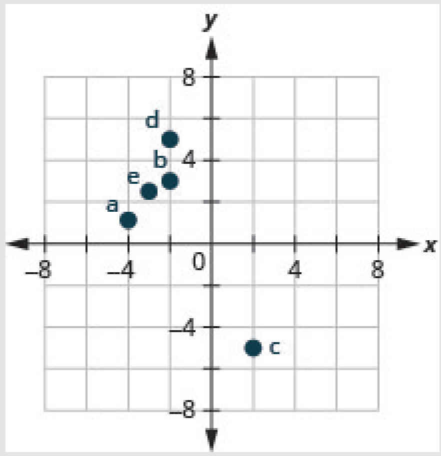


Figure 3.9.6

How do the signs affect the location of the points? You may have noticed some patterns as you graphed the points in the previous example.

For the point in Figure 3.9.4 in Quadrant IV, what do you notice about the signs of the coordinates? What about the signs of the coordinates of points in the third quadrant? The second quadrant? The first quadrant?

Can you tell just by looking at the coordinates in which quadrant the point $(-2, 5)$ is located? In which quadrant is $(2, -5)$ located?

Quadrants

We can summarize sign patterns of the quadrants in this way.

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
(x, y)	(x, y)	(x, y)	(x, y)
$(+, +)$	$(-, +)$	$(-, -)$	$(+, -)$

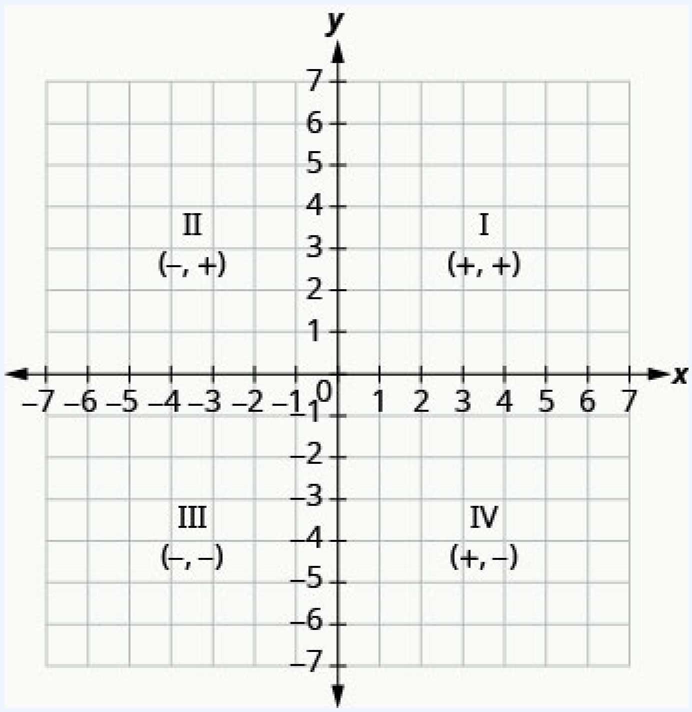


Figure 3.9.7

What if one coordinate is zero as shown in Figure 3.9.8? Where is the point $(0, 4)$ located? Where is the point $(-2, 0)$ located?

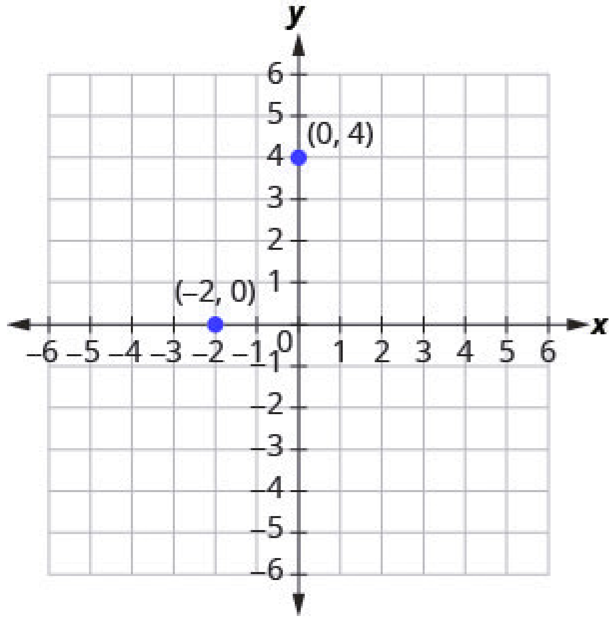


Figure 3.9.8

The point $(0, 4)$ is on the y-axis and the point $(-2, 0)$ is on the x-axis.

Points on the Axes

Points with a y-coordinate equal to 0 are on the x-axis, and have coordinates $(a, 0)$.

Points with an x-coordinate equal to 0 are on the y-axis, and have coordinates $(0, b)$.

Example 3.9.2

Plot each point:

a. $(0, 5)$

b. $(4, 0)$
c. $(-3, 0)$
d. $(0, 0)$
e. $(0, -1)$

Solution

a. Since $x = 0$, the point whose coordinates are $(0, 5)$ is on the y-axis.
b. Since $y = 0$, the point whose coordinates are $(4, 0)$ is on the x-axis.
c. Since $y = 0$, the point whose coordinates are $(-3, 0)$ is on the x-axis.
d. Since $x = 0$ and $y = 0$, the point whose coordinates are $(0, 0)$ is the origin.
e. Since $x = 0$, the point whose coordinates are $(0, -1)$ is on the y-axis.

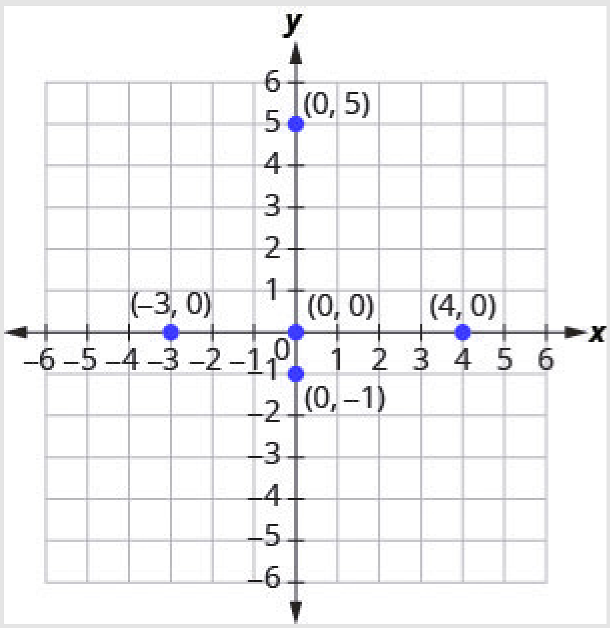


Figure 3.9.9

Try It

9) Plot each point:

a. $(4, 0)$
b. $(-2, 0)$
c. $(0, 0)$
d. $(0, 2)$
e. $(0, -3)$

Solution

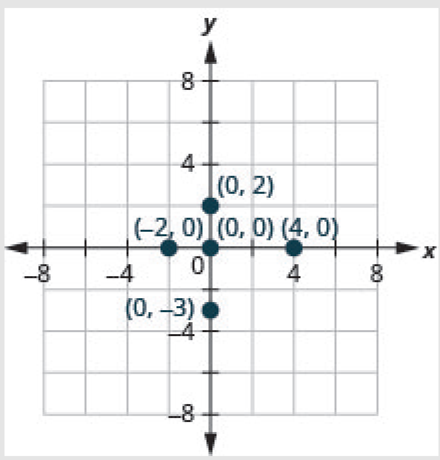


Figure 3.9.10

10) Plot each point:

a. $(-5, 0)$
b. $(3, 0)$
c. $(0, 0)$
d. $(0, -1)$
e. $(0, 4)$

Solution

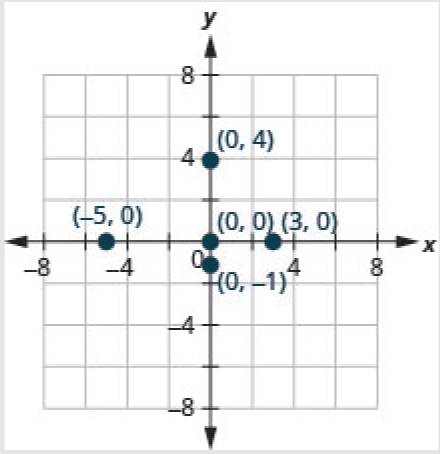


Figure 3.9.11

In algebra, being able to identify the coordinates of a point shown on a graph is just as important as being able to plot points. To identify the x-coordinate of a point on a graph, read the number on the x-axis directly above or below the point. To identify the y-coordinate of a point, read the number on the y-axis directly to the left or right of the point. Remember, when you write the ordered pair use the correct order, (x, y).

Example 3.9.3

Name the ordered pair of each point shown in the rectangular coordinate system.

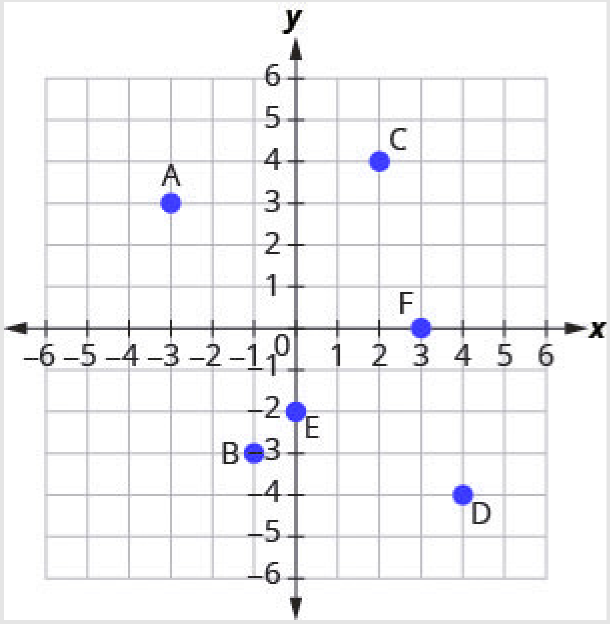


Figure 3.9.12

Solution

Point A is above -3 on the x-axis, so the x-coordinate of the point is -3.

- The point is to the left of 3 on the y-axis, so the y-coordinate of the point is 3.
- The coordinates of the point are $(-3, 3)$.

Point B is below -1 on the x-axis, so the x-coordinate of the point is -1.

- The point is to the left of -3 on the y-axis, so the y-coordinate of the point is -3.
- The coordinates of the point are $(-1, -3)$.

Point C is above 2 on the x-axis, so the x-coordinate of the point is 2.

- The point is to the right of 4 on the y-axis, so the y-coordinate of the point is 4.
- The coordinates of the point are $(2, 4)$.

Point D is below 4 on the x-axis, so the x-coordinate of the point is 4.

- The point is to the right of -4 on the y-axis, so the y-coordinate of the point is -4.
- The coordinates of the point are $(4, -4)$.

Point E is on the y-axis at $y = -2$. The coordinates of point E are $(0, -2)$.

Point F is on the x-axis at $x = 3$. The coordinates of point F are $(3, 0)$.

Try It

11) Name the ordered pair of each point shown in the rectangular coordinate system.

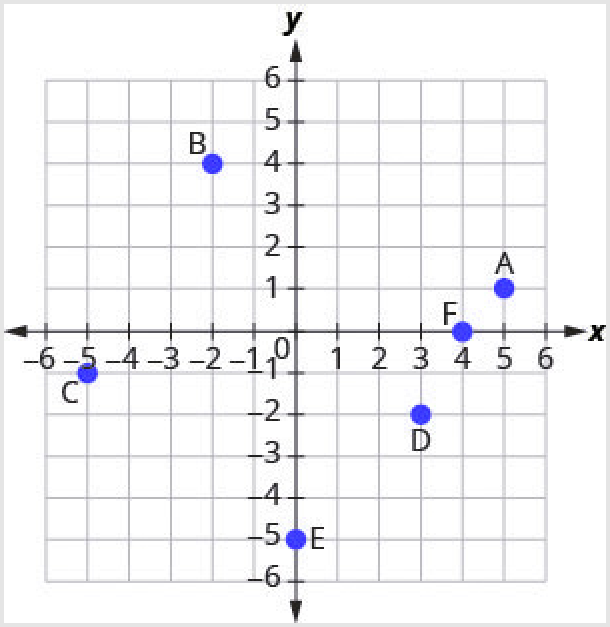


Figure 3.9.13

Solution

a: $(5,1)$
b: $(-2,4)$
c: $(-5,-1)$
d: $(3,-2)$
e: $(0,-5)$
f: $(4,0)$

12) Name the ordered pair of each point shown in the rectangular coordinate system.

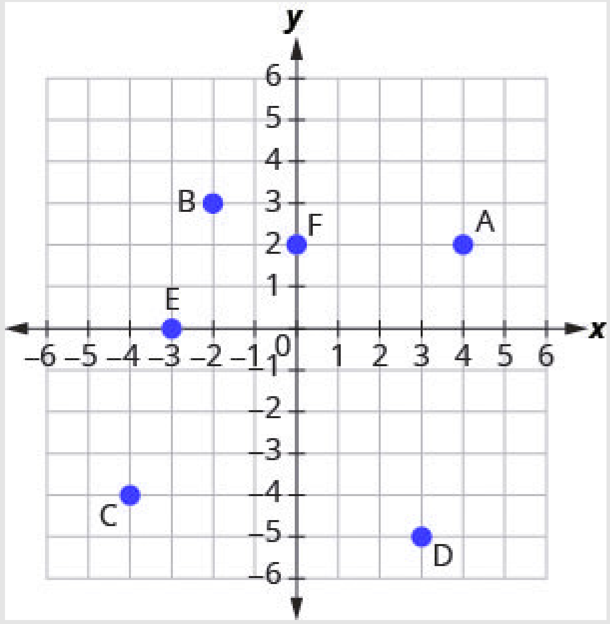


Figure 3.9.14

Solution

a: $(4, 2)$

b: $(-2, 3)$

c: $(-4, -4)$

d: $(3, -5)$

e: $(-3, 0)$

f: $(0, 2)$

Verify Solutions to an Equation in Two Variables

Up to now, all the equations you have solved were equations with just one variable. In almost every case, when you solved the equation you got exactly one solution. The process of solving an equation ended with a statement like $x = 4$. (Then, you checked the solution by substituting back into the equation.)

Here's an example of an equation in one variable, and its one solution.

$$\begin{aligned} 3x + 5 &= 17 \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

But equations can have more than one variable. Equations with two variables may be of the form $Ax + By = C$. Equations of this form are called **linear equations** in two variables.

Linear Equation

An equation of the form $Ax + By = C$, where A and B are not both zero, is called a linear equation *in two variables.*

Notice the word *line* in *linear*. Here is an example of a linear equation in two variables, x and y.

$$rgb]1.0,0.0,0.0Ax + rgb]0.0,0.0,1.0By = rgb]0.5,0.0,0.5C$$

$$x + rgb]0.0,0.0,1.04y = rgb]0.5,0.0,0.58$$

$$rgb]1.0,0.0,0.0Argb]1.0,0.0,0.0 = rgb]1.0,0.0,0.01,\ rgb]0.0,0.0,1.0Brgb]0.0,0.0,1.0 = rgb]0.0,0.0,1.04,\ rgb]0.5,0.0,0.5Crgb]0.5,0.0,0.5 = rgb]0.5,0.0,0.58$$

The equation $y = -3x + 5$ is also a linear equation. But it does not appear to be in the form $Ax + By = C$. We can use the Addition Property of Equality and rewrite it in $Ax + By = C$ form.

	$y = -3x + 5$
Step 1: Add to both sides.	$y + 3x = -3x + 5 + 3x$
Step 2: Simplify.	$y + 3x = 5$
Step 3: Use the Commutative Property to put it in Ax+By=C form.	$3x = y = 5$

By rewriting $y = -3x + 5$ as $3x + y = 5$, we can easily see that it is a linear equation in two variables because it is of the form $Ax + By = C$. When an equation is in the form $Ax + By = C$, we say it is in *standard form*.

Standard Form of Linear Equation

A linear equation is in standard form when it is written $Ax + By = C$.

Most people prefer to have A, B, and C be integers and $A \geq 0$ when writing a linear equation in standard form, although it is not strictly necessary.

Linear equations have infinitely many solutions. For every number that is substituted for x there is a corresponding y value. This pair of values is a *solution* to the linear equation and is represented by the ordered pair (x, y). When we substitute these values of x and y into the equation, the result is a true statement, because the value on the left side is equal to the value on the right side.

Solution of a Linear Equation in Two Variables

An ordered pair (x, y) is a *solution* of the linear equation $Ax + By = C$, if the equation is a true statement when the x and y values of the ordered pair are substituted into the equation.

Example 3.9.4

Determine which ordered pairs are solutions to the equation $x + 4y = 8$.

a. $(0, 2)$
b. $(2, -4)$
c. $(-4, 3)$

Solution
Substitute the x– and y-values from each ordered pair into the equation and determine if the result is a true statement.

(a)	(b)	(c)
$(0, 2)$ $x = rgb]0.0, 0.0, 1.00, y = rgb]1.0, 0.0, 0.02$ $x + 4y = 8$ $rgb]0.0, 0.0, 1.00 + 4 \cdot rgb]1.0, 0.0, 0.02 \stackrel{?}{=} 8$ $0 + 8 \stackrel{?}{=} 8$ $8 = 8✓$	$(2, -4)$ $x = rgb]0.0, 0.0, 1.02, y = rgb]1.0, 0.0, 0.0 - 4$ $x + 4y = 8$ $rgb]0.0, 0.0, 1.02 + 4 \cdot rgb]1.0, 0.0, 0.0 - 4 \stackrel{?}{=} 8$ $2 + (-16) \stackrel{?}{=} 8$ $-14 \neq 8$	$(-4, 3)$ $x = rgb]0.0, 0.0, 1.0 - 4, y = rgb]1.0, 0.0, 0.03$ $x + 4y = 8$ $rgb]0.0, 0.0, 1.0 - 4 + 4 \cdot rgb]1.0, 0.0, 0.03 \stackrel{?}{=} 8$ $-4 + 12 \stackrel{?}{=} 8$ $8 = 8✓$
$(0, 2)$ is a solution	$(2, -4)$ is not a solution	$(-4, 3)$ is a solution

Try It

13) Which of the following ordered pairs are solutions to $2x + 3y = 6$?

a. $(3, 0)$
b. $(2, 0)$
c. $(6, -2)$

Solution
a, c

14) Which of the following ordered pairs are solutions to the equation $4x - y = 8$?

a. $(0, 8)$
b. $(2, 0)$
c. $(1, -4)$

Solution
b, c

Example 3.9.5

Which of the following ordered pairs are solutions to the equation $y = 5x - 1$?

b. $(1, 4)$
c. $(-2, -7)$

Solution
Substitute the x and y values from each ordered pair into the equation and determine if it results in a true statement.

(a)	(b)	(c)
$(0,-1)$	$(1,4)$	$(-2,-7)$
$x = rgb]0.0,0.0,1.00, y = rgb]1.0,0.0,0.0-1$	$x = rgb]0.0,0.0,1.01, y = rgb]1.0,0.0,0.04$	$x = rgb]0.0,0.0,1.0-2, y = rgb]1.0,0.0,0.0-7$
$y = 5x - 1$	$y = 5x - 1$	$y = 5x - 1$
$rgb]1.0,0.0,0.0-1 \stackrel{?}{=} 5(rgb]0.0,0.0,1.00)-1$	$rgb]1.0,0.0,0.04 \stackrel{?}{=} 5(rgb]0.0,0.0,1.01)-1$	$rgb]1.0,0.0,0.0-7 \stackrel{?}{=} 5(rgb]0.0,0.0,1.0-2)-1$
$-1 \stackrel{?}{=} 0-1$	$4 \stackrel{?}{=} 5-1$	$-7 \stackrel{?}{=} -10-1$
$-1 = -1\checkmark$	$4 = 4\checkmark$	$7 \neq -11$
$(0,-1)$ is a solution.	$(1,4)$ is a solution.	$(-2,-7)$ is not a solution.

Try It

15) Which of the following ordered pairs are solutions to the equation $y = 4x - 3$?

a. $(0, 3)$
b. $(1, 1)$
c. $(-1, -1)$

Solution
b

16) Which of the following ordered pairs are solutions to the equation $y = -2x + 6$?

a. $(0, 6)$
b. $(1, 4)$
c. $(-2, -2)$

Solution
a, b

Complete a Table of Solutions to a Linear Equation in Two Variables

In the examples above, we substituted the x and y values of a given ordered pair to determine whether or not it was a solution to a linear equation. But how do you find the ordered pairs if they are not given? It's easier than you might think—you can just pick a value for x and then solve the equation for y. Or, pick a value for y and then solve for x.

We'll start by looking at the solutions to the equation $y = 5x - 1$ that we found in Example 3.9.5. We can summarize this information in a table of solutions, as shown in the below table.

$y = 5x - 1$		
x	y	(x, y)
0	-1	$(0, -1)$
1	4	$(1, 4)$

To find a third solution, we'll let $x = 2$ and solve for y.

$$\begin{array}{ll} & y = 5x - 1 \\ \text{Substitute } x = rgb]0.0, 0.0, 1.02. & y = 5(rgb]0.0, 0.0, 1.02) - 1 \\ \text{Multiply.} & y = 10 - 1 \\ \text{Simplify.} & y = 9 \end{array}$$

The ordered pair $(2, 9)$ is a solution to $y = 5x - 1$. We will add it to below table.

$y = 5x - 1$		
x	y	(x, y)
0	-1	$(0, -1)$
1	4	$(1, 4)$
2	9	$(2, 9)$

We can find more solutions to the equation by substituting in any value of x or any value of y and solving the resulting equation to get another ordered pair that is a solution. There are infinitely many solutions of this equation.

Example 3.9.6

Complete below table to find three solutions to the equation $y = 4x - 2$.

$y = 4x - 2$		
x	y	(x, y)
0		
-1		
2		

Solution

Substitute $x = 0$, $x = -1$, and $x = 2$ into $y = 4x - 2$

$x = rgb]0.0, 0.0, 1.00$
$y = 4x - 2$
$y = 4 \cdot rgb]0.0, 0.0, 1.00 - 2$
$y = -2$
$y = -2$
$(0, -2)$

$x = rgb]0.0, 0.0, 1.0 - 1$
$y = 4x - 2$
$y = 4(rgb]0.0, 0.0, 1.0 - 1) - 2$
$y = -4 - 2$
$y = -6$
$(-1, -6)$

$x = rgb]0.0, 0.0, 1.02$
$y = 4x - 2$
$y = 4 \cdot rgb]0.0, 0.0, 1.02 - 2$
$y = 8 - 2$
$y = 6$
$(2, 6)$

The results are summarized in below table.

$y = 4x - 2$		
x	y	(x, y)
0	-2	$(0, -2)$
-1	-6	$(-1, -6)$
2	6	$(2, 6)$

Try It

17) Complete the table to find three solutions to this equation: $y = 3x - 1$.

$y = 3x - 1$		
x	y	(x, y)
0		
-1		
2		

Solution

$y = 3x - 1$		
x	y	(x, y)
0	-1	$(0, -1)$
-1	-4	$(-1, -4)$
2	5	$(2, 5)$

18) Complete the table to find three solutions to this equation: $y = 6x + 1$.

$y = 6x + 1$		
x	y	(x, y)
0		
-1		
2		

Solution

$y = 6x - 1$		
x	y	(x, y)
0	1	$(0, -1)$
-1	7	$(1, 7)$
2	-11	$(2, -11)$

Example 3.9.7

Complete the below table to find three solutions to the equation $5x - 4y = 20$.

$5x - 4y = 20$		
x	y	(x, y)
0		
	0	
	5	

Solution

Substitute the given value into the equation $5x - 4y = 20$ and solve for the other variable. Then, fill in the values in the table.

$$\begin{aligned} x &= rgb]0.0, 0.0, 1.00 \\ 5x - 4y &= 20 - 2 \\ 5 \cdot rgb]0.0, 0.0, 1.00 &= 20 \\ 0 - 4y &= 20 \\ -4y &= 20 \\ y &= -5 \\ &(0, -5) \end{aligned}$$

$$\begin{aligned} y &= rgb]1.0, 0.0, 0.00 \\ 5x - 4y &= 20 \\ 5x - 4 \cdot rgb]1.0, 0.0, 0.00 &= 20 \\ 5x - 0 &= 20 \\ 5x &= 20 \\ x &= 4 \\ &(4, 0) \end{aligned}$$

$$\begin{aligned} y &= rgb]0.5, 0.0, 0.55 \\ 5x - 4y &= 20 \\ 5x - 4 \cdot rgb]0.5, 0.0, 0.55 &= 20 \\ 5x - 20 &= 20 \\ 5x &= 40 \\ x &= 8 \\ &(8, 5) \end{aligned}$$

The results are summarized in the below table.

$5x - 4y = 20$		
x	y	(x, y)
0	-5	$(0, -5)$
4	0	$(4, 0)$
8	5	$(8, 5)$

Try It

19) Complete the table to find three solutions to this equation: $2x - 5y = 20$.

$2x - 5y = 20$		
x	y	(x, y)
0		
	0	
-5		

Solution

$2x - 5y = 20$		
x	y	(x, y)
0	-4	$(0, -4)$
10	0	$(10, 0)$
-5	-6	$(-5, -6)$

20) Complete the table to find three solutions to this equation: $3x - 4y = 12$.

$3x - 4y = 12$		
x	y	(x, y)
0		
	0	
-4		

Solution

$3x - 4y = 12$		
x	y	(x, y)
0	-3	$(0, -3)$
4	0	$(4, 0)$
-4	-6	$(-4, -6)$

Find Solutions to a Linear Equation

To find a solution to a linear equation, you really can pick *any* number you want to substitute into the equation for x or y. But since you'll need to use that number to solve for the other variable it's a good idea to choose a number that's easy to work with.

When the equation is in y-form, with the y by itself on one side of the equation, it is usually easier to choose values of x and then solve for y.

Example 3.9.8

Find three solutions to the equation $y = -3x + 2$.

Solution

We can substitute any value we want for x or any value for y. Since the equation is in y-form, it will be easier to substitute in values of x.

Let's pick $x = 0$, $x = 1$, and $x = -1$.

$x = rgb]0.0, 0.0, 1.00$ $y = -3x + 2$	$x = rgb]0.0, 0.0, 1.01$ $y = -3x + 2$	$x = rgb]0.0, 0.0, 1.0{-}rgb]0.0, 0.0, 1.01$ $y = -3x + 2$

Step 1: **Substitute the value into the equation.**

$y = -3 \times rgb]0.0, 0.0, 1.00 + 2$	$y = -3 \times rgb]0.0, 0.0, 1.01 + 2$	$y = -3 \times rgb]0.0, 0.0, 1.0(rgb]0.0, 0.0, 1.0{-}rgb]0.0, 0.0, 1.01rgb]0.0, 0.0, 1.0) + 2$

Step 2: **Simplify.**

$y = 0 + 2$	$y = -3 + 2$	$y = 3 + 2$

Step 3: **Simplify.**

$y = 2$	$y = -1$	$y = 5$

Step 4: **Write the ordered pair.**

$(0, 2)$	$(1, -1)$	$(-1, 5)$

Step 5: **Check.**

$$\begin{array}{lll} y = -3x + 2 & y = -3x + 2 & y = -3x + 2 \\ 2 \overset{?}{=} -3(0) + 2 & -1 \overset{?}{=} -3(1) + 2 & 5 \overset{?}{=} -3(-1) + 2 \\ 2 = 2 \checkmark & -1 = -1 \checkmark & 5 = 5 \checkmark \end{array}$$

So, $(0, 2)$, $(1, -1)$ and $(-1, 5)$ are all solutions to $y = -3x + 2$. We show them in the below table.

$y = -3x + 2$		
x	y	(x, y)
0	2	$(0, 2)$
1	-1	$(1, -1)$
-1	5	$(-1, 5)$

Try It

21) Find three solutions to this equation: $y = -2x + 3$.

Solution

Answers will vary.

22) Find three solutions to this equation: $y = -4x + 1$.

Solution

Answers will vary.

We have seen how using zero as one value of x makes finding the value of y easy. When an equation is in standard form, with both the x and y on the same side of the equation, it is usually easier to first find one solution when $x = 0$ find a second solution when $y = 0$, and then find a third solution.

Example 3.9.9

Find three solutions to the equation $3x + 2y = 6$.

Solution

We can substitute any value we want for x or any value for y. Since the equation is in standard form, let's pick first $x = 0$, then $y = 0$, and then find a third point.

$x = rgb]0.0, 0.0, 1.00$	$y = rgb]1.0, 0.0, 0.00$	$x = rgb]0.0, 0.0, 1.01$

Step 1: **Substitute the value into the equation.**

$3x + 2y = 6$	$3x + 2y = 6$	$3x + 2y = 6$

Step 2. **Simplify.**

$3(rgb]0.0, 0.0, 1.00) + 2y = 6$	$3x + 2(rgb]0.0, 0.0, 1.00) = 6$	$3(rgb]0.0, 0.0, 1.01) + 2y = 6$

Step 3: **Solve.**

$\begin{aligned} 0 + 2y &= 6 \\ 2y &= 6 \\ y &= 3 \end{aligned}$	$\begin{aligned} 3x + 0 &= 6 \\ 3x &= 6 \\ x &= 2 \end{aligned}$	$\begin{aligned} 3 + 2y &= 6 \\ 2y &= 3 \\ y &= \frac{3}{2} \end{aligned}$

Step 4: **Write the ordered pair.**

$(0, 3)$	$(2, 0)$	$(1, \frac{3}{2})$

Step 5: **Check.**

$$\begin{aligned} 3x + 2y &= 6 \\ 3(0) + 2(3) &\stackrel{?}{=} 6 \\ 0 + 6 &= 6 \\ 6 &= 6 \checkmark \end{aligned} \qquad \begin{aligned} 3x + 2y &= 6 \\ 3(2) + 2(0) &\stackrel{?}{=} 6 \\ 6 + 0 &= 6 \\ 6 &= 6 \checkmark \end{aligned} \qquad \begin{aligned} 3x + 2y &= 6 \\ 3(1) + 2(\frac{3}{2}) &\stackrel{?}{=} 6 \\ 3 + 3 &\stackrel{?}{=} 6 \\ 6 &= 6 \checkmark \end{aligned}$$

So $(0, 3)$, $(2, 0)$, and $(1, \frac{3}{2})$ are all solutions to the equation $3x + 2y = 6$. We can list these three solutions in below table.

$3x + 2y = 6$		
x	y	(x, y)
0	3	$(0, 3)$
2	0	$(2, 0)$
1	$\frac{3}{2}$	$(1, \frac{3}{2})$

Try It

23) Find three solutions to the equation $2x + 3y = 6$.

Solution

Answers will vary.

24) Find three solutions to the equation $4x + 2y = 8$.

Solution

Answers will vary.

Recognize the Relationship Between the Solutions of an Equation and its Graph

In the previous section, we found several solutions to the equation $3x + 2y = 6$. They are listed in the table below. So, the ordered pairs $(0, 3)$, $(2, 0)$, and $(1, \frac{3}{2})$ are some solutions to the equation $3x + 2y = 6$. We can plot these solutions in the rectangular coordinate system as shown in the below table.

$3x + 2y = 6$		
x	y	(x, y)
0	3	$(0, 3)$
2	0	$(2, 0)$
1	$\frac{3}{2}$	$(1, \frac{3}{2})$

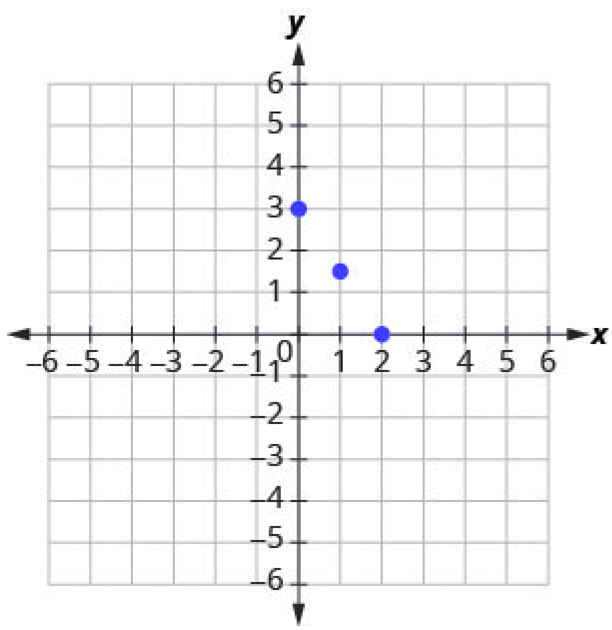


Figure 3.9.15

Notice how the points line up perfectly? We connect the points with a line to get the graph of the equation $3x + 2y = 6$. See Figure 3.9.16. Notice the arrows on the ends of each side of the line. These arrows indicate the line continues.

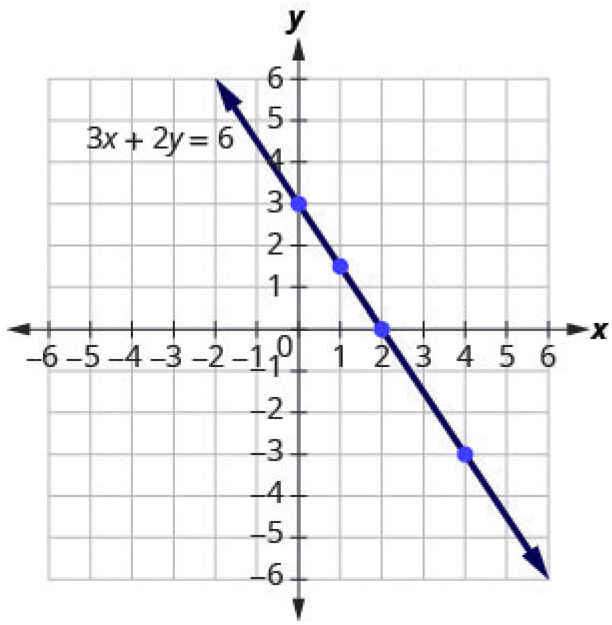


Figure 3.9.16

Every point on the line is a solution of the equation. Also, every solution of this equation is a point on this line. Points *not* on the line are not solutions.

Notice that the point whose coordinates are $(-2, 6)$ is on the line shown in Figure 3.9.17. If you substitute $x = -2$ and $y = 6$ into the equation, you find that it is a solution to the equation.

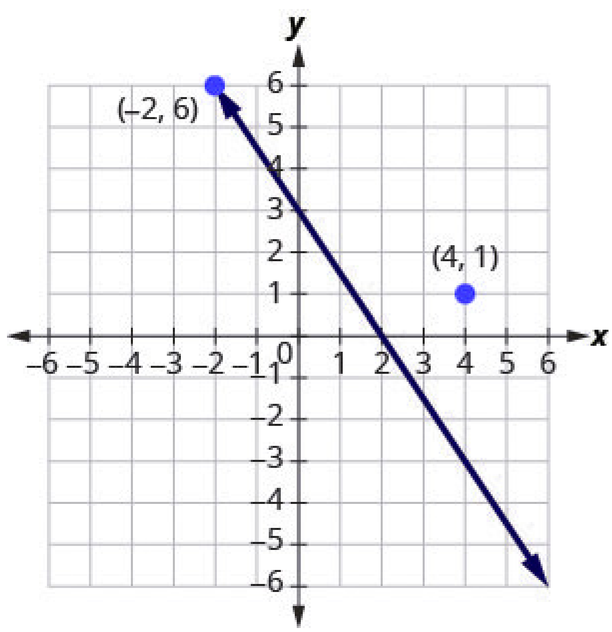


Figure 3.9.17

$$\text{Test}(rgb]0.0,0.0,1.0-2,rgb]1.0,0.0,0.06)$$
$$3x+2y=6$$
$$3(rgb]0.0,0.0,1.0-2)+2(rgb]1.0,0.0,0.06)=6$$
$$-6+12=6$$
$$6=6✓$$

So the point $(-2,6)$ is a solution to the equation $3x+2y=6$. (The phrase "the point whose coordinates are $(-2,6)$" is often shortened to "the point $(-2,6)$.")

$$\text{What about}(rgb]0.0,0.0,1.04,rgb]1.0,0.0,0.01)$$
$$3x+2y=6$$
$$3(rgb]0.0,0.0,1.04)+2(rgb]1.0,0.0,0.01)=6$$
$$12+2\stackrel{?}{=}6$$
$$14\neq 6✓$$

So $(4,1)$ is not a solution to the equation $3x+2y=6$. Therefore, the point $(4,1)$ is not on the line. See Figure 3.9.17. This is an example of the saying, "A picture is worth a thousand words." The line shows you *all* the solutions to the equation. Every point on the line is a solution of the equation. And, every solution of this equation is on this line. This line is called the *graph* of the equation $3x+2y=6$.

Graph of a Linear Equation

The **graph of a linear equation** $Ax + By = C$ is a line.

- Every point on the line is a solution of the equation.
- Every solution of this equation is a point on this line.

Example 3.9.10

The graph of $y = 2x - 3$ is shown.

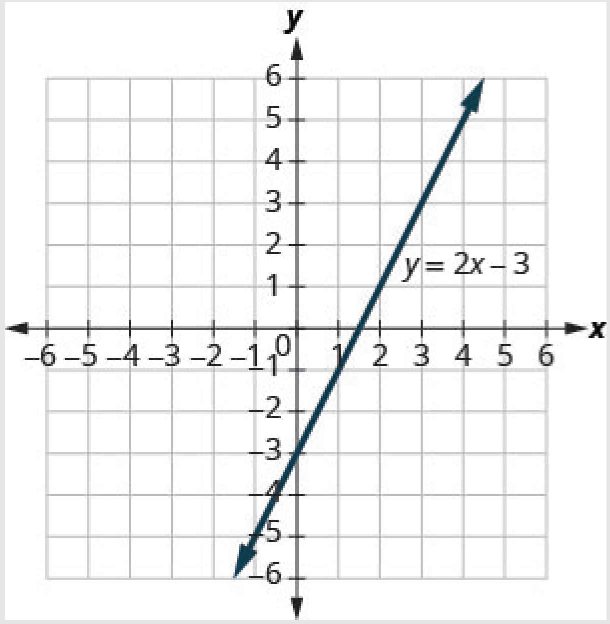


Figure 3.9.18

For each ordered pair, decide:

a. Is the ordered pair a solution to the equation?

b. Is the point on the line?

a. $(0, -3)$

b.$(3, 3)$

c. $(2, -3)$

d.$(-1, -5)$

Solution

Substitute the x– and y– values into the equation to check if the ordered pair is a solution to the equation.

a.

A:	**B:**	**C:**	**D:**
$(rgb]0.0, 0.0, 1.00, rgb]1.0, 0.0, 0.0 - 3)$ $y = 2x - 3$ $rgb]1.0, 0.0, 0.0 - 3 \overset{?}{=} 2(rgb]0.0, 0.0, 1.00) - 3$ $-3 = -3\checkmark$	$(rgb]0.0, 0.0, 1.03, rgb]1.0, 0.0, 0.03)$ $y = 2x - 3$ $rgb]1.0, 0.0, 0.03 \overset{?}{=} 2(rgb]0.0, 0.0, 1.03) - 3$ $3 = 3\checkmark$	$(rgb]0.0, 0.0, 1.02, rgb]1.0, 0.0, 0.0 - 3)$ $y = 2x - 3$ $rgb]1.0, 0.0, 0.0 - 3 \overset{?}{=} 2(rgb]0.0, 0.0, 1.02) - 3$ $-3 \neq 1$	$(rgb]0.0, 0.0, 1.0 - 1, rgb]1.0, 0.0, 0.0 - 5)$ $y = 2x - 3$ $rgb]1.0, 0.0, 0.0 - 5 \overset{?}{=} 2(rgb]0.0, 0.0, 1.0 - 1) - 3$ $-5 = -5\checkmark$
$(0, -3)$ is a solution.	$(3, 3)$ is a solution	$(2, -3)$ is not a solution	$(-1, -5)$ is a solution

b. Plot the points A $(0,3)$, B $(3,3)$, C $(2,-3)$, and D $(-1,-5)$.

The points $(0,3)$, $(3,3)$, and $(-1,-5)$ are on the line $y=2x-3$, and the point $(2,-3)$ is not on the line.

The points that are solutions to $y=2x-3$ are on the line, but the point that is not a solution is not on the line.

Try It

25) Use the graph of $y=3x-1$ to decide whether each ordered pair is:

- a solution to the equation.
- on the line.

a. $(0,-1)$
b. $(2,5)$

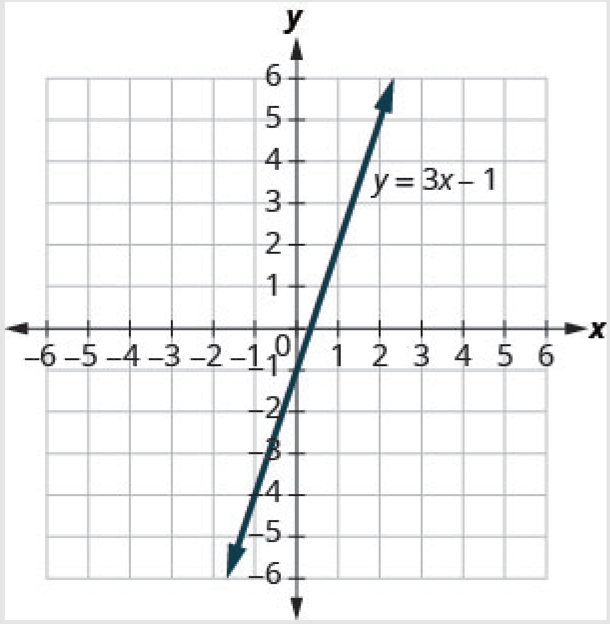


Figure 3.9.19

Solution

a. yes, yes

b. yes, yes

26) Use graph of $y = 3x - 1$ to decide whether each ordered pair is:

- a solution to the equation
- on the line

a. $(3, -1)$

b. $(-1, -4)$

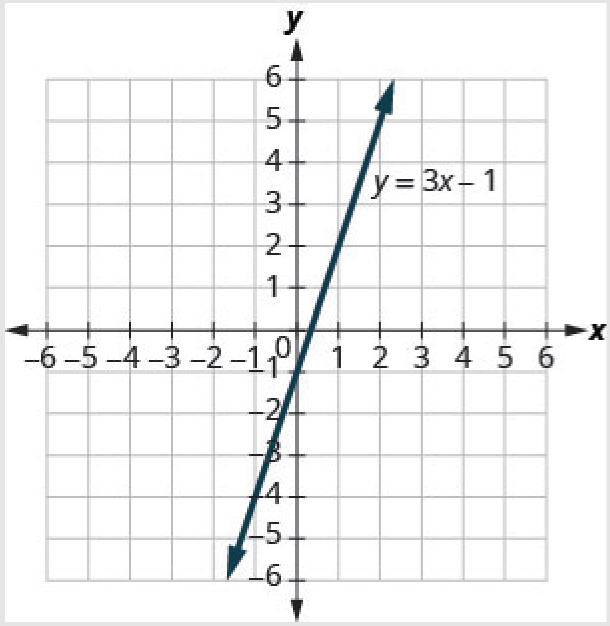


Figure 3.9.20

Solution

a. no, no

b. yes, yes

Graph a Linear Equation by Plotting Points

There are several methods that can be used to graph a linear equation. The method we used to graph $3x + 2y = 6$ is called plotting points, or the Point–Plotting Method.

Example 3.9.11

Graph the equation $y = 2x + 1$ by plotting points.

Solution

Step 1: **Find three points whose coordinates are solutions to the equation.**

You can choose any values for x or y.

In this case, since y is isolated on the left side of the equation, it is easier to choose values for x.

$$y = 2x + 1$$

$$x = rgb]0.0, 0.0, 1.00$$
$$y = 2x + 1$$
$$y = 2 \cdot rgb]0.0, 0.0, 1.00 + 1$$
$$y = 0 + 1$$
$$y = 1$$

$$x = rgb]0.0, 0.0, 1.01$$
$$y = 2x + 1$$
$$y = 2 \cdot rgb]0.0, 0.0, 1.01 + 1$$
$$y = 2 + 1$$
$$y = 3$$

$$x = rgb]0.0, 0.0, 1.0 - 2$$
$$y = 2x + 1$$
$$y = 2 \cdot rgb]0.0, 0.0, 1.0 - 2 + 1$$
$$y = -4 + 1$$
$$y = -3$$

Organize the solutions in a table.

Put the three solutions in a table.

y=2x+1		
x	**y**	**(x,y)**
0	1	(0,1)
1	3	(1,3)
-2	-3	(-2.-3)

Step 2: **Plot the points in a rectangular coordinate system.**

Plot: (0,1), (1, 3), (-2,-3).

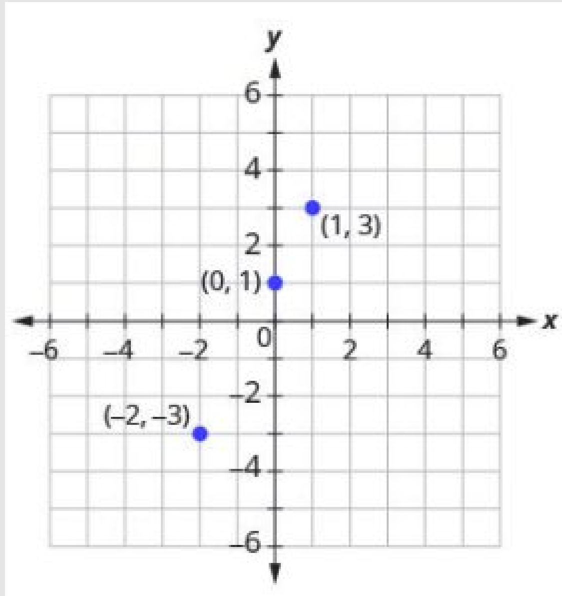


Figure 3.9.21

Check that the points line up. If they do not, carefully check your work!

Do the points line up? Yes, the points line up.

Step 3: **Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.**

This line is the graph of $y = 2x + 1$.

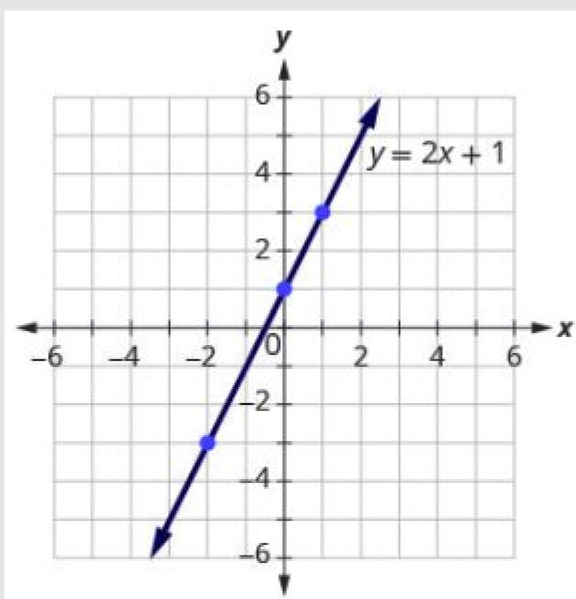


Figure 3.9.22

Try It

27) Graph the equation by plotting points: $y = 2x - 3$.

Solution

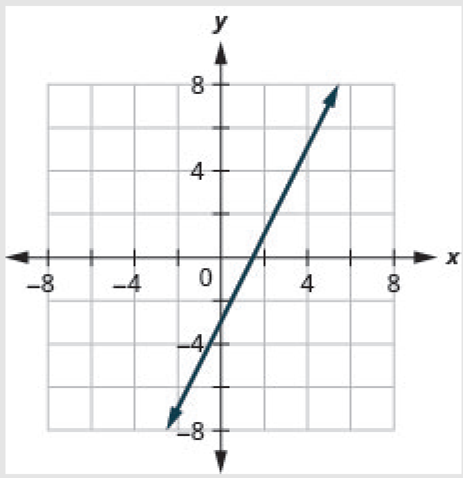


Figure 3.9.23

28) Graph the equation by plotting points: $y = -2x + 4$.

Solution

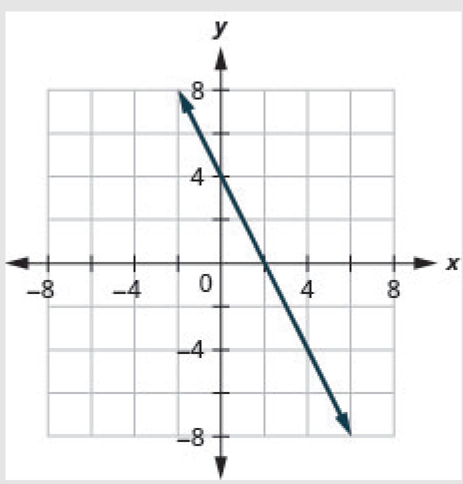


Figure 3.9.24

The steps to take when graphing a linear equation by plotting points are summarized below.

HOW TO

Graph a linear equation by plotting points.

1. Find three points whose coordinates are solutions to the equation. Organize them in a table.
2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.
3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

It is true that it only takes two points to determine a line, but it is a good habit to use three points. If you only plot two points and one of them is incorrect, you can still draw a line but it will not represent the solutions to the equation. It will be the wrong line.

If you use three points, and one is incorrect, the points will not line up. This tells you something is wrong and you need to check your work. Look at the difference between part (a) and part (b) in Figure 3.9.25.

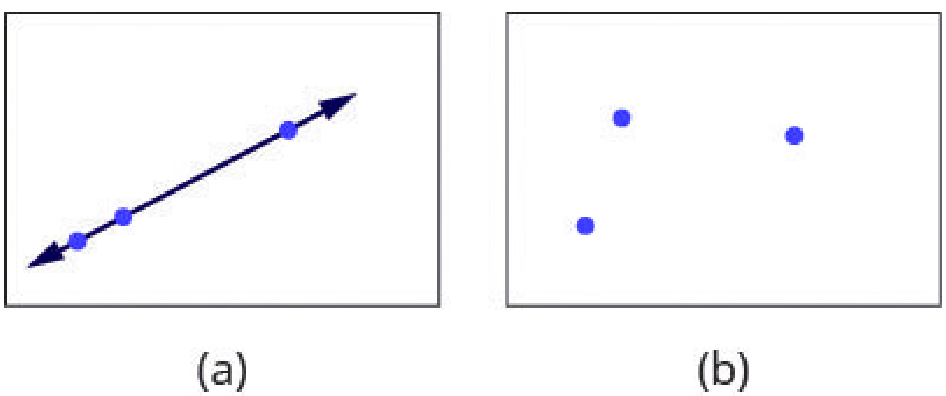


Figure 3.9.25

Let's do another example. This time, we'll show the last two steps all on one grid.

Example 3.9.12

Graph the equation $y = -3x$.

Solution

Step 1: **Find three points that are solutions to the equation.**

Here, again, it's easier to choose values for x. Do you see why?

$$x = rgb]0.0, 0.0, 1.00$$
$$y = -3x$$
$$y = -3 \cdot rgb]0.0, 0.0, 1.00$$
$$y = 0$$

$$x = rgb]0.0, 0.0, 1.01$$
$$y = -3$$
$$y = -3 \cdot rgb]0.0, 0.0, 1.01$$
$$y = -3$$

$$x = rgb]0.0, 0.0, 1.0 - 2$$
$$y = -3$$
$$y = -3(rgb]0.0, 0.0, 1.0 - 2)$$
$$y = 6$$

Step 2: **List the points in the following table.**

$y = -3x$		
x	y	(x, y)
0	0	$(0, 0)$
1	-3	$(1, -3)$
-2	6	$(-2, 6)$

Step 3: **Plot the points, check that they line up, and draw the line.**

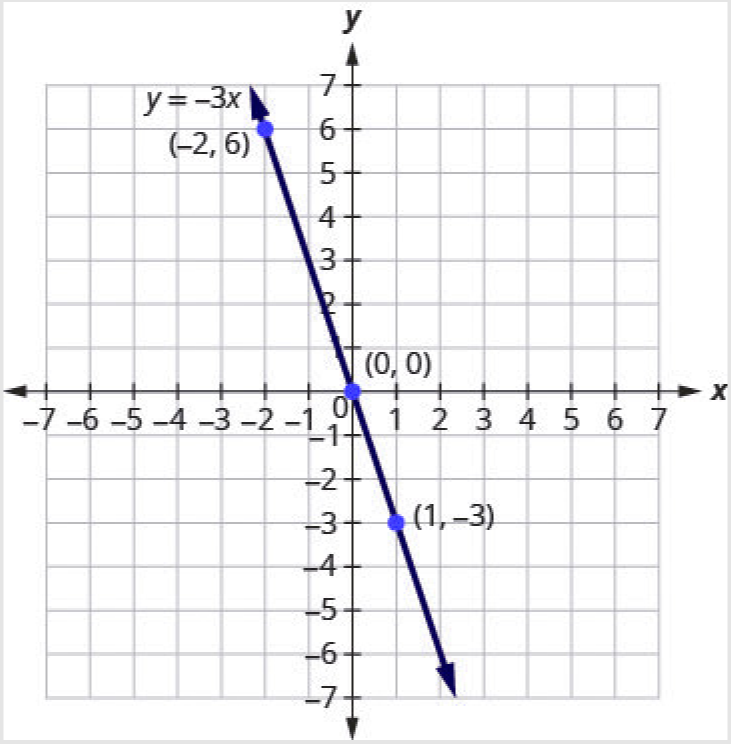


Figure 3.9.26

Try It

29) Graph the equation by plotting points: $y = -4x$.

Solution

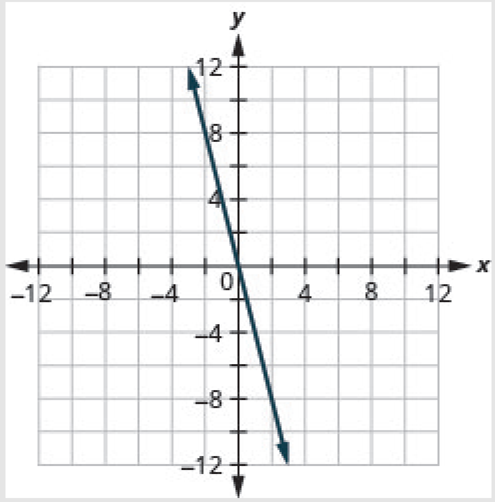


Figure 3.9.27

30) Graph the equation by plotting points: $y = x$.

Solution

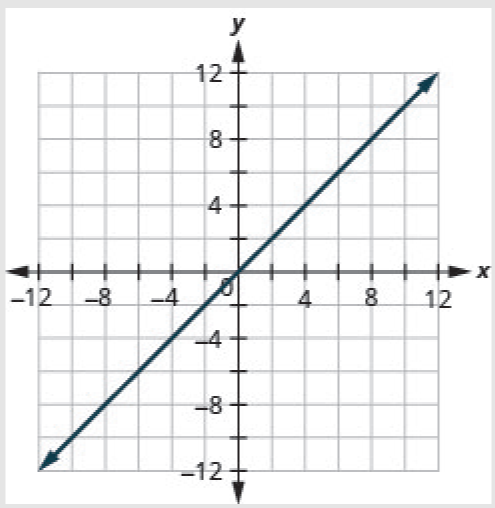


Figure 3.9.28

When an equation includes a fraction as the coefficient of x, we can still substitute any numbers for x. But the math is easier if we make 'good' choices for the values of x. This way we will avoid fraction answers, which are hard to graph precisely.

Example 3.9.13

Graph the equation $y = \frac{1}{2}x + 3$

Solution

Step 1: **Find three points that are solutions to the equation.**

Since this equation has the fraction $\frac{1}{2}$ as a coefficient of x, we will choose values of x carefully. We will use zero as one choice and multiples of 2 for the other choices. Why are multiples of 2 a good choice for values of x?

$$x = rgb]0.0, 0.0, 1.00$$
$$y = \frac{1}{2}x + 3$$
$$y = \frac{1}{2}(rgb]0.0, 0.0, 1.00) + 3$$
$$y = 0 + 3$$
$$y = 3$$

$$x = rgb]0.0, 0.0, 1.02$$
$$y = \frac{1}{2}x + 3$$
$$y = \frac{1}{2}(rgb]0.0, 0.0, 1.02) + 3$$
$$y = 1 + 3$$
$$y = 4$$

$$x = rgb]0.0, 0.0, 1.04$$
$$y = \frac{1}{2}x + 3$$
$$y = \frac{1}{2}(rgb]0.0, 0.0, 1.04) + 3$$
$$y = 2 + 3$$
$$y = 5$$

The points are shown in the below table.

$y = \frac{1}{2}x + 3$		
x	y	(x, y)
0	3	$(0, 3)$
2	4	$(2, 4)$
4	5	$(4, 5)$

Step 2: **Plot the points, check that they line up, and draw the line.**

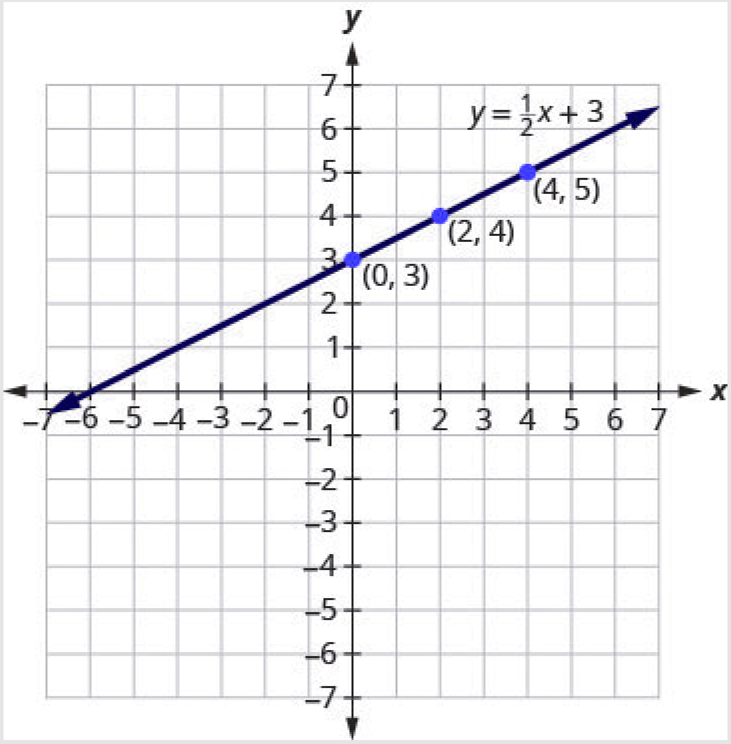


Figure 3.9.29

Try It

31) Graph the equation $y = \frac{1}{3}x - 1$

Solution

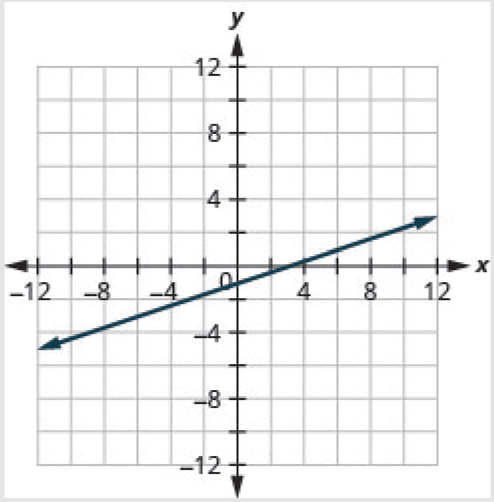


Figure 3.9.30

32) Graph the equation $y = \frac{1}{4}x + 2$.

Solution

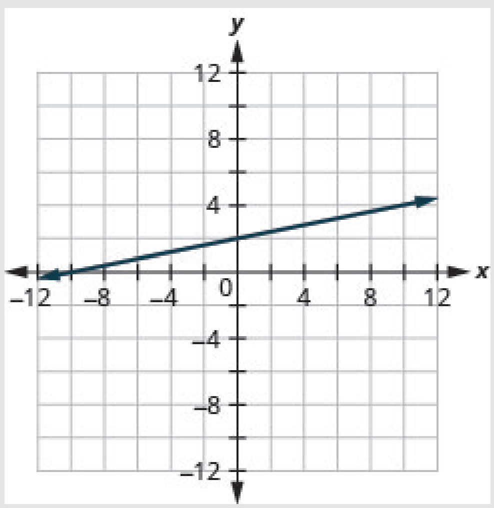


Figure 3.9.31

So far, all the equations we graphed had y given in terms of x. Now we'll graph an equation with x and y on the same side. Let's see what happens in the equation $2x + y = 3$. If $y = 0$ what is the value of x?

$$y = rgb]1.0, 0.0, 0.00$$

$$2x + y = 3$$

$$2x + rgb]1.0, 0.0, 0.00 = 3$$

$$2x = 3$$

$$x = \frac{3}{2}$$

$$\left(\frac{3}{2}, 0\right)$$

This point has a fraction for the $x-$ coordinate and, while we could graph this point, it is hard to be precise graphing fractions. Remember in the example $y = \frac{1}{2}x + 3$, we carefully chose values for x so as not to graph fractions at all. If we solve the equation $2x + y = 3$ for y, it will be easier to find three solutions to the equation.

$$2x + y = 3$$

$$y = -2x + 3$$

The solutions for $x = 0$, $x = 1$, and $x = -1$ are shown in the below table. The graph is shown in the Figure 3.9.32.

$2x + y = 3$		
x	y	(x, y)
0	3	$(0, 3)$
1	1	$(1, 1)$
-1	5	$(-1, 5)$

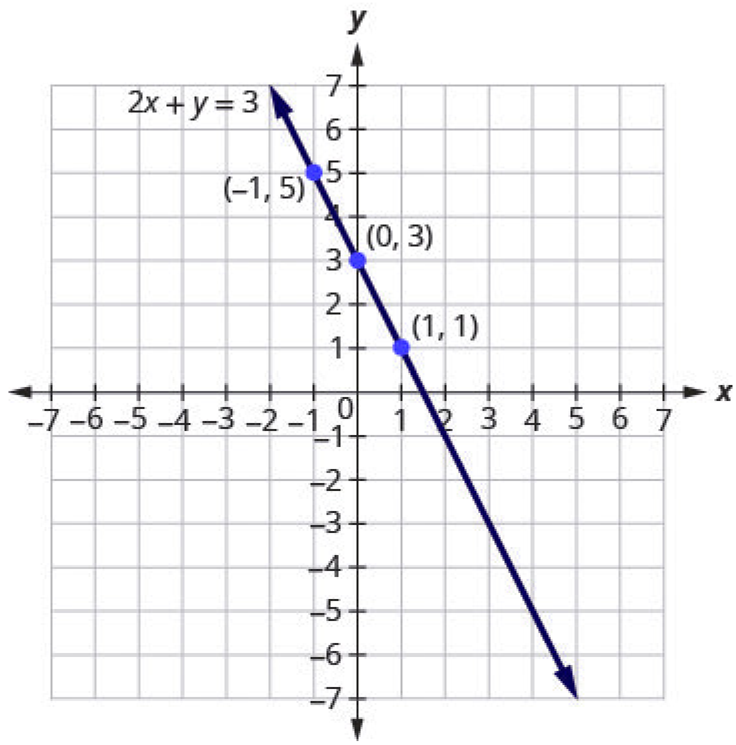


Figure 3.9.32

Can you locate the point $(\frac{3}{2}, 0)$, which we found by letting $y = 0$, on the line?

Example 3.9.14

Graph the equation $3x + y = -1$.

Solution

Step 1: **Find three points that are solutions to the equation.**

$$3x + y = -1$$

Step 2: **First, solve the equation for y.**

$$y = -3x - 1$$

Step 3: **We'll let x be 0, 1, and -1 to find 3 points.**

The ordered pairs are shown in the below table. Plot the points, check that they line up, and draw the line. See Figure 3.9.33.

$3x+y=-1$		
x	y	(x,y)
0	-1	$(0,-1)$
1	-4	$(1,-4)$
-1	2	$(-1,2)$

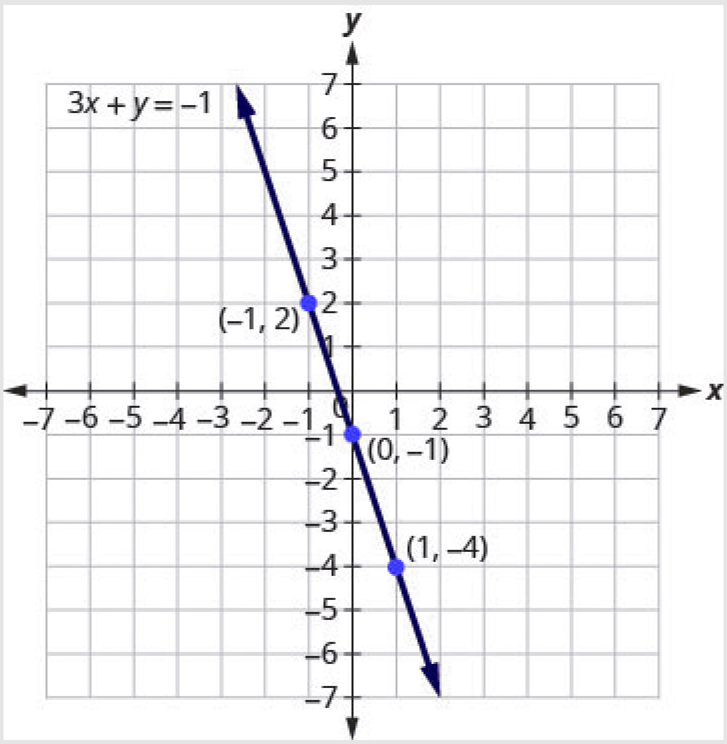


Figure 3.9.33

Try It

33) Graph the equation $2x+y=2$.

Solution

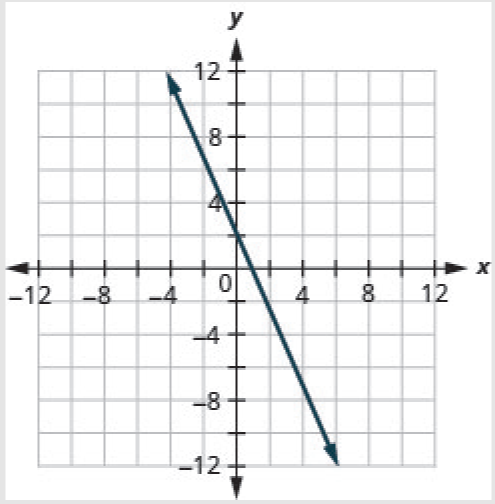


Figure 3.9.34

34) Graph the equation $4x + y = -3$.

Solution

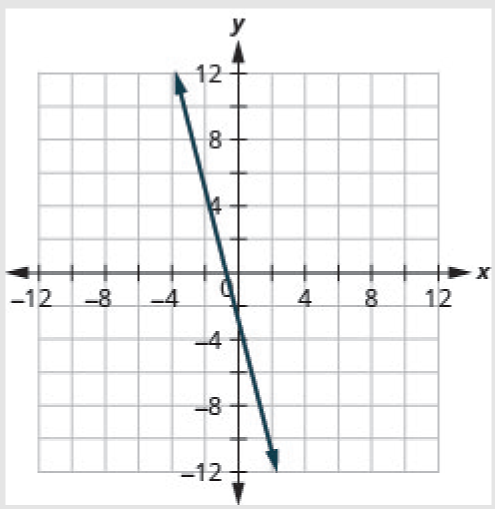


Figure 3.9.35

If you can choose any three points to graph a line, how will you know if your graph matches the one shown in the answers in the book? If the points where the graphs cross the x– and y-axis are the same, the graphs match!

The equation in Example 3.9.14, was written in standard form, with both x and y on the same side. We solved that equation for y in just one step. But for other equations in standard form it is not that easy to solve for y, so we will leave them in standard form. We can still find a first point to plot by letting $x = 0$ and solving for y

. We can plot a second point by letting $y = 0$ and then solving for x. Then we will plot a third point by using some other value for x or y.

Example 3.9.15

Graph the equation $2x - 3y = 6$

Solution

Step 1: **Find three points that are solutions to the equation.**

$$2x - 3y = 6$$

Step 2: **First, let $x = 0$.**

$$2(0) - 3y = 6$$

Step 3: **Solve for y.**

$$\begin{aligned} -3y &= 6 \\ y &= -2 \end{aligned}$$

Step 4: **Now let $y = 0$.**

$$2x - 3(0) = 6$$

Step 5: **Solve for x.**

$$\begin{aligned} 2x &= 6 \\ x &= 3 \end{aligned}$$

Step 6: **We need a third point. Remember, we can choose any value for x or y. We'll let $x = 6$.**

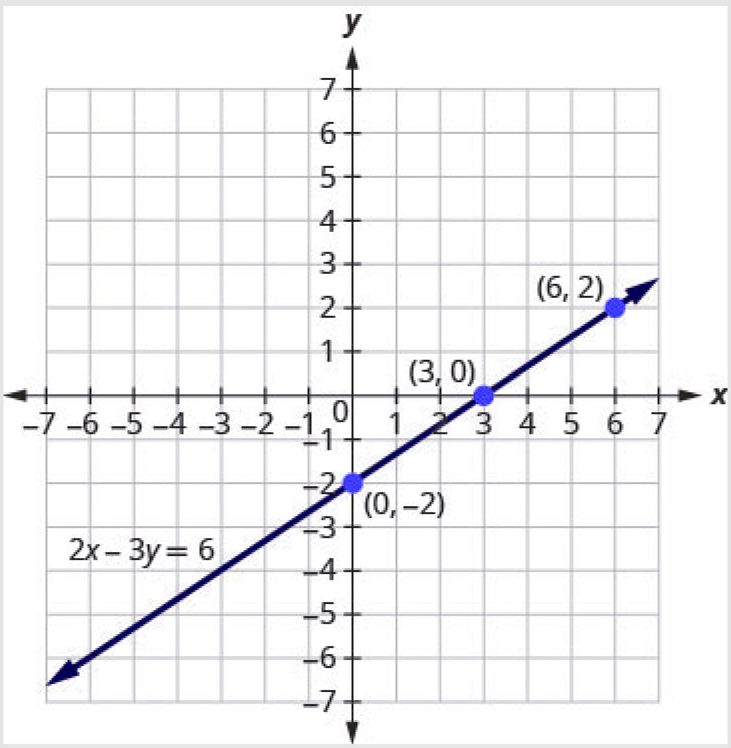
$$2(6) - 3y = 6$$

Step 7: **Solve for y.**

$$\begin{aligned} 12 - 3y &= 6 \\ -3y &= -6 \\ y &= 2 \end{aligned}$$

We list the ordered pairs in the below table. Plot the points, check that they line up, and draw the line. See Figure 3.9.36

$2x - 3y = 6$		
x	y	(x, y)
0	-2	$(0, -2)$
3	0	$(3, 0)$
6	2	$(6, 2)$



3.9.36

Try It

35) Graph the equation $4x + 2y = 8$.

Solution

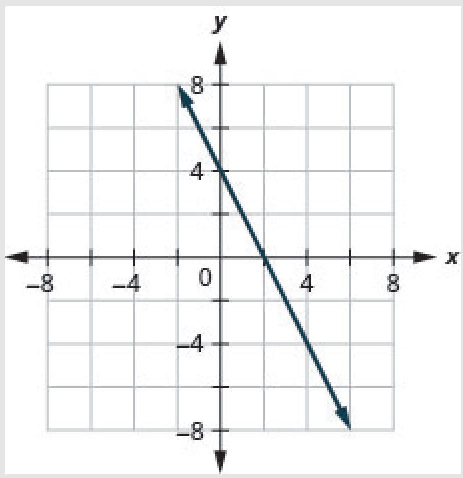


Figure 3.9.37

36) Graph the equation $2x - 4y = 8$.

Solution

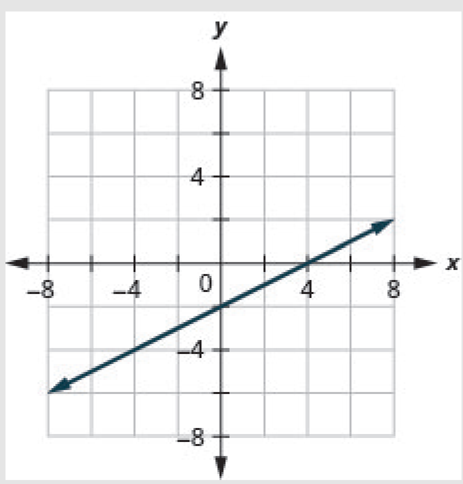


Figure 3.9.38

Graph Vertical and Horizontal Lines

Can we graph an equation with only one variable? Just x and no y, or just y without an x? How will we make a table of values to get the points to plot?

Let's consider the equation $x = -3$. This equation has only one variable, x. The equation says that x is *always* equal to -3, so its value does not depend on y. No matter what y is, the value of x is always -3.

So to make a table of values, write -3 in for all the x values. Then choose any values for y. Since x does not depend on y, you can choose any numbers you like. But to fit the points on our coordinate graph, we'll use 1, 2, and 3 for the y-coordinates. See the table below.

$x = -3$		
x	y	(x, y)
-3	1	$(-3, 1)$
-3	2	$(-3, 2)$
-3	3	$(-3, 3)$

Plot the points from the above table and connect them with a straight line. Notice in Figure 3.9.39 that we have graphed a vertical line.

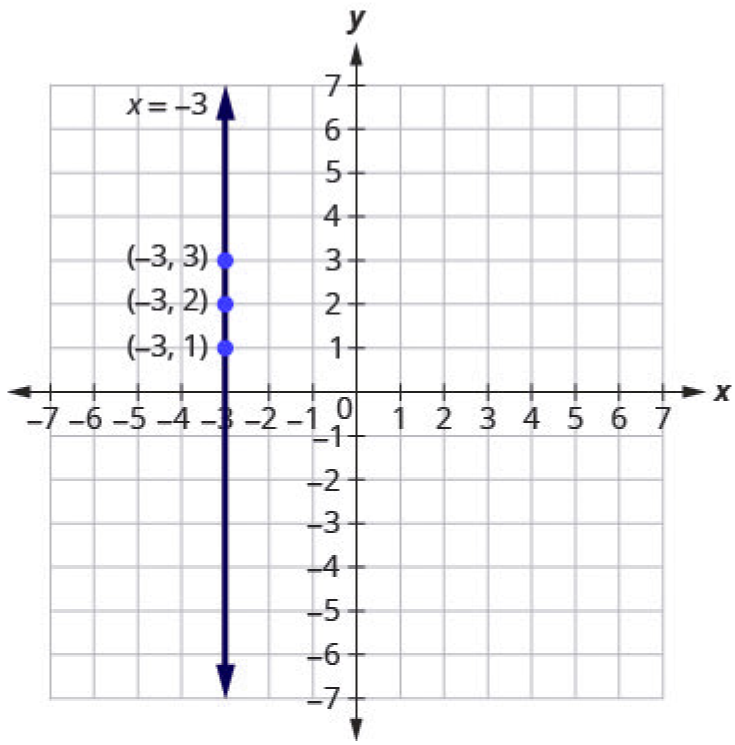


Figure 3.9.39

Vertical Line

A vertical line is the graph of an equation of the form $x = a$.

The line passes through the x-axis at $(a, 0)$.

Example 3.9.16

Graph the equation $x = 2$.

Solution

The equation has only one variable, x, and x is always equal to 2. We create the table below where x is always 2 and then put in any values for y. The graph is a vertical line passing through the x-axis at 2. See Figure 3.9.40.

$x = 2$		
x	y	(x, y)
2	1	$(2, 1)$
2	2	$(2, 2)$
2	3	$(2, 3)$

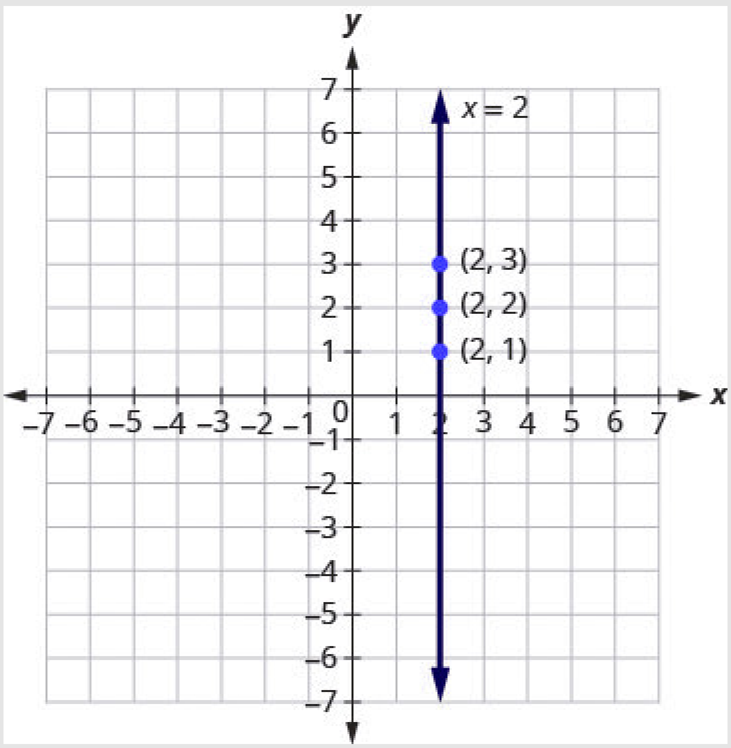


Figure 3.9.40

Try It

37) Graph the equation $x = 5$.

Solution

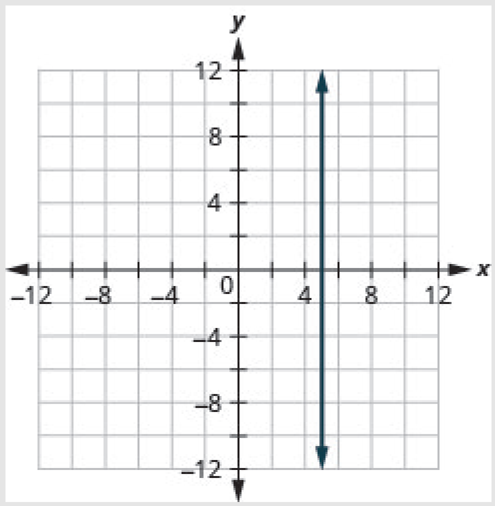


Figure 3.9.41

38) Graph the equation $x = -2$.

Solution

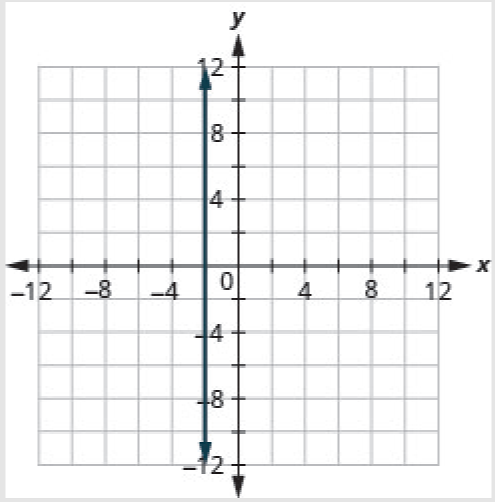


Figure 3.9.42

What if the equation has y but no x? Let's graph the equation $y = 4$. This time the y-value is a constant, so in this equation, y does not depend on x. Fill in 4 for all the y's in the below table and then choose any values for x. We'll use 0, 2, and 4 for the x-coordinates.

$y = 4$		
x	y	(x, y)
0	4	$(0, 4)$
2	4	$(2, 4)$
4	4	$(4, 4)$

The graph is a horizontal line passing through the y-axis at 4. See Figure 3.9.43.

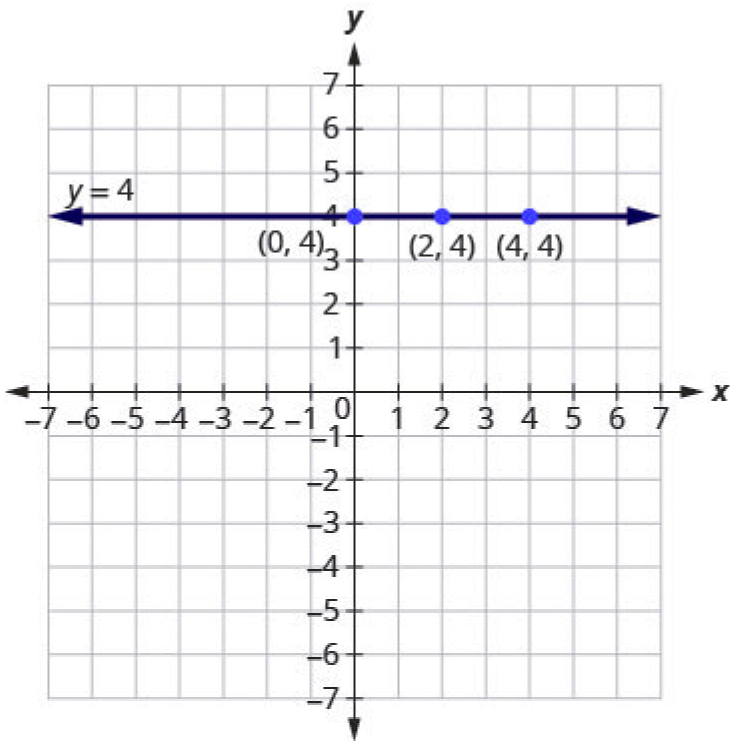


Figure 3.9.43

Horizontal Line

A horizontal line is the graph of an equation of the form $y = b$.

The line passes through the y-axis at $(0, b)$.

Example 3.9.17

Graph the equation $y = -1$

Solution

The equation $y = -1$ has only one variable, y. The value of y is constant. All the ordered pairs in the below table have the same y-coordinate. The graph is a horizontal line passing through the y-axis at -1, as shown in Figure 3.9.44.

$y = -1$		
x	y	(x, y)
0	-1	$(0, -1)$
3	-1	$(3, -1)$
-3	-1	$(-3, -1)$

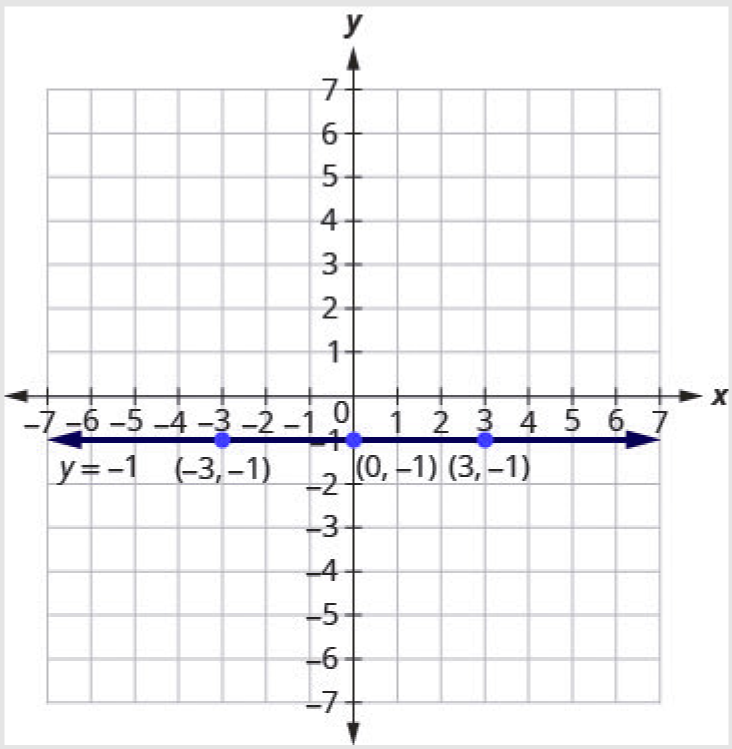


Figure 3.9.44

Try It

39) Graph the equation $y = -4$.

Solution

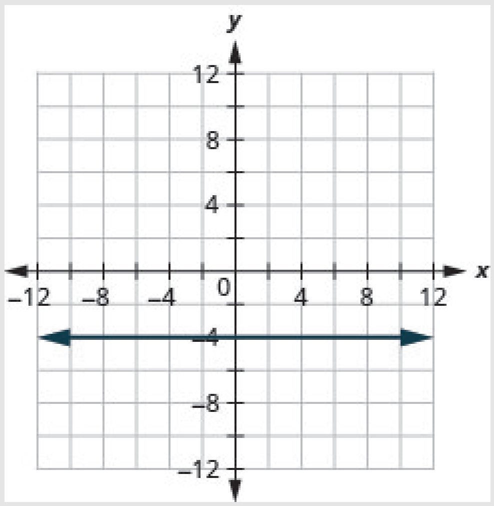


Figure 3.9.45

40) Graph the equation $y = 3$.

Solution

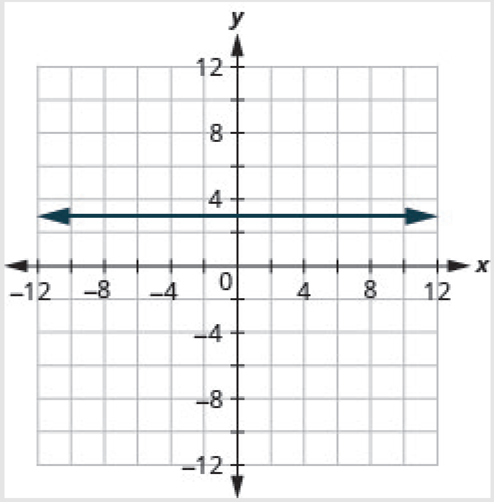


Figure 3.9.46

The equations for vertical and horizontal lines look very similar to equations like $y = 4x$. What is the difference between the equations $y = 4x$ and $y = 4$?

The equation $y = 4x$ has both x and y. The value of y depends on the value of x. The y-coordinate changes according to the value of x. The equation $y = 4$ has only one variable. The value of y is constant. The y-coordinate is always 4. It does not depend on the value of x. See the table below.

$y = 4x$		
x	y	(x, y)
0	0	$(0, 0)$
1	4	$(1, 4)$
2	8	$(2, 8)$

$y = 4$		
x	y	
0	4	
1	4	
2	4	

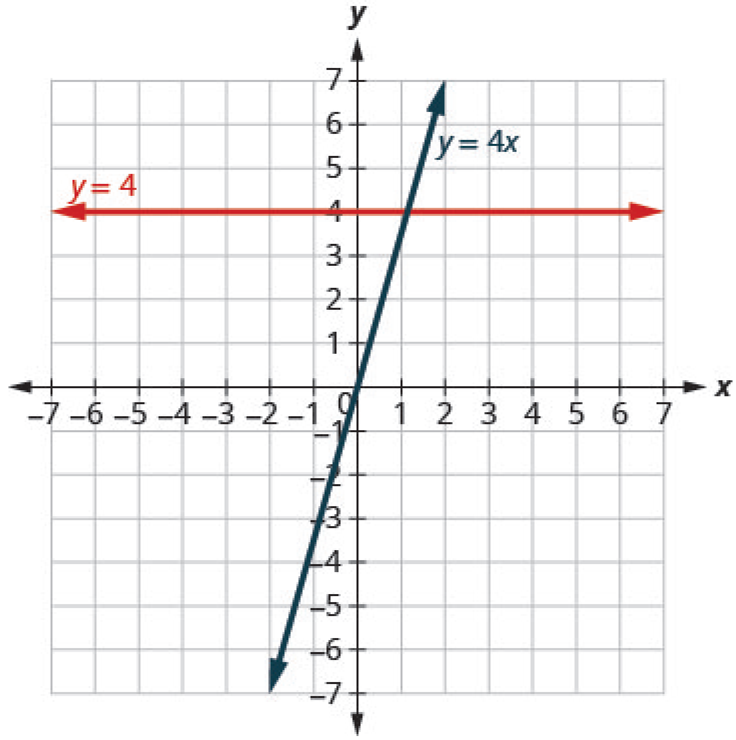


Figure 3.9.47

Notice, in Figure 3.9.47, the equation $y = 4x$ gives a slanted line, while $y = 4$ gives a horizontal line.

Example 3.9.18

Graph $y = -3x$ and $y = -3$ in the same rectangular coordinate system.

Solution

Notice that the first equation has the variable x, while the second does not. See the below table. The two graphs are shown in Figure 3.9.48.

$y=-3x$			$y=-3$		
x	y	(x,y)	x	y	(x,y)
0	0	$(0,0)$	0	-3	$(0,-3)$
1	-3	$(1,-3)$	1	-3	$(1,-3)$
2	-6	$(2,-6)$	2	-3	$(2,-3)$

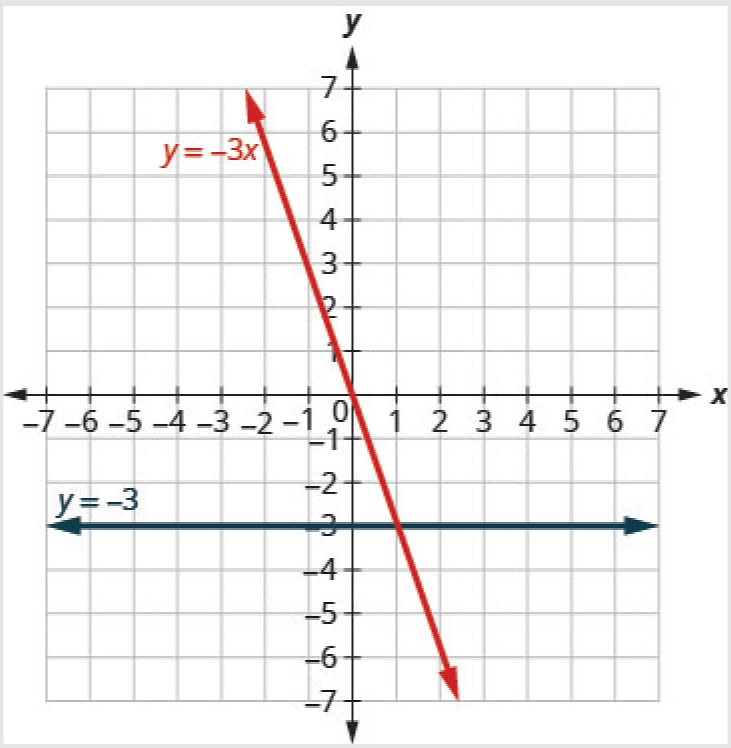


Figure 3.9.48

Try It

41) Graph $y=-4x$ and $y=-4$ in the same rectangular coordinate system.

Solution

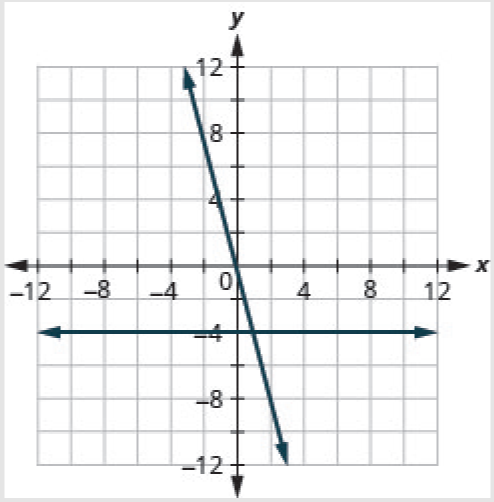


Figure 3.9.49

42) Graph $y = 3$ and $y = 3x$ in the same rectangular coordinate system.

Solution

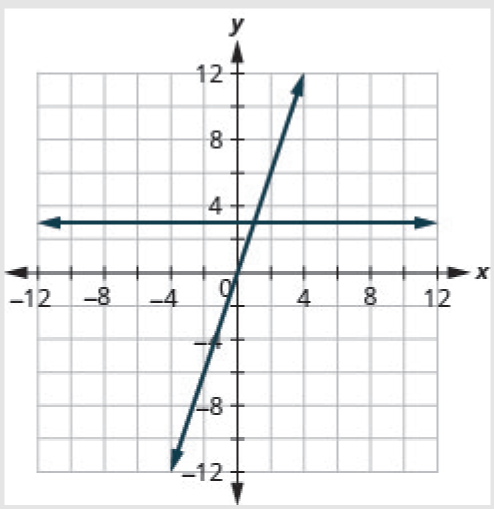


Figure 3.9.50

Intercepts on a Graph

Every linear equation can be represented by a unique line that shows all the solutions of the equation. We have seen that when graphing a line by plotting points, you can use any three solutions to graph. This means that two people graphing the line might use different sets of three points.

At first glance, their two lines might not appear to be the same, since they would have different points labelled. But if all the work was done correctly, the lines should be exactly the same. One way to recognize that they are indeed the same line is to look at where the line crosses the x-axis and the y-axis. These points are called the **intercepts of the line**.

Intercepts of a Line

The points where a line crosses the x-axis and the y-axis are called the intercepts of a line.

Let's look at the graphs of the lines in Figure 3.9.51.

Examples of graphs crossing the x-negative axis.

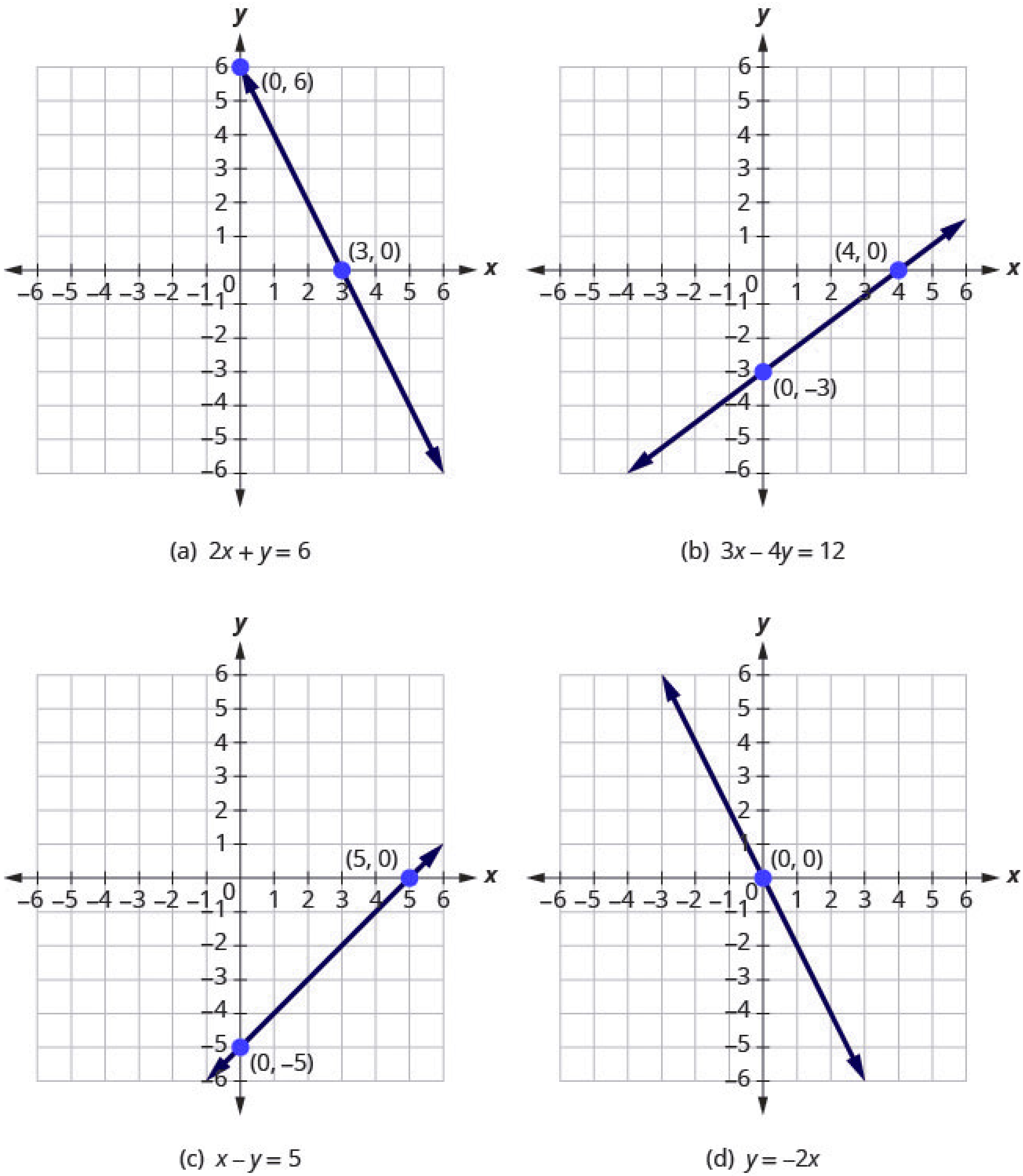


Figure 3.9.51

First, notice where each of these lines crosses the x negative axis. See the table below.

Figure	The line crosses the x-axis at:	Ordered pair of this point
Figure (a)	3	$(3, 0)$
Figure (b)	4	$(4, 0)$
Figure (c)	5	$(5, 0)$
Figure (d)	0	$(0, 0)$

Do you see a pattern?

For each row, the y-coordinate of the point where the line crosses the x-axis is zero. The point where the line crosses the x-axis has the form $(a, 0)$ and is called the **[latex]x[/latex]-intercept** of a line. The x-intercept occurs when y is zero.

Now, let's look at the points where these lines cross the y-axis. See the table below.

Figure	The line crosses the y-axis at:	Ordered pair for this point
Figure (a)	6	$(0, 6)$
Figure (b)	-3	$(0, -3)$
Figure (c)	-5	$(0, 5)$
Figure (d)	0	$(0, 0)$

What is the pattern here?

In each row, the x-coordinate of the point where the line crosses the y-axis is zero. The point where the line crosses the y-axis has the form $(0, b)$ and is called the y–intercept of the line. The y-intercept occurs when x is zero.

The x-intercept is the point $(a, 0)$ where the line crosses the x-axis.

The y-intercept is the point $(0, b)$ where the line crosses the y-axis.

	x	y
The x-intercept occurs when y is zero.	a	0
The y-intercept occurs when x is zero	0	b

Example 3.9.19

Find the x-intercept and y-intercept on each graph.

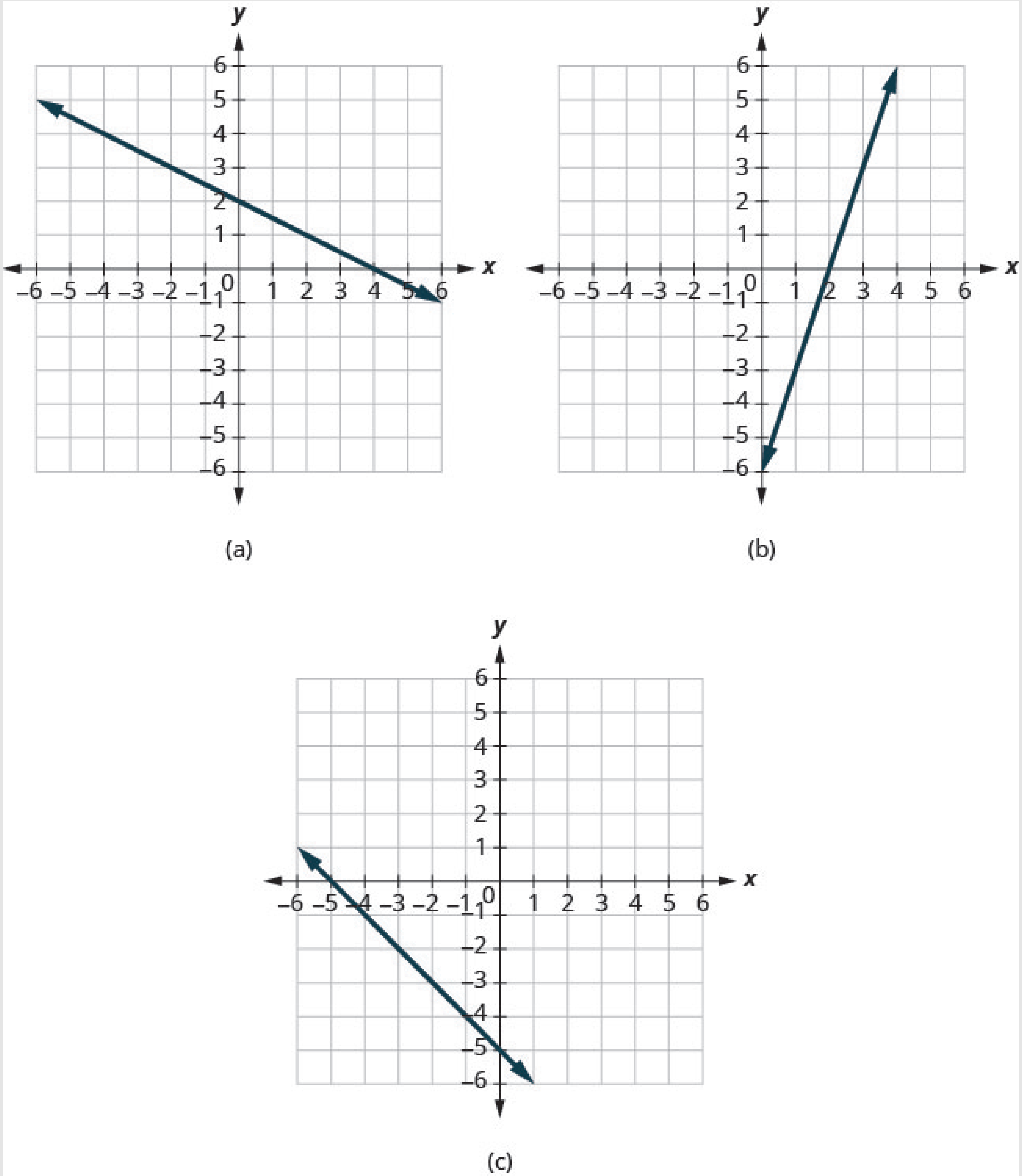


Figure 3.9.52

Solution

a. The graph crosses the x-axis at the point $(4, 0)$. The x-intercept is $(4, 0)$.

The graph crosses the y-axis at the point $(0, 2)$. The y-intercept is $(0, 2)$.

b. The graph crosses the x-axis at the point $(2, 0)$. The x-intercept is $(2, 0)$.
The graph crosses the y-axis at the point $(0, -6)$. The y-intercept is $(0, -6)$.

c. The graph crosses the x-axis at the point $(-5, 0)$. The x-intercept is $(-5, 0)$.
The graph crosses the y-axis at the point $(0, -5)$. The y-intercept is $(0, -5)$.

Try It

43) Find the x-intercept and y-intercept on the graph.

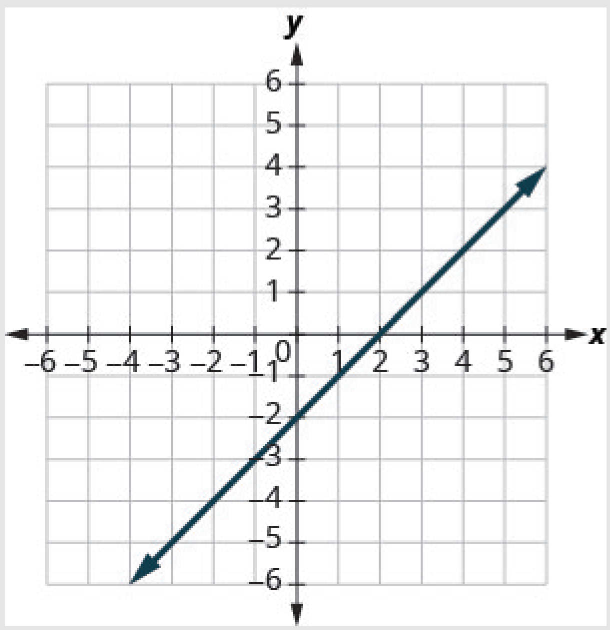


Figure 3.9.53

Solution

x-intercept: $(2, 0)$; y-intercept: $(0, -2)$

44) Find the x-intercept and y-intercept on the graph.

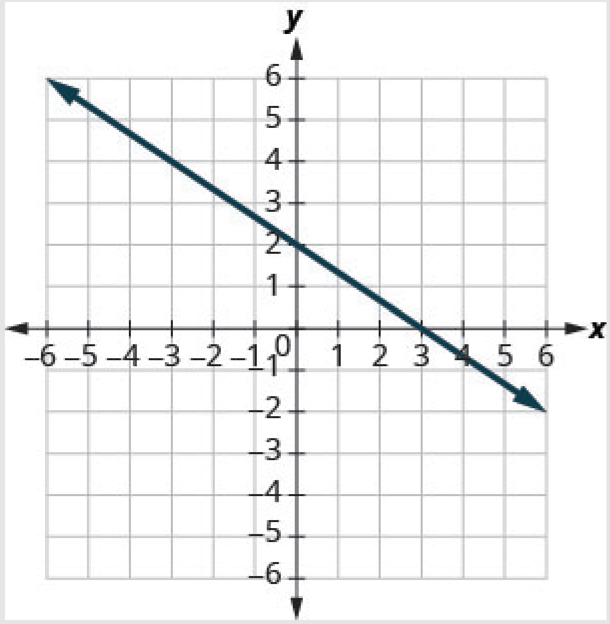


Figure 3.9.54

Solution

x-intercept: $(3, 0)$, y-intercept: $(0, 2)$

Intercepts from an Equation of a Line

Recognizing that the x-intercept occurs when y is zero and that the y-intercept occurs when x is zero, gives us a method to find the intercepts of a line from its equation. To find the x-intercept, let $y = 0$ and solve for x. To find the y-intercept, let $x = 0$ and solve for y.

Find the x-intercept and y-intercept from the *Equation of a Line*

Use the equation of the line. To find:

- the x-intercept of the line, let $y = 0$ and solve for x.
- the y-intercept of the line, let $x = 0$ and solve for y.

Example 3.9.20

Find the intercepts of $2x + y = 6$.

Solution

We will let $y = 0$ to find the x-intercept, and let $x = 0$ to find the y-intercept. We will fill in the table, which reminds us of what we need to find.

$2x + y = 6$		
x	y	
	0	x-intercept
0		y-intercept

To find the x-intercept, let $y = 0$.

Step 1: **Let $y = 0$.**

$$2x + rgb]1.0, 0.0, 0.0y = 6$$
$$2x \ + \ rgb]1.0, 0.0, 0.00 = 6$$

Step 2: **Simplify.**

$$2x = 6$$
$$x = 3$$

Step 3: **The x-intercept is.**

$$(3, 0)$$

Step 4: **To find the y-intercept, let $x = 0$.**

Step 5: **Let $x = 0$.**

$$2rgb]1.0, 0.0, 0.0x \ + \ y \ = \ 6$$
$$2 \ \times \ rgb]1.0, 0.0, 0.00 \ + \ y \ = \ 6$$

Step 6: **Simplify.**

$$0 \ + \ y \ = \ 6$$
$$y \ = \ 6$$

Step 7: **The y-intercept is.**

$$(0,6)$$

The intercepts are the points $(3,0)$ and $(0,6)$ as shown in the below table.

$2x+y=6$	
x	y
3	0
0	6

Try It

45) Find the intercepts of $3x+y=12$.

Solution
x-intercept: $(4,0)$, y-intercept: $(0,12)$

46) Find the intercepts of $x+4y=8$.

Solution
x-intercept: $(8,0)$, y-intercept: $(0,2)$

Example 3.9.21

Find the intercepts of $4x-3y=12$.

Solution
Step 1: **To find the x-intercept, let $y=0$.**

$$4x - 3rgb]1.0, 0.0, 0.0y = 12$$
$$4x - 3 \times (rgb]1.0, 0.0, 0.00) = 12$$

Step 3: **Simplify.**

$$\begin{aligned} 4x - 0 &= 12 \\ 4x &= 12 \\ x &= 3 \end{aligned}$$

Step 4: **The x-intercept is:**

$$(3, 0)$$

Step 5: **To find the y-intercept, let $x = 0$.**

Step 6: **Let $x = 0$.**

$$4rgb]1.0, 0.0, 0.0x - 3y = 12$$
$$4 \times rgb]1.0, 0.0, 0.00 - 3y = 12$$

Step 7: **Simplify.**

$$\begin{aligned} 0 - 3y &= 12 \\ -3y &= 12 \\ y &= -4 \end{aligned}$$

Step 8: **The y-intercept is:**

$$(0, -4)$$

The intercepts are the points $(3, 0)$ and $(0, -4)$ as shown in the following table.

$4x - 3y = 12$	
x	y
3	0
0	-4

Try It

47) Find the intercepts of $3x – 4y = 12$.

Solution
x-intercept: $(4, 0)$, y-intercept: $(0, -3)$

48) Find the intercepts of $2x – 4y = 8$.

Solution
x-intercept: $(4, 0)$, y-intercept: $(0, -2)$

Graph a Line Using the Intercepts

To graph a linear equation by plotting points, you need to find three points whose coordinates are solutions to the equation. You can use the x-intercept and y-intercept as two of your three points. Find the intercepts, and then find a third point to ensure accuracy. Make sure the points line up—then draw the line. This method is often the quickest way to graph a line.

Example 3.9.22

Graph $-x + 2y = 6$ using the intercepts.

Solution
Step 1: **Find the x– and y-intercepts of the line.**

Let $y = 0$ and solve for x.

Let $x = 0$ and solve for y.

Find the x-intercept.

$$\text{Let } y = 0$$
$$-x + 2y = 6$$
$$-x + 2(rgb]1.0, 0.0, 0.00) = 6$$
$$-x = 6$$
$$x = 6$$
$$\text{The x-intercept is (-6,0).}$$

Find the y-intercept.

$$\text{Let } x = 0$$
$$-x + 2y = 6$$
$$-rgb]1.0, 0.0, 0.00 + 2y = 6$$
$$2y = 6$$
$$y = 3$$
$$\text{The y-intercept is (0,3).}$$

Step 2: **Find another solution to the equation.**

We'll use $x = 2$.

$$\text{Let } x = 2$$
$$-x + 2y = 6$$
$$-rgb]1.0, 0.0, 0.02 + 2y = 6$$
$$2y = 8$$
$$y = 4$$
$$\text{A third point is (2,4).}$$

Step 3: **Plot the three points.**

Check that the points line up.

x	y	(x, y)
-6	0	(-6,0)
0	3	(0,3)
2	4	(2,4)

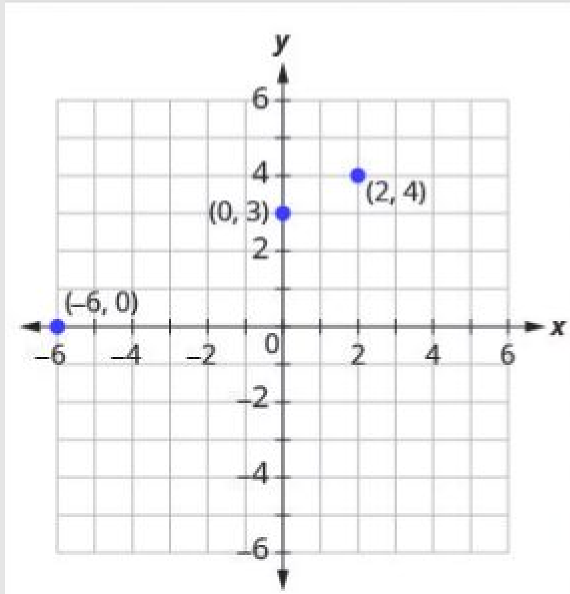


Figure 3.9.55

***Step 4:* Draw the line.**

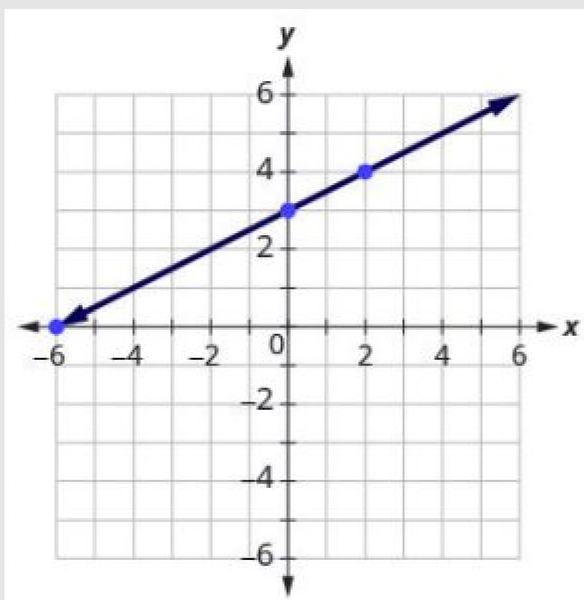


Figure 3.9.56

See the graph.

Try It

49) Graph $x - 2y = 4$ using the intercepts.

Solution

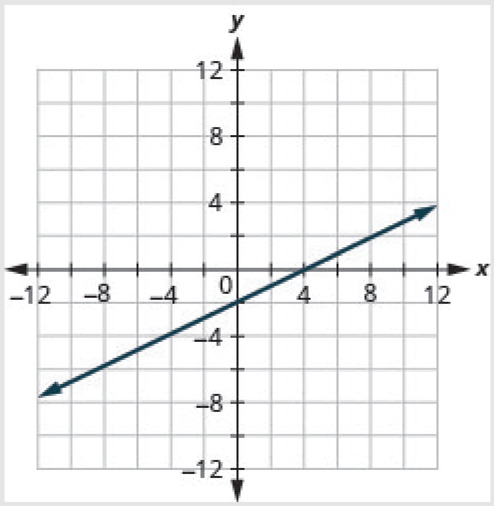


Figure 3.9.57

50) Graph $-x + 3y = 6$ using the intercepts.

Solution

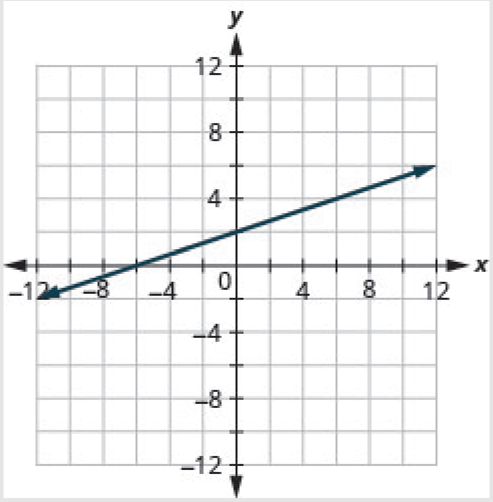


Figure 3.9.58

HOW TO

Graph a linear equation using the intercepts.

The steps to graph a linear equation using the intercepts are summarized below.

1. Find the x-intercept and y-intercept of the line.
   - Let $y = 0$ and solve for x.
   - Let $x = 0$ and solve for y.
2. Find a third solution to the equation.
3. Plot the three points and check that they line up.
4. Draw the line.

Example 3.9.23

Graph $4x – 3y = 12$ using the intercepts.

Solution

$$\begin{aligned} \text{x-intercept, let } y &= 0 \\ 4x - 3y &= 12 \\ 4x - 3(rgb]1.0, 0.0, 0.00) &= 12 \\ 4x &= 12 \\ x &= 3 \end{aligned}$$

$$\begin{aligned} \text{y-intercept, let } x &= 0 \\ 4x - 3y &= 12 \\ 4(rgb]1.0, 0.0, 0.00) - 3y &= 12 \\ -3y &= 12 \\ y &= -4 \end{aligned}$$

$$\begin{aligned} \text{third point, let } y &= 4 \\ 4x - 3y &= 12 \\ 4x - 3(rgb]1.0, 0.0, 0.04) &= 12 \\ 4x - 12 &= 12 \\ 4x &= 24 \\ x &= 6 \end{aligned}$$

Find the intercepts and a third point. We list the points in the table below and show the graph below.

$4x - 3y = 12$		
x	y	(x, y)
3	0	$(3, 0)$
0	-4	$(0, -4)$
6	4	$(6, 4)$

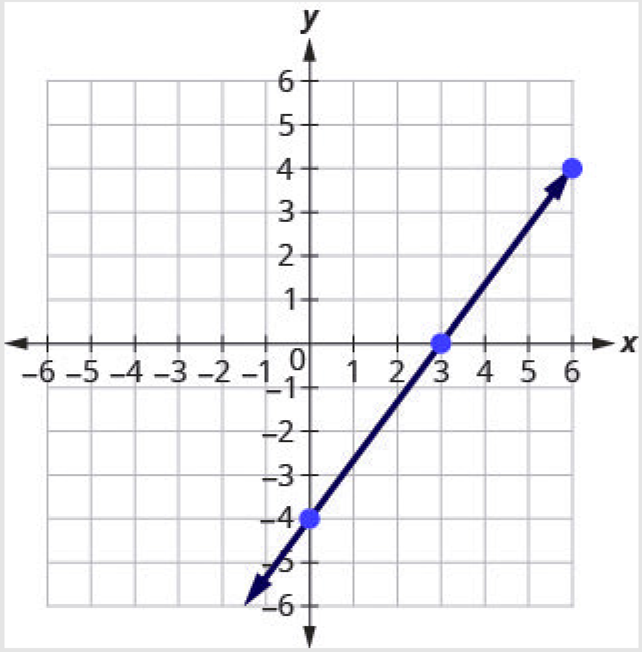


Figure 3.9.59

Try It

51) Graph $5x – 2y = 10$ using the intercepts.

Solution

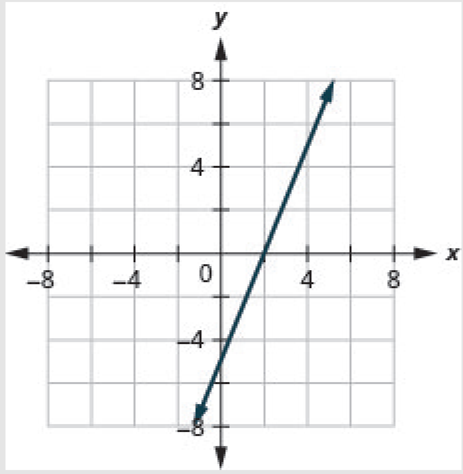


Figure 3.9.60

52) Graph $3x - 4y = 12$ using the intercepts.

Solution

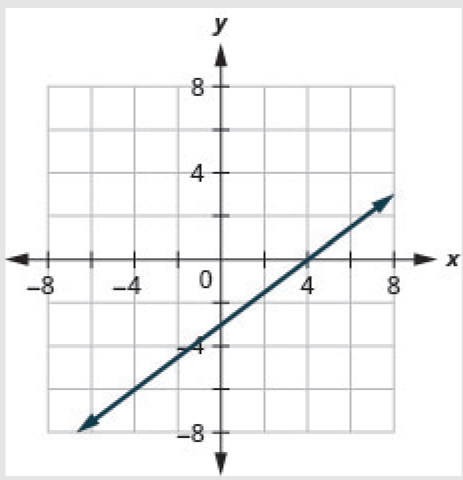


Figure 3.9.61

Example 3.9.24

Graph $y = 5x$ using the intercepts.

Solution

Step 1: **Find the x– and y-intercepts of the line.**

$$\begin{array}{rl} \text{x-intercept} & \text{y-intercept} \\ \text{Let } y = 0 & \text{Let } x = 0 \\ y = 5x & y = 5x \\ rgb]1.0,0.0,0.00 = 5x & y = 5 \cdot rgb]1.0,0.0,0.00 \\ 0 = x & y = 0 \\ \text{(0,0)} & (0,0) \end{array}$$

This line has only one intercept. It is the point $(0, 0)$.

Step 2: **To ensure accuracy we need to plot three points. Since the x-intercept and y-intercept are the same point, we need *two* more points to graph the line.**

$$\begin{array}{ll} \text{Let } x = 1 & \text{Let } x = -1 \\ y = 5x & y = 5x \\ y = 5 \cdot (rgb]1.0,0.0,0.01) & y = 5(rgb]1.0,0.0,0.0-1) \\ y = 5 & y = -5 \end{array}$$

See table below.

$y = 5x$		
x	y	(x, y)
0	0	$(0, 0)$
1	5	$(1, 5)$
-1	-5	$(-1, -5)$

Step 3: **Plot the three points, check that they line up, and draw the line.**

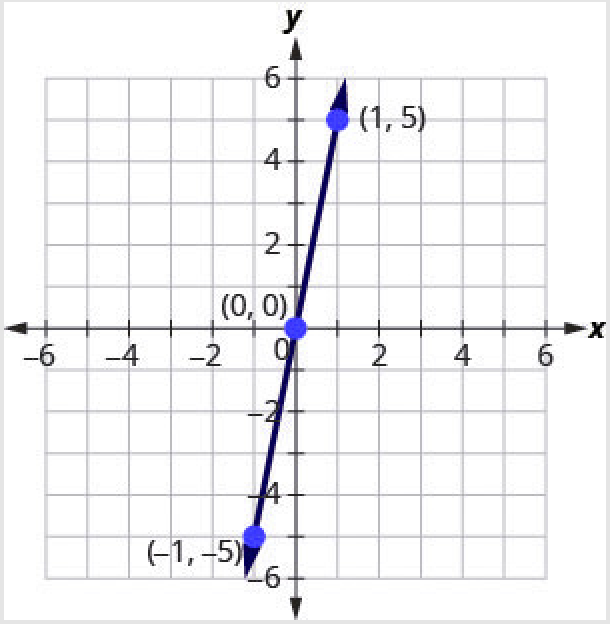


Figure 3.9.362

Try It

53) Graph $y = 4x$ using the intercepts.

Solution

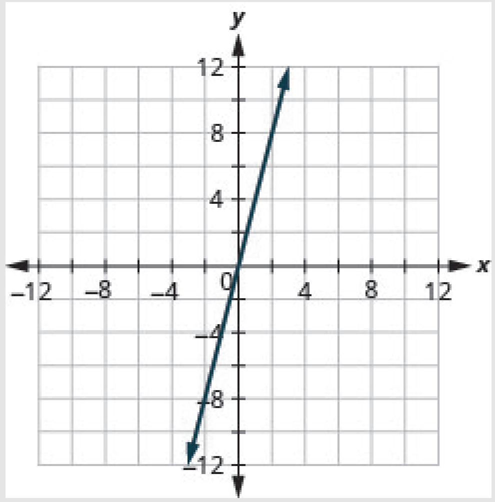


Figure 3.9.63

54) Graph $y = -x$ the intercepts.

Solution

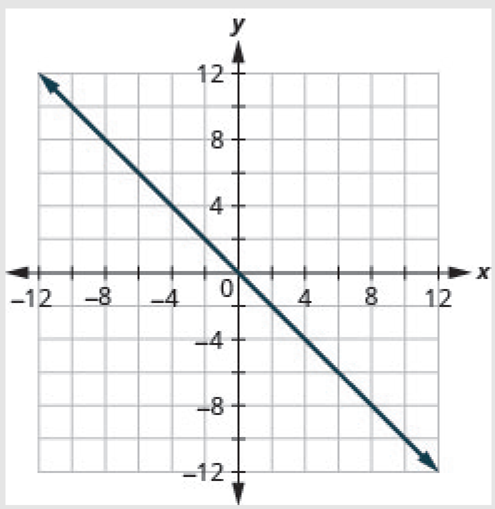


Figure 3.9.64

Key Concepts

- **Sign Patterns of the Quadrants**

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
(x, y)	(x, y)	(x, y)	(x, y)
$(+, +)$	$(-, +)$	$(-, -)$	$(+, -)$

- **Points on the Axes**
  - On the x-axis, $y = 0$. Points with a y-coordinate equal to 0 are on the x-axis, and have coordinates $(a, 0)$.
  - On the y-axis, $x = 0$. Points with an x-coordinate equal to 0 are on the y-axis, and have coordinates $(0, b)$.
- **Solution of a Linear Equation**
  - An ordered pair (x, y) is a solution of the linear equation $Ax + By = C$, if the equation is a true statement when the x and y values of the ordered pair are substituted into the equation.
- **Graph a Linear Equation by Plotting Points**
  1. Find three points whose coordinates are solutions to the equation. Organize them in a table.
  2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work!
  3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.
- **Find the x-intercept and y-Intercept from the Equation of a Line**
  - Use the equation of the line to find the x-intercept of the line, let $y = 0$ and solve for

x.

- Use the equation of the line to find the y-intercept of the line, let $x = 0$ and solve for y.

- **Graph a Linear Equation using the Intercepts**

1. Find the x-intercept and y-intercept of the line.
   Let $y = 0$ and solve for x.
   Let $x = 0$ and solve for y.
2. Find a third solution to the equation.
3. Plot the three points and then check that they line up.
4. Draw the line.

- **Strategy for Choosing the Most Convenient Method to Graph a Line:**
  - Consider the form of the equation.
  - If it only has one variable, it is a vertical or horizontal line.
    $x = a$ is a vertical line passing through the x-axis at a.
    $y = b$ is a horizontal line passing through the y-axis at b.
  - If y is isolated on one side of the equation, graph by plotting points.
  - Choose any three values for x and then solve for the corresponding y-values.
  - If the equation is of the form $Ax + By = C$, find the intercepts. Find the x-intercept and y-intercept, then a third point.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it

online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=1692#h5p-23

b. After reviewing this checklist, what will you do to become confident for all goals?

Glossary

linear equation
A linear equation is of the form $Ax + By = C$, where A and B are not both zero, is called a linear equation in two variables.

ordered pair
An ordered pair (x, y) gives the coordinates of a point in a rectangular coordinate system.

origin
The point $(0, 0)$ is called the origin. It is the point where the x-axis and y-axis intersect.

quadrant
The x-axis and the y-axis divide a plane into four regions, called quadrants.

rectangular coordinate system
A grid system is used in algebra to show a relationship between two variables; also called the xy-plane or the 'coordinate plane'.

***x*-coordinate**
The first number in an ordered pair (x, y).

y-coordinate
The second number in an ordered pair (x, y).

graph of a linear equation
The graph of a linear equation $Ax + By = C$ is a straight line. Every point on the line is a solution of the equation. Every solution of this equation is a point on this line.

horizontal line

A horizontal line is the graph of an equation of the form $y = b$. The line passes through the y -axis at $(0, b)$.

vertical line

A vertical line is the graph of an equation of the form $x = a$. The line passes through the x -axis at $(a, 0)$.

intercepts of a line

The points where a line crosses the x-axis and the y-axis are called the intercepts of the line.

x-intercept

The point $(a, 0)$ where the line crosses the x-axis; the x–intercept occurs when y is zero.

y-intercept

The point $(0, b)$ where the line crosses the y-axis; the y-intercept occurs when x is zero.

3.10 SLOPE OF A LINE

Learning Objectives

By the end of this section, you will be able to:

- Use $m = \dfrac{rise}{run}$ to find the slope of a line from its graph
- Find the slope of horizontal and vertical lines
- Use the slope formula to find the slope of a line between two points
- Graph a line given a point and the slope
- Solve slope applications
- Recognize the relation between the graph and the slope–intercept form of an equation of a line
- Identify the slope and y-intercept form of an equation of a line
- Graph a line using its slope and intercept
- Choose the most convenient method to graph a line
- Graph and interpret applications of slope–intercept
- Use slopes to identify parallel lines
- Use slopes to identify perpendicular lines

Try It

Before you get started, take this readiness quiz:

1) Simplify: $\dfrac{1-4}{8-2}$

2) Divide: $\dfrac{0}{4}, \dfrac{4}{0}$

3) Simplify: $\dfrac{15}{-3}, \dfrac{-15}{3}, \dfrac{-15}{-3}$

4) Add: $\dfrac{x}{4} + \dfrac{1}{4}$

5) Find the reciprocal of $\dfrac{3}{7}$

6) Solve $2x - 3y = 12$

When you graph linear equations, you may notice that some lines tilt up as they go from left to right and some lines tilt down. Some lines are very steep and some lines are flatter. What determines whether a line tilts up or down or if it is steep or flat?

In mathematics, the 'tilt' of a line is called the *slope* of the line. The concept of slope has many applications in the real world. The pitch of a roof, grade of a highway, and a ramp for a wheelchair are some examples where you literally see slopes. And when you ride a bicycle, you feel the slope as you pump uphill or coast downhill.

In this section, we will explore the concept of slope.

Use $m = \dfrac{rise}{run}$ to Find the Slope of a Line from its Graph

Now, we'll look at some graphs on the xy-coordinate plane and see how to find their slopes. The method we will use here will be similar to how one would use **geoboard**s.

To find the slope, we must count out the **rise** and the **run**. But where do we start?

We locate two points on the line whose coordinates are integers. We then start with the point on the left and sketch a right triangle, so we can count the rise and run.

Example 3.10.1

How to Use $m = \dfrac{rise}{run}$ to Find the **Slope of a Line** from its Graph

Find the slope of the line shown.

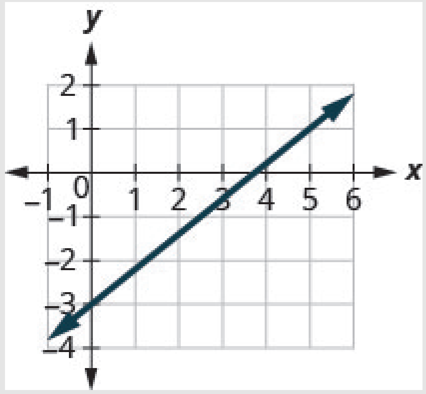


Figure 3.10.1

Solution

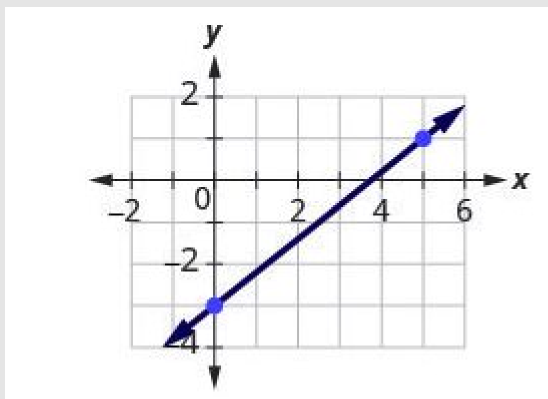


Figure 3.10.2

Step 1: **Locate two points whose coordinates are integers.**

Mark $(0-,3)(5,1)$

Step 2: **Starting with the point on the left, sketch a right triangle, going from the first point to the second point.**

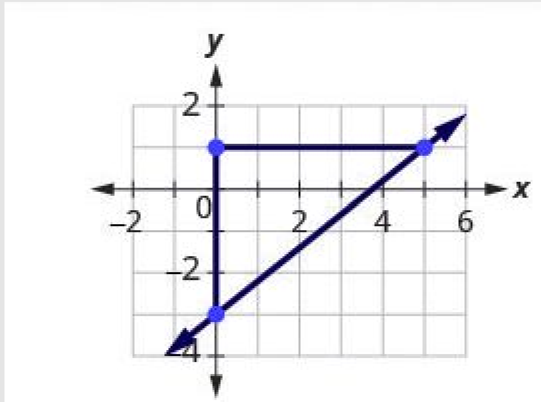


Figure 3.10.3

***Step 3:* Count the rise and the run on the legs of the triangle.**

Count the rise.

Count the run.

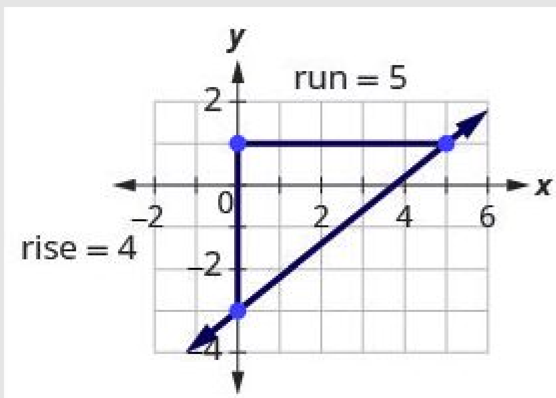


Figure 3.10.4

***Step 4:* Take the ration of rise to run to find the slope.**

Use the slope formula.	$m = \frac{rise}{run}$
Substitute the values of the rise and run.	$m = \frac{4}{5}$

The slope of the line is $\frac{4}{5}$.

Try It

7) Find the slope of the line shown.

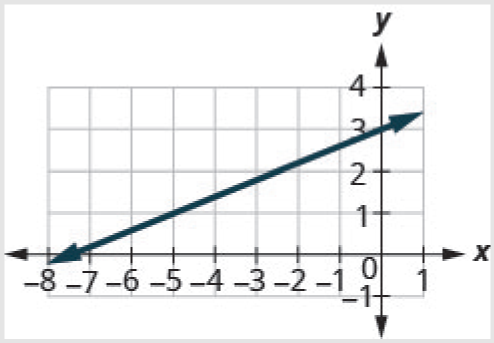


Figure 3.10.5

Solution

$\frac{2}{5}$

8) Find the slope of the line shown.

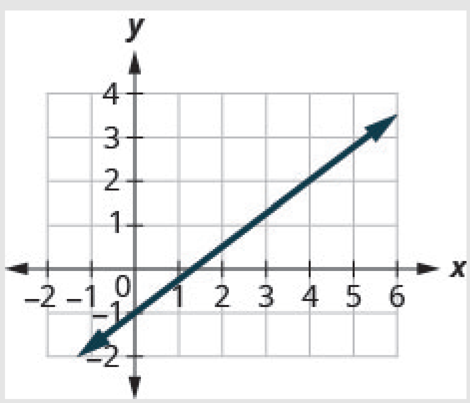


Figure 3.10.6

Solution

$\frac{3}{4}$

HOW TO

Find the slope of a line from its graph using $m = \frac{rise}{run}$

1. Locate two points on the line whose coordinates are integers.
2. Starting with the point on the left, sketch a right triangle, going from the first point to the second point.
3. Count the rise and the run on the legs of the triangle.
4. Take the ratio of rise to run to find the slope, $m = \frac{rise}{run}$

Example 3.10.2

Find the slope of the line shown.

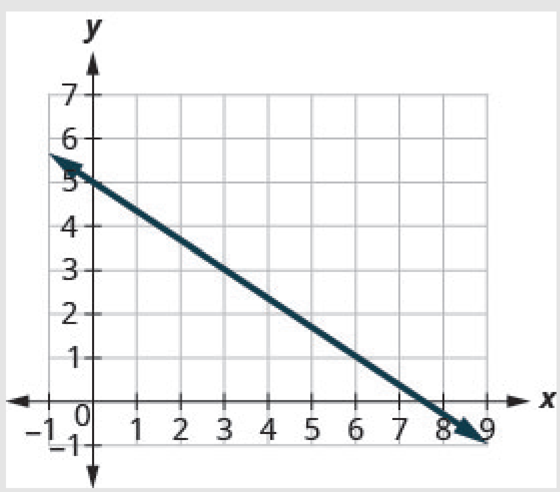


Figure 3.10.7

Solution

m=\frac{rise}{run}

Step 1. **Locate two points on the graph whose coordinates are integers.**

$(0, 5)$ and $(3, 3)$

Step 2: **Which point is on the left?**

$(0, 5)$

Step 3: **Starting at $(0, 5)$, sketch a right triangle to $(3, 3)$.**

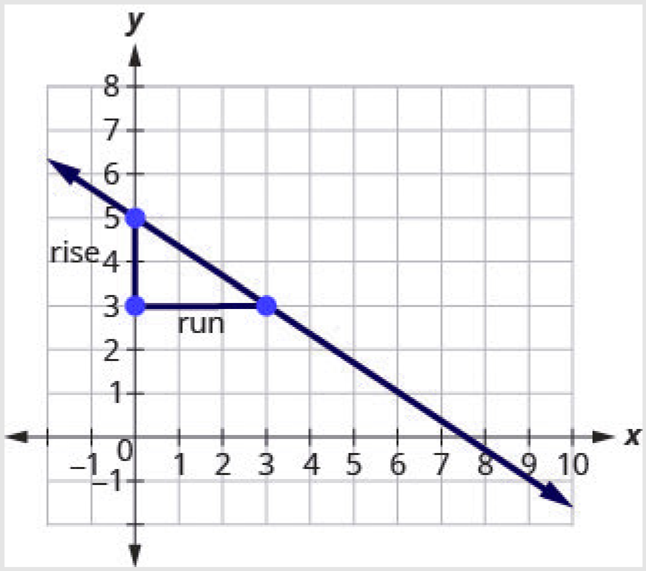


Figure 3.10.8

Step 4: **Count the rise—it is negative.**

The rise is -2.

Step 5: **Count the run.**

The run is 3.

Step 6: Use the slope formula.

$$m = \frac{rise}{run}$$

Step 7: **Substitute the values of the rise and run.**

$$m = \frac{-2}{3}$$

Simplify. $$m = -\frac{2}{3}$$

The slope of the line is $-\frac{2}{3}$.

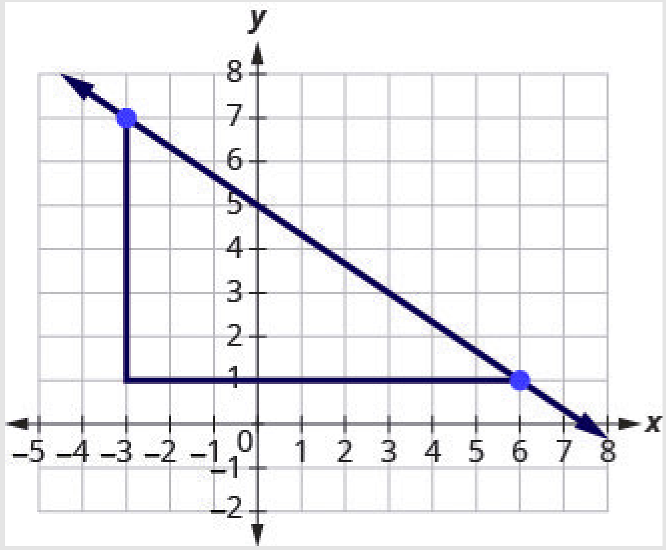


Figure 3.10.9

So y increases by 3 units as x decreases by 2 units. What if we used the points left $(-3,7)$ and $(6,1)$ to find the slope of the line?The rise would be -6 and the run would be 9. Then $m=\frac{-6}{9}$, and that simplifies to $m=-\frac{2}{3}$. Remember, it does not matter which points you use—the slope of the line is always the same.

Try It

9) Find the slope of the line shown.

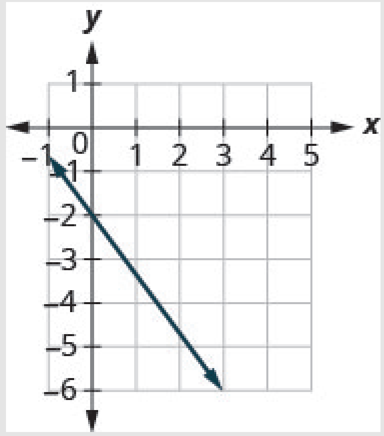


Figure 3.10.10

Solution

$-\frac{4}{3}$

10) Find the slope of the line shown.

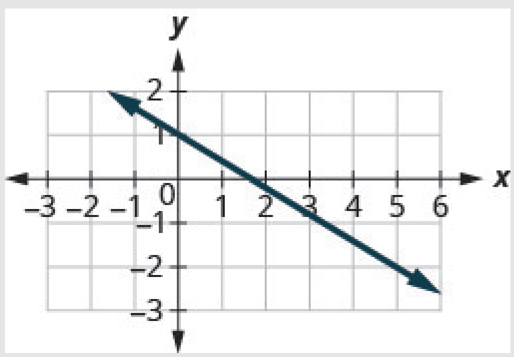


Figure 3.10.11

Solution

$-\frac{3}{5}$

In the last two examples, the lines had y-intercepts with integer values, so it was convenient to use the y-intercept as one of the points to find the slope. In the next example, the y-intercept is a fraction. Instead of using that point, we'll look for two other points whose coordinates are integers. This will make the slope calculations easier.

Example 3.10.3

Find the slope of the line shown.

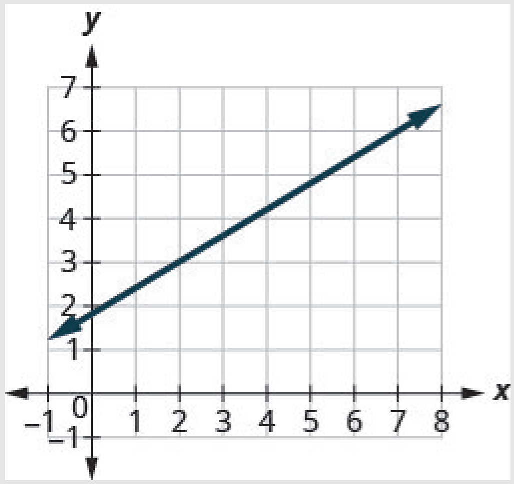


Figure 3.10.12

Solution

Step 1: **Locate two points on the graph whose coordinates are integers.**

$$(2, 3) \text{ and } (7, 6)$$

Step 2: **Which point is on the left?**

$$(2, 3)$$

Step 3: **Starting at $(2, 3)$, sketch a right triangle to $(7, 6)$.**

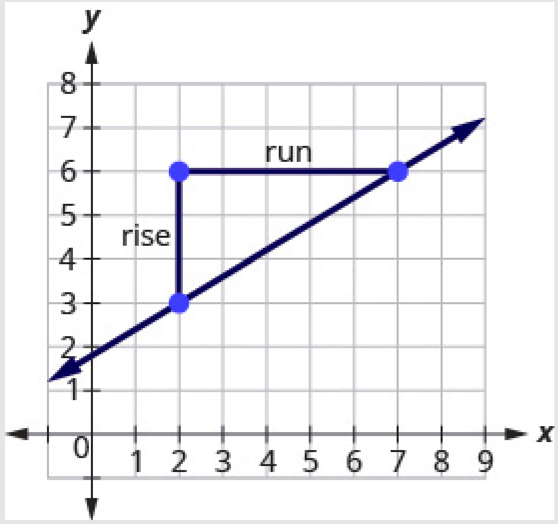


Figure 3.10.13

Step 4: **Count the rise.**

The rise is 3.

Step 5: **Count the run.**

The run is 5.

Step 6: **Use the slope formula.**

$$m = \frac{rise}{run}$$

Step 7: **Substitute the values of the rise and run.**

The slope of the line is $\frac{3}{5}$

This means that y increases 5 units as x increases 3 units. If you used a geoboard to introduce the concept of slope, you would always start with the point on the left and count the rise and the run to get to the point on the right. That way the run was always positive and the rise determined whether the slope was positive or negative. What would happen if we started with the point on the right? Let's use the points $(2, 3)$ and $(7, 6)$ again, but now we'll start at $(7, 6)$.

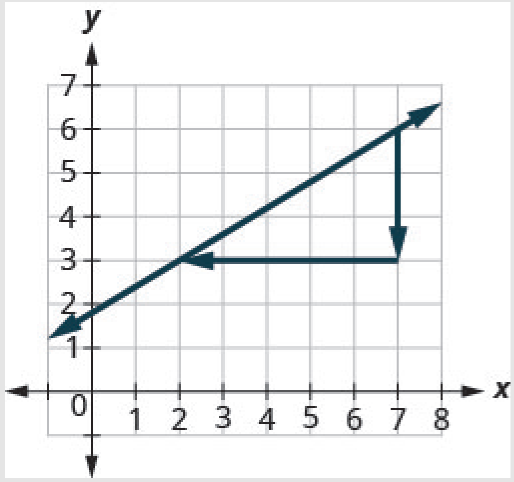


Figure 3.10.14

Step 1: **Count the rise.**

The rise is -3.

Step 2: **Count the run. It goes from right to left, so it is negative.**

The run is -5.

Step 3: **Use the slope formula.**

$$m = \frac{rise}{run}$$

Step 4: **Substitute the values of the rise and run.**

$$m = \frac{-3}{-5}$$

Step 5: **The slope of the line is $\frac{3}{5}$.**

It does not matter where you start—the slope of the line is always the same.

Try It

11) Find the slope of the line shown.

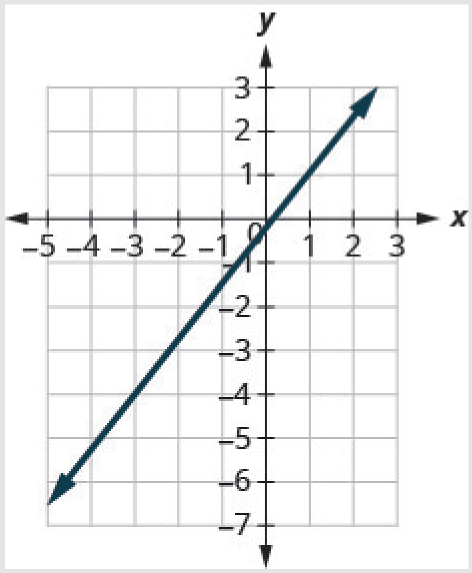


Figure 3.10.15

Solution

$\frac{5}{4}$

12) Find the slope of the line shown.

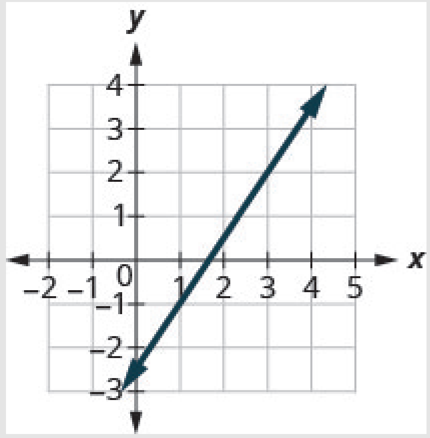


Figure 3.10.16

Solution

$\frac{3}{2}$

Find the Slope of Horizontal and Vertical Lines

Do you remember what was special about horizontal and vertical lines? Their equations had just one variable.

Horizontal line $y = b$	Vertical line $x = a$
y-coordinates are the same.	x-coordinates are the same.

So how do we find the slope of the horizontal line $y = 4$? One approach would be to graph the horizontal line, find two points on it, and count the rise and the run. Let's see what happens when we do this.

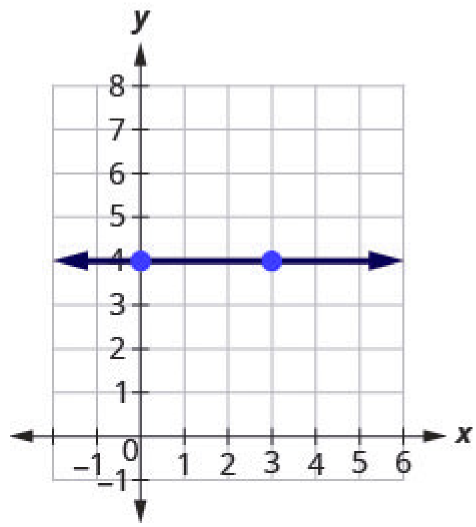


Figure 3.10.17

Step 1: **What is the rise?**

The rise is 0.

Step 2: **Count the run.**

The run is 3.

Step 3: **What is the slope?**

$$m = \frac{rise}{run}$$
$$m = \frac{0}{3}$$
$$m = 0$$

The slope of the horizontal line $y = 4$ is 0.

All horizontal lines have slope 0. When the y-coordinates are the same, the rise is 0.

Slope of a Horizontal Line

The slope of a horizontal line, $y = b$, is 0.

The floor of your room is horizontal. Its slope is 0. If you carefully placed a ball on the floor, it would not roll away.

Now, we'll consider a vertical line, the line.

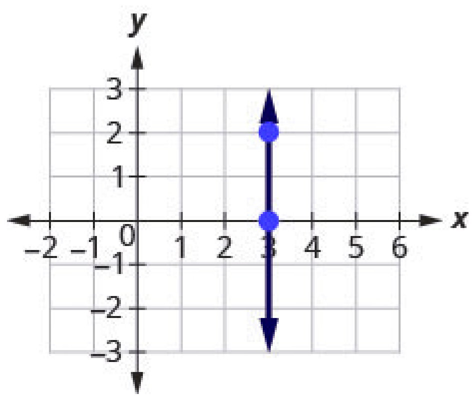


Figure 3.10.18

Step 1: **What is the rise?**

The rise is 2.

Step 2: **Count the run.**

The run is 0.

Step 3: **What is the slope?**

$$m = \frac{rise}{run}$$

$$m = \frac{2}{0}$$

But we can't divide by 0. Division by 0 is not defined. So we say that the slope of the vertical line $x = 3$ is undefined.

The slope of any vertical line is undefined. When the x-coordinates of a line are all the same, the run is 0.

Slope of a Vertical Line

The slope of a vertical line, $x = a$, is undefined.

Example 3.10.4

Find the slope of each line:

a. $x = 8$
b. $y = -5$

Solution

a. $x = 8$
This is a vertical line.
Its slope is undefined.

---

b. $y = -5$
This is a horizontal line.
It has slope 0.

Try It

13) Find the slope of the line: $x = -4$.

Solution
undefined

14) Find the slope of the line: $y = 7$.

Solution
0

Quick Guide to the Slopes of Lines

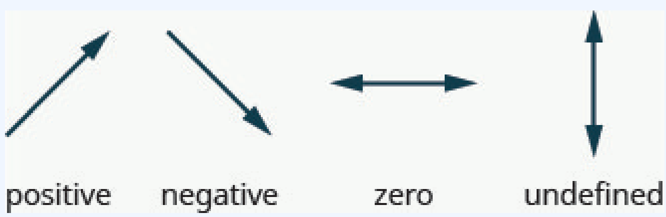


Figure 3.10.19

Remember, we 'read' a line from left to right, just like we read written words in English.

Use the Slope Formula to find the Slope of a Line Between Two Points

Sometimes we'll need to find the slope of a line between two points when we don't have a graph to count out the rise and the run. We could plot the points on grid paper, then count out the rise and the run, but as we'll see, there is a way to find the slope without graphing. Before we get to it, we need to introduce some algebraic notation.

We have seen that an ordered pair (x, y) gives the coordinates of a point. But when we work with slopes, we use two points. How can the same symbol (x, y) be used to represent two different points? Mathematicians use subscripts to distinguish the points.

(x_1, y_1)	read ' x sub 1, y sub 1 '
(x_2, y_2)	read ' x sub 2, y sub 2 '

The use of subscripts in math is very much like the use of last name initials in elementary school. Maybe you remember Laura C. and Laura M. in your third grade class?

We will use (x_1, y_1) to identify the first point and (x_2, y_2) to identify the second point.

If we had more than two points, we could use (x_3, y_3), (x_4, y_4), and so on.

Let's see how the rise and run relate to the coordinates of the two points by taking another look at the slope of the line between the points $(2, 3)$ and $(7, 6)$.

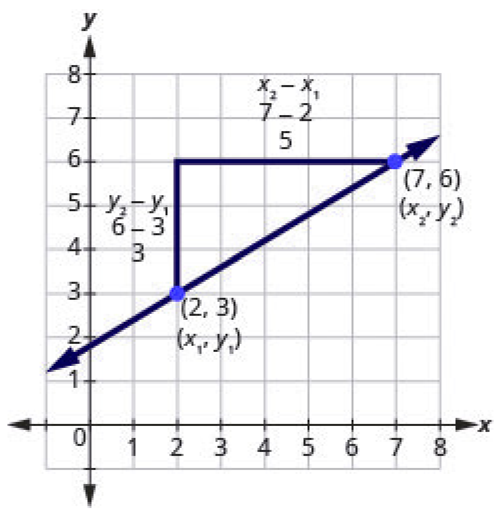


Figure 3.10.20

Since we have two points, we will use subscript notation,

$$\begin{pmatrix} x_1 & y_1 \\ 2 & 3 \end{pmatrix} \quad \begin{pmatrix} x_2 & y_2 \\ 7 & 6 \end{pmatrix}$$

On the graph, we counted the rise of 3 and the run of 5.

Notice that the rise of 3 can be found by subtracting the y-coordinates 6 and 3.

$$3 = 6 - 3$$

And the run of 5 can be found by subtracting the x-coordinates 7 and 2.

$$5 = 7 - 2$$

We know $m = \frac{rise}{run}$. So $m = \frac{3}{5}$.

We rewrite the rise and run by putting in the coordinates $m = \frac{6-3}{7-2}$.

But 6 is y_2, the y-coordinate of the second point and 3 is y_1 the y-coordinate of the first point.

So we can rewrite the slope using subscript notation. $m = \frac{y_2 - y_1}{7-2}$

Also, 7 is x_2, the x-coordinate of the second point and 2 is x_1, the x-coordinate of the first point.

So, again, we rewrite the slope using subscript notation. $m = \frac{y_2 - y_1}{x_2 - x_1}$

We've shown that $m = \frac{y_2 - y_1}{x_2 - x_1}$ is really another version of $m = \frac{rise}{run}$. We can use this formula to find the slope of a line when we have two points on the line.

Slope Formula

The slope of the line between two points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

This is the **slope formula**.

The slope is:

y of the second point minus y of the first point
over
x of the second points minus x of the first point

Example 3.10.5

Use the slope formula to find the slope of the line between the points $(1, 2)$ and $(4, 5)$.

Solution

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Step 1: **We'll call $(1, 2)$ point #1 and $(4, 5)$ point #2.**

$$\begin{pmatrix} x_1 & y_1 \\ 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} x_2 & y_2 \\ 4 & 5 \end{pmatrix}$$

Step 2: **Use the slope formula.**

$$m = \frac{rise}{run}$$

Step 3: **Substitute the values.**

y of the second point minus y of the first point.

$$m = \frac{5 - 2}{x_2 - x_1}$$

x of the second point minus x of the first point.

$$m = \frac{5 - 2}{4 - 1}$$

Step 4: **Simplify the numerator and the denominator.**

$$m = \frac{3}{3}$$

$$\text{Simplify.} \qquad m = 1$$

Let's confirm this by counting out the slope on a graph using $m = \frac{rise}{run}$.

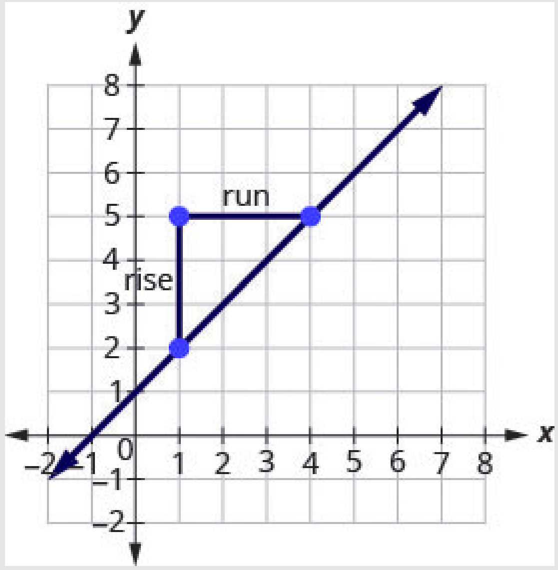


Figure 3.10.21

It doesn't matter which point you call point #1 and which one you call point #2. The slope will be the same. Try the calculation yourself.

Try It

15) Use the slope formula to find the slope of the line through the points: $(8, 5)$ and $(6, 3)$.

Solution

1

16) Use the slope formula to find the slope of the line through the points: $(1, 5)$ and $(5, 9)$.

Solution

1

Example 3.10.6

Use the slope formula to find the slope of the line through the points $(-2, -3)$ and $(-7, 4)$.

Solution

Step 1. **We'll call $(-2, -3)$ point #1 and $(-7, 4)$ point #2.**

$$\begin{pmatrix} x_1 & y_1 \\ -2 & -3 \end{pmatrix} \quad \begin{pmatrix} x_2 & y_2 \\ -7 & 4 \end{pmatrix}$$

Step 2. **Use the slope formula.**

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Step 3. **Substitute the values.**

y of the second point minus y of the first point.

$$m = \frac{4 - (-3)}{x_2 - x_1}$$

x of the second point minus x of the first point.

$$m = \frac{4 - (-3)}{-7 - (-2)}$$

Step 4. **Simplify.**

$$m = \frac{7}{-5}$$
$$m = -\frac{7}{5}$$

Let's verify this slope on the graph shown.

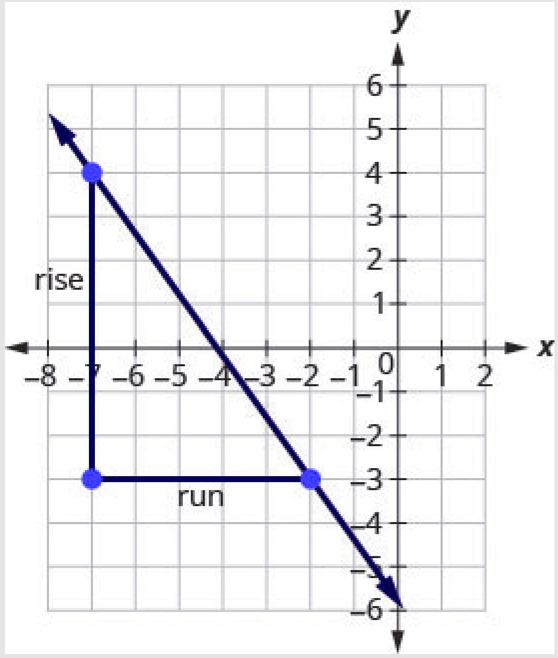


Figure 3.10.22

$$m = \frac{rise}{run}$$
$$m = \frac{-7}{5}$$
$$m = -\frac{7}{5}$$

Try It

17) Use the slope formula to find the slope of the line through the points: $(-3, 4)$ and $(2, -1)$

Solution

-1

18) Use the slope formula to find the slope of the line through the pair of points: $(-2, 6)$ and $(-3, -4)$.

Solution

10

Graph a Line Given a Point and the Slope

Up to now, in this chapter, we have graphed lines by plotting points, by using intercepts, and by recognizing horizontal and vertical lines.

One other method we can use to graph lines is called the *point–slope method*. We will use this method when we know one point and the slope of the line. We will start by plotting the point and then use the definition of slope to draw the graph of the line.

Example 3.10.7

Graph the line passing through the point $(1, -1)$ whose slope is $m = \dfrac{3}{4}$.

Solution

Step 1: **Plot the given point.**

Plot $(1, -1)$.

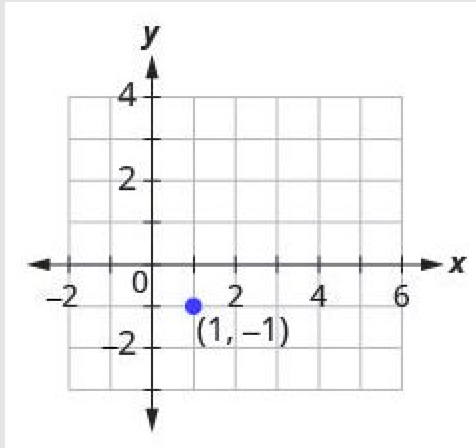


Figure 3.10.23

Step 2: **Use the slope formula** $m = \frac{rise}{run}$ **to identify the rise and the run.**

Identify the rise and the run.

$$m = \frac{3}{4}$$

$$\frac{rise}{run} = \frac{3}{4}$$

$$\text{rise} = 3$$
$$\text{run} = 4$$

Step 3: **Starting at the given point, count out the rise and run to mark the second point**

Start at $(1, -1)$ and count the rise and the run. Up 3 units, right 4 units.

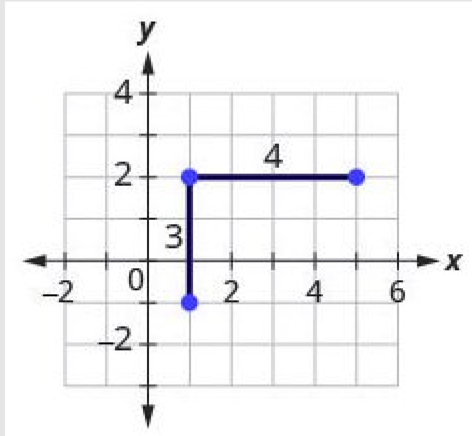


Figure 3.10.24

Step 4: **Connect the points with a line.**

Connect the two points with a line.

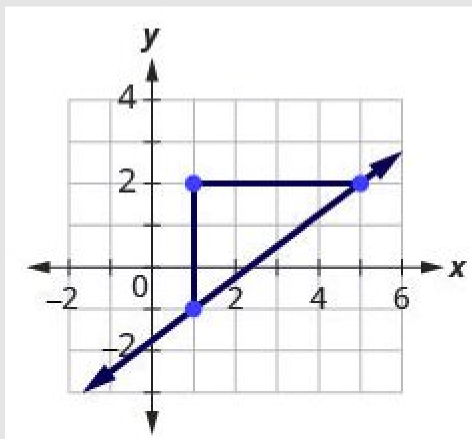


Figure 3.10.25

Try It

19) Graph the line passing through the point $(2, -2)$ with the slope $m = \dfrac{4}{3}$.

Solution

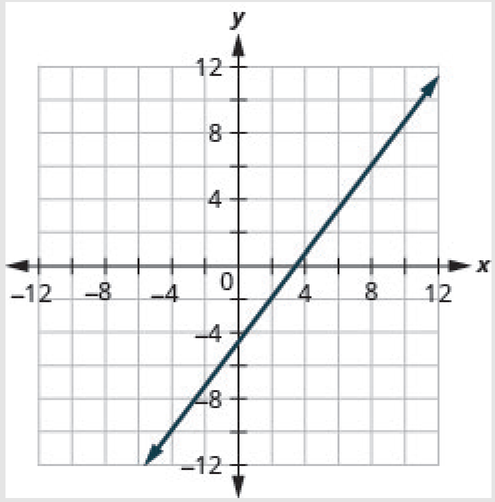


Figure 3.10.26

20) Graph the line passing through the point $(-2, 3)$ with the slope $m = \frac{1}{4}$.

Solution

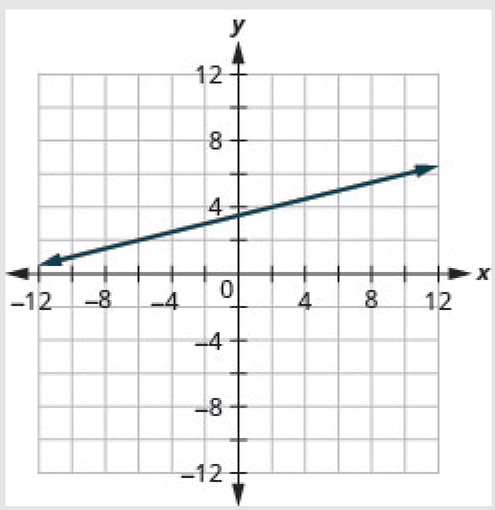


Figure 3.10.27

HOW TO

Graph a line given a point and the slope.

1. Plot the given point.
2. Use the slope formula $m = \dfrac{rise}{run}$ to identify the rise and the run.
3. Starting at the given point, count out the rise and run to mark the second point.
4. Connect the points with a line.

Example 3.10.8

Graph the line with y-intercept 2 whose slope is $m = -\dfrac{2}{3}$.

Solution

Step 1: **Plot the given point, the y-intercept, $(0, 2)$.**

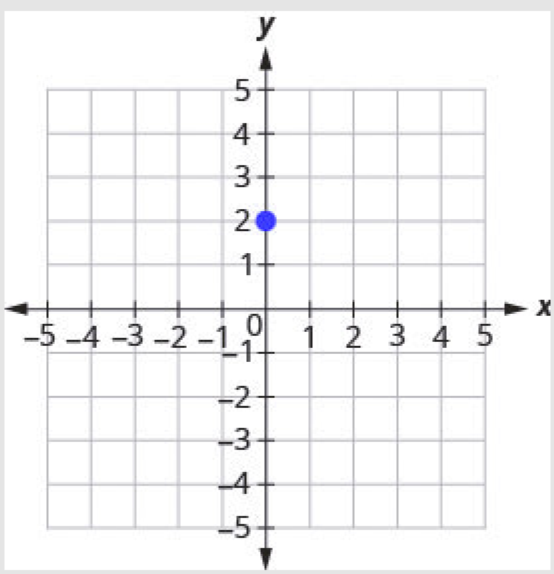


Figure 3.10.28

$$m = -\frac{2}{3}$$

Step 2: **Identify the rise and the run.**

$$\left(\frac{rise}{run}\right) = \frac{-2}{3}$$
$$rise = -2$$
$$run = 3$$

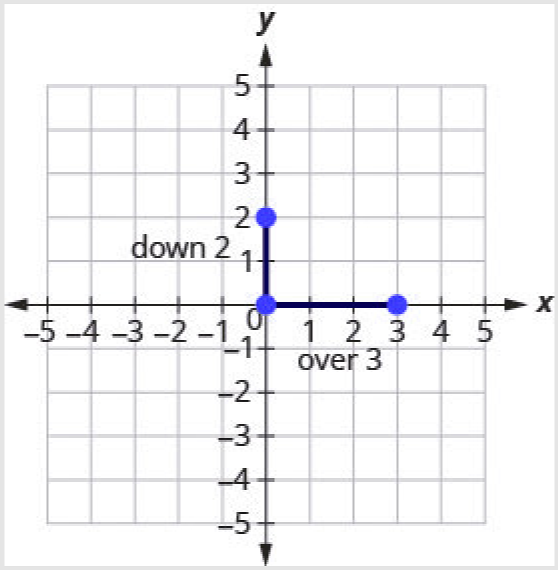


Figure 3.10.29

Step 3: **Count the rise and the run. Mark the second point.**

Step 4: **Connect the two points with a line.**

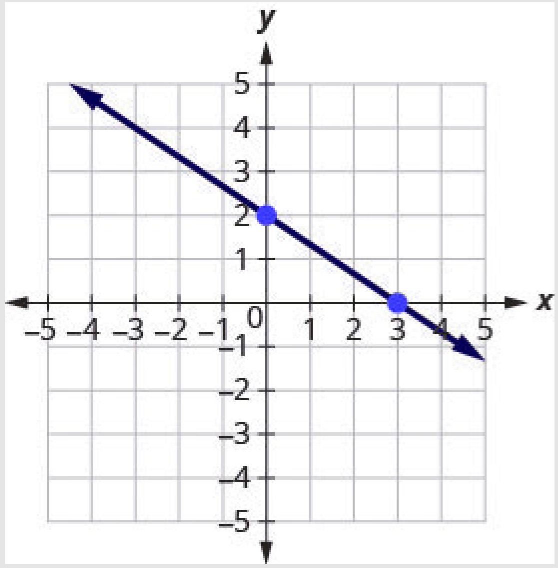


Figure 3.10.30

You can check your work by finding a third point. Since the slope is $m = -\frac{2}{3}$, it can be written as $m = \frac{2}{-3}$. Go back to $(0, 2)$ and count out the rise, 2, and the run, -3

Try It

21) Graph the line with the y-intercept 4 and slope $m = -\frac{5}{2}$.

Solution

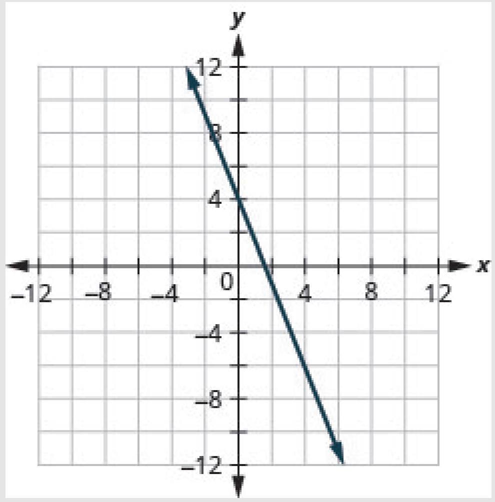


Figure 3.10.31

22) Graph the line with the x-intercept -3 and slope $m = -\frac{3}{4}$.

Solution

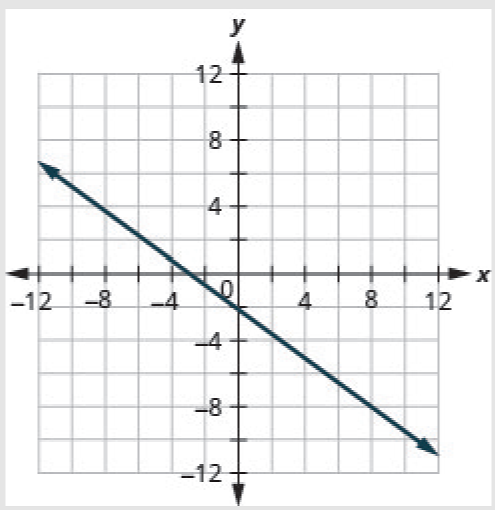


Figure 3.10.32

Example 3.10.9

Graph the line passing through the point $(-1, -3)$ whose slope is $m = 4$.

Solution

Step 1: **Plot the given point.**

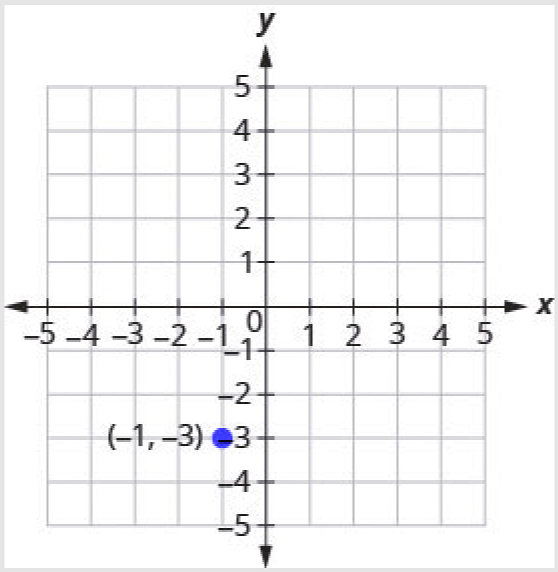


Figure 3.10.33

Step 2: **Identify the rise and the run.**

$$m = 4$$

Step 3: **Write 4 as a fraction.**

$$\frac{rise}{run} = \frac{4}{1}$$
$$rise = 4$$
$$run = 1$$

Step 4: **Count the rise and run and mark the second point.**

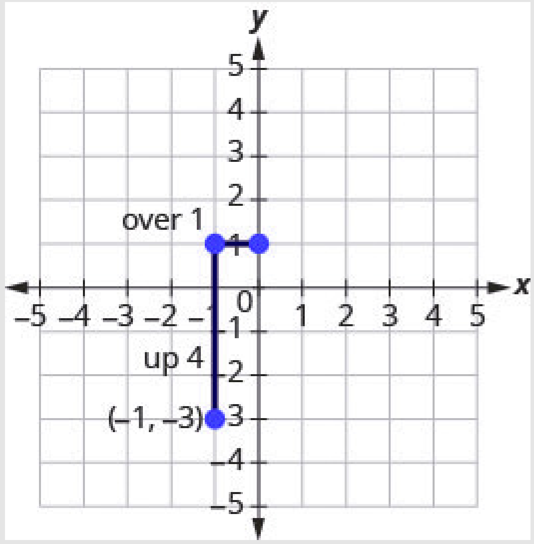


Figure 3.10.34

Step 5: Connect the two points with a line.

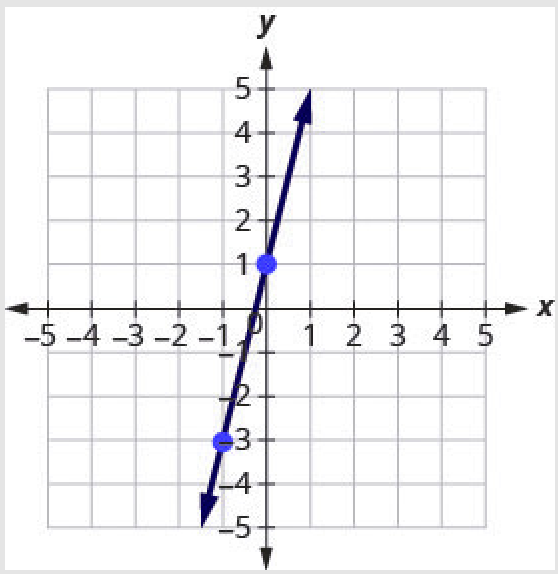


Figure 3.10.35

You can check your work by finding a third point. Since the slope is $m = 4$, it can be written as $m = \frac{-4}{-1}$. Go back to $(-1, -3)$ and count out the rise, -4, and the run, -1.

Try It

23) Graph the line with the point $(-2, 1)$ and slope $m = 3$.

Solution

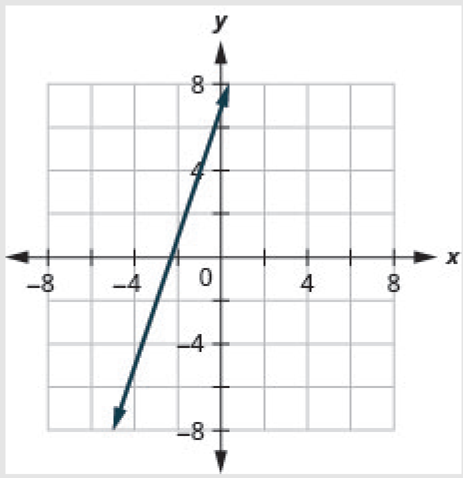


Figure 3.10.36

24) Graph the line with the point $(4, -2)$ and slope $m = -2$.

Solution

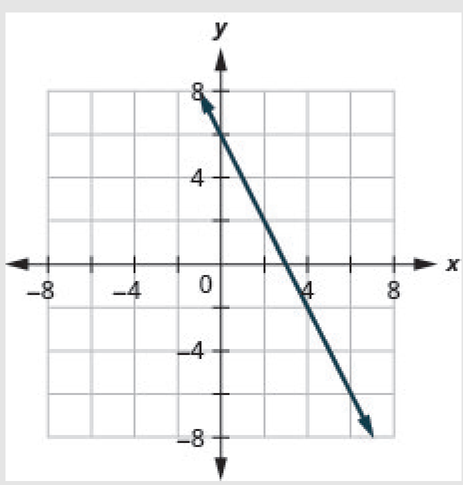


Figure 3.10.37

Solve Slope Applications

At the beginning of this section, we said there are many applications of slope in the real world. Let's look at a few now.

Example 3.10.10

The 'pitch' of a building's roof is the slope of the roof. Knowing the pitch is important in climates where there is heavy snowfall. If the roof is too flat, the weight of the snow may cause it to collapse. What is the slope of the roof shown?

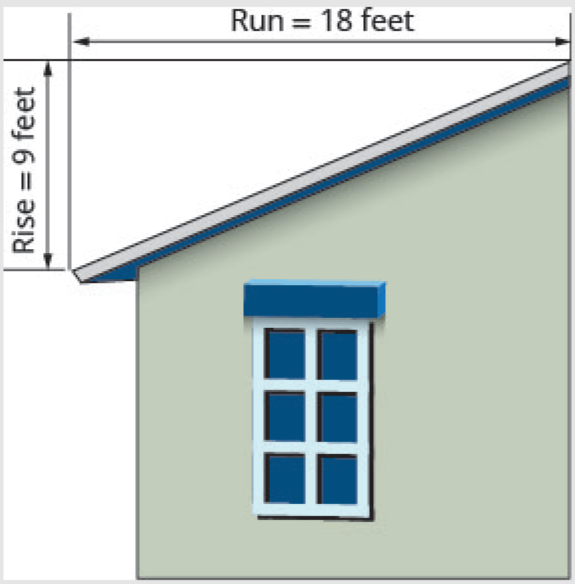


Figure 3.10.38

Solution

Step 1: **Use the slope formula.**

$$m = \frac{rise}{run}$$

Step 2: **Substitute the values for rise and run.**

$$m = \frac{9}{18}$$

Step 3: **Simplify.**

$$m = \frac{1}{2}$$

***Step 4:* The slope of the roof is $\frac{1}{2}$.**

The roof rises 1 foot for every 2 feet of horizontal run.

Try It

25) Use Example 3.10.10, substituting the $rise = 14$ and $run = 24$.

Solution

$\frac{7}{12}$

26) Use Example 3.10.10, substituting $rise = 15$ and $run = 36$.

Solution

$\frac{5}{12}$

Example 3.10.11

Have you ever thought about the sewage pipes going from your house to the street? They must slope down $\frac{1}{4}$ inch per foot in order to drain properly. What is the required slope?

Figure 3.10.39

Solution

***Step 1:* Use the slope formula**

$$m = \frac{rise}{run}$$

$$m = \frac{-\frac{1}{4} inch}{1\ foot}$$

$$m = \frac{-\frac{1}{4} inch}{12\ inches}$$

$$\text{Simplify.} \qquad m = -\frac{1}{48}$$

The slope of the pipe is $-\frac{1}{48}$

The pipe drops 1 inch for every 48 inches of horizontal run.

Try It

27) Find the slope of a pipe that slopes down $\frac{1}{3}$ inch per foot.

Solution

$-\frac{1}{36}$

28) Find the slope of a pipe that slopes down $\frac{3}{4}$ inch per yard.

Solution

$-\frac{1}{48}$

Recognize the Relation Between the Graph and the Slope–Intercept Form of an Equation of a Line

We have graphed linear equations by plotting points, using intercepts, recognizing horizontal and vertical lines, and using the point–slope method. Once we see how an equation in slope–intercept form and its graph are related, we'll have one more method we can use to graph lines.

In Graph Linear Equations in Two Variables, we graphed the line of the equation $y = \frac{1}{2}x + 3$ by plotting points. See Figure 3.10.40. Let's find the slope of this line.

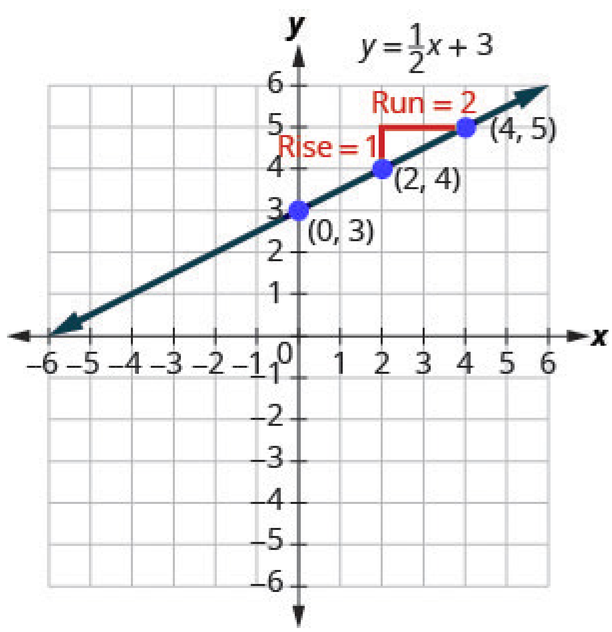


Figure 3.10.40

The red lines show us the rise is 1 and the run is 2. Substituting into the slope formula:

$$m = \frac{rise}{run}$$
$$m = \frac{1}{5}$$

What is the y-intercept of the line? The y-intercept is where the line crosses the y-axis, so y-intercept is $(0, 3)$. The equation of this line is:

$$y = rgb]1.0, 0.0, 0.0\frac{1}{2}x + rgb]0.0, 0.0, 1.03$$

Notice, the line has:

Slope $m = rgb]1.0, 0.0, 0.0\frac{1}{2}$ and y-intercept $(0, rgb]0.0, 0.0, 1.03)$

When a linear equation is solved for y, the coefficient of the x term is the slope and the constant term is the y-coordinate of the y-intercept. We say that the equation $y = \frac{1}{2}x + 3$ is in slope–intercept form.

Slope $m = rgb]1.0, 0.0, 0.0\frac{1}{2}$; y-intercept $(0, rgb]0.0, 0.0, 1.03)$

$$y = rgb]1.0, 0.0, 0.0\frac{1}{2}x + rgb]0.0, 0.0, 1.03$$
$$rgb]0.1, 0.1, 0.1y rgb]0.1, 0.1, 0.1 = rgb]1.0, 0.0, 0.0m rgb]0.1, 0.1, 0.1 + rgb]0.0, 0.0, 1.0b$$

Slope-Intercept Form of an Equation of a Line

The slope–intercept form of an equation of a line with slope m and y-intercept, $(0, b)$ is,

$$y = mx + b$$

Sometimes the slope–intercept form is called the "y-form."

Example 3.10.12

Use the graph to find the slope and y-intercept of the line, $y = 2x + 1$.

Compare these values to the equation $y = mx + b$.

Solution

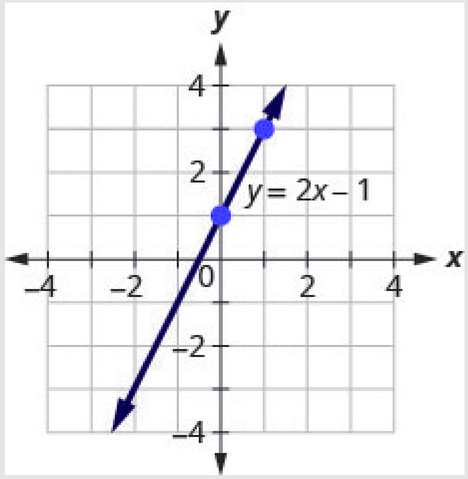


Figure 3.10.41

To find the slope of the line, we need to choose two points on the line. We'll use the points $(0, 1)$ and $(1, 3)$.

Step 1: **Find the rise and run.**

$$m = \frac{rise}{run}$$
$$m = \frac{2}{1}$$
$$m = 2$$

Step 2: **Find the y-intercept of the line.**

The y-intercept is the point $(0, 1)$.

Step 3: **We found the slope $m = 2$ and the y-intercept $(0, 1)$.**

$$y = rgb]1.0, 0.0, 0.02x \; + \; rgb]0.0, 0.0, 1.01$$
$$y = rgb]1.0, 0.0, 0.0mx \; + \; rgb]0.0, 0.0, 1.0b$$

The slope is the same as the coefficient of x and the y-coordinate of the y-intercept is the same as the constant term.

Try It

29) Use the graph to find the slope and y-intercept of the line $y = \frac{2}{3}x - 1$. Compare these values to the equation $y = mx + b$.

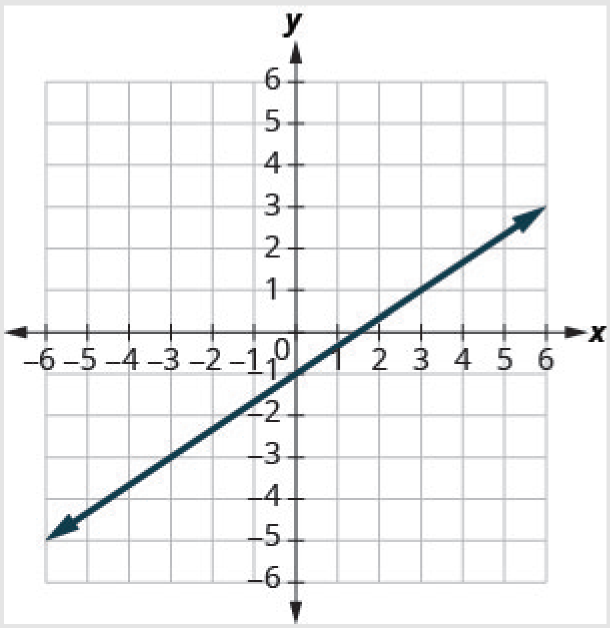


Figure 3.10.42

Solution

slope $m = \frac{2}{3}$ and y-intercept $(0, -1)$

30) Use the graph to find the slope and y-intercept of the line $y = \frac{1}{2}x + 3$. Compare these values to the equation $y = mx + b$.

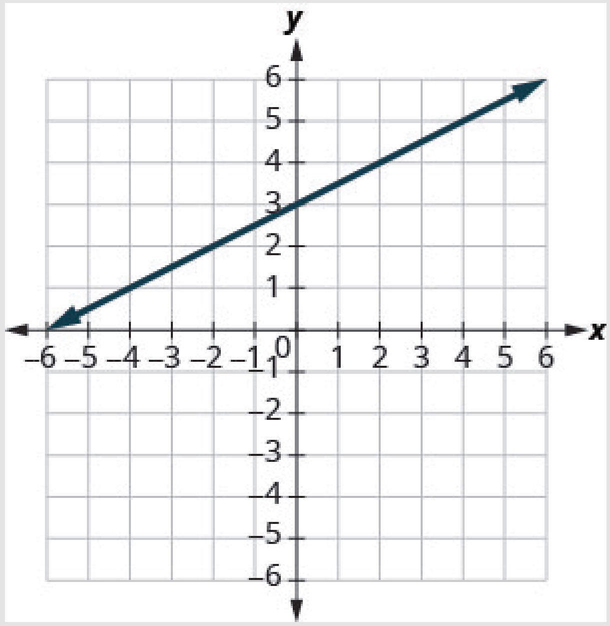


Figure 3.10.43

Solution

slope $m = \frac{1}{2}$ and y-intercept $(0, 3)$

Identify the Slope and y-Intercept From an Equation of a Line

In Understand Slope of a Line, we graphed a line using the slope and a point. When we are given an equation in slope–intercept form, we can use the y-intercept as the point, and then count out the slope from there. Let's practice finding the values of the slope and y-intercept from the equation of a line.

Example 3.10.13

Identify the slope and y-intercept of the line with equation $y = -3x + 5$.

Solution

We compare our equation to the slope–intercept form of the equation.

$$y \;=\; rgb]1.0,0.0,0.0mx \;+\; rgb]0.0,0.0,1.0b$$

Step 1: **Write the equation of the line.**

$$y = rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.03x \;+\; rgb]0.0,0.0,1.05$$

Step 2: **Identify the slope.**

$$m = rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.03$$

Step 3: **Identify the** y**-intercept.**

y-intercept is

$$rgb]0.1,0.1,0.1(rgb]0.1,0.1,0.10rgb]0.1,0.1,0.1,rgb]0.1,0.1,0.1\; rgb]0.0,0.0,1.05rgb]0.1,0.1,0.1)$$

.

Try It

31) Identify the slope and y-intercept of the line $y = \frac{2}{5}x - 1$

Solution

$\frac{2}{5}$; $(0, -1)$

32) Identify the slope and y-intercept of the line $y = -\frac{4}{3}x + 1$

Solution

$-\frac{4}{3}$; $(0, 1)$

When an equation of a line is not given in slope–intercept form, our first step will be to solve the equation for y.

Example 3.10.14

Identify the slope and y-intercept of the line with equation $x + 2y = 6$.

Solution

This equation is not in slope–intercept form. In order to compare it to the slope–intercept form we must first solve the equation for y.

Step 1: **Solve for y.**

$$x + 2y = 6$$

Step 2: **Subtract x from each side.**

$$2y \; = \; -x \; + \; 6$$

Step 3: **Divide both sides by 2.**

$$\frac{2y}{2} = \frac{-x+6}{2}$$

$$\text{Simplify.} \qquad \frac{2y}{2} = \frac{-x}{2} + \frac{6}{2}$$

Remember: $\dfrac{a+b}{c} = \dfrac{a}{c} + \dfrac{b}{c}$

$$y \; = -\frac{1}{2}x \; + \; 3$$

Step 6: **Write the slope–intercept form of the equation of the line.**

$$y \; = \; rgb]1.0, 0.0, 0.0mx \; + \; rgb]0.0, 0.0, 1.0b$$

Step 7: **Write the equation of the line.**

$$y \; = \; -rgb]1.0, 0.0, 0.0\frac{1}{2}x \; + \; rgb]0.0, 0.0, 1.03$$

Step 8: **Identify the slope.**

$$rgb]0.1, 0.1, 0.1mrgb]0.1, 0.1, 0.1 \; rgb]0.1, 0.1, 0.1 \; =rgb]0.0, 0.0, 1.0 \; rgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.0\frac{1}{2}$$

Step 9: **Identify the y-intercept.**

y-intercept = $(0, \; rgb]0.0, 0.0, 1.03)$

Try It

33) Identify the slope and y-intercept of the line $x + 4y = 8$.

Solution

$-\frac{1}{4}$; $(0, 2)$

34) Identify the slope and y-intercept of the line $3x + 2y = 12$.

Solution

$-\frac{3}{2}$; $(0, 6)$

Graph a Line Using its Slope and Intercept

Now that we know how to find the slope and y-intercept of a line from its equation, we can graph the line by plotting the y-intercept and then using the slope to find another point.

Example 3.10.15

Graph the line of the equation $y = 4x - 2$ using its slope and y-intercept.

Solution

Step 1: **Find the slope-intercept form of the equation.**

This equation is in slope-intercept form

$$y = 4x - 2$$

Step 2: **Identify the slope and y-intercept.**

Use $y = rgb]1.0, 0.0, 0.0mx + rgb]0.0, 0.0, 1.0b$

Find the slope.

Find the y-intercept.

$$y = rgb]1.0, 0.0, 0.0mx + rgb]0.0, 0.0, 1.0b$$
$$y = rgb]1.0, 0.0, 0.04x + (rgb]0.0, 0.0, 1.0 - 2)$$
$$m = 4$$
$$b = -2, (0, -2)$$

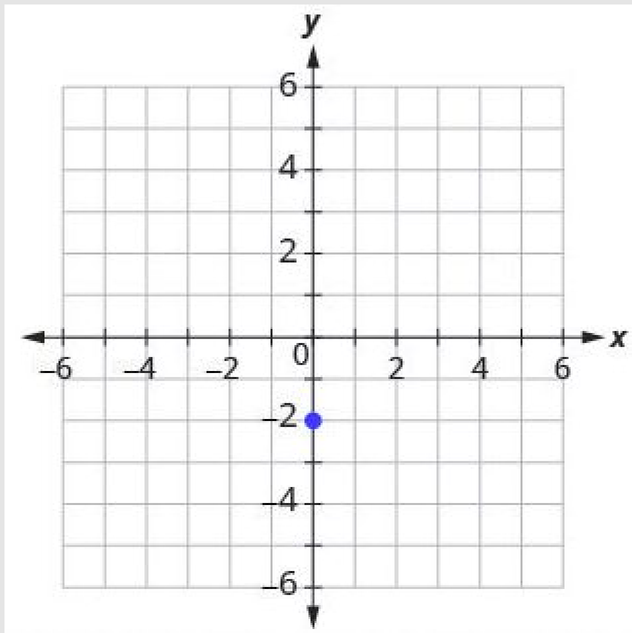


Figure 3.10.44

Step 4: **Use the slope formula** $m = \dfrac{rise}{run}$ **to identify the rise and the run.**

Identify the rise and the run.

$$m = 4$$
$$\frac{rise}{run} = \frac{4}{1}$$
$$rise = 4$$
$$run = 1$$

Step 5: **Starting at the y-intercept, count out the rise and run to mark the second point.**

Start at (0,-2) and count the rise and run.

Up 4, right 1.

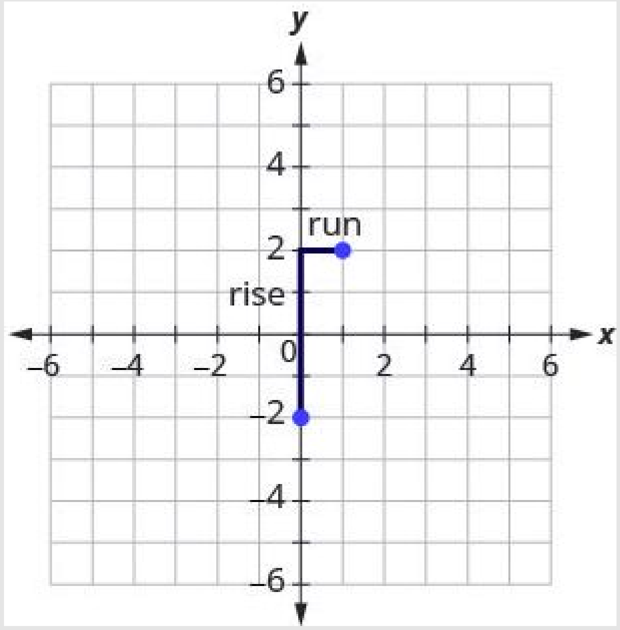


Figure 3.10.45

Step 6: **Connect the points with a line.**

Connect the two points with a line.

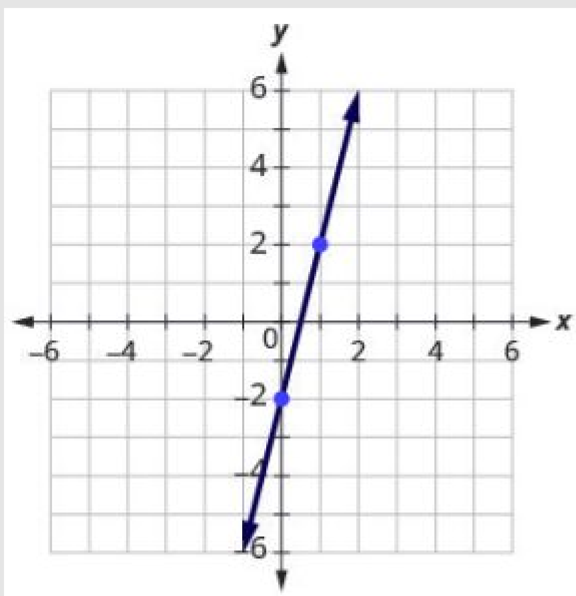


Figure 3.10.46

Try It

35) Graph the line of the equation $y = 4x + 1$ using its slope and y-intercept.

Solution

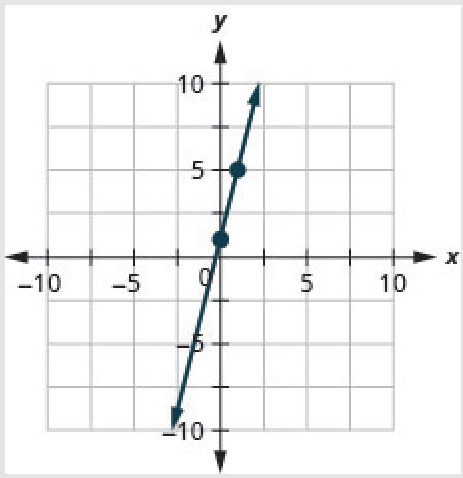


Figure 3.10.47

36) Graph the line of the equation $y = 2x - 3$ using its slope and y-intercept.

Solution

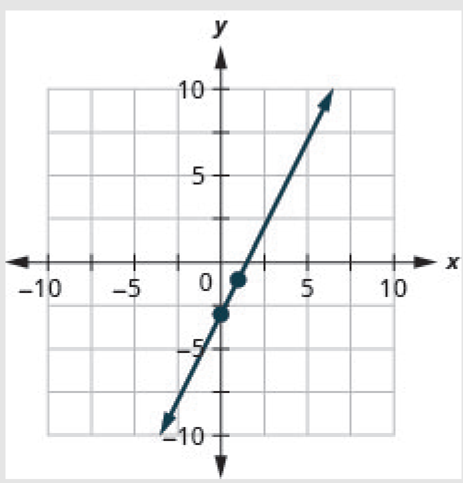


Figure 3.10.48

HOW TO

Graph a line using its slope and y-intercept.

1. Find the slope-intercept form of the equation of the line.
2. Identify the slope and y-intercept.
3. Plot the y-intercept.
4. Use the slope formula $m = \frac{rise}{run}$ to identify the **rise** and the **run**.
5. Starting at the y-intercept, count out the rise and run to mark the second point.
6. Connect the points with a line.

Example 3.10.16

Graph the line of the equation $y = -x + 4$ using its slope and y-intercept.

Solution

$$y = mx + b$$

Step 1: **The equation is in slope–intercept form.**

$$y = -x + 4$$

Step 2: **Identify the slope and y-intercept.**

$$m = -1$$

y-intercept is $(0, 4)$

Step 3: **Plot the y-intercept.**

See graph below.

Step 4: **Identify the rise and the run.**

$$m = \frac{-1}{1}$$

Step 5: **Count out the rise and run to mark the second point.**

$$rise = -1, run = 1$$

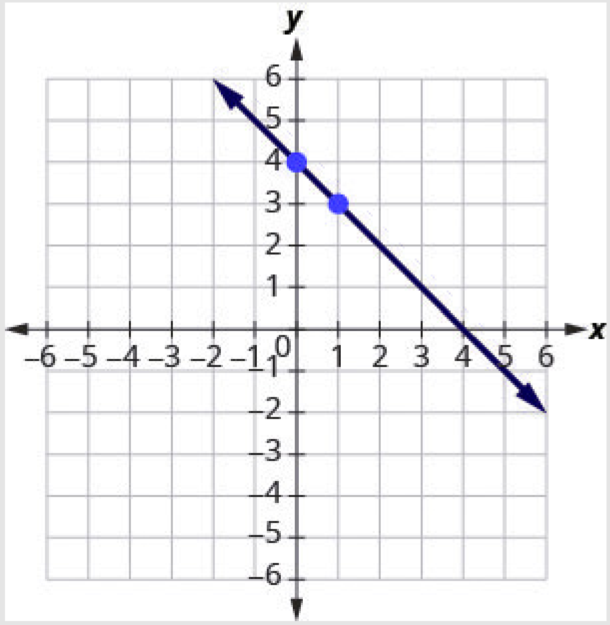


Figure 3.10.49

Step 6: **Draw the line.**

Step 7: **To check your work, you can find another point on the line and make sure it is a solution of the equation.**

In the graph we see the line goes through $(4, 0)$.

Step 8: **Check.**

$$y = -x + 4$$
$$0 \stackrel{?}{=} -4 + 4$$
$$0 = 0 \checkmark$$

Try It

37) Graph the line of the equation $y = -x - 3$ using its slope and y-intercept.

Solution

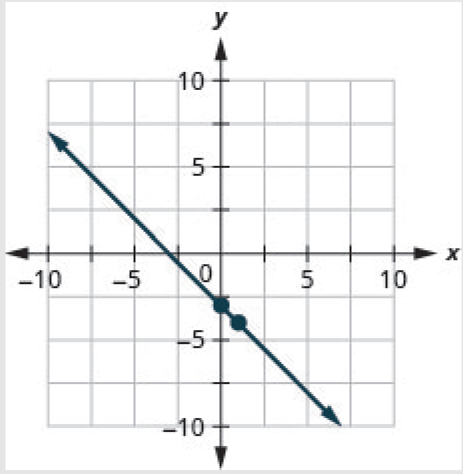


Figure 3.10.50

38) Graph the line of the equation $y = -x - 1$ using its slope and y-intercept.

Solution

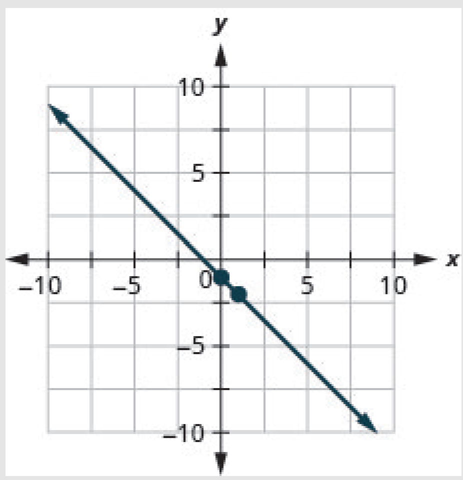


Figure 3.10.51

Example 3.10.17

Graph the line of the equation $y = -\frac{2}{3}x - 3$ using its slope and y-intercept.

Solution

$$y = mx + b$$

Step 1: **The equation is in slope–intercept form.**

$$m = -\frac{2}{3}; \text{ } y\text{-intercept is } (0, -3)$$

Step 2: **Identify the slope and y-intercept.**

See graph below.

Step 3: **Plot the y-intercept.**

Step 4: **Identify the rise and the run.**

Step 5: **Count out the rise and run to mark the second point.**

Step 6: **Draw the line.**

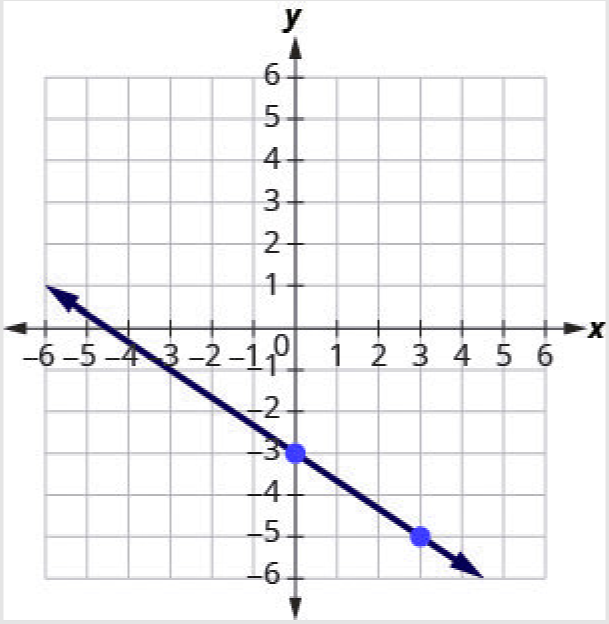


Figure 3.10.52

Try It

39) Graph the line of the equation $y = -\frac{5}{2}x + 1$ using its slope and y-intercept.

Solution

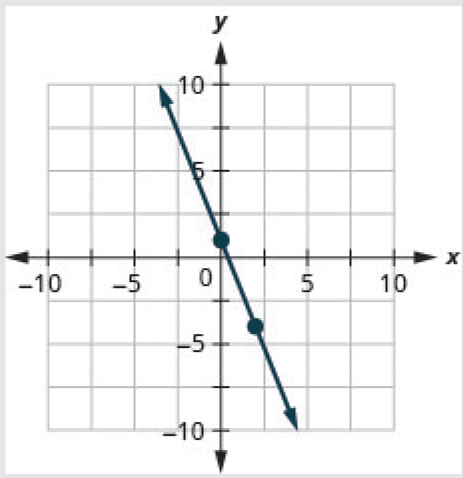


Figure 3.10.53

40) Graph the line of the equation $y = -\frac{3}{4}x - 2$ using its slope and y-intercept.

Solution

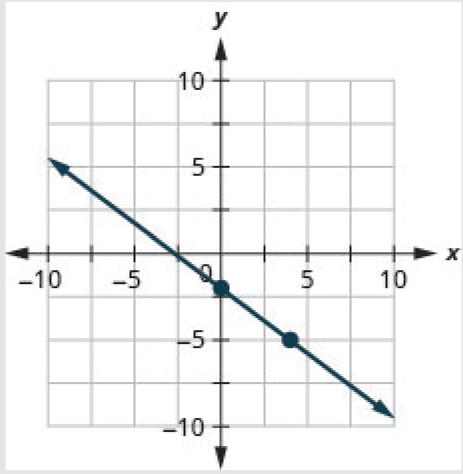


Figure 3.10.54

Example 3.10.18

Graph the line of the equation $4x - 3y = 12$ using its slope and y-intercept.

Solution

$$4x - 3y = 12$$

Step 1: **Find the slope–intercept form of the equation.**

$$\begin{aligned} 4x - 3y &= 12 \\ -3y &= -4x + 12 \\ -\frac{3y}{3} &= \frac{-4x + 12}{-3} \end{aligned}$$

Step 2: **The equation is now in slope–intercept form.**

$$y = \frac{4}{3}x - 4$$

Step 3: **Identify the slope and y-intercept.**

$$m = \frac{4}{3}$$

y-intercept is $(0, -4)$

See graph below.

Step 4: **Plot the y-intercept.**

Step 5: **Identify the rise and the run; count out the rise and run to mark the second point.**

Step 6: **Draw the line.**

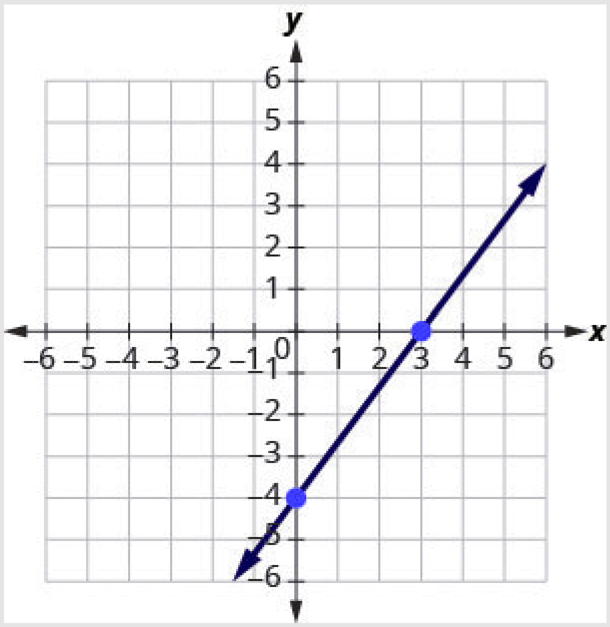


Figure 3.10.55

Try It

41) Graph the line of the equation $2x - y = 6$ using its slope and y-intercept.

Solution

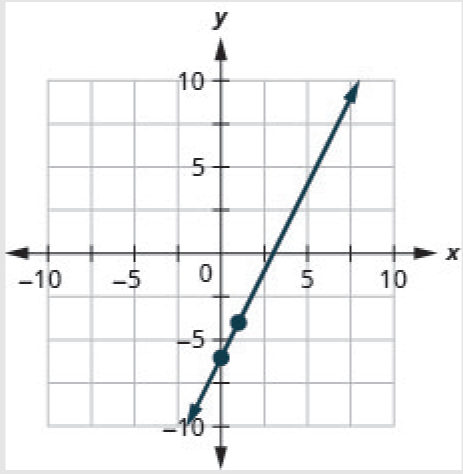


Figure 3.10.56

42) Graph the line of the equation $3x - 2y = 8$ using its slope and y-intercept.

Solution

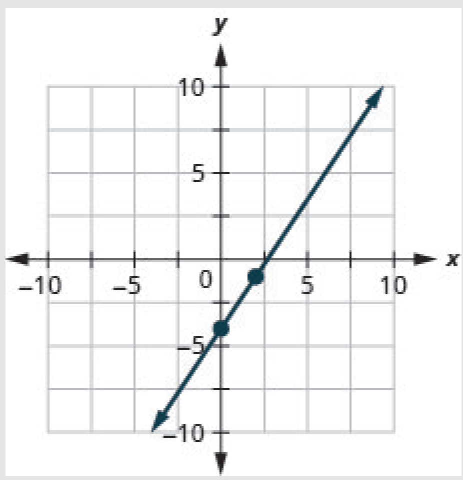


Figure 3.10.57

We have used a grid with x and y both going from about -10 to 10 for all the equations we've graphed so far. Not all linear equations can be graphed on this small grid. Often, especially in applications with real-world data, we'll need to extend the axes to bigger positive or smaller negative numbers.

Example 3.10.19

Graph the line of the equation $y = 0.2x + 45$ using its slope and y-intercept.

Solution

We'll use a grid with the axes going from about -80 to 80.

$$y = mx + b$$

Step 1: **The equation is in slope–intercept form.**

$$y = 0.2x + 45$$

Step 2: **Identify the slope and y-intercept.**

$$m = 0.2$$

The y-intercept is $(0, 45)$.

Step 3: **Plot the y-intercept.**

See graph below.

Step 4: **Count out the rise and run to mark the second point.**

The slope is $m = 0.2$; in fraction form this means $m = \frac{2}{10}$. Given the scale of our graph, it would be easier to use the equivalent fraction $m = \frac{10}{50}$.

Step 5: **Draw the line.**

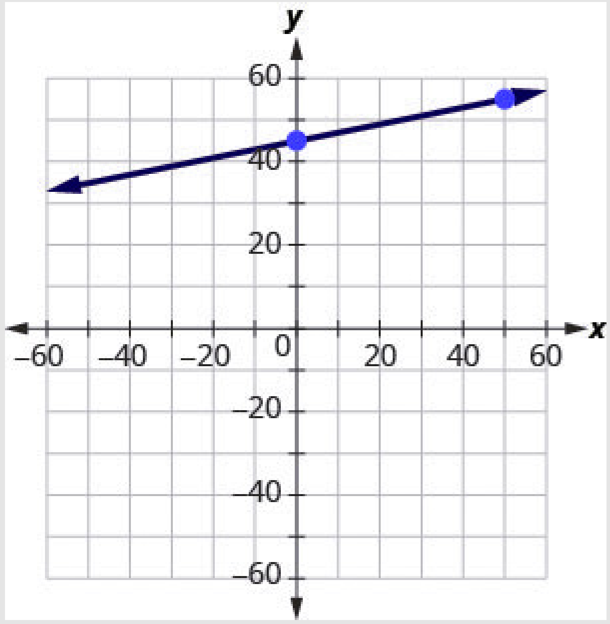


Figure 3.10.58

Try It

43) Graph the line of the equation $y = 0.5x + 25$ using its slope and y-intercept.

Solution

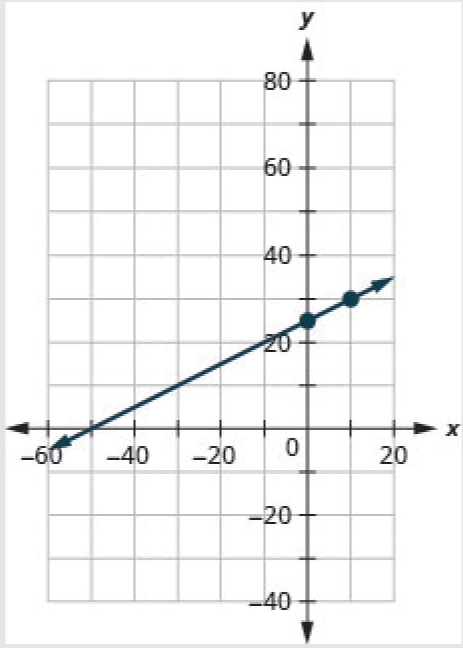


Figure 3.10.59

44) Graph the line of the equation $y = 0.1x - 30$ using its slope and y-intercept.

Solution

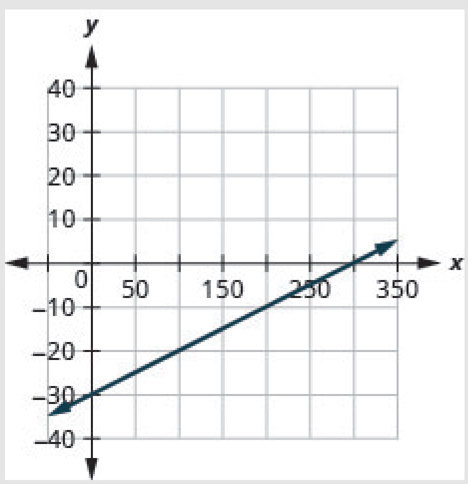


Figure 3.10.60

Now that we have graphed lines by using the slope and y-intercept, let's summarize all the methods we have used to graph lines. See Figure 3.10.61.

Methods to Graph Lines			
Point Plotting	**Slope–Intercept**	**Intercepts**	**Recognize Vertical and Horizontal Lines**
$\begin{array}{c\|c} x & y \\ \hline & \\ \hline & \\ \hline & \end{array}$	$y = mx + b$	$\begin{array}{c\|c} x & y \\ \hline 0 & \\ \hline & 0 \\ \hline & \end{array}$	
Find three points. Plot the points, make sure they line up, then draw the line.	Find the slope and *y*-intercept. Start at the *y*-intercept, then count the slope to get a second point.	Find the intercepts and a third point. Plot the points, make sure they line up, then draw the line.	The equation has only one variable. $x = a$ vertical $y = b$ horizontal

Figure 3.10.61

Choose the Most Convenient Method to Graph a Line

Now that we have seen several methods we can use to graph lines, how do we know which method to use for a given equation?

While we could plot points, use the slope–intercept form, or find the intercepts for *any* equation, if we recognize the most convenient way to graph a certain type of equation, our work will be easier. Generally, plotting points is not the most efficient way to graph a line. We saw better methods in previous sections and earlier in this section. Let's look for some patterns to help determine the most convenient method to graph a line.

Here are six equations we graphed in this chapter, and the method we used to graph each of them.

	Equation	Method
#1	$x = 2$	Vertical Line
#2	$y = 4$	Horizontal Line
#3	$-x + 2y = 6$	Intercepts
#4	$4x - 3y = 12$	Intercepts
#5	$y = 4x - 2$	Slope-intercept
#6	$y = -x + 4$	Slope-intercept

Equations #1 and #2 each have just one variable. Remember, in equations of this form the value of that one variable is constant; it does not depend on the value of the other variable. Equations of this form have graphs that are vertical or horizontal lines.

In equations #3 and #4, both x and y are on the same side of the equation. These two equations are of the form $Ax + By = C$. We substituted $y = 0$ to find the x-intercept and $x = 0$ to find the y-intercept, and then found a third point by choosing another value for x or y.

Equations #5 and #6 are written in slope–intercept form. After identifying the slope and y-intercept from the equation we used them to graph the line.

This leads to the following strategy.

Strategy for Choosing the Most Convenient Method to Graph a Line

Consider the form of the equation.

- **If it only has one variable, it is a vertical or horizontal line.**
    - $x = a$ is a vertical line passing through the x-axis at a.
    - $y = b$ is a horizontal line passing through the y-axis at b.
- **If y is isolated on one side of the equation, in the form $y = mx + b$, graph by using the slope and y-intercept.**
    - Identify the slope and y-intercept and then graph.

- **If the equation is of the form $Ax + By = C$, find the intercepts**.
    - Find the x– and y-intercepts, a third point, and then graph.

Example 3.10.20

Determine the most convenient method to graph each line.

a. $y = -6$
b. $5x - 3y = 15$
c. $x = 7$
d. $y = \frac{2}{5}x - 1$

Solution

a. $y = -6$
This equation has only one variable, y. Its graph is a horizontal line crossing the y-axis at -6.

---

b. $5x - 3y = 15$
This equation is of the form $Ax + By = C$. The easiest way to graph it will be to find the intercepts and one more point.

---

c. $x = 7$
There is only one variable, x. The graph is a vertical line crossing the x-axis at 7.

---

d. $y = \frac{2}{5}x - 1$

Since this equation is in $y = mx + b$ form, it will be easiest to graph this line by using the slope and y -intercept.

Try It

45) Determine the most convenient method to graph each line:

a. $3x + 2y = 12$
b. $y = 4$
c. $y = \frac{1}{5}x - 4$
d. $x = -7$

Solution
a. intercepts
b. horizontal line
c. slope–intercept
d. vertical line

46) Determine the most convenient method to graph each line:

a. $x = 6$
b. $y = -\frac{3}{4}x + 1$
c. $y = -8$
d. $4x - 3y = -1$

Solution
a. vertical line
b. slope–intercept
c. horizontal line
d. intercepts

Graph and Interpret Applications of Slope–Intercept

Many real-world applications are modelled by linear equations. We will take a look at a few applications here so you can see how equations written in slope–intercept form relate to real-world situations.

Usually when a linear equation models a real-world situation, different letters are used for the variables, instead of x and y. The variable names remind us of what quantities are being measured.

Example 3.10.21

The equation $F = \frac{9}{5}C + 32$ is used to convert temperatures, C, on the Celsius scale to temperatures, F, on the Fahrenheit scale.

a. Find the Fahrenheit temperature for a Celsius temperature of 0.
b. Find the Fahrenheit temperature for a Celsius temperature of 20.
c. Interpret the slope and F-intercept of the equation.
d. Graph the equation.

Solution

a.

Step 1: **Find the Fahrenheit temperature for a Celsius temperature of 0.**

Step 2: **Find F when $C = 0$.**

$$F7 = \frac{9}{5}C + 32$$
$$F = \frac{9}{5} \times 0 + 32$$
$$F = 32$$

---

b.

Step 1: **Find the Fahrenheit temperature for a Celsius temperature of 20.**

Step 2: **Find F when $C = 20$.**

$$F = \frac{9}{5}C + 32$$
$$F = \frac{9}{5}(20) + 32$$
$$F = 36 + 32$$
$$F = 68$$

---

c.

Interpret the slope and F-intercept of the equation. Even though this equation uses F and C, it is still in slope–intercept form.

$$rgb]1.0, 0.0, 0.0y = mrgb]0.0, 0.0, 1.0x + b$$
$$rgb]1.0, 0.0, 0.0F = mrgb]0.0, 0.0, 1.0C + b$$
$$F = \frac{9}{5}C + 32$$

The slope, $\frac{9}{5}$, means that the temperature Fahrenheit (F) increases 9 degrees when the temperature Celsius (C) increases 5 degrees. The F-intercept means that when the temperature is (0) on the Celsius scale, it is 32° on the Fahrenheit scale.

---

d.

Graph the equation. We'll need to use a larger scale than our usual. Start at the F-intercept $(0, 32)$ then count out the rise of 9 and the run of 5 to get a second point. See Figure 3.10.62

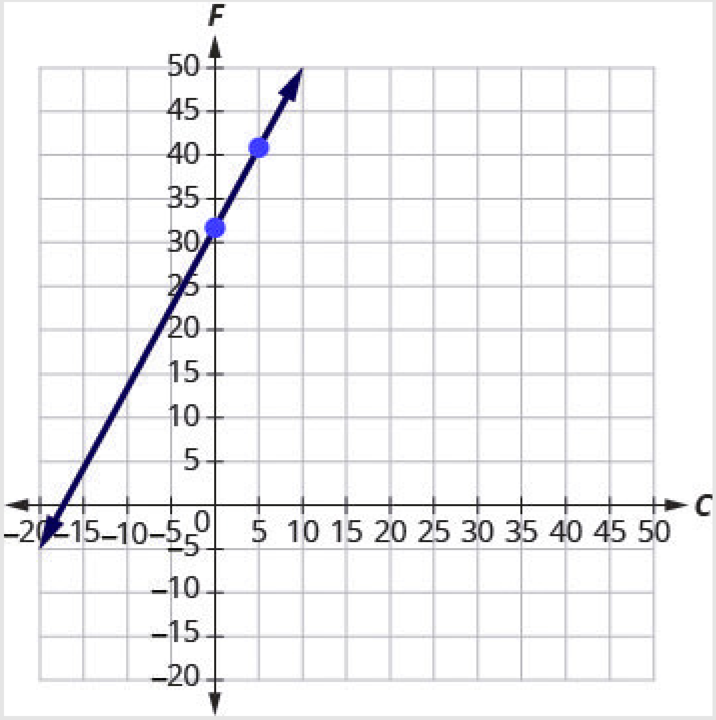


Figure 3.10.62

Try It

47) The equation $h = 2s + 50$ is used to estimate a woman's height in inches, h, based on her shoe size, s.

a. Estimate the height of a child who wears women's shoe size 0.

b. Estimate the height of a woman with shoe size 8.

c. Interpret the slope and h-intercept of the equation.

d. Graph the equation.

Solution

a. 50 inches

b. 66 inches

c. The slope, 2, means that the height, h, increases by 2 inches when the shoe size, s, increases by 1. The h-intercept means that when the shoe size is 0, the height is 50 inches.

d.

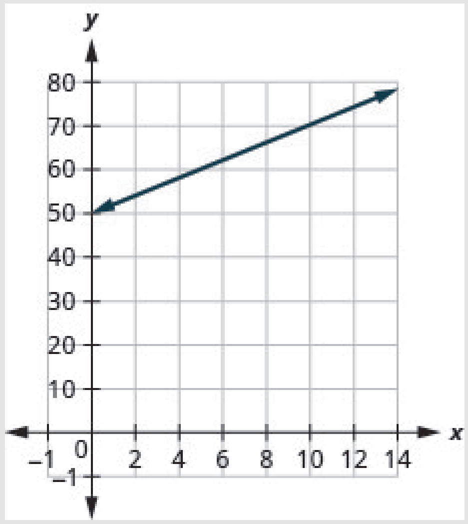


Figure 3.10.63

48) The equation $T = \frac{1}{4}n + 40$ is used to estimate the temperature in degrees Fahrenheit, T, based on the number of cricket chirps, n, in one minute.

a. Estimate the temperature when there are no chirps.
b. Estimate the temperature when the number of chirps in one minute is 100.
c. Interpret the slope and T-intercept of the equation.
d. Graph the equation.

Solution

a. 40 degrees
b. 65 degrees
c. The slope, $\frac{1}{4}$, means that the temperature Fahrenheit (F) increases 1 degree when the number of chirps, n, increases by 4. The T-intercept means that when the number of chirps is 0, the temperature is $40°$

d.

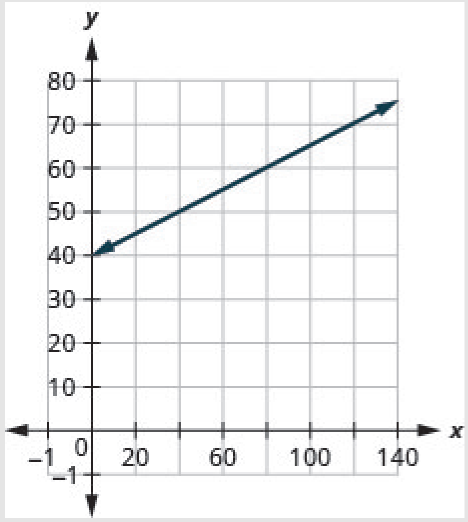


Figure 3.10.64

The cost of running some types business has two components—a *fixed cost* and a *variable cost*. The fixed cost is always the same regardless of how many units are produced. This is the cost of rent, insurance, equipment, advertising, and other items that must be paid regularly. The variable cost depends on the number of units produced. It is for the material and labour needed to produce each item.

Example 3.10.22

Stella has a home business selling gourmet pizzas. The equation $C = 4p + 25$ models the relation between her weekly cost, C, in dollars and the number of pizzas, p, that she sells.

a. Find Stella's cost for a week when she sells no pizzas.
b. Find the cost for a week when she sells 15 pizzas.
c. Interpret the slope and C-intercept of the equation.
d. Graph the equation.

Solution

a.

Step 1: **Find Stella's cost for a week when she sells no pizzas.**

$$C = 4p + 25$$

Step 2: **Find C when $p = 0$.**

$$\begin{array}{ll} & C = 4p + 25 \\ & C = 4(0) + 25 \\ \text{Simplify.} & C = 25 \end{array}$$

Stella's fixed cost is $\$25$ when she sells no pizzas.

---

b.

Step 1: **Find the cost for a week when she sells 15 pizzas.**

$$C = 4p + 25$$

Step 2: **Find C when $p = 15$.**

$$\begin{array}{ll} & C = 4p + 25 \\ & C = 4(15) + 25 \\ \text{Simplify.} & C = 60 + 25 \\ & C = 85 \end{array}$$

Stella's costs are $\$85$ when she sells 15 pizzas.

---

c.

Step 1: **Interpret the slope and C-intercept of the equation.**

$$rgb]1.0,0.0,0.0y = mrgb]0.0,0.0,1.0x + b$$
$$rgb]1.0,0.0,0.0C = 4rgb]0.0,0.0,1.0p + 25$$

The slope, 4, means that the cost increases by $\$4$ for each pizza Stella sells. The C-intercept means that even when Stella sells no pizzas, her costs for the week are $\$25$.

---

d.

Step 1: **Graph the equation.**

We'll need to use a larger scale than our usual. Start at the C-intercept $(0, 25)$ then count out the rise of 4 and the run of 1 to get a second point

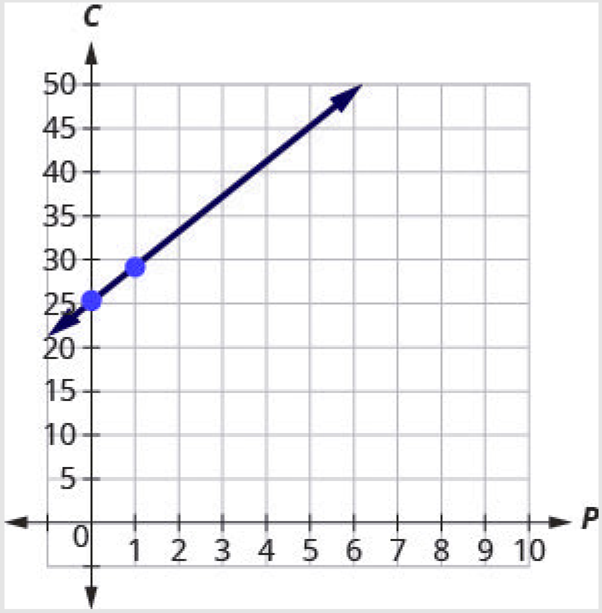


Figure 3.10.65

Try It

49) Sam drives a delivery van. The equation $C = 0.5m + 60$ models the relation between his weekly cost, C, in dollars and the number of miles, m, that he drives.

a. Find Sam's cost for a week when he drives 0 miles.

b. Find the cost for a week when he drives 250 miles.

c. Interpret the slope and C-intercept of the equation.

d. Graph the equation.

Solution

a. $\$60$

b. $\$185$

c. The slope, 0.5, means that the weekly cost, C, increases by $\$0.50$ when the number of miles driven, n, increases by 1. The C-intercept means that when the number of miles driven is 0, the weekly cost is $\$60$.

d.

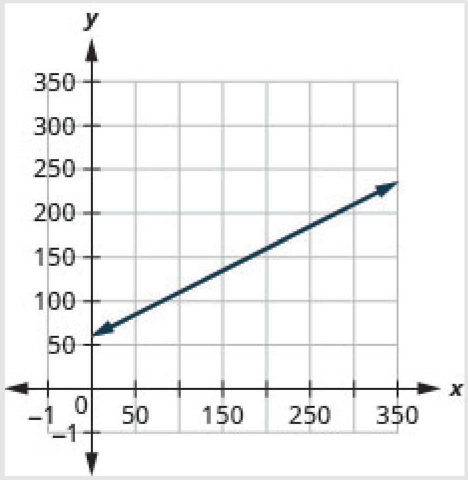


Figure 3.10.66

50) Loreen has a calligraphy business. The equation $C = 1.8n + 35$ models the relation between her weekly cost, C, in dollars and the number of wedding invitations, n, that she writes.

a. Find Loreen's cost for a week when she writes no invitations.

b. Find the cost for a week when she writes 75 invitations.

c. Interpret the slope and C-intercept of the equation.

d. Graph the equation.

Solution

a. $\$35$

b. $\$170$

c. The slope, 1.8, means that the weekly cost, C, increases by $\$1.80$ when the number of invitations, n, increases by 1.80.

The C-intercept means that when the number of invitations is 0, the weekly cost is $\$35$.

d.

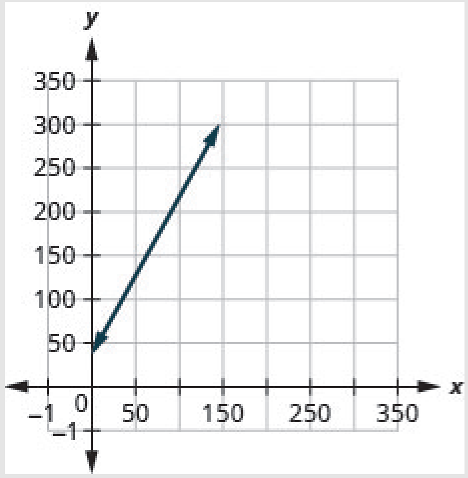


Figure 3.10.67

Use Slopes to Identify Parallel Lines

The slope of a line indicates how steep the line is and whether it rises or falls as we read it from left to right. Two lines that have the same slope are called **parallel lines**. Parallel lines never intersect.

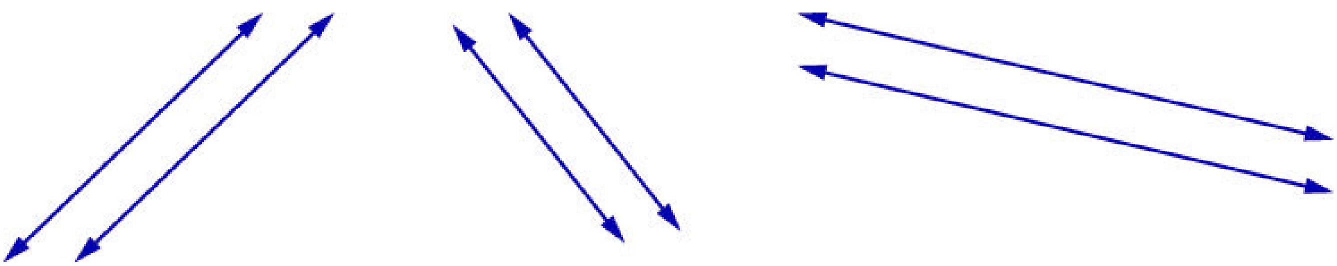


Figure 3.10.68

We say this more formally in terms of the rectangular coordinate system. Two lines that have the same slope and different y-intercepts are called parallel lines. See Figure 3.10.69.

Verify that both lines have the same slope, $m = \frac{2}{5}$, and different y-intercepts.

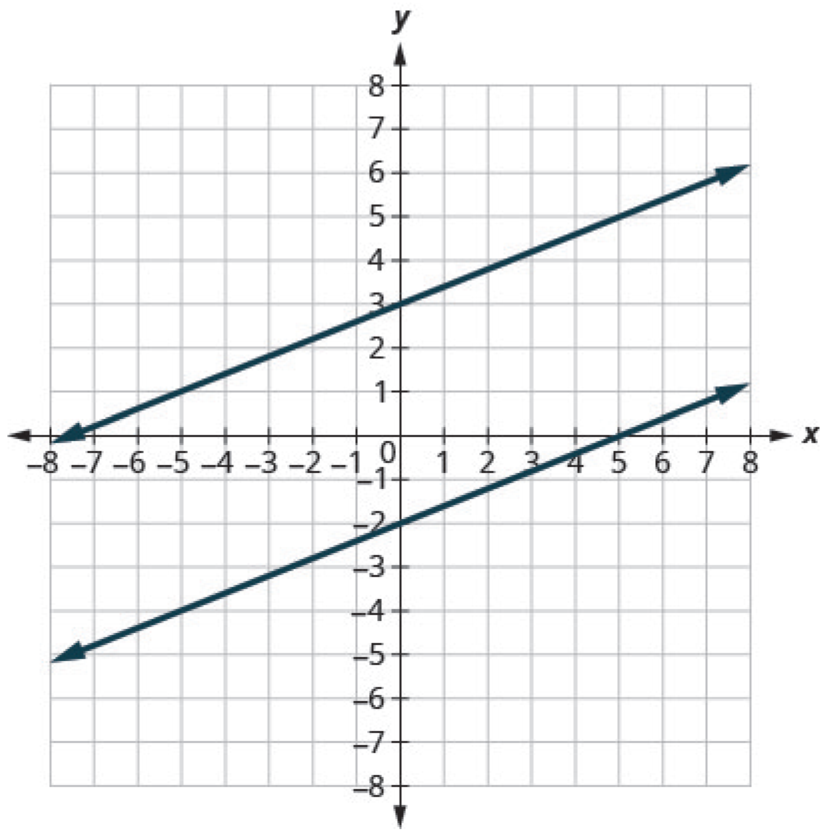


Figure 3.10.69

What about vertical lines? The slope of a vertical line is undefined, so vertical lines don't fit in the definition above. We say that vertical lines that have different x-intercepts are parallel. See Figure 3.10.70.

Vertical lines with different x-intercepts are parallel.

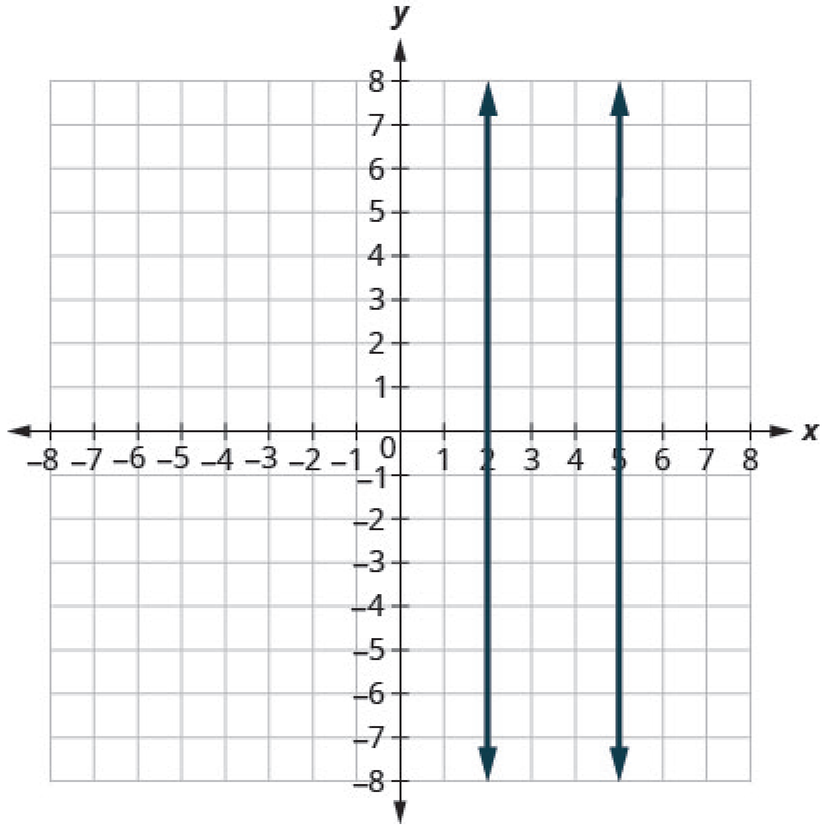


Figure 3.10.70

Parallel Lines

Parallel lines are lines in the same plane that do not intersect.

- Parallel lines have the same slope and different y-intercepts.
- If m_1 and m_2 are the slopes of two parallel lines then $m_1 = m_2$.
- Parallel vertical lines have different x-intercepts.

Let's graph the equations $y = -2x + 3$ and $2x + y = -1$ on the same grid. The first equation is already in slope–intercept form: $y = -2x + 3$. We solve the second equation for y:

$$2x + y = -1$$
$$y = -2x - 1$$

Graph the lines.

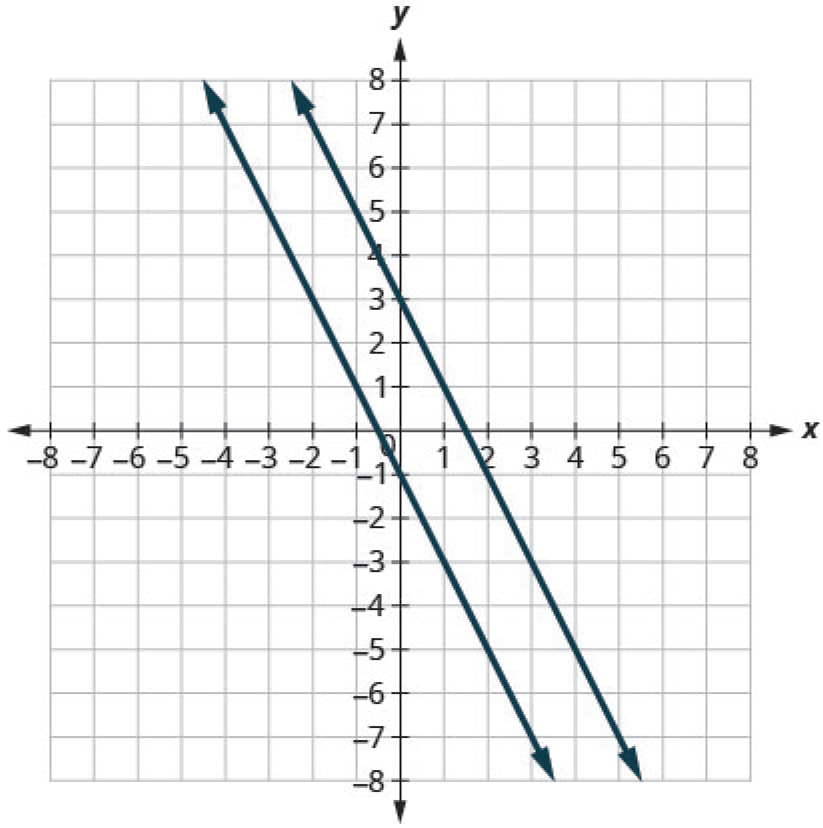


Figure 3.10.71

Notice the lines look parallel. What is the slope of each line? What is the y-intercept of each line?

$y = mx + b$	$y = mx + b$
$y = -2x + 3$	$y = -2x - 1$
$m = -2$	$m = -2$
$b = 3, (0, 3)$	$b = -1, (0, -1)$

The slopes of the lines are the same and the y-intercept of each line is different. So we know these lines are parallel.

Since parallel lines have the same slope and different y-intercepts, we can now just look at the slope–intercept form of the equations of lines and decide if the lines are parallel.

Example 3.10.23

Use slopes and y-intercepts to determine if the lines $3x - 2y = 6$ and $y = \frac{3}{2}x + 1$ are parallel.

Solution

Step 1: **Solve the first equation for y.**

$3x - 2y = 6$ $-2y = -3x + 6$ $\frac{-2y}{-2} = \frac{3x+6}{-2}$ $y = \frac{3}{2}x - 3$	$y = \frac{3}{2}x + 1$

Step 2: **The equation is now in slope-intercept form.**

$y = \frac{3}{2}x - 3$	

Step 3: **The equation of the second line is already in slope-intercept form.**

	$y = \frac{3}{2}x + 1$

Step 4: **Identify the slope and y-intercept of both lines.**

$y = mx + b$ $m = \frac{3}{2}$ $y = \frac{3}{2}x - 3$	$y = mx + b$ $m = \frac{3}{2}$ $y = \frac{3}{2}x + 1$ $y = \frac{3}{2}x + 1$
y-intercept is $(0, -3)$	y-intercept is $(0, 1)$

The lines have the same slope and different y-intercepts and so they are parallel. You may want to graph the lines to confirm whether they are parallel.

Try It

51) Use slopes and y-intercepts to determine if the lines $2x + 5y = 5$ and $y = -\frac{2}{5}x - 4$ are parallel.

Solution
parallel

52) Use slopes and y-intercepts to determine if the lines $4x - 3y = 6$ and $y = \frac{4}{3}x - 1$are parallel.

Solution
parallel

Example 3.10.24

Use slopes and y-intercepts to determine if the lines $y = -4$ and $y = 3$ are parallel.

Solution

$y = -4$	$y = 3$

Step 1: **Write each equation in slope-intercept form.**

Since there is no x term we write $0x$.

$y = 0x - 4$	$y = 0x + 3$

Step 2: **Identify the slope and y-intercept of both lines.**

$m = 0$	$m = 0$
y-intercept is $(0, 4)$	y-intercept is $(0, 3)$

The lines have the same slope and different y-intercepts and so they are parallel.

There is another way you can look at this example. If you recognize right away from the equations that these are horizontal lines, you know their slopes are both 0. Since the horizontal lines cross the y-axis at $y = -4$ and at $y = 3$, we know the y-intercepts are $(0, -4)$ and $(0, 3)$. The lines have the same slope and different y-intercepts and so they are parallel.

Try It

53) Use slopes and y-intercepts to determine if the lines $y = 8$ and $y = -6$ are parallel.

Solution

parallel

54) Use slopes and y-intercepts to determine if the lines $y = 1$ and $y = -5$ are parallel.

Solution

parallel

Example 3.10.25

Use slopes and y-intercepts to determine if the lines $x = -2$ and $x = -5$ are parallel.

Solution

$$x = -2 \text{ and } x = -5$$

Since there is no y, the equations cannot be put in slope–intercept form. But we recognize them as equations of vertical lines. Their x-intercepts are -2 and -5. Since their x-intercepts are different, the vertical lines are parallel.

Try It

55) Use slopes and y-intercepts to determine if the lines $x = 1$ and $x = -5$ are parallel.

Solution

parallel

56) Use slopes and y-intercepts to determine if the lines $x = 8$ and $x = -6$ are parallel.

Solution

parallel

Example 3.10.26

Use slopes and y-intercepts to determine if the lines $y = 2x - 3$ and $-6x + 3y = -9$ are parallel. You may want to graph these lines, too, to see what they look like.

Solution

$y = 2x - 3$	$-6x + 3y = -9$

Step 1: **The first equation is already in slope-intercept form.**

$y = 2x - 3$	

Step 2: **Solve the second equation for y.**

	$-6x + 3y = -9$ $3y = 6x - 9$ $\frac{3y}{3} = \frac{6x - 9}{3}$ $y = 2x - 3$

Step 3: **The second equation is now in slope-intercept form.**

$y = 2x - 3$	$y = 2x - 3$

Step 4: **Identify the slope and y-intercept of both lines.**

$y = 2x - 3$ $y = mx + b$ $m = 2$	$y = 2x - 3$ $y = mx + b$ $m = 2$
y-intercept is $(0, -3)$	y-intercept is $(0, -3)$

The lines have the same slope, but they also have the same y-intercepts. Their equations represent the same line. They are not parallel; they are the same line.

Try It

57) Use slopes and y-intercepts to determine if the lines $y = -\frac{1}{2}x - 1$ and $x + 2y = 2$ are parallel.

Solution
not parallel; same line

58) Use slopes and y-intercepts to determine if the lines $y = \frac{3}{4}x - 3$ and $3x - 4y = 12$ are parallel.

Solution
not parallel; same line

Use Slopes to Identify Perpendicular Lines

Let's look at the lines whose equations are $y = \frac{1}{4}x - 1$ and $y = -4x + 2$, shown in Figure 3.10.72.

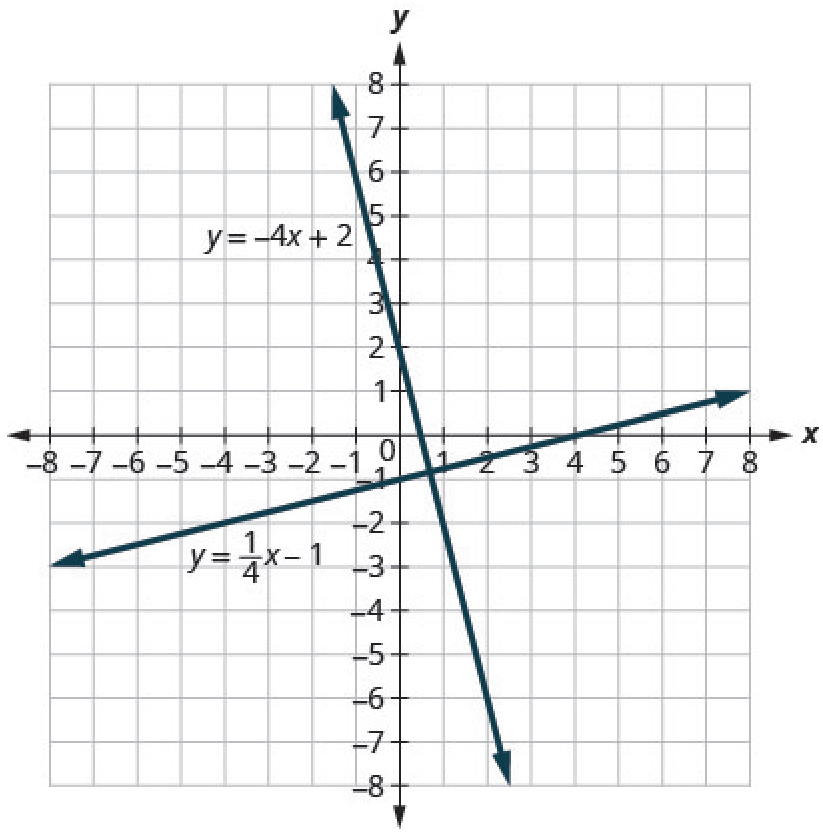


Figure 3.10.72

These lines lie in the same plane and intersect in right angles. We call these lines **perpendicular**.

What do you notice about the slopes of these two lines? As we read from left to right, the line $y = \frac{1}{4}x - 1$ rises, so its slope is positive. The line $y = -4x + 2$ drops from left to right, so it has a **negative slope**. Does it make sense to you that the slopes of two perpendicular lines will have opposite signs?

If we look at the slope of the first line, $m_1 = \frac{1}{4}$, and the slope of the second line, $m_2 = -4$, we can see that they are *negative reciprocals* of each other. If we multiply them, their product is -1.

$$m_1 \cdot m_2$$
$$\frac{1}{4}(-4)$$
$$-1$$

This is always true for perpendicular lines and leads us to this definition.

Perpendicular Lines

Perpendicular lines are lines in the same plane that form a right angle.

If m_1 and m_2 are the slopes of two perpendicular lines, then:

$m_1 \cdot m_2 = -1$ and $m_1 = -\dfrac{1}{m_2}$

Vertical lines and horizontal lines are always perpendicular to each other.

We were able to look at the slope–intercept form of linear equations and determine whether or not the lines were parallel. We can do the same thing for perpendicular lines.

We find the slope–intercept form of the equation, and then see if the slopes are negative reciprocals. If the product of the slopes is -1, the lines are perpendicular. Perpendicular lines may have the same y-intercepts.

Example 3.10.27

Use slopes to determine if the lines, $y = -5x - 4$ and $x - 5y = 5$ are perpendicular.

Solution

Step 1: **The first equation is already in slope-intercept form.**

$y = -5x - 4$	

Step 2: **Solve the second equation for y.**

	$x - 5y = 5$ $-5y = x + 5$ $\frac{-5y}{-5} = \frac{x+5}{-5}$ $y = \frac{1}{-5}x - 1$

Step 3: **Identify the slope of each line.**

$y = -5x - 4$ $y = mx + b$ $m_1 = -5$	$y = \frac{1}{5}x - 4$ $y = mx + b$ $m_2 = \frac{1}{5}$

The slopes are negative reciprocals of each other, so the lines are perpendicular. We check by multiplying the slopes,

$$m_1 \cdot m_2$$
$$= -5(\frac{1}{5})$$
$$= -1$$

Try It

59) Use slopes to determine if the lines $y = -3x + 2$ and $x - 3y = 4$ are perpendicular.

Solution

perpendicular

60) Use slopes to determine if the lines $y = 2x - 5$ and $x + 2y = -6$ are perpendicular.

Solution

perpendicular

Example 3.10.28

Use slopes to determine if the lines, $7x + 2y = 3$ and $2x + 7y = 5$ are perpendicular.

Solution

Step 1: **Solve the equations for y.**

$\begin{aligned} 7x + 2y &= 3 \\ 2y &= -7x + 3 \\ \frac{2y}{2} &= \frac{-7x+3}{2} \\ y &= \frac{-7}{2}x + \frac{3}{2} \end{aligned}$	$\begin{aligned} 2x + 7y &= 5 \\ 2y &= -2x + 5 \\ \frac{7y}{7} &= \frac{-2x+5}{7} \\ y &= \frac{-2}{7}x + \frac{5}{7} \end{aligned}$

Step 2: **Identify the slope of each line.**

$\begin{aligned} y &= mx + b \\ m_1 &= -\frac{7}{2} \end{aligned}$	$\begin{aligned} y &= mx + b \\ m_2 &= -\frac{2}{7} \end{aligned}$

The slopes are reciprocals of each other, but they have the same sign. Since they are not negative reciprocals, the lines are not perpendicular.

Try It

61) Use slopes to determine if the lines $5x + 4y = 1$ and $4x + 5y = 3$ are perpendicular.

Solution

not perpendicular

62) Use slopes to determine if the lines $2x - 9y = 3$ and $9x - 2y = 1$ are perpendicular.

Solution

not perpendicular

Key Concepts

- **Find the Slope of a Line from its Graph using** $m = \frac{rise}{run}$

1. Locate two points on the line whose coordinates are integers.
2. Starting with the point on the left, sketch a right triangle, going from the first point to the second point.
3. Count the rise and the run on the legs of the triangle.
4. Take the ratio of rise to run to find the slope.

- **Graph a Line Given a Point and the Slope**

1. Plot the given point.
2. Use the slope formula $m = \frac{rise}{run}$ to identify the rise and the run.
3. Starting at the given point, count out the rise and run to mark the second point.

4. Connect the points with a line.

- **Slope of a Horizontal Line**
    - The slope of a horizontal line, $y = b$, is 0.
- **Slope of a vertical line**
    - The slope of a vertical line, $x = a$, is undefined

Graph a Line Using its Slope and y-Intercept

1. The slope–intercept form of an equation of a line with slope m and y-intercept, $(0, b)$ is, $y = mx + b$.
2. Find the slope-intercept form of the equation of the line.
3. Identify the slope and y-intercept.
4. Plot the y-intercept.
5. Use the slope formula $m = \dfrac{rise}{run}$ to identify the rise and the run.
6. Starting at the y-intercept, count out the rise and run to mark the second point.
7. Connect the points with a line.

- **Strategy for Choosing the Most Convenient Method to Graph a Line:** Consider the form of the equation.
    - If it only has one variable, it is a vertical or horizontal line.
      $x = a$ is a vertical line passing through the x-axis at a.
      $y = b$ is a horizontal line passing through the y-axis at b.
    - If y is isolated on one side of the equation, in the form $y = mx + b$, graph by using the slope and y-intercept.
      Identify the slope and y-intercept and then graph.
    - If the equation is of the form $Ax + By = C$, find the intercepts.
      Find the x-intercept and y-intercept, a third point, and then graph.
- **Parallel lines** are lines in the same plane that do not intersect.
    - Parallel lines have the same slope and different y-intercepts.
        - If m_1 and m_2 are the slopes of two parallel lines then $m_1 = m_2$.

- Parallel vertical lines have different x-intercepts.

- **Perpendicular lines** are lines in the same plane that form a right angle.
  - If m_1 and m_2 are the slopes of two perpendicular lines, then $m_1 \cdot m_2 = -1$ and $m_1 = -\frac{1}{m_2}$
  - Vertical lines and horizontal lines are always perpendicular to each other.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=1863#h5p-24

b. On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

geoboard
A geoboard is a board with a grid of pegs on it.

negative slope

A negative slope of a line goes down as you read from left to right.

positive slope

A positive slope of a line goes up as you read from left to right.

rise

The rise of a line is its vertical change.

run

The run of a line is its horizontal change.

slope formula

The slope of the line between two points (x_1, y_1) and (x_2, y_2) is $m = \frac{y_2 - y_1}{x_2 - x_1}$

slope of a line

The slope of a line is $m = \frac{rise}{run}$. The rise measures the vertical change and the run measures the horizontal change.

parallel lines

Lines in the same plane that do not intersect.

perpendicular lines

Lines in the same plane that form a right angle.

slope-intercept form of an equation of a line

The slope–intercept form of an equation of a line with slope m and y-intercept, $(0, b)$ is, $y = mx + b$.

3.11 FIND THE EQUATION OF A LINE

Learning Objectives

By the end of this section, you will be able to:

- Find an equation of the line given the slope and y-intercept
- Find an equation of the line given the slope and a point
- Find an equation of the line given two points
- Find an equation of a line parallel to a given line
- Find an equation of a line perpendicular to a given line

Try It

Before you get started, take this readiness quiz:

1) Solve: $\frac{2}{3}=\frac{x}{5}$

2) Simplify: $-\frac{2}{5}(x-15)$

How do online retailers know that 'you may also like' a particular item based on something you just ordered? How can economists know how a rise in the minimum wage will affect the unemployment rate? How do medical researchers create drugs to target cancer cells? How can traffic engineers predict the effect on your commuting time of an increase or decrease in gas prices? It's all mathematics.

You are at an exciting point in your mathematical journey as the mathematics you are studying has interesting applications in the real world.

The physical sciences, social sciences, and the business world are full of situations that can be modelled with linear equations relating two variables. Data is collected and graphed. If the data points appear to form a straight line, an equation of that line can be used to predict the value of one variable based on the value of the other variable.

To create a mathematical model of a linear relation between two variables, we must be able to find the equation of the line. In this section we will look at several ways to write the equation of a line. The specific method we use will be determined by what information we are given.

Find an Equation of the Line Given the Slope and y -Intercept

We can easily determine the slope and intercept of a line if the equation was written in slope–intercept form, $y = mx + b$. Now, we will do the reverse—we will start with the slope and y-intercept and use them to find the equation of the line.

Example 3.11.1

Find an equation of a line with slope -7 and y-intercept $(0, -1)$.

Solution
Since we are given the slope and y-intercept of the line, we can substitute the needed values into the slope–intercept form, $y = mx + b$.

Step 1: **Name the slope.**

$$m = 7$$

Step 2: **Name the y-intercept.**

y-intercept = $(0,\ rgb]0.0, 0.0, 1.0{-}rgb]0.0, 0.0, 1.01)$

Step 3: **Substitute the values into $y = mx + b$.**

$$y = rgb]1.0, 0.0, 0.0mx + rgb]0.0, 0.0, 1.0b$$
$$y = rgb]1.0, 0.0, 0.0{-}rgb]1.0, 0.0, 0.07x + (rgb]0.0, 0.0, 1.0{-}rgb]0.0, 0.0, 1.01)$$
$$y = -7x - 1$$

Try It

3) Find an equation of a line with slope $\frac{2}{5}$ and y-intercept $(0, 4)$.

Solution

$y = \frac{2}{5}x + 4$

4) Find an equation of a line with slope -1 and y-intercept $(0, -3)$.

Solution

$y = -x - 3$

Sometimes, the slope and intercept need to be determined from the graph.

Example 3.11.2

Find the equation of the line shown.

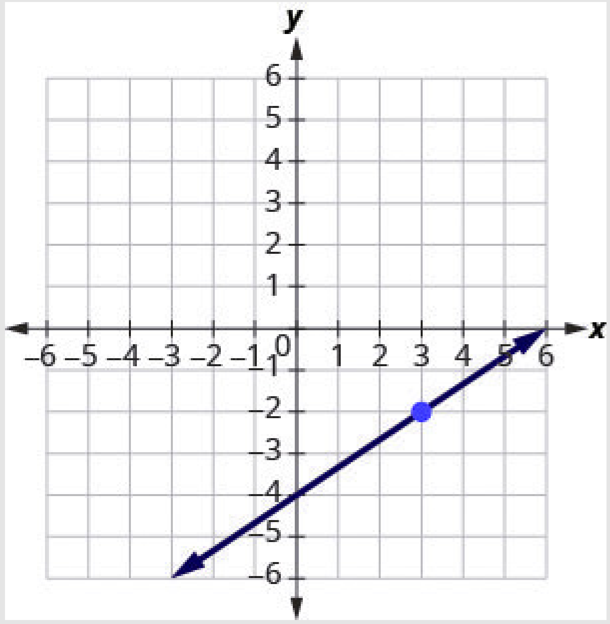


Figure 3.11.1

Solution

We need to find the slope and y-intercept of the line from the graph so we can substitute the needed values into the slope–intercept form, $y = mx + b$.

To find the slope, we choose two points on the graph.

The y-intercept is $(0, -4)$ and the graph passes through $(3, -2)$.

Step 1: **Find the slope by counting the rise and run.**

$$m = \frac{rise}{run}$$
$$m = rgb]1.0, 0.0, 0.0\frac{2}{3}$$

Step 2: **Find the y-intercept.**

$$y - intercept \ = (0, rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.04)$$

Step 3: **Substitute the values into $y = mx + b$.**

$$y = rgb]1.0, 0.0, 0.0mx + rgb]0.0, 0.0, 1.0b$$
$$y = rgb]1.0, 0.0, 0.0\frac{2}{3}xrgb]0.0, 0.0, 1.0 - rgb]0.0, 0.0, 1.0\ rgb]0.0, 0.0, 1.04$$

Try It

5) Find the equation of the line shown in the graph.

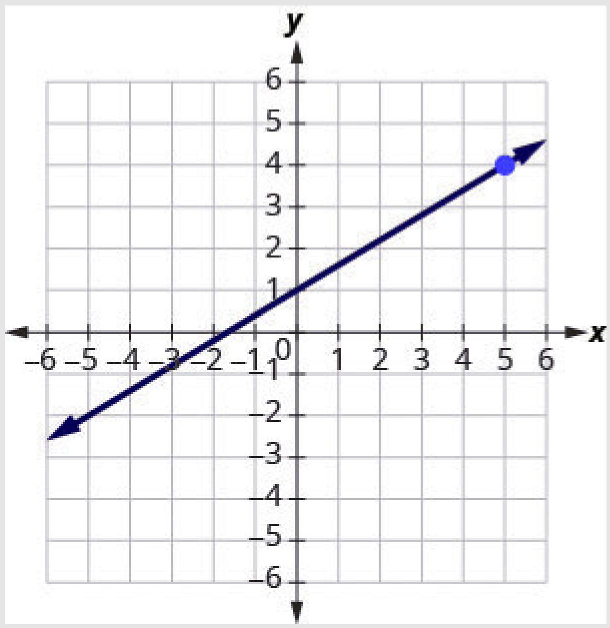


Figure 3.11.2

Solution

$$y = \frac{3}{5}x + 1$$

6) Find the equation of the line shown in the graph.

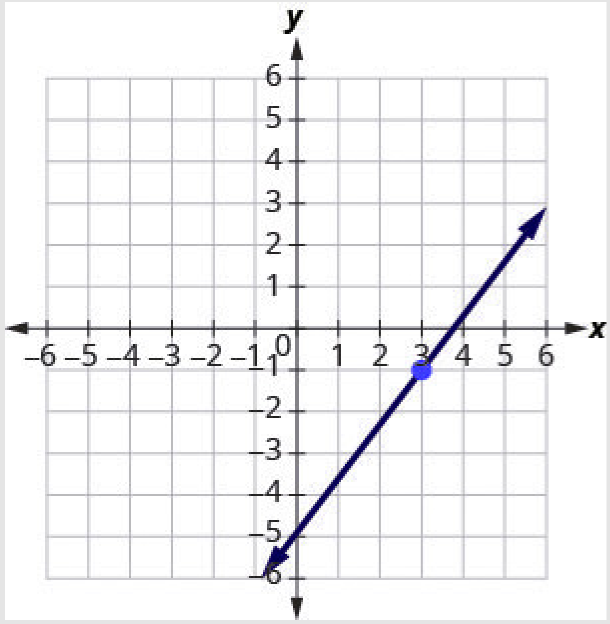


Figure 3.11.3

Solution

$y = \frac{4}{3}x - 5$

Find an Equation of the Line Given the Slope and a Point

Finding an equation of a line using the slope–intercept form of the equation works well when you are given the slope and y-intercept or when you read them off a graph. But what happens when you have another point instead of the y-intercept?

We are going to use the slope formula to derive another form of an equation of the line. Suppose we have a line that has slope m and that contains some specific point (x_1, y_1) and some other point, which we will just call (x, y). We can write the slope of this line and then change it to a different form.

$$m = \frac{y - y_1}{x - x_1}$$

Step 1: Multiply both sides of the equation by$(x - x_1)$. $\quad m(x - x_1) = (\frac{y - y_1}{x - x_1})(x - x_1)$

Step 2: Simplify. $\quad m(x - x_1) = y - y_1$

Step 3: Rewrite the equation with the y terms on the left. $\quad (y - y_1) = m(x - x_1)$

This format is called the **point–slope form** of an equation of a line.

Point–slope Form of an Equation of a Line

The point–slope form of an equation of a line with slope m and containing the point (x_1, y_1) is
$y - rgb]0.0,0.0,1.0y_{rgb]0.0,0.0,1.01} = rgb]1.0,0.0,0.0m(x - rgb]0.0,0.0,1.0x_{rgb]0.0,0.0,1.01})$
.

We can use the point–slope form of an equation to find an equation of a line when we are given the slope and one point. Then we will rewrite the equation in slope–intercept form. Most applications of linear equations use the the slope–intercept form.

Example 3.11.3

Find an equation of a line with slope $m = \frac{2}{5}$ that contains the point $(10, 3)$. Write the equation in slope–intercept form.

Solution

Step 1: **Identify the slope.**

The slope is given.

$$rgb]0.0,0.0,1.0m = \frac{2}{5}$$

Step 2: **Identify the point.**

The point is given.

$$\begin{pmatrix} rgb]1.0,0.0,0.0x_1 & rgb]1.0,0.0,0.0y_1 \\ 10, & 3 \end{pmatrix}$$

Step 3: **Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.**

$$y - rgb]1.0,0.0,0.0y_{rgb]1.0,0.0,0.01} = rgb]0.0,0.0,1.0m(x - rgb]1.0,0.0,0.0x_{rgb]1.0,0.0,0.01})$$

$$y - rgb]1.0,0.0,0.03 = rgb]0.0,0.0,1.0\frac{2}{5}(x - rgb]1.0,0.0,0.010)$$

Simplify. $$y - 3 = \frac{2}{5}x - 4$$

Step 4: **Write the equation in slope-intercept form.**

$$y = \frac{2}{5}x - 1$$

Try It

7) Find an equation of a line with slope $m = \frac{5}{6}$ and containing the point left $(6, 3)$.

Solution

$y = \frac{5}{6}x - 2$

8) Find an equation of a line with slope $m = \frac{2}{3}$ and containing the point $(9, 2)$.

Solution

$y = \frac{2}{3}x - 4$

HOW TO

Find an equation of a line given the slope and a point.

1. Identify the slope.

2. Identify the point.
3. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.
4. Write the equation in slope–intercept form.

Example 3.11.4

Find an equation of a line with slope $m = -\frac{1}{3}$ that contains the point $(6, -4)$. Write the equation in slope–intercept form.

Solution

Since we are given a point and the slope of the line, we can substitute the needed values into the point–slope form, $y - y_1$

Step 1: **Identify the slope.**

$$m \ = \ rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.0\frac{1}{3}$$

Step 2: **Identify the point.**

$$(\overset{rgb]0.0,0.0,1.0x_{rgb]0.0,0.0,0.0,1.01}}{6}, \overset{rgb]0.0,0.0,1.0y_{rgb]0.0,0.0,0.0,1.01}}{-4})$$

Step 3: **Substitute the values into** $y - y_1 = m(x - x_1)$

$$y - rgb]0.0, 0.0, 1.0y_{rgb]0.0,0.0,0.0,1.01} = rgb]1.0, 0.0, 0.0m(x - rgb]0.0, 0.0, 1.0x_{rgb]0.0,0.0,0.0,1.01})$$

$$y - (rgb]0.0, 0.0, 1.0 - rgb]0.0, 0.0, 1.04) = rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.0\frac{1}{3}(x - rgb]0.0, 0.0, 1.06)$$

Step 4: **Simplify.**

$$y \ - \ 4 \ = \ rgb]0.1, 0.1, 0.1 - rgb]0.1, 0.1, 0.1\frac{1}{3}x \ - \ 6$$

Step 5: **Write in slope–intercept form.**

$$y \ = \ rgb]0.1, 0.1, 0.1 - rgb]0.1, 0.1, 0.1\frac{1}{3}x \ - \ 2$$

Try It

9) Find an equation of a line with slope $m = -\frac{2}{5}$ and containing the point $(10, -5)$.

Solution

$y = -\frac{2}{5}x - 1$

10) Find an equation of a line with slope $m = -\frac{3}{4}$, and containing the point $(4, -7)$.

Solution

$y = -\frac{3}{4}x - 4$

Example 3.11.5

Find an equation of a horizontal line that contains the point $(-1, 2)$. Write the equation in slope–intercept form.

Solution

Every horizontal line has slope 0. We can substitute the slope and points into the point–slope form, $y - y_1 = m(x - x_1)$.

Step 1: **Identify the slope.**

$$m = rgb]1.0, 0.0, 0.00$$

Step 2: **Identify the point.**

$$\left(\overset{rgb]0.0,0.0,1.0x_{rgb]0.0,0.0,1.01}}{-1}, \overset{rgb]0.0,0.0,1.0y_{rgb]0.0,0.0,1.01}}{2} \right)$$

Step 3: **Substitute the values into $y - y_1 = m(x - x_1)$.**

$$y - rgb]0.0, 0.0, 1.0y_{rgb]0.0,0.0,1.01} = rgb]1.0, 0.0, 0.0m(x - rgb]0.0, 0.0, 1.0x_{rgb]0.0,0.0,1.01})$$

$$y - rgb]0.0, 0.0, 1.02 = rgb]1.0, 0.0, 0.00(x - (rgb]0.0, 0.0, 1.0 - rgb]0.0, 0.0, 1.01))$$

$$\text{Simplify.} \quad y - 2 = (x + 1)$$

$$y - 2 = 0$$

$$y = 2$$

Step 4: **Write in slope–intercept form.**

It is in y-form, but could be written $y = 0x + 2$.

Did we end up with the form of a horizontal line, $y = a$?

Try It

11) Find an equation of a horizontal line containing the point $(-3, 8)$.

Solution

$y = 8$

12) Find an equation of a horizontal line containing the point $(-1, 4)$.

Solution

$y = 4$

Find an Equation of the Line Given Two Points

When real-world data is collected, a linear model can be created from two data points. In the next example we'll see how to find an equation of a line when just two points are given.

We have two options so far for finding an equation of a line: slope–intercept or point–slope. Since we will know two points, it will make more sense to use the point–slope form.

But then we need the slope. Can we find the slope with just two points? Yes. Then, once we have the slope, we can use it and one of the given points to find the equation.

Example 3.11.6

Find an equation of a line that contains the points $(5,4)$ and $(3,6)$. Write the equation in slope–intercept form.

Solution

Step 1: Find the slope using the given points.

Find the slope of the line through (5,4) and (3,6)

$$m=\frac{y_2-y_1}{x_2-x_1}$$
$$m=\frac{6-4}{3-5}$$
$$m=\frac{2}{-2}$$
$$m=-1$$

Step 2: Choose one point.

Chose either point.

$$\begin{pmatrix} rgb]1.0,0.0,0.0x_1 & rgb]1.0,0.0,0.0y_1 \\ 5, & 4 \end{pmatrix}$$

Step 3: Substitute the values into the point-slope form, $y-y_1=m(x-x_1)$**.**

$$y-y_1=m(x-x_1)$$
$$y-4=-1(x-5)$$
$$y-4=-1x+5$$
$$y=-1x+9$$

Step 4: **Write the equation in slope-intercept form.**

$$y=-1x+9$$

Use the point $(3,6)$ and see that you get the same equation.

Try It

13) Find an equation of a line containing the points $(3, 1)$ and $(5, 6)$.

Solution

$$y = \frac{5}{2}x - \frac{13}{2}$$

14) Find an equation of a line containing the points $(1, 4)$ and $(6, 2)$.

Solution

$$y = -\frac{2}{5}x + \frac{22}{5}$$

HOW TO

Find an equation of a line given two points.

1. Find the slope using the given points.
2. Choose one point.
3. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.
4. Write the equation in slope–intercept form.

Example 3.11.7

Find an equation of a line that contains the points $(-3, -1)$ and $(2, -2)$. Write the equation in slope–intercept form.

Solution

Since we have two points, we will find an equation of the line using the point–slope form. The first step will be to find the slope.

Step 1: **Find the slope of the line through $(-3, -1)$ and $(2, -2)$.**

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-2-(-1)}{2-(-3)}$$

$$m = \frac{-1}{5}$$

$$rgb]1.0, 0.0, 0.0m = rgb]0.0, 0.0, 1.0\frac{-1}{5}$$

Step 2: **Choose either point.**

$$\left(\frac{rgb]0.0,0.0,1.0x_{rgb]0.0,0.0,1.01}}{2}, -\frac{rgb]0.0,0.0,1.0y_{rgb]0.0,0.0,1.01}}{2}\right)$$

Step 3: **Substitute the values into $y - y_1 = m(x - x_1)$.**

$$y - rgb]0.0, 0.0, 1.0y_{rgb]0.0,0.0,1.01} = rgb]1.0, 0.0, 0.0m(x - rgb]0.0, 0.0, 1.0x_{rgb]0.0,0.0,1.01})$$

$$y - rgb]0.0, 0.0, 1.0(rgb]0.0, 0.0, 1.0 - rgb]0.0, 0.0, 1.02rgb]0.0, 0.0, 1.0) = rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.0\frac{1}{5}(x - rgb]0.0, 0.0, 1.02)$$

$$y + 2 = -\frac{1}{5}x + \frac{2}{5}$$

Step 4: **Write in slope–intercept form.**

$$y = -\frac{1}{5}x - \frac{8}{5}$$

Try It

15) Find an equation of a line containing the points $(-2, -4)$ and $(1, -3)$.

Solution

$$y = \frac{1}{3}x - \frac{10}{3}$$

16) Find an equation of a line containing the points $(-4, -3)$ and $(1, -5)$.

Solution

$$y = -\frac{2}{5}x - \frac{23}{5}$$

Example 3.11.8

Find an equation of a line that contains the points $(-2, 4)$ and $(-2, -3)$. Write the equation in slope–intercept form.

Solution

Again, the first step will be to find the slope.

Find the slope of the line through $(-2, 4)$ and $(-2, -3)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-3 - 4}{-2 - (-2)}$$

$$m = \frac{-7}{0}$$

The slope is undefined.

This tells us it is a vertical line. Both of our points have an x-coordinate of -2. So our equation of the line is $x = -2$. Since there is no y, we cannot write it in slope–intercept form.

You may want to sketch a graph using the two given points. Does the graph agree with our conclusion that this is a vertical line?

Try It

17) Find an equation of a line containing the points $(5, 1)$ and $(5, -4)$.

Solution

$x = 5$

18) Find an equation of a line containing the points $(-4, 4)$ and $(-4, 3)$.

Solution

$x = -4$

We have seen that we can use either the slope–intercept form or the point–slope form to find an equation of a line. Which form we use will depend on the information we are given. This is summarized in the table below.

To Write an Equation of a Line		
If given:	**Use:**	**Form:**
Slope and y-intercept	slope–intercept	$y = mx + b$
Slope and a point	point–slope	$y - y_1 = m(x - x_1)$
Two points	point–slope	$y - y_1 = m(x - x_1)$

Find an Equation of a Line Parallel to a Given Line

Suppose we need to find an equation of a line that passes through a specific point and is parallel to a given line.

We can use the fact that parallel lines have the same slope. So we will have a point and the slope—just what we need to use the point–slope equation.

First let's look at this graphically.

The graph shows the graph of $y = 2x - 3$. We want to graph a line parallel to this line and passing through the point $(-2, 1)$.

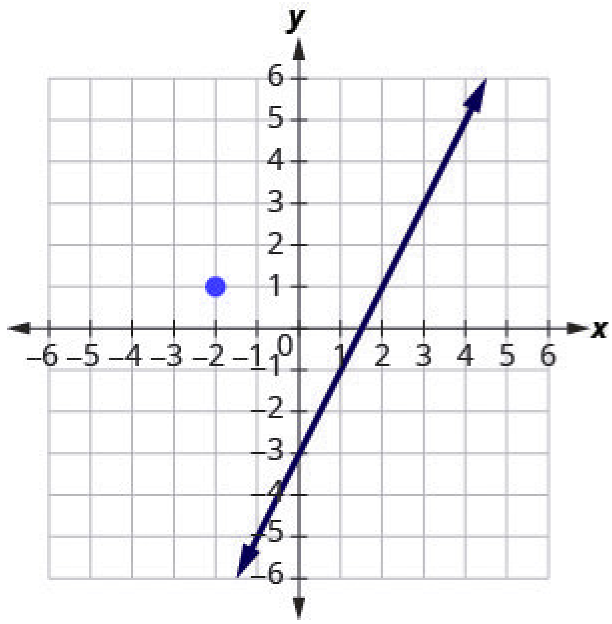


Figure 3.11.4

We know that parallel lines have the same slope. So the second line will have the same slope as $y = 2x - 3$. That slope is $m_{||} = 2$. We'll use the notation m to represent the slope of a line parallel to a line with slope m. (Notice that the subscript || looks like two parallel lines.)

The second line will pass through $(-2, 1)$ and have $m = 2$. To graph the line, we start at $(-2, 1)$ and count out the rise and run. With $m = 2$ (or $m = \frac{2}{1}$), we count out the rise 2 and the run 1. We draw the line.

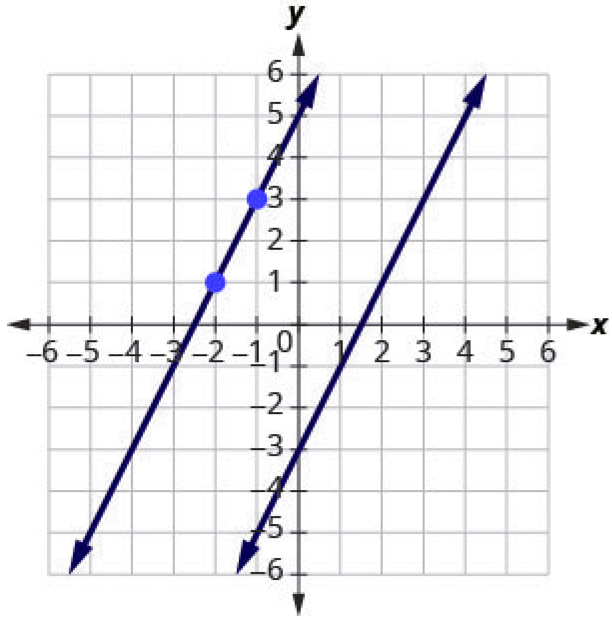


Figure 3.11.5

Do the lines appear parallel? Does the second line pass through $(-2, 1)$?

Now, let's see how to do this algebraically.

We can use either the slope–intercept form or the point–slope form to find an equation of a line. Here we know one point and can find the slope. So we will use the point–slope form.

Example 3.11.9

Find an equation of a line parallel to $y = 2x - 3$ that contains the point $(-2, 1)$. Write the equation in slope–intercept form.

Solution

***Step 1:* Find the slope of the given line.**

The line is in slope-intercept form, $y = 2x - 3$.

$$m = 2$$

***Step 2:* Find the slope of the parallel line.**

Parallel lines have the same slope.

$$m_2 = 2$$

Step 3: **Identify the point.**

The given point is, (-2,1).

$$\begin{pmatrix} x_1 & y_1 \\ -2, & 1 \end{pmatrix}$$

Step 4: **Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.**

Simplify.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 2(x - (-2)) \\ y - 1 &= 2(x + 2) \\ y - 1 &= 2x + 4 \end{aligned}$$

Step 5: **Write the equation in slope-intercept form.**

$$y = 2x + 5$$

Does this equation make sense? What is the y-intercept of the line? What is the slope?

Try It

19) Find an equation of a line parallel to the line $y = 3x + 1$ that contains the point $(4, 2)$. Write the equation in slope–intercept form.

Solution

$y = 3x - 10$

20) Find an equation of a line parallel to the line $y = \frac{1}{2}x - 3$ that contains the point $(6, 4)$.

Solution

$y = \frac{1}{2}x + 1$

HOW TO

Find an equation of a line parallel to a given line.

1. Find the slope of the given line.
2. Find the slope of the parallel line.
3. Identify the point.
4. Substitute the values into the point–slope form, $y - y_1 = m(x - x_1)$.
5. Write the equation in slope–intercept form.

Find an Equation of a Line Perpendicular to a Given Line

Now, let's consider perpendicular lines. Suppose we need to find a line passing through a specific point and which is perpendicular to a given line. We can use the fact that perpendicular lines have slopes that are negative reciprocals. We will again use the point–slope equation, like we did with parallel lines.

The graph shows the graph of $y = 2x - 3$. Now, we want to graph a line perpendicular to this line and passing through $(-2, 1)$.

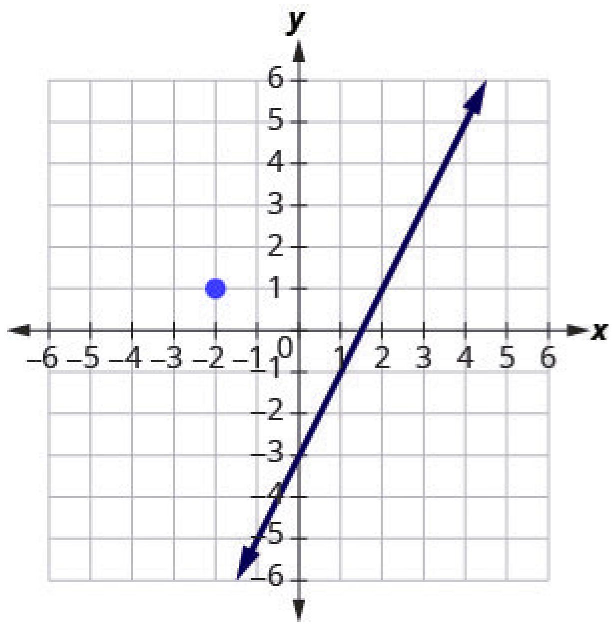


Figure 3.11.6

We know that perpendicular lines have slopes that are negative reciprocals. We'll use the notation $m_{\perp}$ to represent the slope of a line perpendicular to a line with slope m. (Notice that the subscript $\perp$ looks like the right angles made by two perpendicular lines.)

$$y = 2x - 3 \qquad \text{perpendicular line}$$

$$m = 2 \qquad m - \perp = \frac{1}{2}$$

We now know the perpendicular line will pass through $(-2, 1)$ with $m_{\perp} = -\frac{1}{2}$

To graph the line, we will start at $(-2, 1)$ and count out the rise -1 and the run 2. Then we draw the line.

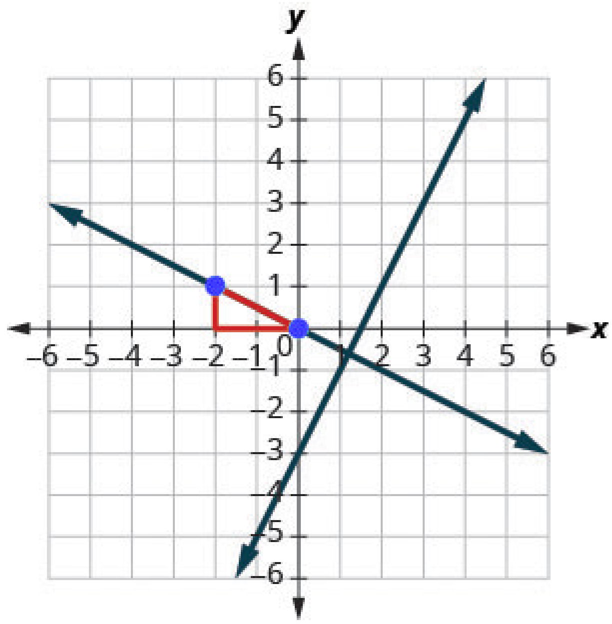


Figure 3.11.7

Do the lines appear perpendicular? Does the second line pass through $(-2, 1)$?

Now, let's see how to do this algebraically. We can use either the slope–intercept form or the point–slope form to find an equation of a line. In this example we know one point, and can find the slope, so we will use the point–slope form.

Example 3.11.10

Find an equation of a line perpendicular to $y = 2x - 3$ that contains the point $(-2, 1)$. Write the equation in slope–intercept form.

Solution

Step 1: **Find the slope of the given line.**

The line is in slope-intercept form, $y = 2x - 3$.

$$m = 2$$

Step 2: **Find the slope of the perpendicular line.**

The slopes of perpendicular lines are negative reciprocals.

$$m_{\perp} = -\frac{1}{2}$$

Step 3: **Identify the point.**

The given point is, (-2,1)

$$\begin{pmatrix} x_1 & y_1 \\ -2, & 1 \end{pmatrix}$$

Step 4: **Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.**

Simplify.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= -\frac{1}{2}(x - (-2)) \\ y - 1 &= -\frac{1}{2}(x + 2) \\ y - 1 &= -\frac{1}{2}x - 1 \end{aligned}$$

Step 5: **Write in slope-intercept form.**

$$y = -\frac{1}{2}x$$

Try It

21) Find an equation of a line perpendicular to the line $y = 3x + 1$ that contains the point $(4, 2)$. Write the equation in slope–intercept form.

Solution

$y = -\frac{1}{3}x + \frac{10}{3}$

22) Find an equation of a line perpendicular to the line $y = \frac{1}{2}x - 3$ that contains the point $(6, 4)$.

Solution

$y = -2x + 16$

HOW TO

Find an equation of a line perpendicular to a given line.

1. Find the slope of the given line.
2. Find the slope of the perpendicular line.
3. Identify the point.
4. Substitute the values into the point–slope form, $y - y_1 = m(x - x_1)$.
5. Write the equation in slope–intercept form.

Example 3.11.11

Find an equation of a line perpendicular to $x = 5$ that contains the point $(3, -2)$. Write the equation in slope–intercept form.

Solution

Again, since we know one point, the point–slope option seems more promising than the slope–intercept option. We need the slope to use this form, and we know the new line will be perpendicular to $x = 5$. This line is vertical, so its perpendicular will be horizontal. This tells us the $m_{\perp} = 0$.

Step 1: **Identify the point.**

$$(3, -2)$$

Step 2: **Identify the slope of the perpendicular line.**

$$m_{\perp} = 0$$

Step 3: **Substitute the values into $y - y_1 = m(x - x_1)$.**

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-2) &= 0(x - 3) \\ y + 2 &= 0 \\ y &= -2 \end{aligned}$$

Step 4: **Simplify.**

$$y = -2$$

Sketch the graph of both lines. Do they appear to be perpendicular?

Try It

23) Find an equation of a line that is perpendicular to the line $x = 4$ that contains the point $(4, -5)$. Write the equation in slope–intercept form.

Solution

$y = -5$

24) Find an equation of a line that is perpendicular to the line $x = 2$ that contains the point $(2, -1)$. Write the equation in slope–intercept form.

Solution

$y = -1$

In Example 3.11.11, we used the point–slope form to find the equation. We could have looked at this in a different way.

We want to find a line that is perpendicular to $x = 5$ that contains the point $(3, -2)$. The graph shows us the line $x = 5$ and the point $(3, -2)$.

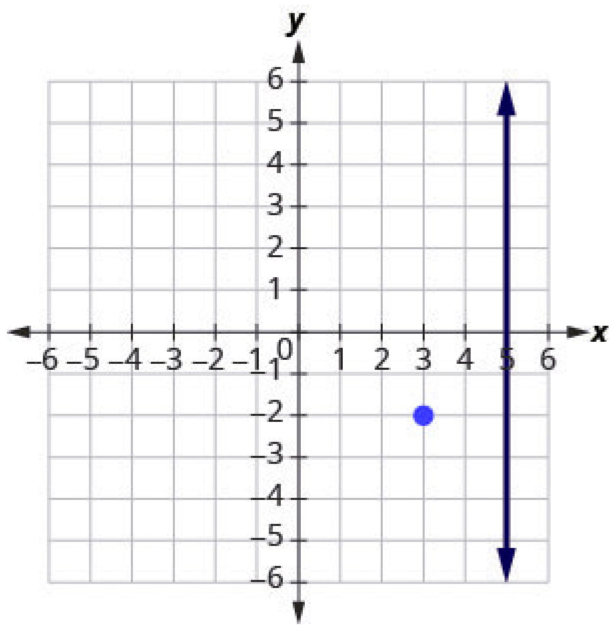


Figure 3.11.8

We know every line perpendicular to a vertical line is horizontal, so we will sketch the horizontal line through $(3, -2)$.

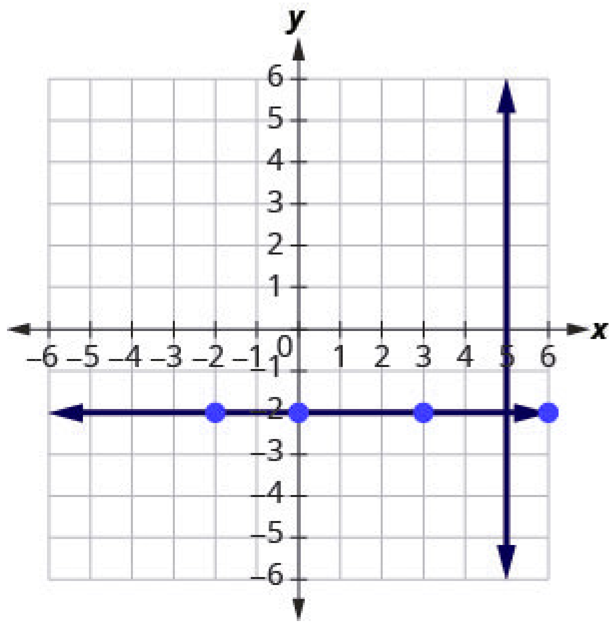


Figure 3.11.9

Do the lines appear perpendicular?

If we look at a few points on this horizontal line, we notice they all have y-coordinates of -2. So, the equation of the line perpendicular to the vertical line $x = 5$ is $y = -2$.

Example 3.11.12

Find an equation of a line that is perpendicular to $y = -4$ that contains the point $(-4, 2)$. Write the equation in slope–intercept form.

Solution

The line $y = -4$ is a horizontal line. Any line perpendicular to it must be vertical, in the form $x = a$. Since the perpendicular line is vertical and passes through $(-4, 2)$, every point on it has an x-coordinate of -4. The equation of the perpendicular line is $x = -4$. You may want to sketch the lines. Do they appear perpendicular?

Try It

25) Find an equation of a line that is perpendicular to the line $y = 1$ that contains the point $(-5, 1)$. Write the equation in slope–intercept form.

Solution

$x = -5$

26) Find an equation of a line that is perpendicular to the line $y = -5$ that contains the point $(-4, -5)$.

Solution

$x = -4$

Access this online resource for additional instruction and practice with finding the equation of a line.

- Use the Point-Slope Form of an Equation of a Line

Key Concepts

- **To Find an Equation of a Line Given the Slope and a Point**
  1. Identify the slope.

2. Identify the point.
3. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.
4. Write the equation in slope-intercept form.

- **To Find an Equation of a Line Given Two Points**
  1. Find the slope using the given points.
  2. Choose one point.
  3. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.
  4. Write the equation in slope-intercept form.
- **To Write and Equation of a Line**
  - If given slope and y-intercept, use slope–intercept form $y = mx + b$.
  - If given slope and a point, use point–slope form $y - y_1 = m(x - x_1)$.
  - If given two points, use point–slope form $y - y_1 = m(x - x_1)$.
- **To Find an Equation of a Line Parallel to a Given Line**
  1. Find the slope of the given line.
  2. Find the slope of the parallel line.
  3. Identify the point.
  4. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.
  5. Write the equation in slope-intercept form.
- **To Find an Equation of a Line Perpendicular to a Given Line**
  - Find the slope of the given line.
  - Find the slope of the perpendicular line.
  - Identify the point.
  - Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.
  - Write the equation in slope-intercept form.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=2019#h5p-25

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

point–slope form
The point–slope form of an equation of a line with slope m and containing the point (x_1, y_1) is $y - y_1 = m(x - x_1)$.

3.12 LINEAR FUNCTIONS AND APPLICATIONS OF LINEAR FUNCTIONS

Learning Objectives

In this section you will:

- Represent a linear function.
- Write and interpret an equation for a linear function.
- Model real-world problems with linear functions.
- Build linear models from verbal descriptions.
- Model a set of data with a linear function.


Figure 3.12.1 Shanghai MagLev Train by Jody McIntyre CC-BY-SA 2.0

Just as with the growth of a bamboo plant, there are many situations that involve constant change over time. Consider, for example, the first commercial maglev train in the world, the Shanghai MagLev Train Figure 3.12.1. It carries passengers comfortably for a 30-kilometer trip from the airport to the subway station in only eight minutes[1] .

Suppose a maglev train travels a long distance, and maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. How can we analyze the train's distance from the station as a function of time? In this section, we will investigate a kind

1. http://www.chinahighlights.com/shanghai/transportation/maglev-train.htm

of function that is useful for this purpose, and use it to investigate real-world situations such as the train's distance from the station at a given point in time.

Representing Linear Functions

The function describing the train's motion is a **linear function**, which is defined as a function with a constant rate of change. This is a polynomial of degree 1. There are several ways to represent a linear function, including word form, function notation, tabular form, and graphical form. We will describe the train's motion as a function using each method.

Representing a Linear Function in Word Form

Let's begin by describing the linear function in words. For the train problem, we just considered, the following word sentence may be used to describe the function relationship.

- *The train's distance from the station is a function of the time during which the train moves at a constant speed plus its original distance from the station when it began moving at constant speed.*

The speed is the rate of change. Recall that a rate of change is a measure of how quickly the dependent variable changes with respect to the independent variable. The rate of change for this example is constant, which means that it is the same for each input value. As the time (input) increases by 1 second, the corresponding distance (output) increases by 83 meters. The train began moving at this constant speed at a distance of 250 meters from the station.

Representing a Linear Function in Function Notation

Another approach to representing linear functions is by using function notation. One example of function notation is an equation written in the slope-intercept form of a line, where x is the input value, m is the rate of change, and b is the initial value of the dependent variable.

$$\begin{array}{ll} \text{Equation form} & y = mx + b \\ \text{Function notation} & f(x) = mx + b \end{array}$$

In the example of the train, we might use the notation D where the total distance D is a function of the time

t. The rate, m, is 83 meters per second. The initial value of the dependent variable b is the original distance from the station, 250 meters. We can write a generalized equation to represent the motion of the train.

$$D = 83t + 250.$$

Representing a Linear Function in Tabular Form

A third method of representing a linear function is through the use of a table. The relationship between the distance from the station and the time is represented in Figure 3.12.2. From the table, we can see that the distance changes by 83 meters for every 1 second increase in time.

1 second 1 second 1 second

t	0	1	2	3
$D(t)$	250	333	416	499

83 meters 83 meters 83 meters

Figure 3.12.2 Tabular representation of the function[latex]\,D\,[/latex]showing selected input and output values

Can the input in the previous example be any real number?

No. The input represents time so while nonnegative rational and irrational numbers are possible, negative real numbers are not possible for this example. The input consists of non-negative real numbers.

Representing a Linear Function in Graphical Form

Another way to represent linear functions is visually, using a graph. We can use the function relationship from above, $D(t) = 83t + 250$, to draw a graph as represented in Figure 3.12.3 Notice the graph is a line**.** When we plot a linear function, the graph is always a line.

The rate of change, which is constant, determines the slant, or slope of the line. The point at which the input value is zero is the vertical intercept, or y-intercept, of the line. We can see from the graph that the y

-intercept in the train example we just saw is $(0, 250)$ and represents the distance of the train from the station when it began moving at a constant speed.

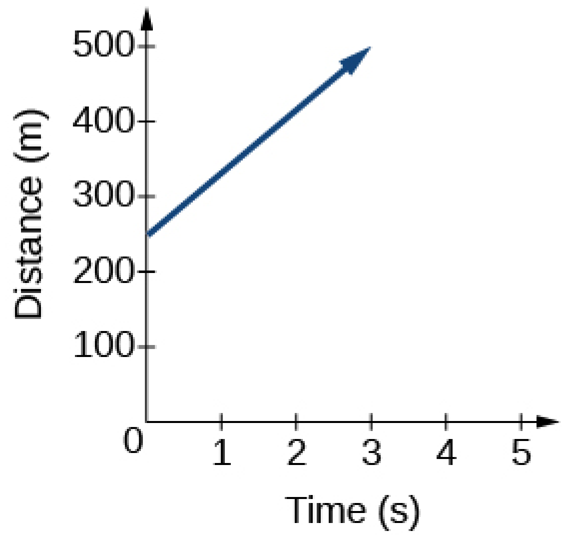


Figure 3.12.3 The graph of [latex]t=83t+250[/latex]. Graphs of linear functions are lines because the rate of change is constant.

Notice that the graph of the train example is restricted, but this is not always the case. Consider the graph of the line $f(x) = 2x + 1$.

Ask yourself what numbers can be input to the function. In other words, what is the domain of the function? The domain is comprised of all real numbers because any number may be doubled, and then have one added to the product.

Linear Function

A linear function is a function whose graph is a line. Linear functions can be written in the slope-intercept form of a line

$$f(x) = mx + b$$

where b is the initial or starting value of the function (when input, $x = 0$), and m, is the constant rate of change, or slope of the function. The y-intercept is at $(0, b)$.

Example 3.12.1.

Using a Linear Function to Find the Pressure on a Diver

The pressure, P in pounds per square inch *(PSI)* on the diver in Figure 3.12.4 depends upon her depth below the water surface, d in feet. This relationship may be modelled by the equation, $P(d) = 0.434d + 14.696$. Restate this function in words.

Figure 3.12.4 Photo by Adam Reeder CC-BY-NC 2.0

Solution

To restate the function in words, we need to describe each part of the equation. The pressure as a function of depth equals four hundred thirty-four thousandths times depth plus fourteen and six hundred ninety-six thousandths.

Analysis

The initial value, 14.696, is the pressure in PSI on the diver at a depth of 0 feet, which is the surface of the water. The rate of change, or slope, is 0.434 PSI per foot. This tells us that the pressure on the diver increases 0.434 PSI for each foot her depth increases.

Modelling Real-World Problems with Linear Functions

In the real world, problems are not always explicitly stated in terms of a function or represented with a graph. Fortunately, we can analyze the problem by first representing it as a linear function and then interpreting the components of the function. As long as we know, or can figure out, the initial value and the rate of change of a **linear function**, we can solve many different kinds of real-world problems.

HOW TO

Given a linear function f and the initial value and rate of change, evaluate f.

1. Determine the initial value and the rate of change (slope).
2. Substitute the values into $f(x) = mx + b$.
3. Evaluate the function at $x = c$.

Example 3.12.2

Using a Linear Function to Determine the Number of Songs in a Music Collection

Marcus currently has 200 songs in his music collection. Every month, he adds 15 new songs. Write a formula for the number of songs, N in his collection as a function of time, t the number of months. How many songs will he own at the end of one year?

Solution

The initial value for this function is 200 because he currently owns 200 songs, so $N(0) = 200$, which means that $b = 200$.

The number of songs increases by 15 songs per month, so the rate of change is 15 songs per month. Therefore we know that $m = 15$. We can substitute the initial value and the rate of change into the slope-intercept form of a line.

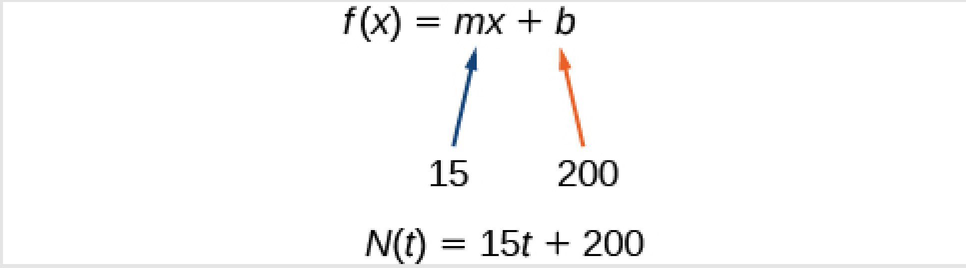


Figure 3.12.5

We can write the formula $N(t) = 15t + 200$.

With this formula, we can then predict how many songs Marcus will have at the end of one year (12 months). In other words, we can evaluate the function at, $t = 12$.

$$\begin{aligned} \mathrm{N}(12) &= 15(12) + 200 \\ &= 180 + 200 \\ &= 380 \end{aligned}$$

Marcus will have 380 songs in 12 months.

Analysis

Notice that N is an **increasing linear function**. As the input (the number of months) increases, the output (number of songs) increases as well.

Example 3.12.3

Using a Linear Function to Calculate Salary Based on Commission

Working as an insurance salesperson, Ilya earns a base salary plus a commission on each new policy. Therefore, Ilya's weekly income depends on the number of new policies, n he sells during the week. Last week he sold 3 new policies, and earned $\$760$ for the week. The week before, he sold 5 new policies and earned $\$920$. Find an equation for I and interpret the meaning of the components of the equation.

Solution

The given information gives us two input-output pairs: $(3, \$760)$ and $(5, \$920)$ We start by finding the rate of change.

$$\mathrm{m} = \frac{\$920 - \$760}{5 - 3}$$

$$= \frac{\$160}{2\ policies}$$

$$= \$80\ per\ policy$$

Keeping track of units can help us interpret this quantity. Income increased by $\$160$ when the number of policies increased by 2, so the rate of change is $\$80$ per policy. Therefore, Ilya earns a commission of $\$80$ for each policy sold during the week.

We can then solve for the initial value.

\begin{eqnarray*} \mbox{I(n)} & = & {80n + b} \\ \\ \ 760 & = & {80(3) +b} \mbox{ when n=3, I(3)=760} \\ \\ \ \mbox{760 - 80(3)} & = & {b} \\ \\ \ 520 & = & {b} \end{eqnarray*}

The value of b is the starting value for the function and represents Ilya's income when $n = 0$ or when no new policies are sold. We can interpret this as Ilya's base salary for the week, which does not depend upon the number of policies sold.

We can now write the final equation.

$$I = 80n + 520$$

Our final interpretation is that Ilya's base salary is $\$520$ per week and he earns an additional $\$80$ commission for each policy sold.

Example 3.12.4

Using Tabular Form to Write an Equation for a Linear Function

Below table relates the number of rats in a population to time, in weeks. Use the table to write a linear equation.

number of weeks, w	0	2	4	6
number of rats, $P(w)$	1000	1080	1160	1240

Solution

We can see from the table that the initial value for the number of rats is 1000, so $b = 1000$.

Rather than solving for m we can tell from looking at the table that the population increases by 80 for every 2 weeks that pass. This means that the rate of change is 80 rats per 2 weeks, which can be simplified to 40 rats per week.

$$P(w) = 40w + 1000$$

If we did not notice the rate of change from the table we could still solve for the slope using any two points from the table. For example, using $(2, 1080)$ and $(6, 1240)$

$$\begin{aligned} \mathrm{m} &= \frac{1240 - 1080}{6 - 2} \\ &= \frac{160}{4} \\ &= 40 \end{aligned}$$

Is the initial value always provided in a table of values like Example 3.12.4?

No. Sometimes the initial value is provided in a table of values, but sometimes it is not. If you see an input of 0*, then the initial value would be the corresponding output. If the initial value is not provided because there is no value of input on the table equal to* 0*, find the slope, substitute one coordinate pair and the slope into* $f(x) = mx + b$ *and solve for* b*.*

Try It

1) A new plant food was introduced to a young tree to test its effect on the height of the tree. The table shows the height of the tree, in feet, x months since the measurements began. Write a linear function, $H(x)$, where x is the number of months since the start of the experiment.

x	0	2	4	8	12
$H(x)$	12.5	13.5	14.5	16.5	18.5

Solution

$H(x) = 0.5x + 12.5$

Building Linear Models from Verbal Descriptions

When building linear models to solve problems involving quantities with a constant rate of change, we typically follow the same problem strategies that we would use for any type of function. Let's briefly review them:

1. Identify changing quantities, and then define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
2. Carefully read the problem to identify important information. Look for information that provides values for the variables or values for parts of the functional model, such as slope and initial value.
3. Carefully read the problem to determine what we are trying to find, identify, solve, or interpret.
4. Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table, or even finding a formula for the function being used to model the problem.
5. When needed, write a formula for the function.
6. Solve or evaluate the function using the formula.
7. Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
8. Clearly convey your result using appropriate units, and answer in full sentences when necessary.

Now let's take a look at the student in Seattle. In her situation, there are two changing quantities: time and money. The amount of money she has remaining while on vacation depends on how long she stays. We can use this information to define our variables, including units.

Output: M, money remaining in dollars.
Input: t, time in weeks.

So, the amount of money remaining depends on the number of weeks: $M(t)$.
We can also identify the initial value and the rate of change.

Initial Value: She saved $\$3,500$, so $\$3,500$ is the initial value for M.
Rate of Change: She anticipates spending $\$400$ each week, so $\$400$ per week is the rate of change, or slope.

Notice that the unit of dollars per week matches the unit of our output variable divided by our input variable. Also, because the slope is negative, the linear function is decreasing. This should make sense because she is spending money each week.

The rate of change is constant, so we can start with the linear model $M(t) = mt + b$. Then we can substitute the intercept and slope provided.

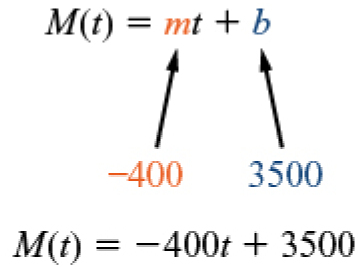


Figure 3.12.6

To find the x–intercept, we set the output to zero, and solve for the input.

$$0 = -400t + 3500$$

$$t = \frac{3500}{400}$$

$$= 8.75$$

The x-intercept is 8.75 weeks. Because this represents the input value when the output will be zero, we could say that Emily will have no money left after 8.75 weeks.

When modelling any real-life scenario with functions, there is typically a limited domain over which that model will be valid—almost no trend continues indefinitely. Here the domain refers to the number of weeks. In this case, it doesn't make sense to talk about input values less than zero. A negative input value could refer to a number of weeks before she saved $\$3,500$, but the scenario discussed poses the question once she saved $\$3,500$ because this is when her trip and subsequent spending starts. It is also likely that this model is not valid after the x-intercept, unless Emily uses a credit card and goes into debt. The domain represents the set of input values, so the reasonable domain for this function is $0 \leq t \leq 8.75$

In this example, we were given a written description of the situation. We followed the steps of modelling a problem to analyze the information. However, the information provided may not always be the same. Sometimes we might be provided with an intercept. Other times we might be provided with an output value. We must be careful to analyze the information we are given, and use it appropriately to build a linear model.

Using a Given Intercept to Build a Model

Some real-world problems provide the y-intercept, which is the constant or initial value. Once the y-intercept is known, the x-intercept can be calculated. Suppose, for example, that Hannah plans to pay off a no-interest loan from her parents. Her loan balance is $\$1,000$. She plans to pay $\$250$ per month until her balance is $\$0$. The y-intercept is the initial amount of her debt, or $\$1,000$. The rate of change, or slope, is $-\$250$ per month. We can then use the slope-intercept form and the given information to develop a linear model.

$$\text{f(x)} = mx + b$$

$$= -250x + 1000$$

Now we can set the function equal to 0, and solve for x to find the x-intercept.

$$0 = -250x + 1000$$

$$1000 = 250x$$

$$4 = x$$

$$x = 4$$

The x-intercept is the number of months it takes her to reach a balance of $\$0$. The x-intercept is 4 months, so it will take Hannah four months to pay off her loan.

Using a Given Input and Output to Build a Model

Many real-world applications are not as direct as the ones we just considered. Instead they require us to identify some aspect of a linear function. We might sometimes instead be asked to evaluate the linear model at a given input or set the equation of the linear model equal to a specified output.

HOW TO

Given a word problem that includes two pairs of input and output values, use the linear function to solve a problem.

1. Identify the input and output values.
2. Convert the data to two coordinate pairs.
3. Find the slope.
4. Write the linear model.
5. Use the model to make a prediction by evaluating the function at a given x-value.
6. Use the model to identify an x-value that results in a given y-value.
7. Answer the question posed.

Example 3.12.5

Using a Linear Model to Investigate a Town's Population

A town's population has been growing linearly. In 2004, the population was 6,200. By 2009, the population had grown to 8,100. Assume this trend continues.

a. Predict the population in 2013.
b. Identify the year in which the population will reach 15,000.

Solution

The two changing quantities are the population size and time. While we could use the actual year value as the input quantity, doing so tends to lead to very cumbersome equations because the y-intercept would correspond to the year 0, more than 2000 years ago!

To make computation a little nicer, we will define our input as the number of years since 2004.

Input: t, years since 2004

Output: $P(t)$, the town's population

To predict the population in 2013 $t=9$, we would first need an equation for the population. Likewise, to find when the population would reach 15,000, we would need to solve for the input that would provide an output of 15,000. To write an equation, we need the initial value and the rate of change, or slope.

To determine the rate of change, we will use the change in output per change in input.

$$m = \frac{change\ \ in\ \ output}{change\ \ in\ \ input}$$

The problem gives us two input-output pairs. Converting them to match our defined variables, the year 2004 would correspond to $t=0$ giving the point $(0, 6200)$. Notice that through our clever choice of variable definition, we have "given" ourselves the y-intercept of the function. The year 2009 would correspond to, $t=5$ giving the point $(5, 8100)$.

The two coordinate pairs are $(0, 6200)$ and $(5, 8100)$. Recall that we encountered examples in which we were provided two points earlier in the chapter. We can use these values to calculate the slope.

\begin{eqnarray*} \mbox{m} & = & \frac{8100-6200}{5-0} \\ \\ \ & = & \frac{1900}{5} \\ \\ \ & = & 380 \end{eqnarray*}

We already know the y-intercept of the line, so we can immediately write the equation:

$$P(t) = 380t + 6200$$

To predict the population in 2013, we evaluate our function at, $t = 9$.

\begin{eqnarray*} \mbox{P(9)} & = & 380(9) + 6,200 \\ \\ \ & = & 9,620 \end{eqnarray*}

if the trend continues, our model predicts a population of 9,620 in 2013.

To find when the population will reach 15,000, we can set

\begin{eqnarray*} \mbox{P(t)} & = & 15000 \\ \\ \ 150000 & = & 380t + 6200 \\ \\ \ 8800 & = & 380t \\ \\ \ t & \approx & 23.158 \end{eqnarray*}

Our model predicts the population will reach 15,000 in a little more than 23 years after 2004, or somewhere around the year 2027.

Try It

2) A company sells doughnuts. They incur a fixed cost of $25,000 for rent, insurance, and other expenses. It costs $0.25 to produce each doughnut.

a. Write a linear model to represent the cost C of the company as a function of x, the number of doughnuts produced.
b. Find and interpret the y-intercept.

Solution

a. $C(x) = 0.25x + 25,000$
b. The y-intercept is $(0, 25,000)$. If the company does not produce a single doughnut, they still incur a cost of 25,000.

3) A city's population has been growing linearly. In 2008, the population was 28,200. By 2012, the population was 36,800. Assume this trend continues.

a. Predict the population in 2014.
b. Identify the year in which the population will reach 54,000.

Solution

a. 41,100
b. 2020

Key Concepts

- Linear functions can be represented in words, function notation, tabular form, and graphical form.
- The equation for a linear function can be written if the slope m and initial value b are known.
- A linear function can be used to solve real-world problems given information in different forms.
- We can use the same problem strategies that we would use for any type of function.
- When modelling and solving a problem, identify the variables and look for key values, including the slope and y-intercept.
- Draw a diagram, where appropriate.
- Check for reasonableness of the answer.
- Linear models may be built by identifying or calculating the slope and using the y-intercept.
  - The x-intercept may be found by setting, $y=0$, which is setting the expression $mx+b$ equal to 0.
  - The point of intersection of a system of linear equations is the point where the x– and y-values are the same.
  - A graph of the system may be used to identify the points where one line falls below (or above) the other line.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=5804#h5p-35

b. Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

Glossary

decreasing linear function
a function with a negative slope: If $f(x) = mx + b,\ m < 0$.

increasing linear function
a function with a positive slope: If $f(x) = mx + b,\ m > 0$.

linear function
a function with a constant rate of change that is a polynomial of degree 1, and whose graph is a straight line

3.13 UNIT SOURCES

Unit 3 Sources

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UNIT 4: SYSTEMS OF LINEAR EQUATIONS

Chapter Outline

4.0 INTRODUCTION

Designing the number and sizes of windows in a home can pose challenges for an architect.

Figure 4.0.1: The number of windows designed by an architect can be challenging. Photo by Mike Kononov Unsplash License

An architect designing a home may have restrictions on both the area and perimeter of the windows because of energy and structural concerns. The length and width chosen for each window would have to satisfy two equations: one for the area and the other for the perimeter. Similarly, a banker may have a fixed amount of money to put into two investment funds. A restaurant owner may want to increase profits, but in order to do that, he will need to hire more staff. A job applicant may compare salary and costs of commuting for two job offers.

In this chapter, we will look at methods to solve situations like these using equations with two variables.

4.1 SOLVE SYSTEMS OF EQUATIONS BY GRAPHING

Learning Objectives

By the end of this section, you will be able to:

- Determine whether an ordered pair is a solution of a system of equations
- Solve a system of linear equations by graphing
- Determine the number of solutions of linear system
- Solve applications of systems of equations by graphing

Try It

Before you get started, take this readiness quiz:

1) For the equation $y = \frac{2}{3}x - 4$

a. is $(6, 0)$ a solution? b. is $(-3, -2)$ a solution?

2) Find the slope and y-intercept of the line $3x - y = 12$.

3) Find the x– and y-intercepts of the line $2x - 3y = 12$.

Determine Whether an Ordered Pair is a Solution of a System of Equations

In Solving Linear Equations and Applications of Linear Functions we learned how to solve linear equations with one variable. Remember that the solution of an equation is a value of the variable that makes a true statement when substituted into the equation.

Now we will work with systems of linear equations, two or more linear equations grouped together.

System of Linear Equations

When two or more linear equations are grouped together, they form a **system of linear equations**.

We will focus our work here on systems of two linear equations in two unknowns. Later, you may solve larger systems of equations.

An example of a system of two linear equations is shown below. We use a brace to show the two equations are grouped together to form a system of equations.

$$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$$

A linear equation in two variables, like $2x + y = 7$, has an infinite number of solutions. Its graph is a line. Remember, every point on the line is a solution to the equation and every solution to the equation is a point on the line.

To solve a system of two linear equations, we want to find the values of the variables that are solutions to both equations. In other words, we are looking for the ordered pairs (x, y) that make both equations true. These are called the solutions to a system of equations.

Solutions of a System of Equations

Solutions of a system of equations are the values of the variables that make all the equations true. A solution of a system of two linear equations is represented by an ordered pair (x, y).

To determine if an ordered pair is a solution to a system of two equations, we substitute the values of the variables into each equation. If the ordered pair makes both equations true, it is a solution to the system.

Let's consider the system below:

$$\begin{cases} 3x - y = 7 \\ x - 2y = 4 \end{cases}$$

Is the ordered pair $(2, -1)$ a solution?

We substitute $rgb]1.0, 0.0, 0.0x rgb]1.0, 0.0, 0.0 \; rgb]1.0, 0.0, 0.0 = rgb]1.0, 0.0, 0.0 \; rgb]1.0, 0.0, 0.02$ and $rgb]0.0, 0.0, 1.0y rgb]0.0, 0.0, 1.0 \; rgb]0.0, 0.0, 1.0 = rgb]0.0, 0.0, 1.0 \; rgb]0.0, 0.0, 1.0 - rgb]0.0, 0.0, 1.01$ into both equations.

$$\begin{aligned} 3rgb]1.0, 0.0, 0.0x \; - \; rgb]0.0, 0.0, 1.0y &= 7 \\ 3(rgb]1.0, 0.0, 0.02) \; - \; (rgb]0.0, 0.0, 1.0 - rgb]0.0, 0.0, 1.01) &\stackrel{?}{=} 7 \\ 7 &= 7 \checkmark \end{aligned}$$

$$\begin{aligned} rgb]1.0, 0.0, 0.0x \; - \; 2rgb]0.0, 0.0, 1.0y &= 4 \\ rgb]1.0, 0.0, 0.02 \; - \; 2(rgb]0.0, 0.0, 1.0 - rgb]0.0, 0.0, 1.01) &\stackrel{?}{=} 4 \\ 4 &= 4 \checkmark \end{aligned}$$

The ordered pair $(2, -1)$ made both equations true. Therefore $(2, -1)$ is a solution to this system.

Let's try another ordered pair. Is the ordered pair $(3, 2)$ a solution?

We substitute $rgb]1.0, 0.0, 0.0x rgb]1.0, 0.0, 0.0 \; rgb]1.0, 0.0, 0.0 = rgb]1.0, 0.0, 0.0 \; rgb]1.0, 0.0, 0.03$ and $rgb]0.0, 0.0, 1.0y rgb]0.0, 0.0, 1.0 \; rgb]0.0, 0.0, 1.0 = rgb]0.0, 0.0, 1.0 \; rgb]0.0, 0.0, 1.02$ into both equations.

$$\begin{aligned} 3rgb]1.0,0.0,0.0x &- rgb]0.0,0.0,1.0y &= 7 \\ 3(rgb]1.0,0.0,0.03) &- rgb]0.0,0.0,1.02 &\stackrel{?}{=} 7 \\ && 7 = 7\checkmark \end{aligned}$$

$$\begin{aligned} rgb]1.0,0.0,0.0x &- 2rgb]0.0,0.0,1.0y &\stackrel{?}{=} 4 \\ rgb]1.0,0.0,0.02 &- 2(rgb]0.0,0.0,1.02) &= 4 \\ && -2 = 4\ False \end{aligned}$$

The ordered pair $(3, 2)$ made one equation true, but it made the other equation false. Since it is not a solution to **both** equations, it is not a solution to this system.

Example 4.1.1

Determine whether the ordered pair is a solution to the system: $\begin{cases} x - y = -1 \\ 2x - y = -5 \end{cases}$

a. $(-2, -1)$ b. $(-4, -3)$

Solution

a.

$$\begin{cases} x - y = -1 \\ 2x - y = -5 \end{cases}$$

We substitute x = $rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.02$ and y = $rgb]0.0,0.0,1.0{-}rgb]0.0,0.0,1.01$ into both equations.

$$\begin{aligned} rgb]1.0,0.0,0.0x &- rgb]0.0,0.0,1.0y &= -1 \\ rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.02 &- (rgb]0.0,0.0,1.0{-}rgb]0.0,0.0,1.01) &\stackrel{?}{=} -1 \\ && -1 = -1\ \checkmark \end{aligned}$$

$$\begin{aligned} 2rgb]1.0,0.0,0.0x &- rgb]0.0,0.0,1.0y &= -5 \\ 2(rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.02rgb]0.1,0.1,0.1) &- (rgb]0.0,0.0,1.0{-}rgb]0.0,0.0,1.01) &\stackrel{?}{=} -5 \\ && 5 \neq -5 \end{aligned}$$

$(-2, -1)$ does not make both equations true. $(-2, -1)$ is not a solution.

---

b.

We substitute $x = rgb]1.0, 0.0, 0.0{-}rgb]1.0, 0.0, 0.04$ and $y = rgb]0.0, 0.0, 1.0{-}rgb]0.0, 0.0, 1.03$ into both equations.

$$\begin{aligned} rgb]1.0,0.0,0.0x - rgb]0.0,0.0,1.0y &= -1 \\ rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.04 - (rgb]0.0,0.0,1.0{-}rgb]0.0,0.0,1.03) &\stackrel{?}{=} -1 \\ -1 &= -1\checkmark \end{aligned}$$

$$\begin{aligned} rgb]0.1,0.1,0.12rgb]1.0,0.0,0.0x - rgb]0.0,0.0,1.0y &= -5 \\ 2(rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.04rgb]0.1,0.1,0.1) - (rgb]0.0,0.0,1.0{-}rgb]0.0,0.0,1.03) &\stackrel{?}{=} -5 \\ -5 &= -5\checkmark \end{aligned}$$

$(-4, -3)$ does make both equations true. $(-4, -3)$ is a solution.

Try It

4) Determine whether the ordered pair is a solution to the system: $\begin{cases} 3x + y = 0 \\ x + 2y = -5 \end{cases}$

a. $(1, -3)$ b. $(0, 0)$

Solution

a. yes b. no

5) Determine whether the ordered pair is a solution to the system: $\begin{cases} x - 3y = -8 \\ -3x - y = 4 \end{cases}$

a. $(2, -2)$ b. $(-2, 2)$

Solution

a. no b. yes

Solve a System of Linear Equations by Graphing

In this chapter we will use three methods to solve a system of linear equations. The first method we'll use is graphing.

The graph of a linear equation is a line. Each point on the line is a solution to the equation. For a system of two equations, we will graph two lines. Then we can see all the points that are solutions to each equation. And, by finding what the lines have in common, we'll find the solution to the system.

Most linear equations in one variable have one solution, but we saw that some equations, called contradictions, have no solutions and for other equations, called identities, all numbers are solutions.

Similarly, when we solve a system of two linear equations represented by a graph of two lines in the same plane, there are three possible cases, as shown in Figure 4.1.1:

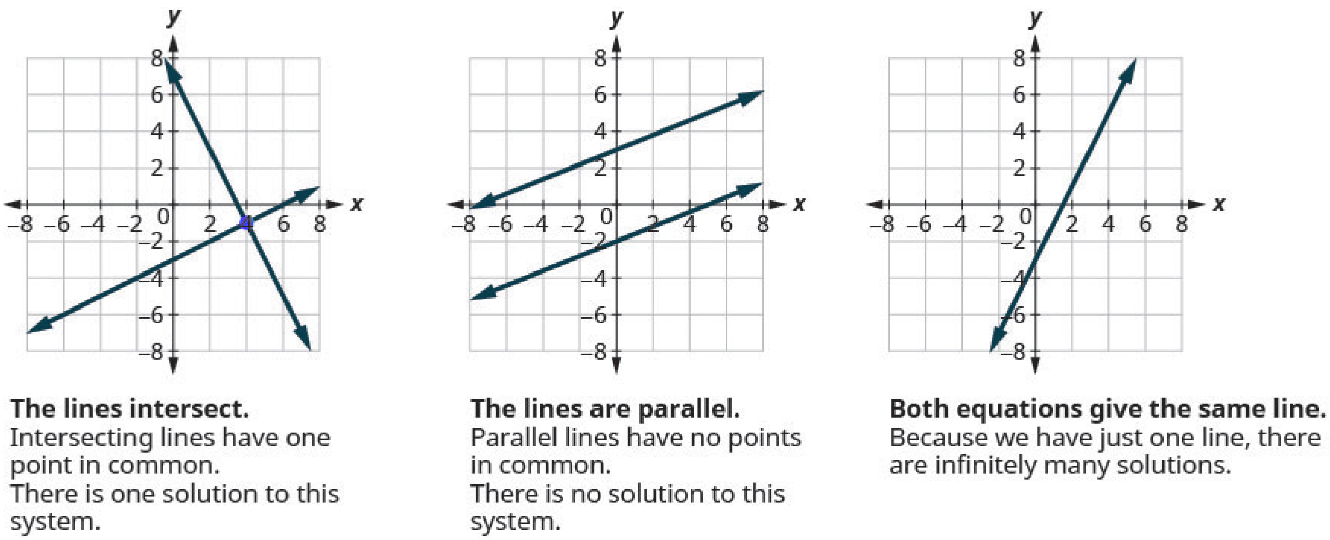


Figure 4.1.1

For the first example of solving a system of linear equations in this section and in the next two sections, we will solve the same system of two linear equations. But we'll use a different method in each section. After seeing the third method, you'll decide which method was the most convenient way to solve this system.

Example 4.1.2

Solve the system by graphing: $\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$

Solution

***Step 1:* Graph the first equation.**

To graph the first line, write the equation in slope-intercept form.

$$rgb]1.0, 0.0, 0.02x + yrgb]1.0, 0.0, 0.0 \; = 7$$
$$y = -2x + 7$$
$$m = -2$$
$$b = 7$$

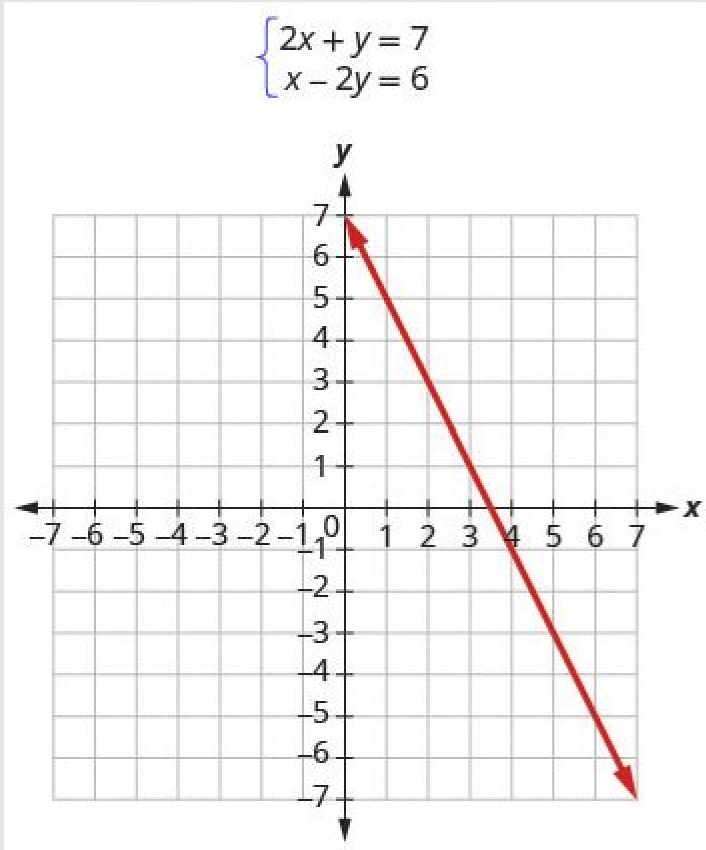


Figure 4.1.2

Step 2: **Graph the second equation on the same rectangular coordinate system.**

To graph the second line, use intercepts.

$x - 2y = 6$

$(0, -3) \;\; (6, 0)$

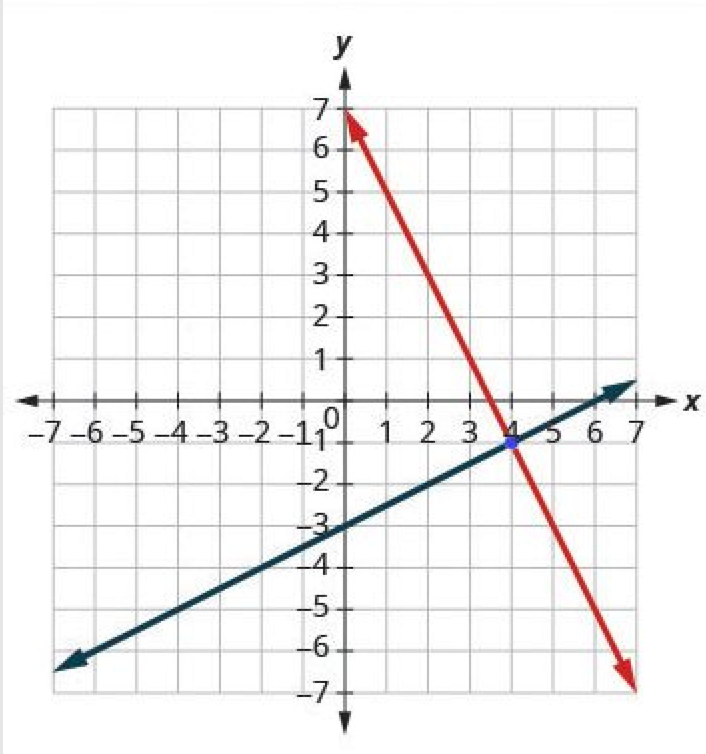


Figure 4.1.3

Step 3: **Determine whether the lines intersect, are parallel, or are the same line.**

Look at the graph of the lines.

The lines intersect.

Step 4: **Identify the solution to the system.**

If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system.

If the lines are parallel, the system has no solution. If the lines are the same, the system has an infinite number of solutions.

Since the lines intersect, find the point of intersection.

Check the point in both equations.

The lines intersect at $(4, -1)$

$$2x + y = 7$$
$$2(4) + (-1) \stackrel{?}{=} 7$$
$$8 - 1 \stackrel{?}{=} 7$$
$$7 = 7 \checkmark$$

$$x - 2y = 6$$
$$4 - 2(-1) \stackrel{?}{=} 6$$
$$6 = 6 \checkmark$$

The solution is $(4, -1)$.

Try It

6) Solve each system by graphing: $\begin{cases} x - 3y = -3 \\ x + y = 5 \end{cases}$

Solution

$(3, 2)$

7) Solve each system by graphing: $\begin{cases} -x + y = 1 \\ 3x + 2y = 12 \end{cases}$

Solution

$(2, 3)$

The steps to use to solve a system of linear equations by graphing are shown below.

HOW TO

Solve a system of linear equations by graphing.

1. Graph the first equation.
2. Graph the second equation on the same rectangular coordinate system.
3. Determine whether the lines intersect, are parallel, or are the same line.
4. Identify the solution to the system.
    - ***If the lines intersect***, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system.
    - ***If the lines are parallel***, the system has no solution.
    - ***If the lines are the same***, the system has an infinite number of solutions.

Example 4.1.3

Solve the system by graphing: $\begin{cases} y = 2x + 1 \\ y = 4x - 1 \end{cases}$

Solution

Both of the equations in this system are in slope-intercept form, so we will use their slopes and y-intercepts to graph them. $\begin{cases} y = 2x + 1 \\ y = 4x - 1 \end{cases}$

Step 1: **Find the slope and y-intercept of the first equation.**

$$\begin{aligned} y &= 2x + 1 \\ m &= 2 \\ &= 1 \end{aligned}$$

Step 2: **Find the slope and y-intercept of the second equation.**

$$y = 4x - 1$$
$$m = 4$$
$$= -1$$

***Step 3:* Graph the two lines.**

***Step 4:* Determine the point of intersection.**

The lines intersect at $(1, 3)$.

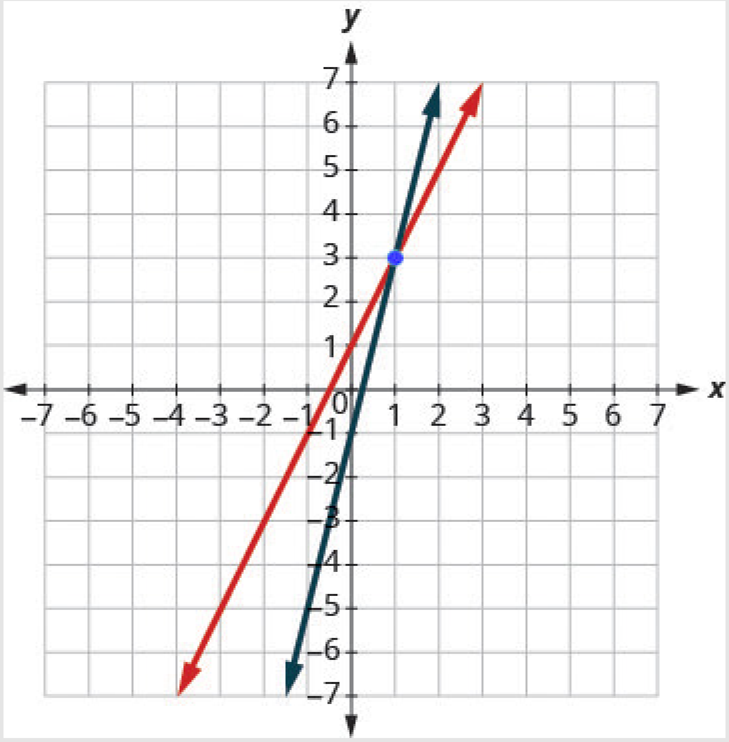


Figure 4.1.4

***Step 5:* Check the solution in both equations.**

$$\begin{aligned} y &= 2x + 1 \\ 3 &\stackrel{?}{=} 2 \cdot 1 + 1 \\ 3 &= 3\checkmark \end{aligned} \qquad \begin{aligned} y &= 4x - 1 \\ 3 &\stackrel{?}{=} 4 \cdot 1 - 1 \\ 3 &= 3\checkmark \end{aligned}$$

***Step 6:* Write solution as point (x, y).**

The solution is $(1, 3)$.

Try It

8) Solve each system by graphing: $\begin{cases} y = 2x + 2 \\ y = -x - 4 \end{cases}$

Solution

$(-2, -2)$

9) Solve each system by graphing: $\begin{cases} y = 3x + 3 \\ y = -x + 7 \end{cases}$

Solution

$(1, 6)$

Both equations in Example 4.1.3 were given in slope–intercept form. This made it easy for us to quickly graph the lines. In the next example, we'll first re-write the equations into slope–intercept form.

Example 4.1.4

Solve the system by graphing: $\begin{cases} 3x + y = -1 \\ 2x + y = 0 \end{cases}$

Solution

We'll solve both of these equations for y so that we can easily graph them using their slopes and y -intercepts. $\begin{cases} 3x + y = -1 \\ 2x + y = 0 \end{cases}$

Step 1: **Solve the first equation for y.**

Step 2: **Find the slope and y-intercept.**

$$\begin{aligned} 3x + y &= -1 \\ y &= -3x - 1 \end{aligned}$$

$$\begin{aligned} m &= -3 \\ b &= -1 \end{aligned}$$

$$\begin{aligned} 2x + y &= 0 \\ y &= -2x \end{aligned}$$

$$\begin{aligned} m &= -2 \\ b &= 0 \end{aligned}$$

Step 3: **Solve the second equation for y.**

Step 4: **Find the slope and y-intercept.**

Step 5: **Graph the lines.**

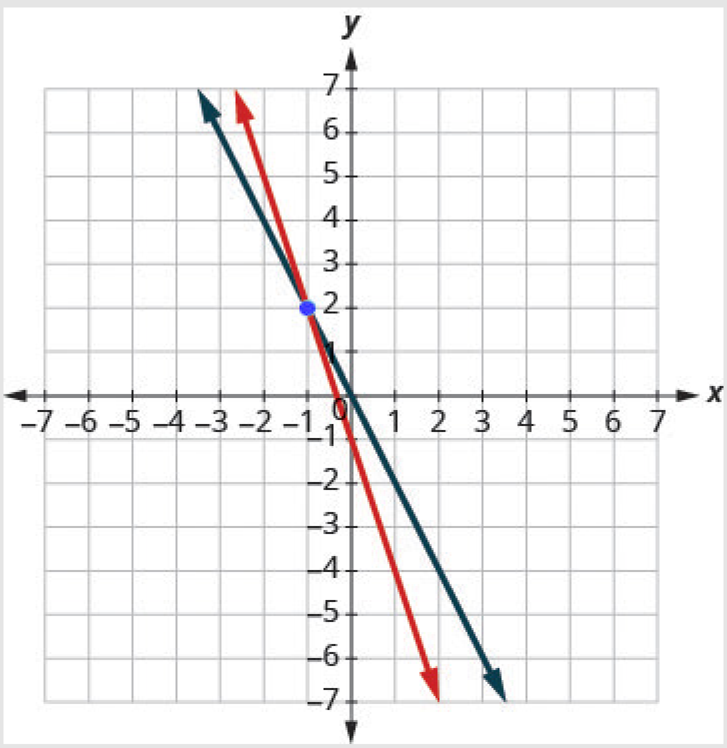


Figure 4.1.5

Step 6: **Determine the point of intersection.**

The lines intersect at $(-1, 2)$.

Step 7: **Check the solution in both equations.**

$$\begin{array}{rcl} 3x+y & = & -1 \\ 3(-1)+2 & \stackrel{?}{=} & -1 \\ -1 & = & -1\checkmark \end{array} \qquad \begin{array}{rcl} 2x+y & = & 0 \\ 2(-1)+2 & \stackrel{?}{=} & 0 \\ 0 & = & 0\checkmark \end{array}$$

The solution is $(-1, 2)$.

Try It

10) Solve each system by graphing: $\begin{cases} -x+y=1 \\ 2x+y=10 \end{cases}$

Solution

$(3, 4)$

11) Solve each system by graphing: $\begin{cases} 2x+y=6 \\ x+y=1 \end{cases}$

Solution

$(5, -4)$

Usually when equations are given in standard form, the most convenient way to graph them is by using the intercepts. We'll do this in Example 4.1.5.

Example 4.1.5

Solve the system by graphing: $\begin{cases} x+y=2 \\ x-y=4 \end{cases}$

Solution

We will find the x– and y-intercepts of both equations and use them to graph the lines.

$$x+y=2$$

Step 1:**To find the intercepts, let $x=0$ and solve for y, then let $y=0$ and solve for x.**

$$\begin{aligned} x+y &= 2 \\ 0+y &= 2 \\ y &= 2 \end{aligned} \qquad \begin{aligned} x+y &= 2 \\ x+0 &= 2 \\ x &= 2 \end{aligned}$$

x	y
0	2
2	0

Step 2: **To find the intercepts, let $x=0$ then let $y=0$.**

$$\begin{aligned} x-y &= 4 \\ 0-y &= 4 \\ -y &= 4 \\ y &= -4 \end{aligned} \qquad \begin{aligned} x-y &= 4 \\ x-0 &= 4 \\ x &= 4 \end{aligned}$$

x	y
0	-4
4	0

Step 3: **Graph the line.**

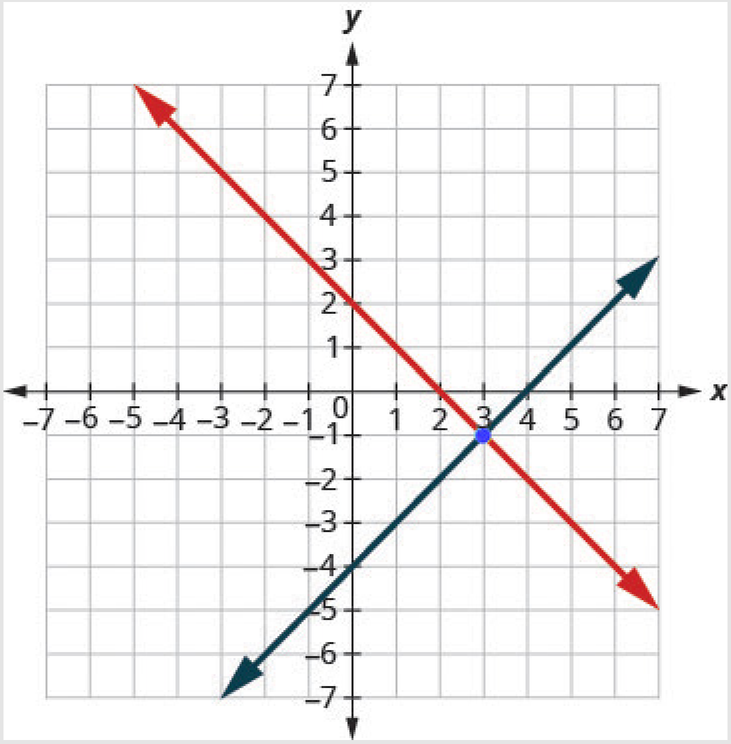


Figure 4.1.6

Step 4: **Determine the point of intersection.**

The lines intersect at $(3, -1)$.

Step 5: **Check the solution in both equations.**

$$\begin{aligned} x+y &= 2 \\ 3+(-1) &\stackrel{?}{=} 2 \\ 2 &= 2\checkmark \end{aligned} \qquad \begin{aligned} x-y &= 4 \\ 3-(-1) &\stackrel{?}{=} 4 \\ 4 &= 4\checkmark \end{aligned}$$

The solution is $(3, -1)$.

Try It

12) Solve each system by graphing: $\begin{cases} x+y=6 \\ x-y=2 \end{cases}$

Solution

$(4, 2)$

13) Solve each system by graphing: $\begin{cases} x+y=2 \\ x-y=-8 \end{cases}$

Solution

$(5, -3)$

Do you remember how to graph a linear equation with just one variable? It will be either a vertical or a horizontal line.

Example 4.1.6

Solve the system by graphing: $\begin{cases} y=6 \\ 2x+3y=12 \end{cases}$

Solution

$$\begin{cases} y=6 \\ 2x+3y=12 \end{cases}$$

Step 1: **We know the first equation represents a horizontal line whose y-intercept is 6.**

y = 6

Step 2: **The second equation is most conveniently graphed using intercepts.**

$$2x + 3y = 12$$

Step 3: **To find the intercepts, let $x = 0$ and then $y = 0$.**

x	y
0	4
6	0

Step 4: **Graph the lines.**

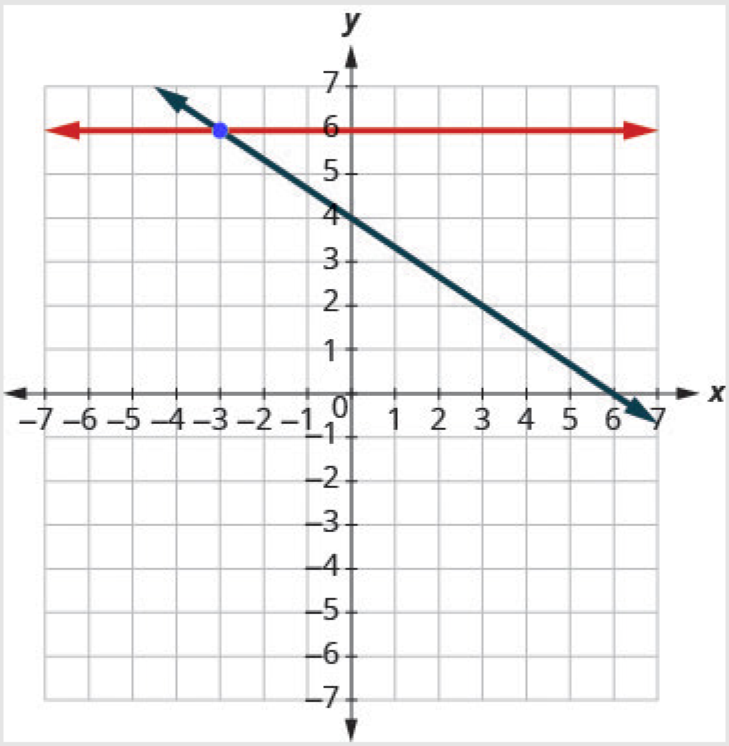


Figure 4.1.7

Step 5: **Determine the point of intersection.**

The lines intersect at $(-3, 6)$.

Step 6: **Check the solution to both equations.**

$$\begin{aligned} y &= 6 \\ 6 &\stackrel{?}{=} 6\checkmark \\ 2 &= 2 \end{aligned} \qquad \begin{aligned} 2x+3y &= 12 \\ 2(-3)+3(6) &\stackrel{?}{=} 12 \\ -6+18 &\stackrel{?}{=} 12 \\ 12 &= 12\checkmark \end{aligned}$$

The solution is $(-3, 6)$.

Try It

14) Solve each system by graphing: $\begin{cases} y = -1 \\ x + 3y = 6 \end{cases}$

Solution

$(9, -1)$

15) Solve each system by graphing: $\begin{cases} x = 4 \\ 3x - 2y = 24 \end{cases}$

Solution

$(4, -6)$

In all the systems of linear equations so far, the lines intersected and the solution was one point. In the next two examples, we'll look at a system of equations that has no solution and at a system of equations that has an infinite number of solutions.

Example 4.1.7

Solve the system by graphing: $\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$

Solution

$$\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$$

Step 1: **To graph the first equation, we will use its slope and y-intercept.**

$$y = \frac{1}{2}x - 3$$
$$m = \frac{1}{2}$$
$$b = -3$$

Step 2: **To graph the second equation, we will use the intercepts.**

$$x - 2y = 4$$

x	y
0	-2
4	0

Step 3: **Graph the lines.**

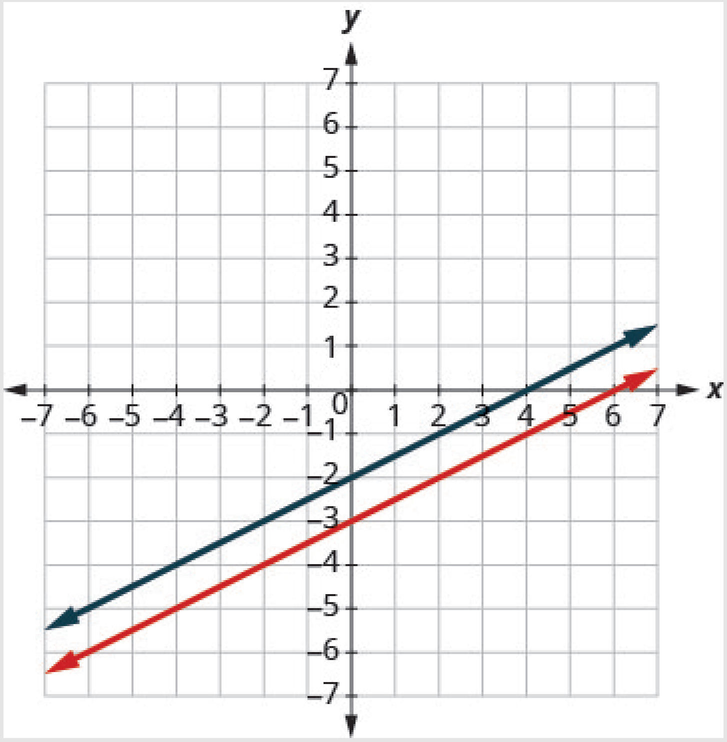


Figure 4.1.8

Step 4: **Determine the point of intersection.**

The lines are parallel.

Since no point is on both lines, there is no ordered pair that makes both equations true. There is no solution to this system.

Try It

16) Solve each system by graphing: $\begin{cases} y = -\frac{1}{4}x + 2 \\ x + 4y = -8 \end{cases}$

Solution

no solution

17) Solve each system by graphing: $\begin{cases} y = 3x - 1 \\ 6x - 2y = 6 \end{cases}$

Solution

no solution

Example 4.1.8

Solve the system by graphing: $\begin{cases} y = 2x - 3 \\ -6x + 3y = -9 \end{cases}$

Solution

$$\begin{cases} y = 2x - 3 \\ -6x + 3y = -9 \end{cases}$$

Step 1: **Find the slope and y-intercept of the first equation.**

$$\begin{aligned} y &= 2x - 3 \\ m &= 2 \\ b &= -3 \end{aligned}$$

Step 2: **Find the intercepts of the second equation.**

$$-6x + 3y = -9$$

x	y
0	-3
\frac{3}{2}	0

Step 3: **Graph the lines.**

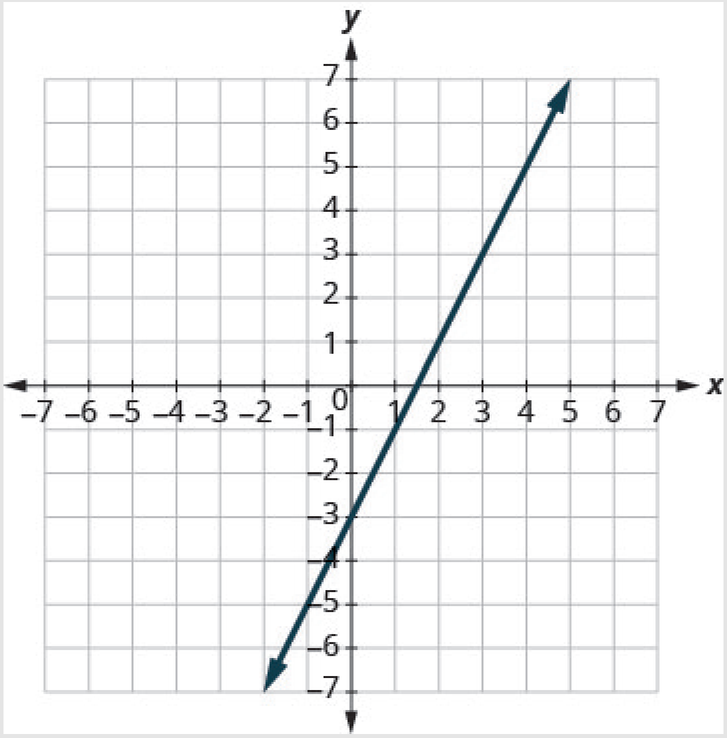


Figure 4.1.9

Step 4: **Determine the point of intersection.**

The lines are the same!

Since every point on the line makes both equations true, there are infinitely many ordered pairs that make both equations true.

There are infinitely many solutions to this system.

Try It

18) Solve each system by graphing: $\begin{cases} y = -3x - 6 \\ 6x + 2y = -12 \end{cases}$

Solution

infinitely many solutions

19) Solve each system by graphing: $\begin{cases} y = \frac{1}{2}x - 4 \\ 2x - 4y = 16 \end{cases}$

Solution

infinitely many solutions

If you write the second equation in Example 4.1.8 in slope-intercept form, you may recognize that the equations have the same slope and same y-intercept.

When we graphed the second line in the last example, we drew it right over the first line. We say the two lines are coincident. **Coincident lines** have the same slope and same y-intercept.

Coincident Lines

Coincident lines have the same slope and same y-intercept.

Determine the Number of Solutions of a Linear System

There will be times when we will want to know how many solutions there will be to a system of linear equations, but we might not actually have to find the solution. It will be helpful to determine this without graphing.

We have seen that two lines in the same plane must either intersect or are parallel. The systems of equations in Example 4.1.2 through Example 4.1.6 all had two intersecting lines. Each system had one solution.

A system with parallel lines, like Example 4.1.7, has no solution. What happened in Example 4.1.8? The equations have coincident lines, and so the system had infinitely many solutions.

We'll organize these results in Figure 4.2.10 below:

Figure 4.1.10

Graph	Number of solutions
2 intersecting lines	1
Parallel lines	None
Same line	Infinitely many

Parallel lines have the same slope but different y-intercepts. So, if we write both equations in a system of linear equations in slope–intercept form, we can see how many solutions there will be without graphing! Look at the system we solved in Example 4.1.7.

$$\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$$

The first line is in slope–intercept form.	If we solve the second equation for y, we get
$y = \frac{1}{2}x - 3$	$\begin{aligned} x - 2y &= 4 \\ -2y &= -x + 4 \\ y &= \frac{1}{2}x - 2 \end{aligned}$
$m = \frac{1}{2}, b = -3$	$m = \frac{1}{2}, b = -2$

The two lines have the same slope but different y-intercepts. They are parallel lines.

Figure 4.1.11 shows how to determine the number of solutions of a linear system by looking at the slopes and intercepts.

Number of Solutions of a Linear System of Equations

Figure 4.1.11

Slopes	Intercepts	Type of Lines	Number of Solutions
Different		Intersecting	1 point
Same	Different	Parallel	No solution
Same	Same	Coincident	Infinitely many solutions

Let's take one more look at our equations in Example 4.1.7 that gave us parallel lines.

$$\begin{cases} y = \frac{1}{2}x - 3 \\ x - 2y = 4 \end{cases}$$

When both lines were in slope-intercept form we had:

$$y = \frac{1}{2}x - 3 \qquad y = \frac{1}{2}x - 2$$

Do you recognize that it is impossible to have a single ordered pair (x, y) that is a solution to both of those equations?

We call a system of equations like this an **inconsistent system**. It has no solution.

A system of equations that has at least one solution is called a **consistent system**.

Consistent and Inconsistent Systems

A consistent system of equations is a system of equations with at least one solution.

An inconsistent system of equations is a system of equations with no solution.

We also categorize the equations in a system of equations by calling the equations *independent* or *dependent*. If two equations are **independent equations**, they each have their own set of solutions. Intersecting lines and parallel lines are independent.

If two equations are dependent, all the solutions of one equation are also solutions of the other equation. When we graph two **dependent equations**, we get coincident lines.

Independent and Dependent Equations

Two equations are independent if they have different solutions.

Two equations are dependent if all the solutions of one equation are also solutions of the other equation.

Let's sum this up by looking at the graphs of the three types of systems. See Figure 4.1.12 and Figure 4.2.13.

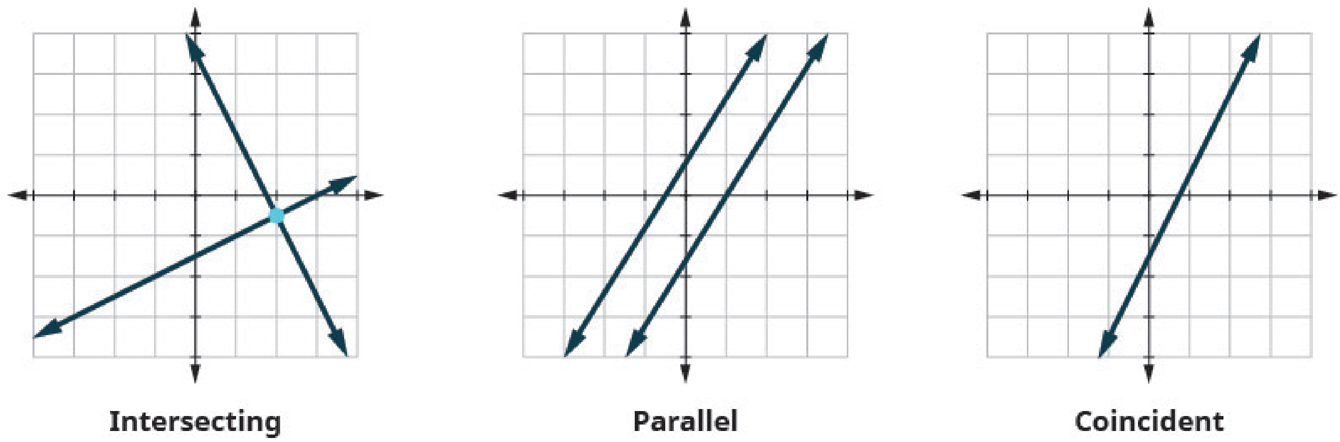


Figure 4.1.12

Figure 4.1.13

Lines	Intersecting	Parallel	Coincident
Number of solutions	1 point	No solution	Infinitely many
Consistent/ Inconsistent	Consistent	Inconsistent	Consistent
Dependent/ Independent	Independent	Independent	Dependent

Example 4.1.9

Without graphing, determine the number of solutions and then classify the system of equations:

$$\begin{cases} y = 3x - 1 \\ 6x - 2y = 12 \end{cases}$$

Solution

Step 1: **We will compare the slopes and intercepts of the two lines.**

$$\begin{cases} y = 3x - 1 \\ 6x - 2y = 12 \end{cases}$$

Step 2: **The first equation is already in slope-intercept form.**

$$y = 3x - 1$$

***Step 3:* Write the second equation in slope-intercept form.**

$$\begin{aligned} 6x - 2y &= 12 \\ -2y &= -6x + 12 \\ \frac{-2y}{-2} &= \frac{-6x+12}{-2} \\ y &= 3x - 6 \end{aligned}$$

***Step 4:* Find the slope and intercept of each line.**

$$\begin{aligned} y &= 3x - 1 \\ m &= 3 \\ b &= -1 \end{aligned} \qquad \begin{aligned} y &= 3x - 6 \\ m &= 3 \\ b &= -6 \end{aligned}$$

Since the slopes are the same and y-intercepts are different, the lines are parallel.

A system of equations whose graphs are parallel lines has no solution and is inconsistent and independent.

Try It

20) Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} y = -2x - 4 \\ 4x + 2y = 9 \end{cases}$$

Solution

no solution, inconsistent, independent

21) Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} y = \frac{1}{3}x - 5 \\ x - 3y = 6 \end{cases}$$

Solution

no solution, inconsistent, independent

Example 4.1.10

Without graphing, determine the number of solutions and then classify the system of equations:

$$\begin{cases} 2x + y = -3 \\ x - 5y = 5 \end{cases}$$

Solution

Step 1: **We will compare the slope and intercepts of the two lines.**

$$\begin{cases} 2x + y = -3 \\ x - 5y = 5 \end{cases}$$

Step 2: **Write both equations in slope-intercept form.**

$$\begin{aligned} 2x + y &= -3 \\ y &= -2x - 3 \\ x - 5y = 5 &= 5 \\ -5y &= -x + 5 \\ \frac{-5y}{-5} &= \frac{-x+5}{-5} \\ y &= \frac{1}{5}x - 1 \end{aligned}$$

Step 3: **Find the slope and intercept of each line.**

$$\begin{aligned} y &= -2x - 3 \\ m &= -2 \\ b &= -3 \\ y &= \frac{1}{5}x - 1 \\ m &= \frac{1}{5} \\ b &= -1 \end{aligned}$$

Since the slopes are different, the lines intersect.

A system of equations whose graphs are intersect has 1 solution and is consistent and independent.

Try It

22) Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} 3x + 2y = 2 \\ 2x + y = 1 \end{cases}$$

Solution
one solution, consistent, independent

23) Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} x + 4y = 12 \\ -x + y = 3 \end{cases}$$

Solution
one solution, consistent, independent

Example 4.1.11

Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} 3x - 2y = 4 \\ y = \frac{3}{2}x - 2 \end{cases}$$

Solution

Step 1: **We will compare the slopes and intercepts of the two lines.**

$$\begin{cases} 3x - 2y = 4 \\ y = \frac{3}{2}x - 2 \end{cases}$$

Step 2: **Write the first equation in slope-intercept form.**

$$\begin{aligned} 3x - 2y &= 4 \\ -2y &= -3x + 4 \\ \frac{-2y}{-2} &= \frac{-3x+4}{-2} \\ y &= \frac{3}{2}x - 2 \end{aligned}$$

Step 3: **The second equation is already in slope-intercept form.**

$$y = \frac{3}{2}x - 2$$

Since the slopes are the same, they have the same slope and same y-intercept and so the lines are coincident.

A system of equations whose graphs are coincident lines has infinitely many solutions and is consistent and dependent.

Try It

24) Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} 4x - 5y = 20 \\ y = \frac{4}{5}x - 4 \end{cases}$$

Solution

infinitely many solutions, consistent, dependent

25) Without graphing, determine the number of solutions and then classify the system of equations.

$$\begin{cases} -2x - 4y = 8 \\ y = -\frac{1}{2}x - 2 \end{cases}$$

Solution

infinitely many solutions, consistent, dependent

Solve Applications of Systems of Equations by Graphing

We will use the same problem solving strategy we used in Math Models to set up and solve applications of systems of linear equations. We'll modify the strategy slightly here to make it appropriate for systems of equations.

HOW TO

Use a problem solving strategy for systems of linear equations.

1. Read the problem. Make sure all the words and ideas are understood.
2. Identify what we are looking for.
3. Name what we are looking for. Choose variables to represent those quantities.
4. Translate into a system of equations.
5. Solve the system of equations using good algebra techniques.
6. Check the answer in the problem and make sure it makes sense.
7. Answer the question with a complete sentence.

Step 5 is where we will use the method introduced in this section. We will graph the equations and find the solution.

Example 4.1.12

Sondra is making 10 quarts of punch from fruit juice and club soda. The number of quarts of fruit juice is 4 times the number of quarts of club soda. How many quarts of fruit juice and how many quarts of club soda does Sondra need?

Solution

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

We are looking for the number of quarts of fruit juice and the number of quarts of club soda that Sondra will need.

Step 3: **Name what we are looking for. Choose variables to represent those quantities.**

Let f = number of quarts of fruit juice.
Let c = number of quarts of club soda

***Step 4:* Translate into a system of equations.**

$$\text{The } \underbrace{\text{number of quarts of fruit juice}}_{f} \underbrace{\text{and}}_{+} \text{ the } \underbrace{\text{number of quarts of club soda}}_{c} \underbrace{\text{is 10}}_{=10}$$

$$\text{The } \underbrace{\text{number of quarts of fruit juice}}_{f} \underbrace{\text{is}}_{=} \underbrace{\text{four times the number of quarts of club soda}}_{4c}$$

We now have the system. $\begin{cases} f + c = 10 \\ f = 4c \end{cases}$

***Step 5:* Solve the system of equations using good algebra techniques.**

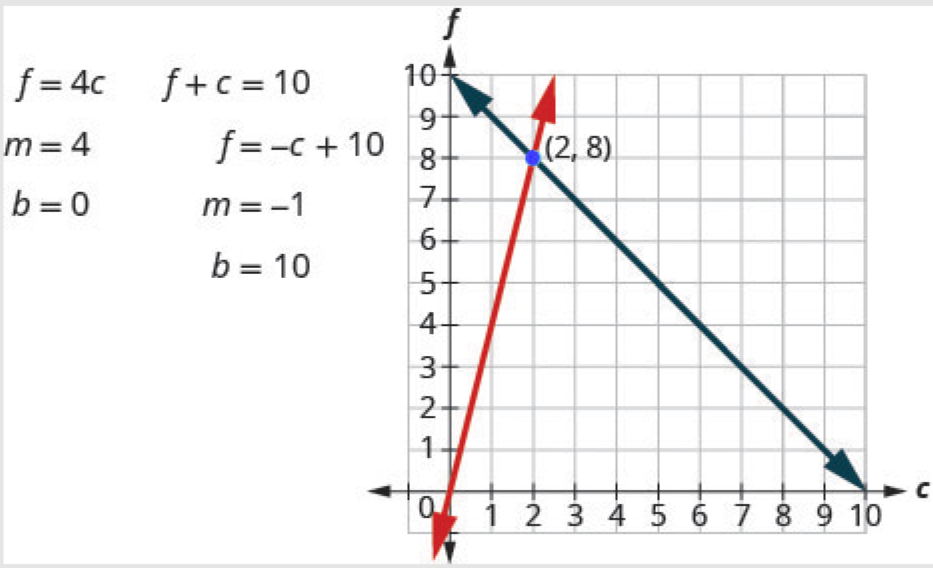


Figure 4.1.14

The point of intersection $(2, 8)$ is the solution. This means Sondra needs 2 quarts of club soda and 8 quarts of fruit juice.

***Step 6:* Check the answer in the problem and make sure it makes sense.**

Does this make sense in the problem?

Yes, the number of quarts of fruit juice, 8 is 4 times the number of quarts of club soda, 2.

Yes, 10 quarts of punch is 8 quarts of fruit juice plus 2 quarts of club soda.

***Step 7:* Answer the question with a complete sentence.**

Sondra needs 8 quarts of fruit juice and 2 quarts of soda.

Try It

26) Manny is making 12 quarts of orange juice from concentrate and water. The number of quarts of water is 3 times the number of quarts of concentrate. How many quarts of concentrate and how many quarts of water does Manny need?

Solution

Manny needs 3 quarts juice concentrate and 9 quarts water.

27) Alisha is making an 18 ounce coffee beverage that is made from brewed coffee and milk. The number of ounces of brewed coffee is 5 times greater than the number of ounces of milk. How many ounces of coffee and how many ounces of milk does Alisha need?

Solution

Alisha needs 15 ounces of coffee and 3 ounces of milk.

Access these online resources for additional instruction and practice with solving systems of equations by graphing.

- Instructional Video Solving Linear Systems by Graphing
- Instructional Video Solve by Graphing

Key Concepts

- **To solve a system of linear equations by graphing**

  1. Graph the first equation.
  2. Graph the second equation on the same rectangular coordinate system.
  3. Determine whether the lines intersect, are parallel, or are the same line.
  4. Identify the solution to the system.
     If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system.
     If the lines are parallel, the system has no solution.
     If the lines are the same, the system has an infinite number of solutions.
  5. Check the solution in both equations.

- Determine the number of solutions from the graph of a linear system

Graph	Number of solutions
2 intersecting lines	1
Parallel lines	None
Same line	Infinitely many

- Determine the number of solutions of a linear system by looking at the slopes and intercepts

Number of Solutions of a Linear System of Equations

Slopes	**Intercepts**	**Type of Lines**	**Number of Solutions**
Different		Intersecting	1 point
Same	Different	Parallel	No solution
Same	Same	Coincident	Infinitely many solutions

- Determine the number of solutions and how to classify a system of equations

Lines	**Intersecting**	**Parallel**	**Coincident**
Number of solutions	1 point	No colution	Infinitely many
Consistent/ Inconsistent	Consistent	Inconsistent	Consistent
Dependent/ Independent	Independent	Independent	Dependent

- **Problem Solving Strategy for Systems of Linear Equations**

1. Read the problem. Make sure all the words and ideas are understood.
2. Identify what we are looking for.
3. Name what we are looking for. Choose variables to represent those quantities.
4. Translate into a system of equations.
5. Solve the system of equations using good algebra techniques.
6. Check the answer in the problem and make sure it makes sense.
7. Answer the question with a complete sentence.

Self Check

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=2205#h5p-9

Glossary

coincident lines
Coincident lines are lines that have the same slope and same y-intercept.

consistent system
A consistent system of equations is a system of equations with at least one solution.

dependent equations
Two equations are dependent if all the solutions of one equation are also solutions of the other equation.

inconsistent system
An inconsistent system of equations is a system of equations with no solution.

independent equations
Two equations are independent if they have different solutions.

solutions of a system of equations
Solutions of a system of equations are the values of the variables that make all the equations true. A solution of a system of two linear equations is represented by an ordered pair (x, y).

system of linear equations
When two or more linear equations are grouped together, they form a system of linear equations.

4.2 SOLVE SYSTEMS OF EQUATIONS BY SUBSTITUTION

Learning Objectives

By the end of this section, you will be able to:

- Solve a system of equations by substitution
- Solve applications of systems of equations by substitution

Try It

Before you get started, take this readiness quiz:

1) Simplify $-5(3-x)$.
2) Simplify $4-2(n+5)$.
3) Solve for y: $8y-8=32-2y$.
4) Solve for x: $3x-9y=-3$.

Solving systems of linear equations by graphing is a good way to visualize the types of solutions that may result. However, there are many cases where solving a system by graphing is inconvenient or imprecise. If the graphs extend beyond the small grid with x and y both between -10 and 10, graphing the lines may be cumbersome. And if the solutions to the system are not integers, it can be hard to read their values precisely from a graph.

In this section, we will solve systems of linear equations by the substitution method.

Solve a System of Equations by Substitution

We will use the same system we used first for graphing.

$$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$$

We will first solve one of the equations for either x or y. We can choose either equation and solve for either variable—but we'll try to make a choice that will keep the work easy.

Then we substitute that expression into the other equation. The result is an equation with just one variable—and we know how to solve those!

After we find the value of one variable, we will substitute that value into one of the original equations and solve for the other variable. Finally, we check our solution and make sure it makes both equations true.

We'll fill in all these steps now in Example 4.2.1

Example 4.2.1

How to Solve a System of Equations by Substitution

Solve the system by substitution. $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$

Solution

Step 1. Solve one of the equations for either variable.	We'll solve the first equation for y.	$2x+y=7$ $y=7-2x$

Figure 4.2.1

Step 2. Substitute the expression from Step 1 into the other equation.	We replace y in the second equation with the expression $7 - 2x$.	$x - 2y = 6$ $x - 2(7 - 2x) = 6$

Figure 4.2.2

Step 3. Solve the resulting equation.	Now we have an equation with just 1 variable. We know how to solve this!	$x - 2(7 - 2x) = 6$ $x - 14 + 4x = 6$ $5x = 20$ $x = 4$

Figure 4.2.3

Step 4. Substitute the solution in Step 3 into one of the original equations to find the other variable.	We'll use the first equation and replace x with 4.	$2x + y = 7$ $2(4) + y = 7$ $8 + y = 7$ $y = -1$

Figure 4.2.4

Step 5. Write the solution as an ordered pair.	The ordered pair is (x, y).	$(4, -1)$

Figure 4.2.5

Step 6. Check that the ordered pair is a solution to **both** original equations.	Substitute $(4, -1)$ into both equations and make sure they are both true.	$2x + y = 7$ $x - 2y = 6$ $2(4) + (-1) \stackrel{?}{=} 7$ $4 - 2(-1) \stackrel{?}{=} 6$ $7 = 7$ ✓ $6 = 6$ ✓ Both equations are true. $(4, -1)$ is the solution to the system.

Figure 4.2.6

Try It

5) Solve the system by substitution.

$$\begin{cases} -2x + y = -11 \\ x + 3y = 9 \end{cases}$$

Solution

$(6, 1)$

6) Solve the system by substitution.

$$\begin{cases} x + 3y = 10 \\ 4x + y = 18 \end{cases}$$

Solution

$(4, 2)$

HOW TO

Solve a system of equations by substitution.

1. Solve one of the equations for either variable.
2. Substitute the expression from Step 1 into the other equation.
3. Solve the resulting equation.
4. Substitute the solution in Step 3 into one of the original equations to find the other variable.
5. Write the solution as an ordered pair.
6. Check that the ordered pair is a solution to **both** original equations.

If one of the equations in the system is given in slope–intercept form, Step 1 is already done! We'll see this in Example 4.2.2

Example 4.2.2

Solve the system by substitution.

$$\begin{cases} x + y = -1 \\ y = x + 5 \end{cases}$$

Solution

The second equation is already solved for y. We will substitute the expression in place of y in the first equation.

The second equation is already solved for y. We will substitute into the first equation.

$$\begin{cases} x \ + \ y \ = \ -1 \\ y \ = \ x \ + \ 5 \end{cases}$$

***Step 1:* Replace the y with $x + 5$**

$$y = rgb]1.0, 0.0, 0.0\boxed{x + 5}$$
$$x + rgb]1.0, 0.0, 0.0y = -1$$
$$x + rgb]1.0, 0.0, 0.0 \ rgb]1.0, 0.0, 0.0x rgb]1.0, 0.0, 0.0 \ rgb]1.0, 0.0, 0.0 + rgb]1.0, 0.0, 0.0 \ rgb]1.0, 0.0, 0.05 = -1$$

***Step 2:* Solve the resulting equation for x**

$$x + rgb]1.0, 0.0, 0.0 \ rgb]1.0, 0.0, 0.0x rgb]1.0, 0.0, 0.0 \ rgb]1.0, 0.0, 0.0 + rgb]1.0, 0.0, 0.0 \ rgb]1.0, 0.0, 0.05 = -1$$
$$2x + 5 = -1$$
$$2x = -6$$

***Step 3:* Substitute $x = -3$ into $y = x + 5$ to find y**

$$rgb]1.0, 0.0, 0.0x = rgb]1.0, 0.0, 0.0 \ rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.03$$
$$y = rgb]1.0, 0.0, 0.0x + 5$$
$$y = rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.03 + 5$$
$$y = 2$$

The ordered pair is $(-3, 2)$.

***Step 4:* Check the ordered pair in both equations:**

$x+y=-1$ $-3+2\stackrel{?}{=}-1$ $-1=-1\checkmark$	$y=x+5$ $2\stackrel{?}{=}-3+5$ $2=2\checkmark$

The solution is $(-3,2)$.

Try It

7) Solve the system by substitution.

$$\begin{cases} x+y=6 \\ y=3x-2 \end{cases}$$

Solution

$(2,4)$

8) Solve the system by substitution.

$$\begin{cases} 2x-y=1 \\ y=-3x-6 \end{cases}$$

Solution

$(-1,-3)$

If the equations are given in standard form, we'll need to start by solving for one of the variables. In this next example, we'll solve the first equation for y.

Example 4.2.3

Solve the system by substitution.

$$\begin{cases} 3x + y = 5 \\ 2x + 4y = -10 \end{cases}$$

Solution

We need to solve one equation for one variable. Then we will substitute that expression into the other equation.

Solve for y

$$3x \;+\; y \;=\; 5$$

Step 1: **Substitute into the other equation.**

Step 2: **Replace the y with $-3x+5$.**

$$\begin{aligned} 3x + y &= 5 \\ y &= -3 + 5 \\ 2x + 4y &= -10 \\ 2x + 4(-3+5) &= -10 \end{aligned}$$

Step 3: **Solve the resulting equation for x.**

$$\begin{aligned} 2x - 12x + 20 &= -10 \\ -10x + 20 &= -10 \\ -10x &= -30 \\ x &= 3 \end{aligned}$$

Step 4: **Substitute $x = 3$ into $3x + y = 5$ to find y.**

$$\begin{aligned} rgb]1.0, 0.0, 0.0x rgb]1.0, 0.0, 0.0 &= rgb]1.0, 0.0, 0.0\ rgb]1.0, 0.0, 0.03 \\ 3 rgb]1.0, 0.0, 0.0x \;+\; y &= 5 \\ 3(rgb]1.0, 0.0, 0.03) \;+\; y &= 5 \\ 9 \;+\; y &= 5 \\ y &= -4 \end{aligned}$$

The ordered pair is $(3, -4)$.

Step 5: **Check the ordered pair in both equations:**

$3x + y = 5$ $-3 \times 3 + (-4) \stackrel{?}{=} 5$ $9 - 4 \stackrel{?}{=} 5$ $5 = 5 \checkmark$	$2x + 4y = -10$ $2 \times 3 + 4(-4) = -10$ $6 - 16 \stackrel{?}{=} -10$ $-10 = -10 \checkmark$

The solution is $(3, -4)$.

Try It

9) Solve the system by substitution.

$$\begin{cases} 4x + y = 2 \\ 3x + 2y = -1 \end{cases}$$

Solution

$(1, -2)$

10) Solve the system by substitution.

$$\begin{cases} -x + y = 4 \\ 4x - y = -2 \end{cases}$$

Solution

$(2, 6)$

In Example 4.2.3 it was easiest to solve for y in the first equation because it had a coefficient of 1. In Example 4.2.4 it will be easier to solve for x.

Example 4.2.4

Solve the system by substitution.

$$\begin{cases} x - 2y = -2 \\ 3x + 2y = 34 \end{cases}$$

Solution

We will solve the first equation for x and then substitute the expression into the second equation.

$$x \; - \; 2y \; = \; -2$$

Step 1: **Solve for x.**

Step 2: **Substitute into the other equation.**

Step 3: **Replace the x with $2y - 2$.**

$$3rgb]1.0,0.0,0.0x \; + \; 2y \; = \; 34$$
$$3(rgb]1.0,0.0,0.02rgb]1.0,0.0,0.0yrgb]1.0,0.0,0.0 \; rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0 \; rgb]1.0,0.0,0.02) \; + \; 2y \; = \; 34$$

Step 4: **Solve the resulting equation for y.**

$$\begin{aligned} 6y \; - \; 6 \; + \; 2y \; &= \; 34 \\ 8y \; - \; 6 \; &= \; 34 \\ 8y \; &= \; 40 \\ y \; &= \; 5 \end{aligned}$$

Step 5: **Substitute $y = 5$ into $x - 2y = -2$ to find x.**

$$\begin{aligned} x \; - \; 2rgb]1.0,0.0,0.0y \; &= \; -2 \\ x \; - \; 2 \; \times \; rgb]1.0,0.0,0.05 \; &= \; -2 \\ x \; - \; 10 \; &= \; -2 \\ x \; &= \; 8 \end{aligned}$$

The ordered pair is $(8, 5)$.

Step 6: **Check the ordered pair in both equations:**

$x-2y=-2$	$3x+2y=34$
$8-2\times 5\stackrel{?}{=}-2$	$3\times 8+2\times 5\stackrel{?}{=}34$
$8-10\stackrel{?}{=}-2$	$24+10\stackrel{?}{=}34$
$-2=-2\checkmark$	$34=34\checkmark$

The solution is $(8,5)$.

Try It

11) Solve the system by substitution.

$$\begin{cases} x-5y=13 \\ 4x-3y=1 \end{cases}$$

Solution

$(-2,-3)$

12) Solve the system by substitution.

$$\begin{cases} x-6y=-6 \\ 2x-4y=4 \end{cases}$$

Solution

$(6,2)$

When both equations are already solved for the same variable, it is easy to substitute!

Example 4.2.5

Solve the system by substitution.

$$\begin{cases} y = -2x + 5 \\ y = \frac{1}{2}x \end{cases}$$

Solution

Since both equations are solved for y, we can substitute one into the other.

Step 1: **Substitute $\frac{1}{2}x$ for y in the first equation.**

$$\begin{aligned} rgb]1.0, 0.0, 0.0y &= rgb]1.0, 0.0, 0.0\frac{1}{2}rgb]1.0, 0.0, 0.0x \\ rgb]1.0, 0.0, 0.0y &= -2x + 5 \end{aligned}$$

Step 2: **Replace the y with $\frac{1}{2}x$**

$$rgb]1.0, 0.0, 0.0\frac{1}{2}rgb]1.0, 0.0, 0.0x = -2x + 5$$

Step 3: **Solve the resulting equation. Start by clearing the fraction.**

$$2\left(\frac{1}{2}x\right) = 2(-2x + 5)$$

Step 4: **Solve for x.**

$$\begin{aligned} x &= -4x + 10 \\ 5x &= 10 \\ x &= 2 \end{aligned}$$

Step 5: **Substitute $x = 2$ into $y = \frac{1}{2}x$ to find y.**

$$
\begin{aligned}
rgb]1.0,0.0,0.0x &= rgb]1.0,0.0,0.02 \\
y &= \frac{1}{2}rgb]1.0,0.0,0.0x \\
y &= \frac{1}{2} \times rgb]1.0,0.0,0.02 \\
y &= 1
\end{aligned}
$$

The ordered pair is $(2, 1)$.

***Step 6:* Check the ordered pair in both equations**

$y = \frac{1}{2}x$ $1 \stackrel{?}{=} \frac{1}{2} \times 2$ $1 = 1 \checkmark$	$y = -2x + 5$ $1 \stackrel{?}{=} -2 \times 2 + 5$ $1 = 1 \checkmark$

The solution is $(2, 1)$.

Try It

13) Solve the system by substitution.

$$\begin{cases} x + y = 6 \\ y = 3x - 2 \end{cases}$$

Solution

$(6, 2)$

14) Solve the system by substitution.

$$\begin{cases} 4x - y = 0 \\ 2x - 3y = 5 \end{cases}$$

Solution

$(8, 2)$

Be very careful with the signs in the next example.

Example 4.2.6

Solve the system by substitution.

$$\begin{cases} 4x + 2y = 4 \\ 6x - y = 8 \end{cases}$$

Solution

We need to solve one equation for one variable. We will solve the first equation for y.

$$4x \; + \; 2y \; = \; 4$$

Step 1: **Solve the first equation for y.**

$$2y \; = \; -4x \; + \; 4$$

Step 2: **Substitute $-2x+2$ for y in the second equation.**

$$\begin{array}{l} rgb]1.0,0.0,0.0y \; = \; rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.02rgb]1.0,0.0,0.0xrgb]1.0,0.0,0.0 \; rgb]1.0,0.0,0.0{+}rgb]1.0,0.0,0.0 \; rgb]1.0,0.0,0.02 \\ 6x \; - \; rgb]1.0,0.0,0.0y \; = \; 8 \end{array}$$

Step 3: **Replace the y with $-2x+2$.**

$$6x \; - \; (rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.02rgb]1.0,0.0,0.0xrgb]1.0,0.0,0.0 \; rgb]1.0,0.0,0.0{+}rgb]1.0,0.0,0.0 \; rgb]1.0,0.0,0.02) \; = \; 8$$

Step 4: **Solve the equation for x.**

$$\begin{aligned} 6x \; + \; 2x \; - \; 2 &= 8 \\ 8x \; - \; 2 &= 8 \\ 8x &= 10 \\ x &= \frac{5}{2} \end{aligned}$$

Step 5: **Substitute $x = \frac{5}{4}$ into $4x + 2y = 4$ to find y.**

$$\begin{aligned} x &= \frac{5}{2} \\ 4rgb]1.0,0.0,0.0x + 2y &= 4 \\ 4(rgb]1.0,0.0,0.0\frac{5}{2}) + 2y &= 4 \\ 5 + 2y &= 4 \\ 2y &= -1 \\ y &= -\frac{1}{2} \end{aligned}$$

The ordered pair is $(\frac{5}{4}, -\frac{1}{2})$

Step 6: **Check the ordered pair in both equations.**

$4x + 2y = 4$ $4(\frac{5}{4}) + 2(-\frac{1}{2}) \stackrel{?}{=} 4$ $5 - 1 \stackrel{?}{=} 4$ $4 = 4 ✓$	$6x - y = 8$ $6\left(\frac{5}{4}\right) - \left(-\frac{1}{2}\right) \stackrel{?}{=} 8$ $\frac{15}{4} - \left(-\frac{1}{2}\right) \stackrel{?}{=} 8$ $\frac{16}{2} \stackrel{?}{=} 8$ $8 = 8 ✓$

The solution is $(\frac{5}{4}, -\frac{1}{2})$

Try It

15) Solve the system by substitution.

$$\begin{cases} 3x + y = 5 \\ 2x + 4y = -10 \end{cases}$$

Solution

$(2, \frac{3}{2})$

16) Solve the system by substitution.

$$\begin{cases} 4x - y = 0 \\ 2x - 3y = 5 \end{cases}$$

Solution

$(-\frac{1}{2}, -2)$

In Example 4.2.7, it will take a little more work to solve one equation for x or y.

Example 4.2.7

Solve the system by substitution.

$$\begin{cases} 4x - 3y = 6 \\ 15y - 20x = -30 \end{cases}$$

Solution

We need to solve one equation for one variable. We will solve the first equation for x.

$$x = \frac{5}{2}$$
$$4x - 3y = 6$$

Step 1: **Solve the first equation for x.**

$$4x = 3y + 6$$

Step 2: **Substitute $\frac{3}{4}y+\frac{3}{2}$ for x in the second equation.**

$$\begin{aligned} rgb]1.0,0.0,0.0x &= rgb]1.0,0.0,0.0\frac{3}{4}rgb]1.0,0.0,0.0y + rgb]1.0,0.0,0.0\frac{3}{2} \\ 15y \; - \; 20rgb]1.0,0.0,0.0x &= -30 \end{aligned}$$

Step 3: **Replace the x with $\frac{3}{4}y+\frac{3}{2}$**

$$15y \; - \; 20rgb]1.0,0.0,0.0\left(\frac{3}{4}y \; + \; \frac{3}{2}\right) \; = \; -30$$

Step 4: **Solve for y.**

$$\begin{aligned} 15y \; - \; 15y \; - \; 30 \; &= \; -30 \\ 0 \; - \; 30 \; &= \; -30 \\ 0 \; &= \; 0 \end{aligned}$$

Since $0=0$ is a true statement, the system is consistent. The equations are dependent. The graphs of these two equations would give the same line. The system has infinitely many solutions.

Try It

17) Solve the system by substitution.

$$\begin{cases} 2x-3y=12 \\ -12y+8x=48 \end{cases}$$

Solution

Infinitely many solutions.

18) Solve the system by substitution.

$$\begin{cases} 5x+2y=12 \\ -4y-10x=-24 \end{cases}$$

Solution

Infinitely many solutions.

Look back at the equations in Example 4.2.7. Is there any way to recognize that they are the same line?

Let's see what happens in the next example.

Example 4.2.8

Solve the system by substitution.

$$\begin{cases} 5x - 2y = -10 \\ y = \frac{5}{2}x \end{cases}$$

Solution

The second equation is already solved for y, so we can substitute for y in the first equation.

Step 1: **Substitute x for y in the first equation.**

$$rgb]1.0,0.0,0.0y \quad = rgb]1.0,0.0,0.0\frac{5}{2}rgb]1.0,0.0,0.0x$$
$$5x - 2rgb]1.0,0.0,0.0y = -10$$

Step 2: **Replace the y with $\frac{5}{2}x$.**

$$5x \quad - \quad 2rgb]1.0,0.0,0.0\left(\frac{5}{2}x\right) \quad = \quad -10$$

Step 3: **Solve for x.**

$$\begin{aligned} 5x \quad - \quad 5x \quad &= \quad -10 \\ 0 \quad &\neq \quad -10 \end{aligned}$$

Since $0 = -10$ is a false statement the equations are inconsistent. The graphs of the two equation would be parallel lines. The system has no solutions.

Try It

19) Solve the system by substitution.

$$\begin{cases} 3x + 2y = 9 \\ y = -\frac{3}{2}x + 1 \end{cases}$$

Solution

No solution.

20) Solve the system by substitution.

$$\begin{cases} 5x - 3y = 2 \\ y = \frac{5}{3}x - 4 \end{cases}$$

Solution

No solution.

Solve Applications of Systems of Equations by Substitution

We'll copy here the problem solving strategy we used in the Solving Systems of Equations by Graphing section for solving systems of equations. Now that we know how to solve systems by substitution, that's what we'll do in Step 5:

How to use a problem solving strategy for systems of linear equations.

How To:

How to use a problem-solving strategy for systems of linear equations.

1. Read the problem. Make sure all the words and ideas are understood.
2. Identify what we are looking for.
3. Name what we are looking for. Choose variables to represent those quantities.
4. Translate into a system of equations.
5. Solve the system of equations using good algebra techniques.
6. Check the answer in the problem and make sure it makes sense.
7. Answer the question with a complete sentence.

Some people find setting up word problems with two variables easier than setting them up with just one variable. Choosing the variable names is easier when all you need to do is write down two letters. Think about this in the next example—how would you have done it with just one variable?

Example 4.2.9

The sum of two numbers is zero. One number is nine less than the other. Find the numbers.

Solution

***Step 1:* Read the problem.**

***Step 2:* Identify what we are looking for.**

We are looking for two numbers.

***Step 3:* Name what we are looking for.**

Let $n =$ the first number

Let $m =$ the second number

Step 4: **Translate into a system of equations.**

The sum of two numbers is zero.

$$n \;+\; m \;=\; 0$$

One number is nine less than the other.

$$n \;=\; m \;-\; 9$$

The system is:

$$\begin{cases} n + m = 0 \\ n = m - 9 \end{cases}$$

Step 5: **Solve the system of equations.**

We will use substitution since the second equation is solved for n.

Substitute $m - 9$ for n in the first equation.

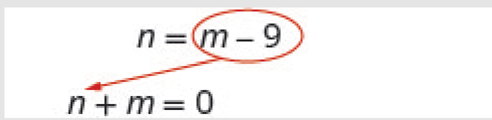


Figure 4.2.7

Solve for m.

$$\begin{aligned} rgb]1.0, 0.0, 0.0 m rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.0 9 + m &= 0 \\ 2m - 9 &= 0 \\ 2m &= 9 \end{aligned}$$

Substitute $m = \dfrac{9}{2}$ into the second equation and then solve for n.

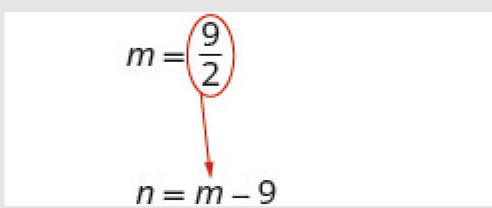


Figure 4.2.8

$$m = rgb]1.0,0.0,0.0\frac{9}{2} - 9$$
$$m = \frac{9}{2} - \frac{18}{2}$$
$$n = -\frac{9}{2}$$

Step 6: **Check the answer in the problem.**

Do these numbers make sense in the problem? We will leave this to you!

Step 7: **Answer the question.**

The numbers are $\frac{9}{2}$ and $-\frac{9}{2}$

Try It

21) The sum of two numbers is 10. One number is 4 less than the other. Find the numbers.

Solution

The numbers are 3 and 7.

22) The sum of two number is -6. One number is 10 less than the other. Find the numbers.

Solution

The numbers are 2 and -8.

In the Example 4.2.10, we'll use the formula for the perimeter of a rectangle, $P = 2L + 2W$.

Example 4.2.10

The perimeter of a rectangle is 88. The length is five more than twice the width. Find the length and the width.

Solution

***Step 1:* Read the problem.**

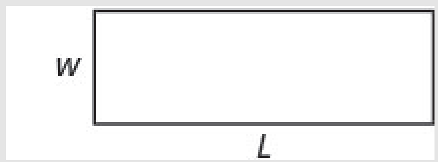


Figure 4.2.9

***Step 2:* Identify what you are looking for.**

We are looking for the length and width.

***Step 3:* Name what we are looking for.**

Let L = the length

Let W = the width

***Step 4:* Translate into a system of equations.**

The perimeter of a rectangle is 88.

$$2L+2W=P$$

$$2L+2W=88$$

The length is five more than twice the width.

$$L=2W+5$$

The system is:

$$\begin{cases} 2L+2W=88 \\ L=2W+5 \end{cases}$$

***Step 5:* Solve the system of equations.**

We will use substitution since the second equation is solved for L.

Substitute $2W + 5$ for L in the first equation.

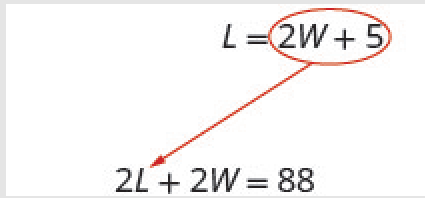


Figure 4.2.10

Solve for W.

$$\begin{aligned} 2(rgb]1.0, 0.0, 0.02rgb]1.0, 0.0, 0.0Wrgb]1.0, 0.0, 0.0+rgb]1.0, 0.0, 0.05) + 2W &= 88 \\ 4W + 10 + 2W &= 88 \\ 6W + 10 &= 88 \\ 6W &= 78 \end{aligned}$$

Substitute $W = 13$ into the second equation and then solve for L.

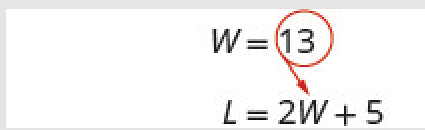


Figure 4.2.11

$$\begin{aligned} L &= 2 \times 13 + 5 \\ L &= 31 \end{aligned}$$

***Step 6:* Check the answer in the problem.**

Does a rectangle with length 31 and width 13 have perimeter 88? Yes.

***Step 7:* Answer the equation.**

The length is 31 and the width is 13.

Try It

23) The perimeter of a rectangle is 40. The length is 4 more than the width. Find the length and width of the rectangle.

Solution

The length is 12 and the width is 8.

24) The perimeter of a rectangle is 58. The length is 5 more than three times the width. Find the length and width of the rectangle.

Solution

The length is 23 and the width is 6.

For Example 4.2.11, we need to remember that the sum of the measures of the angles of a triangle is 180 degrees and that a right triangle has one 90 degree angle.

Example 4.2.11

The measure of one of the small angles of a right triangle is ten more than three times the measure of the other small angle. Find the measures of both angles.

Solution

We will draw and label a figure.

Step 1: **Read the problem.**

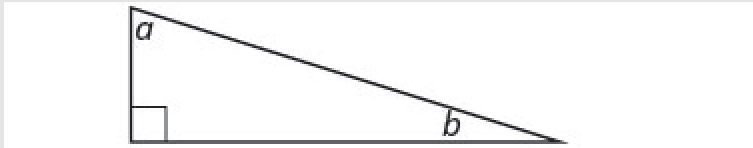


Figure 4.2.12

Step 2: **Identify what you are looking for.**

We are looking for the measures of the angles.

Step 3: **Name what we are looking for.**

Let a = the measure of the 1st angle.

Let b = the measure of the 2nd angle.

Step 4: **Translate into a system of equations.**

The measure of one of the small angles of a right triangle is ten more than three times the measure of the other small angle.

$$a = 3b + 10$$

The sum of the measures of the angles of a triangle is 180.

$$a + b + 90 = 180$$

The system is:

$$\begin{cases} a = 3b + 10 \\ a + b + 90 = 180 \end{cases}$$

Step 5: **Solve the system of equations.**

We will use substitution since the first equation is solved for a.

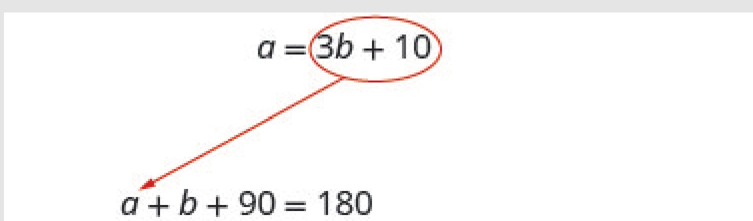


Figure 4.2.13

Substitute $3b + 10$ for a in the second equation.

$$(rgb]1.0,0.0,0.03rgb]1.0,0.0,0.0brgb]1.0,0.0,0.0+rgb]1.0,0.0,0.010) + b + 90 = 180$$

Solve for b.

$$4b + 100 = 180$$
$$4b = 80$$

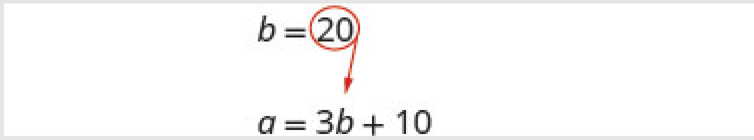


Figure 4.2.14

Substitute $b = 20$ into the first equation and then solve for a.

$$a = 3 \times rgb]1.0, 0.0, 0.020 + 10$$
$$a = 70$$

Step 6: **Check the answer in the problem.**

We will leave this to you!

Step 7: **Answer the question.**

The measures of the small angles are 20 and 70.

Try It

25) The measure of one of the small angles of a right triangle is 2 more than 3 times the measure of the other small angle. Find the measure of both angles.

Solution

The measure of the angles are 22 degrees and 68 degrees.

26) The measure of one of the small angles of a right triangle is 18 less than twice the measure of the other small angle. Find the measure of both angles.

Solution

The measure of the angles are 36 degrees and 54 degrees.

Example 4.2.12

Heather has been offered two options for her salary as a trainer at the gym. Option A would pay her $\$25,000$ plus $\$15$ for each training session. Option B would pay her $\$10,000+\$40$ for each training session. How many training sessions would make the salary options equal?

Solution

Step 1: **Read the problem.**

Step 2: **Identify what you are looking for.**

We are looking for the number of training sessions that would make the pay equal.

Step 3: **Name what we are looking for.**

Let $s=$ Heather's salary.

Let $n=$the number of training sessions

Step 4: **Translate into a system of equations.**

Option A would pay her $\$25,000$ plus 15 for each training session.

$$s=25,000+15n$$

Option B would pay her $\$10,000+\$40$ for each training session

$$s=10,000+40n$$

The system is:

$$\begin{cases} s &= 25,000+15n \\ s &= 10,000+40n \end{cases}$$

Step 5: **Solve the system of equations.**

We will use substitution.

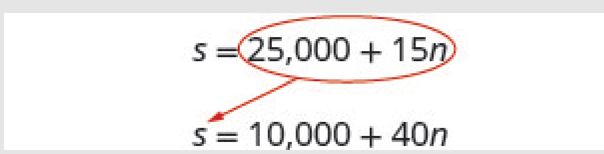


Figure 4.2.15

Substitute $25,000+15n$ for s in the second equation.

$rgb]1.0,0.0,0.025,000rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.015rgb]1.0,0.0,0.0n=10,000+40n$

Solve for n.

$$\begin{aligned} 25,000 &= 10,000+25n \\ 15,000 &= 25n \\ 600 &= n \end{aligned}$$

Step 6: **Check the answer.**

Are 600 training sessions a year reasonable?

Are the two options equal when $n=600$?

Step 7: **Answer the question.**

The salary options would be equal for 600 training sessions.

Try It

27) Geraldine has been offered positions by two insurance companies. The first company pays a salary of $\$12,000$ plus a commission of $\$100$ for each policy sold. The second pays a salary of $\$20,000$ plus a commission of $\$50$ for each policy sold. How many policies would need to be sold to make the total pay the same?

Solution

There would need to be 160 policies sold to make the total pay the same.

28) Kenneth currently sells suits for company A at a salary of $\$22,000$ plus a $\$10$ commission for each suit sold. Company B offers him a position with a salary of $\$28,000$ plus a $\$4$ commission for each suit sold. How many suits would Kenneth need to sell for the options to be equal?

Solution

Kenneth would need to sell $1,000$ suits.

Access these online resources for additional instruction and practice with solving systems of equations by substitution.

Instructional Video-Solve Linear Systems by Substitution

Instructional Video-Solve by Substitution

Key Concepts

- **Solve a system of equations by substitution**
    1. Solve one of the equations for either variable.
    2. Substitute the expression from Step 1 into the other equation.
    3. Solve the resulting equation.
    4. Substitute the solution in Step 3 into one of the original equations to find the other variable.
    5. Write the solution as an ordered pair.
    6. Check that the ordered pair is a solution to both original equations.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=2314#h5p-10

b. After reviewing this checklist, what will you do to become confident for all objectives?

4.3 SOLVE SYSTEMS OF EQUATIONS BY ELIMINATION

Learning Objectives

By the end of this section, you will be able to:

- Solve a system of equations by elimination
- Solve applications of systems of equations by elimination
- Choose the most convenient method to solve a system of linear equations

Try It

Before you get started, take this readiness quiz.

1) Simplify $-5\left(6-3a\right)$.

2) Solve the equation $\frac{1}{3}x+\frac{5}{8}=\frac{31}{24}$.

We have solved systems of linear equations by graphing and by substitution. Graphing works well when the variable coefficients are small and the solution has integer values. Substitution works well when we can easily solve one equation for one of the variables and not have too many fractions in the resulting expression.

The third method of solving systems of linear equations is called the Elimination Method. When we solved a system by substitution, we started with two equations and two variables and reduced it to one equation with one variable. This is what we'll do with the elimination method, too, but we'll have a different way to get there.

Solve a System of Equations by Elimination

The Elimination Method is based on the Addition Property of Equality. The Addition Property of Equality says that when you add the same quantity to both sides of an equation, you still have equality. We will extend the Addition Property of Equality to say that when you add equal quantities to both sides of an equation, the results are equal.

For any expressions a, b, c, and d,

$$\begin{array}{lrcl} \text{if} & a & = & b \\ \text{and} & c & = & d \\ \text{then} & a+c & = & b+d \end{array}$$

To solve a system of equations by elimination, we start with both equations in standard form. Then we decide which variable will be easiest to eliminate. How do we decide? We want to have the coefficients of one variable be opposites, so that we can add the equations together and eliminate that variable.

Notice how that works when we add these two equations together:

$$\begin{array}{rcl} 3x+y & = & 5 \\ 2x-y & = & 0 \\ \hline 5x & = & 5 \end{array}$$

The y's add to zero and we have one equation with one variable.

Let's try another one:

$$\begin{array}{rcl} x+4y & = & 2 \\ 2x+5y & = & -2 \end{array}$$

This time we don't see a variable that can be immediately eliminated if we add the equations.

But if we multiply the first equation by -2, we will make the coefficients of x opposites. We must multiply every term on both sides of the equation by -2.

$$\begin{array}{r} -2(x+4y)=-2(2) \\ 2x+5y=-2 \\ -2x-8y=-4 \\ 2x+5y=-2 \end{array}$$

Now we see that the coefficients of the x terms are opposites, so x will be eliminated when we add these two equations.

Add the equations yourself—the result should be $-3y = -6$. And that looks easy to solve, doesn't it? Here is what it would look like.

$$\begin{cases} -\ 2x - 8y = -4 \\ \underline{\ \ \ 2x + 5y = -2} \end{cases}$$

$$-3y = -6$$

We'll do one more:

$$\begin{aligned} 4x - 3y &= 10 \\ 3x + 5y &= -7 \end{aligned}$$

It doesn't appear that we can get the coefficients of one variable to be opposites by multiplying one of the equations by a constant, unless we use fractions. So instead, we'll have to multiply both equations by a constant.

We can make the coefficients of x be opposites if we multiply the first equation by 3 and the second by -4, so we get $12x$ and $-12x$.

$$\begin{aligned} (4x - 3y) &= 3(10) \\ -4(3x + 5y) &= -4(-7) \end{aligned}$$

This gives us these two new equations:

$$\begin{aligned} 12x - 9y &= 30 \\ -12x + 20y &= 28 \end{aligned}$$

When we add these equations,

$$\begin{aligned} 12x - 9y &= 30 \\ \underline{-12x + 20y} &\underline{= 28} \\ -29 &= 58 \end{aligned}$$

the x's are eliminated and we just have $-29y = 58$.

Once we get an equation with just one variable, we solve it. Then we substitute that value into one of the original equations to solve for the remaining variable. And, as always, we check our answer to make sure it is a solution to both of the original equations.

Now we'll see how to use elimination to solve the same system of equations we solved by graphing and by substitution.

Example 4.3.1

How to Solve a System of Equations by Elimination

Solve the system by elimination.

$$2x + y = 7$$
$$x - 2y = 6$$

Solution

Step 1. Write both equations in standard form. If any coefficients are fractions, clear them.	Both equations are in standard form, $Ax + By = C$. There are no fractions.	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$

Figure 4.3.1

Step 2. Make the coefficients of one variable opposites. Decide which variable you will eliminate. Multiply one or both equations so that the coefficients of that variable are opposites.	We can eliminate the y's by multiplying the first equation by 2.	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$
	Multiply both sides of $2x + y = 7$ by 2.	$\begin{cases} 2(2x + y) = 2(7) \\ x - 2y = 6 \end{cases}$

Figure 4.3.2

Step 3. Add the equations resulting from Step 2 to eliminate one variable.	We add the x's, y's, and constants.	$\begin{cases} 4x + 2y = 14 \\ x - 2y = 6 \end{cases}$ $5x = 20$

Figure 4.3.3

Step 4. Solve for the remaining variable.	Solve for x.	$x = 4$

Figure 4.3.4

Step 5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.	Substitute $x = 4$ into the second equation, $x - 2y = 6$. Then solve for y.	$x - 2y = 6$ $4 - 2y = 6$ $-2y = 2$ $y = -1$

Figure 4.3.5

Step 6. Write the solution as an ordered pair.	Write it as (x, y).	$(4, -1)$

Figure 4.3.6

Step 7. Check that the ordered pair is a solution to **both** original equations.	Substitute $(4, -1)$ into $2x + y = 7$ and $x - 2y = 6$ Do they make both equations true? Yes!	$2x + y = 7$; $x - 2y = 6$ $2(4) + (-1) \stackrel{?}{=} 7$; $4 - 2(-1) \stackrel{?}{=} 6$ $7 = 7$ ✓ ; $6 = 6$ ✓ The solution is $(4, -1)$.

Figure 4.3.7

Try It

3) Solve the system by elimination.

$$\begin{aligned} 3x + y &= 5 \\ 2x - 3y &= 7 \end{aligned}$$

Solution

$(2, -1)$

4) Solve the system by elimination.

$$\begin{aligned} 4x + y &= -5 \\ -2x - 2y &= -2 \end{aligned}$$

Solution

$(-2, 3)$

HOW TO

How to solve a system of equations by elimination.

1. Write both equations in standard form. If any coefficients are fractions, clear them.
2. Make the coefficients of one variable opposites.
   - Decide which variable you will eliminate.
   - Multiply one or both equations so that the coefficients of that variable are opposites.
3. Add the equations resulting from Step 2 to eliminate one variable.
4. Solve for the remaining variable.
5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.
6. Write the solution as an ordered pair.
7. Check that the ordered pair is a solution to both original equations.

First we'll do an example where we can eliminate one variable right away.

Example 4.3.2

Solve the system by elimination.

$$x + y = 10$$
$$x - y = 12$$

Solution

$$\begin{cases} x + y = 10 \\ x - y = 12 \end{cases}$$

Both equations are in standard form.

The coefficients of y are already opposites.

Step 1: **Add the two equations to eliminate y.**

The resulting equation has only 1 variable, x.

$$\begin{cases} x + y &= 10 \\ x - y &= 12 \end{cases}$$
$$2x = 22$$

Step 2: **Solve for x, the remaining variable.**

Step 3: **Substitute $x = 11$ into one of the original equations.**

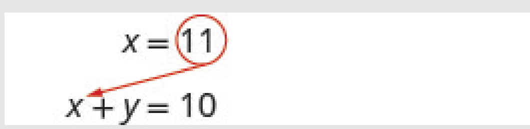


Figure 4.3.8

$$rgb]1.0, 0.0, 0.011 + y = 10$$

Step 4: **Solve for the other variable, y.**

$$y = -1$$

Step 5: **Write the solution as an ordered pair.**

The ordered pair is $(11, -1)$.

***Step 6*. Check that the ordered pair is a solution to both original equations.**

$$\begin{aligned} \text{Equation 1: } x + y &= 10 \\ 11 + (-1) &\stackrel{?}{=} 10 \\ 10 &= 10 \checkmark \end{aligned}$$

$$\begin{aligned} \text{Equation 2: } x - y &= 12 \\ 11 - (-1) &\stackrel{?}{=} 12 \\ 12 &= 12 \checkmark \end{aligned}$$

The solution is $(11, -1)$.

Try It

5) Solve the system by elimination.

$$\begin{aligned} 2x + y &= 5 \\ x - y &= 4 \end{aligned}$$

Solution

$(3, -1)$

6) Solve the system by elimination.

$$\begin{aligned} x + y &= 3 \\ -2x - y &= -1 \end{aligned}$$

Solution

$(-2, 5)$

In Example 4.3.3, we will be able to make the coefficients of one variable opposites by multiplying one equation by a constant.

Example 4.3.3

Solve the system by elimination.

$$3x - 2y = -2$$
$$5x - 6y = 10$$

Solution

$$\begin{cases} 3x - 2y = -2 \\ 5x - 6y = 10 \end{cases}$$

Both equations are in standard form.

None of the coefficients are opposites.

Step 1: **We can make the coefficients of y opposites by multiplying the first equation by -3.**

$$\begin{cases} rgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.03(3x - 2y) = rgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.03(-2) \\ 5x - 6y = 10 \end{cases}$$

Step 2: **Simplify.**

$$\begin{cases} -9x + 6y = 6 \\ 5x - 6y = 10 \end{cases}$$

Step 3: **Add the two equations to eliminate y.**

$$\begin{cases} -9x + 6y = 6 \\ 5x - 6y = 10 \end{cases}$$
$$-4x = 16$$

Step 4: **Solve for the remaining variable, x.**

Step 5: **Substitute $x = -4$ into one of the original equations.**

Figure 4.3.9

$$3(rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.04)-2y=-2$$

Step 6: **Solve for y.**

$$\begin{aligned} -12-2y &= -2 \\ -2y &= 10 \\ y &= -5 \end{aligned}$$

Step 7: **Write the solution as an ordered pair.**

The ordered pair is $(-4,-5)$.

Step 8: **Check that the ordered pair is a solution to both original equations.**

$$\begin{aligned} \text{Equation 1: } 3x-2y &= -2 \\ 3(-4)-2(-5) &\stackrel{?}{=} -2 \\ -12+10 &\stackrel{?}{=} -2 \\ -2 &= -2 \checkmark \end{aligned}$$

$$\begin{aligned} \text{Equation 2: } 5x-6y &= 10 \\ 3(-4)-6(-5) &\stackrel{?}{=} 10 \\ -20+30 &\stackrel{?}{=} 10 \\ 10 &= 10\checkmark \end{aligned}$$

The solution is $(-4,-5)$.

Try It

7) Solve the system by elimination.

$$\begin{aligned} 4x - 3y &= 1 \\ 5x - 9y &= -4 \end{aligned}$$

Solution

$(1, 1)$

8) Solve the system by elimination.

$$\begin{aligned} 3x + 2y &= 2 \\ 6x + 5y &= 8 \end{aligned}$$

Solution

$(-2, 4)$

Now we'll do an example where we need to multiply both equations by constants in order to make the coefficients of one variable opposites.

Example 4.3.4

Solve the system by elimination. $\begin{aligned} 4x - 3y &= 9 \\ 7x + 2y &= -6 \end{aligned}$

Solution

In this example, we cannot multiply just one equation by any constant to get opposite coefficients. So we will strategically multiply both equations by a constant to get the opposites.

$$\begin{cases} 4x - 3y = 9 \\ 7x + 2y = -6 \end{cases}$$

Step 1: **Both equations are in standard form. To get opposite coefficients of y, we will multiply the first equation by 2 and the second equation by 3.**

$$\begin{cases} rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.03(3x - 2y) = rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.03(-2) \\ 5x - 6y = 10 \end{cases}$$

Step 2: **Simplify.**

$$\begin{cases} 8x - 6y = 18 \\ 21x + 6y = -18 \end{cases}$$

Step 3: **Add the two equations to eliminate y.**

$$\begin{cases} 8x - 6y = 18 \\ 21x + 6y = -18 \end{cases}$$
$$39x = 0$$

Step 4: **Solve for x.**

Step 5: **Substitute $x = 0$ into one of the original equations.**

Figure 4.3.10

$$7 \times rgb]1.0,0.0,0.00 + 2y = -6$$

Step 6: **Solve for y.**

$$2y = -6$$
$$y = -3$$

Step 7: **Write the solution as an ordered pair.**

The ordered pair is $(0, -3)$.

Step 8: **Check that the ordered pair is a solution to both original equations.**

$$\begin{array}{rcl} 4x-3y & = & 9 \\ 4(0)-3(-3) & \stackrel{?}{=} & 9 \\ 9 & = & 9\checkmark \end{array} \qquad \begin{array}{rcl} 7x+2y & = & -6 \\ 7(0)+2(-3) & \stackrel{?}{=} & -6 \\ -6 & = & -6\checkmark \end{array}$$

The solution is $(0,-3)$.

What other constants could we have chosen to eliminate one of the variables? Would the solution be the same?

Try It

9) Solve the system by elimination.

$$c3x-4y=-9$$
$$5x+3y=14$$

Solution

$(1,3)$

10) Solve the system by elimination.

$$c7x+8y=4$$
$$3x-5y=27$$

Solution

$(4,-3)$

When the system of equations contains fractions, we will first clear the fractions by multiplying each equation by its LCD.

Example 4.3.5

Solve the system by elimination.

$$x + \frac{1}{2}y = 6$$
$$\frac{3}{2}x + \frac{2}{3}y = \frac{17}{2}$$

Solution

In this example, both equations have fractions. Our first step will be to multiply each equation by its LCD to clear the fractions.

$$\begin{cases} x + \frac{1}{2}y = 6 \\ \frac{3}{2}x + \frac{2}{3}y = \frac{17}{2} \end{cases}$$

Step 1: **To clear the fractions, multiply each equation by its LCD.**

$$\begin{cases} rgb]1.0, 0.0, 0.02(x + \frac{1}{2}y) = rgb]1.0, 0.0, 0.02(6) \\ rgb]1.0, 0.0, 0.06(\frac{3}{2}x + \frac{2}{3}y) = rgb]1.0, 0.0, 0.06(\frac{17}{2}) \end{cases}$$

Step 2: **Simplify.**

$$\begin{cases} 2x + y = 12 \\ 9x + 4y = 51 \end{cases}$$

Now we are ready to eliminate one of the variables. Notice that both equations are in standard form.

Step 3: **We can eliminate y multiplying the top equation by -4.**

$$\begin{cases} rgb]1.0, 0.0, 0.0{-}rgb]1.0, 0.0, 0.04(2x + y) = rgb]1.0, 0.0, 0.0{-}rgb]1.0, 0.0, 0.04(12) \\ 9x + 4y = 51 \end{cases}$$

Step 4: **Simplify and add.**

Step 5: **Substitute $x = 3$ into one of the original equations.**

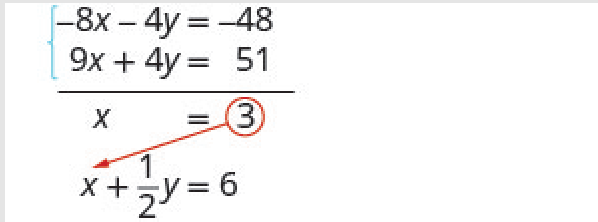


Figure 4.3.11

Step 6: **Solve for y.**

$$\begin{aligned} rgb]1.0,0.0,0.03 + \frac{1}{2}y &= 6 \\ \frac{1}{2}y &= 3 \\ y &= 6 \end{aligned}$$

Step 7: **Write the solution as an ordered pair.**

The ordered pair is $(3, 6)$.

Step 8: **Check that the ordered pair is a solution to both original equations.**

$$\begin{aligned} \text{Equation 1: } x + \frac{1}{2}y &= 6 \\ 3 + \frac{1}{2}(6) &\stackrel{?}{=} 6 \\ 3 + 3 &= 6 \\ 6 &= 6 \\ \text{Equation 2: } \frac{3}{2}x + \frac{2}{3}y &= \frac{17}{2} \\ \frac{3}{2}(3) + \frac{2}{3}(6) &\stackrel{?}{=} \frac{17}{2} \\ \frac{9}{2} + 4 &\stackrel{?}{=} \frac{17}{2} \\ \frac{9}{2} + \frac{8}{2} &\stackrel{?}{=} \frac{17}{2} \\ \frac{17}{2} &= \frac{17}{2} \end{aligned}$$

The solution is $(3, 6)$.

Try It

11) Solve the system by elimination.

$$\frac{1}{3}x - \frac{1}{2}y = 1$$
$$\frac{3}{4}x - y = \frac{5}{2}$$

Solution

$(6, 2)$

12) Solve the system by elimination.

$$x + \frac{3}{5}y = -\frac{1}{5}$$
$$-\frac{1}{2}x - \frac{2}{3}y = \frac{5}{6}$$

Solution

$(1, -2)$

In the Solving Systems of Equations by Graphing we saw that not all systems of linear equations have a single ordered pair as a solution. When the two equations were really the same line, there were infinitely many solutions. We called that a consistent system. When the two equations described parallel lines, there was no solution. We called that an inconsistent system.

Example 4.3.6

Solve the system by elimination.
$$\begin{aligned} 3x + 4y &= 12 \\ y &= 3 - \frac{3}{4}x \end{aligned}$$

Solution

$$\begin{cases} 23x + 4y = 12 \\ y = 3 - \frac{3}{4}x \end{cases}$$

Step 1: **Write the second equation in standard form.**

$$\begin{cases} 23x + 4y = 12 \\ y = 3 - \frac{3}{4}x \end{cases}$$

Step 2: **Clear the fractions by multiplying the second equation by 4.**

$$\begin{cases} 3x + 4y = 12 \\ rgb]1.0,0.0,0.04\left(\frac{3}{4}x + y\right) = rgb]1.0,0.0,0.04(3) \end{cases}$$

Step 3: **Simplify.**

$$\begin{cases} 3x + 4y = 12 \\ 3x + 4y = 12 \end{cases}$$

Step 4: **To eliminate a variable, we multiply the second equation by -1.**

Step 5: **Simplify and add.**

$$\begin{cases} 3x + 4y = 12 \\ \underline{-3x - 4y = 12} \end{cases}$$
$$0 = 0$$

This is a true statement. The equations are consistent but dependent. Their graphs would be the same line. The system has infinitely many solutions.

After we cleared the fractions in the second equation, did you notice that the two equations were the same? That means we have coincident lines.

Try It

13) Solve the system by elimination.

$$\begin{aligned} 5x - 3y &= 15 \\ y &= -5 + \frac{5}{3}x \end{aligned}$$

Solution

Infinitely many solutions.

14) Solve the system by elimination.

$$\begin{aligned} x + 2y &= 6 \\ y &= -\frac{1}{2}x + 3 \end{aligned}$$

Solution

Infinitely many solutions.

Example 4.3.7

Solve the system by elimination. $\begin{aligned} -6x + 15y &= 10 \\ 2x - 5y &= -5 \end{aligned}$

Solution

***Step 1:* The equations are in standard form.**

$$\begin{cases} -6x + 15y = 10 \\ 2x - 5y = -5 \end{cases}$$

Step 2: **Multiply the second equation by 3 to eliminate a variable.**

$$\begin{cases} -6x + 15y = 10 \\ rgb]1.0, 0.0, 0.03(2x - 5y) = rgb]1.0, 0.0, 0.03(-5) \end{cases}$$

Step 3: **Simplify and add.**

$$\begin{cases} -6x + 15y = 10 \\ \underline{-6x - 15y = -15} \end{cases}$$
$$0 \neq -5$$

This statement is false. The equations are inconsistent and so their graphs would be parallel lines.

The system does not have a solution.

Try It

15) Solve the system by elimination.

$$-3x + 2y = 8$$
$$9x - 6y = 13$$

Solution

No solution.

16) Solve the system by elimination.

$$7x - 3y = -2$$
$$-14x + 6y = 8$$

Solution

No solution.

Solve Applications of Systems of Equations by Elimination

Some applications problems translate directly into equations in standard form, so we will use the elimination method to solve them. As before, we use our Problem Solving Strategy to help us stay focused and organized.

Example 4.3.8

The sum of two numbers is 39. Their difference is 9. Find the numbers.

Solution

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

We are looking for two numbers.

Step 3: **Name what we are looking for.**

Choose a variable to represent that quantity.

Let $n =$ the first number.

Let $m =$ the second number.

Step 4: **Translate into a system of equations.**

The system is:

The sum of two numbers is 39.

$$n + m = 39$$

Their difference is 9.

$$\begin{aligned} n - m &= 9 \\ n + m &= 39 \\ n - m &= 9 \end{aligned}$$

Step 5: **Solve the system of equations.**

To solve the system of equations, use elimination.

The equations are in standard form and the coefficients of m are opposites. Add.

Solve for n.

Substitute $n = 24$ into one of the original equations and solve for m.

$$\begin{cases} n + m = 39 \\ \underline{-n - m = 9} \end{cases}$$
$$2n = 48$$

$$n = 24$$
$$n + m = 39$$
$$24 + m = 39$$
$$m = 15$$

Step 6: **Check the answer.**

Since $24 + 15 = 39$ and $24 - 15 = 9$, the answers check.

Step 7: **Answer the question.**

The numbers are 24 and 15.

Try It

17) The sum of two numbers is 42. Their difference is 8. Find the numbers.

Solution

The numbers are 25 and 17.

18) The sum of two numbers is -15. Their difference is -35. Find the numbers.

Solution

The numbers are -25 and 10.

Example 4.3.9

Joe stops at a burger restaurant every day on his way to work. Monday he had one order of medium fries and two small sodas, which had a total of 620 calories. Tuesday he had two orders of medium fries and one small soda, for a total of 820 calories. How many calories are there in one order of medium fries? How many calories in one small soda?

Solution

***Step 1:* Read the problem.**

***Step 2:* Identify what we are looking for.**

We are looking for the number of calories in one order of medium fries and in one small soda.

***Step 3:* Name what we are looking for.**

Let f = the number of calories in 1 order of medium fries.

Let s = the number of calories in 1 small soda.

***Step 4:* Translate into a system of equations:**

Our system is:

one medium fries and two small sodas had a total of 620 calories

$$f + 2s = 620$$

two medium fries and one small soda had a total of 820 calories.

$$2f + s = 820$$

$$\begin{cases} f + 2s = 620 \\ 2f + s = 820 \end{cases}$$

***Step 5:* Solve the system of equations.**

To solve the system of equations, use elimination. The equations are in standard form. To get opposite coefficients of f, multiply the top equation by -2.Simplify and add.Solve for s .Substitute $s = 140$ into one of the original equations and then solve for f.

$$\begin{cases} rgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.02(f + 2s) = rgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.02(620) \\ 2f + s = 820 \end{cases}$$

$$\begin{cases} -2f - 4s = -1240 \\ 2f + s = 820 \end{cases}$$
$$-3s = -420$$

$$\begin{aligned} rgb]1.0,0.0,0.0s &= rgb]1.0,0.0,0.0140 \\ f + 2s &= 620 \\ f + 2 \times rgb]1.0,0.0,0.0140 &= 620 \\ f + 280 &= 620 \\ f &= 340 \end{aligned}$$

Step 6: **Check the answer.**

Verify that these numbers make sense in the problem and that they are solutions to both equations. We leave this to you!

Step 7: **Answer the question.**

The small soda has 140 calories and the fries have 340 calories.

Try It

19) Malik stops at the grocery store to buy a bag of diapers and 2 cans of formula. He spends a total of $\$37$. The next week he stops and buys 2 bags of diapers and 5 cans of formula for a total of $\$87$. How much does a bag of diapers cost? How much is one can of formula?

Solution

The bag of diapers costs $\$11$ and the can of formula costs $\$13$.

20) To get her daily intake of fruit for the day, Sasha eats a banana and 8 strawberries on Wednesday for a calorie count of 145. On the following Wednesday, she eats two bananas and 5 strawberries for a total of 235 calories for the fruit. How many calories are there in a banana? How many calories are in a strawberry?

Solution

There are 105 calories in a banana and 5 calories in a strawberry.

Choose the Most Convenient Method to Solve a System of Linear Equations

When you will have to solve a system of linear equations in a later math class, you will usually not be told which method to use. You will need to make that decision yourself. So you'll want to choose the method that is easiest to do and minimizes your chance of making mistakes.

Graphing	Substitution	Elimination
Use when you need a picture of the situation.	Use when one equation is already solved for one variable.	Use when the equations are in standard form.

Figure 4.1.1

Example 4.3.10

For each system of linear equations decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

a.

$$\begin{aligned} 3x + 8y &= 40 \\ 7x - 4y &= -32 \end{aligned}$$

b.

$$\begin{aligned} 5x + 6y &= 12 \\ y &= \frac{2}{3}x - 1 \end{aligned}$$

Solution

a.

$$\begin{aligned}3x+8y&=40\\7x-4y&=-32\end{aligned}$$

Since both equations are in standard form, using elimination will be most convenient.

---

b.

$$\begin{aligned}5x+6y&=12\\y&=\frac{2}{3}x-1\end{aligned}$$

Since one equation is already solved for y, using substitution will be most convenient.

Try It

21) For each system of linear equations, decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

a.

$$\begin{aligned}4x-5y&=-32\\3x+2y&=-1\end{aligned}$$

b.

$$\begin{aligned}x&=2y-1\\3x-5y&=-7\end{aligned}$$

Solution

a. Since both equations are in standard form, using elimination will be most convenient.

b. Since one equation is already solved for x, using substitution will be most convenient.

22) For each system of linear equations, decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

a.

$$\begin{aligned} y &= 2x - 1 \\ 3x - 4y &= -6 \end{aligned}$$

b.

$$\begin{aligned} 6x - 2y &= 12 \\ 3x + 7y &= -13 \end{aligned}$$

Solution

a. Since one equation is already solved for y, using substitution will be most convenient

b. Since both equations are in standard form, using elimination will be most convenient.

Access these online resources for additional instruction and practice with solving systems of linear equations by elimination.

- Instructional Video-Solving Systems of Equations by Elimination
- Instructional Video-Solving by Elimination
- Instructional Video-Solving Systems by Elimination

Key Concepts

- **To Solve a System of Equations by Elimination**
    1. Write both equations in standard form. If any coefficients are fractions, clear them.

2. Make the coefficients of one variable opposites.
    - Decide which variable you will eliminate.
    - Multiply one or both equations so that the coefficients of that variable are opposites.
3. Add the equations resulting from Step 2 to eliminate one variable.
4. Solve for the remaining variable.
5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.
6. Write the solution as an ordered pair.
7. Check that the ordered pair is a solution to both original equations.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=2369#h5p-12

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

4.4 SOLVE APPLICATIONS WITH SYSTEMS OF EQUATIONS

Learning Objectives

By the end of this section, you will be able to:

- Translate to a system of equations
- Solve direct translation applications
- Solve geometry applications
- Solve uniform motion applications

Try It

Before you get started, take this readiness quiz:

1) The sum of twice a number and nine is 31. Find the number.
2) Twins Jon and Ron together earned $\$96,000$ last year. Ron earned $\$8,000$ more than three times what Jon earned. How much did each of the twins earn?
3) Alessio rides his bike $3\frac{1}{2}$ hours at a rate of 10 miles per hour. How far did he ride?

Previously in this chapter we solved several applications with systems of linear equations. In this section, we'll look at some specific types of applications that relate two quantities. We'll translate the words into linear equations, decide which is the most convenient method to use, and then solve them.

We will use our Problem Solving Strategy for Systems of Linear Equations.

HOW TO

Use a problem solving strategy for systems of linear equations.

1. Read the problem. Make sure all the words and ideas are understood.
2. Identify what we are looking for.
3. Name what we are looking for. Choose variables to represent those quantities.
4. Translate into a system of equations.
5. Solve the system of equations using good algebra techniques.
6. Check the answer in the problem and make sure it makes sense.
7. Answer the question with a complete sentence.

Translate to a System of Equations

Many of the problems we solved in earlier applications related two quantities. Here are two of the examples from the chapter on **Math Models**.

- The sum of two numbers is negative fourteen. One number is four less than the other. Find the numbers.
- A married couple together earns $110,000 a year. The wife earns $16,000 less than twice what her husband earns. What does the husband earn?

In that chapter, we translated each situation into one equation using only one variable. Sometimes it was a bit of a challenge figuring out how to name the two quantities, wasn't it?

Let's see how we can translate these two problems into a system of equations with two variables. We'll focus on Steps 1 through 4 of our Problem Solving Strategy.

Example 4.4.1

How to Translate to a System of Equations

Translate to a system of equations:

The sum of two numbers is negative fourteen. One number is four less than the other. Find the numbers.

Solution

Step 1. Read the problem. Make sure you understand all the words and ideas.	This is a number problem.	The sum of two numbers is negative fourteen. One number is four less than the other. Find the numbers.

Figure 4.4.1

Step 2. Identify what you are looking for.	"Find the numbers."	We are looking for 2 numbers.

Figure 4.4.2

Step 3. Name what you are looking for. Choose variables to represent those quantities.	We will use two variables, m and n.	Let $m =$ one number $n =$ second number

Figure 4.4.3

Step 4. Translate into a system of equations.	We will write one equation for each sentence.	$\underbrace{\text{The sum of the numbers}}_{m+n}\ \underbrace{\text{is}}_{=}\ \underbrace{-14}_{-14}$ $\underbrace{\text{One number}}_{m}\ \underbrace{\text{is}}_{=}\ \underbrace{\text{four less than the other}}_{n-4}$ The system is: $\begin{cases} m+n=-14 \\ m=n-4 \end{cases}$

Figure 4.4.4

Try It

4) Translate to a system of equations: The sum of two numbers is negative twenty-three. One number is 7 less than the other. Find the numbers.

Solution

$$\begin{cases} m+n=-23 \\ m=n-7 \end{cases}$$

5) Translate to a system of equations: The sum of two numbers is negative eighteen. One number is 40 more than the other. Find the numbers.

Solution

$$\begin{cases} m+n=-18 \\ m=n+40 \end{cases}$$

We'll do another example where we stop after we write the system of equations.

Example 4.4.2

Translate to a system of equations:

A married couple together earns $\$110,000$ a year. The wife earns $\$16,000$ less than twice what her husband earns. What does the husband earn?

Solution

***Step 1:* We are looking for the amount that the husband and wife each earn.**

Let $h =$ the amount the husband earns.

Let $w =$ the amount the wife earns.

***Step 2:* Translate.**

A married couple together earns $\$110,000$.

$$w + h = 110,000$$

The wife earns $\$16,000$ less than twice what the husband earns.

$$w = 2h - 16,000$$

***Step 3:* The system of equations is:**

$$\begin{cases} w + h = 110,000 \\ w = 2h - 16,000 \end{cases}$$

Try It

6) Translate to a system of equations: A couple has a total household income of $84,000. The husband earns $18,000 less than twice what the wife earns. How much does the wife earn?

Solution

$$\begin{cases} w + h = 84,000 \\ h = 2w - 18,000 \end{cases}$$

7) Translate to a system of equations: A senior employee makes $5 less than twice what a new employee makes per hour. Together they make $43 per hour. How much does each employee make per hour?

Solution

$$\begin{cases} s = 2n - 5 \\ s + n = 43 \end{cases}$$

Solve Direct Translation Applications

We set up but did not solve, the systems of equations in Examples 4.2.1 and 4.2.2 Now we'll translate a situation to a system of equations and then solve it.

Example 4.4.3

Translate to a system of equations and then solve: Devon is 26 years older than his son Cooper. The sum of their ages is 50. Find their ages.

Solution

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

We are looking for the ages of Devon and Cooper.

Step 3: **Name what we are looking for.**

Let $d =$ Devon's age.

Let $c =$ Cooper's age

Step 4: **Translate into a system of equations.**

Devon is 26 years older than Cooper.

$$d = c + 26$$

The sum of their ages is 50.

$$d + c = 50$$

The system is:

$$\begin{cases} d = c + 26 \\ d + c = 50 \end{cases}$$

***Step 5:* Solve the system of equations.**

Solve by substitution.

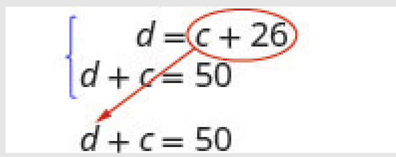


Figure 4.4.5

Substitute $c + 26$ into the second equation.

$$rgb]1.0,0.0,0.0c rgb]1.0,0.0,0.0+ rgb]1.0,0.0,0.026 + c = 50$$

Solve for c.

$$\begin{array}{rrl} & 2c + 26 &= 50 \\ & 2c &= 24 \\ & rgb]1.0,0.0,0.0c &= rgb]1.0,0.0,0.012 \\ \text{Substitute c = 12.} & d &= rgb]1.0,0.0,0.0c + 26 \\ \text{Solve for d.} & d &= rgb]1.0,0.0,0.012 + 26 \\ & d &= 38 \end{array}$$

***Step 6:* Check the answer in the problem.**

Is Devon's age 26 more than Cooper's?
Yes, 38 is 26 more than 12.
Is the sum of their ages 50?
Yes, 38 plus 12 is 50.

***Step 7:* Answer the question.**

Devon is 38 and Cooper is 12 years old.

Try It

8) Translate to a system of equations and then solve: Ali is 12 years older than his youngest sister, Jameela. The sum of their ages is 40. Find their ages.

Solution
Ali is 28 and Jameela is 16.

9) Translate to a system of equations and then solve: Jake's dad is 6 more than 3 times Jake's age. The sum of their ages is 42. Find their ages.

Solution
Jake is 9 and his dad is 33.

Example 4.4.4

Translate to a system of equations and then solve:

When Jenna spent 10 minutes on the elliptical trainer and then did circuit training for 20 minutes, her fitness app says she burned 278 calories. When she spent 20 minutes on the elliptical trainer and 30 minutes circuit training she burned 473 calories. How many calories does she burn for each minute on the elliptical trainer? How many calories does she burn for each minute of circuit training?

Solution
Step 1: **Read the problem.**
Step 2: **Identify what we are looking for.**

> We are looking for the number of calories burned each minute on the elliptical trainer and each minute of circuit training.

Step 3: **Name what we are looking for.**

Let $e =$ number of calories burned per minute on the elliptical trainer.

Let $c =$ number of calories burned per minute while circuit training.

Step 4: **Translate into a system of equations.**

10 minutes on the elliptical and circuit training for 20 minutes, burned 278 calories

$$10e + 20c = 278$$

20 minutes on the elliptical and 30 minutes of circuit training burned 473 calories

$$20e + 30c = 473$$

The system is:

$$\begin{cases} 10e + 20c = 278 \\ 20e + 30c = 473 \end{cases}$$

Step 5: **Solve the system of equations.**

Multiply the first equation by -2 to get opposite coefficients of e.

$$\begin{cases} rgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.02(10e + 20c) = rgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.02(278) \\ 20e + 30c = 473 \end{cases}$$

Simplify and add the equations.

Solve for c.

$$\begin{cases} -20e - 40c = -556 \\ 20e + 30c = 473 \end{cases}$$
$$-10c = -83$$
$$c = 8.3$$

Substitute $c = 8.3$ into one of the original equations to solve for e.

$$\begin{aligned} 10e + 20rgb]1.0, 0.0, 0.0c &= 278 \\ 10e + 20(rgb]1.0, 0.0, 0.08rgb]1.0, 0.0, 0.0.rgb]1.0, 0.0, 0.03) &= 278 \\ 10e + 166 &= 278 \\ 10e &= 112 \\ e &= 11.2 \end{aligned}$$

Step 6: **Check the answer in the problem.**

Check the math on your own.

$$\begin{cases} 10(11.2)+20(8.3)\stackrel{?}{=}278 \\ 20(11.2)+30(8.3)\stackrel{?}{=}473 \end{cases}$$

***Step 7:* Answer the question.**

Jenna burns 8.3 calories per minute circuit training and 11.2 calories per minute while on the elliptical trainer.

Try It

10) *Translate to a system of equations and then solve:* Mark went to the gym and did 40 minutes of Bikram hot yoga and 10 minutes of jumping jacks. He burned 510 calories. The next time he went to the gym, he did 30 minutes of Bikram hot yoga and 20 minutes of jumping jacks burning 470 calories. How many calories were burned for each minute of yoga? How many calories were burned for each minute of jumping jacks?

Solution

Mark burned 11 calories for each minute of yoga and 7 calories for each minute of jumping jacks.

11) *Translate to a system of equations and then solve:* Erin spent 30 minutes on the rowing machine and 20 minutes lifting weights at the gym and burned 430 calories. During her next visit to the gym she spent 50 minutes on the rowing machine and 10 minutes lifting weights and burned 600 calories. How many calories did she burn for each minutes on the rowing machine? How many calories did she burn for each minute of weight lifting?

Solution

Erin burned 11 calories for each minute on the rowing machine and 5 calories for each minute of weight lifting.

Solve Geometry Applications

When we learned about **Math Models**, we solved geometry applications using properties of triangles and rectangles. Now we'll add to our list some properties of angles.

The measures of two **complementary angles** add to 90 degrees. The measures of two **supplementary angles** add to 180 degrees.

Complementary and Supplementary Angles

Two angles are **complementary** if the sum of the measures of their angles is 90 degrees.
Two angles are **supplementary** if the sum of the measures of their angles is 180 degrees.

If two angles are complementary, we say that *one angle is the **complement** of the other.*
If two angles are supplementary, we say that *one angle is the **supplement** of the other.*

Example 4.4.5

Translate to a system of equations and then solve: The difference of two complementary angles is 26 degrees. Find the measures of the angles.

Solution

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

We are looking for the measure of each angle.

Step 3: **Name what we are looking for.**

Let $x =$ the measure of the first angle.

Let m =the measure of the second angle.

***Step 4:* Translate into a system of equations.**

The angles are complementary.

$$x + y = 90$$

The difference of the two angles is *26* degrees.

$$x - y = 26$$

The system is

$$\begin{cases} x + y = 90 \\ x - y = 26 \end{cases}$$

***Step 5:* Solve the system of equations by elimination.**

$$\begin{cases} x + y = 90 \\ x - y = 26 \end{cases}$$

$$\begin{aligned} 2x &= 116 \\ x &= 158 \end{aligned}$$

Substitute $x = 58$ into the first equation.

$$\begin{aligned} x + y &= 90 \\ 58 + y &= 90 \\ y &= 32 \end{aligned}$$

***Step 6:* Check the answer in the problem.**

$$\begin{aligned} 58 + 32 &= 90 \checkmark \\ 58 - 32 &= 26 \checkmark \end{aligned}$$

***Step 7:* Answer the question.**

The angle measures are 58 degrees and 32 degrees.

Try It

12) Translate to a system of equations and then solve: The difference of two complementary angles is 20 degrees. Find the measures of the angles.

Solution

The angle measures are 55 degrees and 35 degrees.

13) Translate to a system of equations and then solve: The difference of two complementary angles is 80 degrees. Find the measures of the angles.

Solution

The angle measures are 5 degrees and 85 degrees.

Example 4.4.6

Translate to a system of equations and then solve:

Two angles are supplementary. The measure of the larger angle is twelve degrees less than five times the measure of the smaller angle. Find the measures of both angles.

Solution

Step 1: **Read the problem.**

Step 2: **Identify what we are looking for.**

We are looking for the measure of each angle.

Step 3: **Name what we are looking for.**

Let $x =$ the measure of the first angle.

Let $y =$ the measure of the second angle

Step 4: **Translate into a system of equations.**

The angles are supplementary.

$$x + y = 180$$

The larger angle is twelve less than five times the smaller angle

$$y = 5x - 12$$

The system is:

$$\begin{cases} x + y = 180 \\ y = 5x - 12 \end{cases}$$

Step 5: **Solve the system of equations substitution.**

Figure 4.4.6

Substitute $5x - 12$ for y in the first equation.

$$x + rgb]1.0, 0.0, 0.05rgb]1.0, 0.0, 0.0xrgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.012 = 180$$

Solve for x.

$$\begin{aligned} 6x - 12 &= 180 \\ 6x &= 192 \\ \text{Substitute in x=32.} \qquad y &= 5rgb]1.0, 0.0, 0.0x - 12 \\ \text{Solve for y.} \qquad y &= 5 \times rgb]1.0, 0.0, 0.032 - 12 \\ y &= 160 - 12 \\ y &= 148 \end{aligned}$$

Step 6: **Check the answer in the problem.**

$$\begin{aligned} 32 + 158 &= 180 \checkmark \\ 5 \times 32 - 12 &= 147 \checkmark \end{aligned}$$

Step 7: **Answer the question.**

The angle measures are 148 and 32.

Try It

14) *Translate to a system of equations and then solve:* Two angles are supplementary. The measure of the larger angle is 12 degrees more than three times the smaller angle. Find the measures of the angles.

Solution

The angle measures are 42 degrees and 138 degrees.

15) *Translate to a system of equations and then solve:* Two angles are supplementary. The measure of the larger angle is 18 less than twice the measure of the smaller angle. Find the measures of the angles.

Solution

The angle measures are 66 degrees and 114 degrees.

Example 4.4.7

Translate to a system of equations and then solve:

Randall has 125 feet of fencing to enclose the rectangular part of his backyard adjacent to his house. He will only need to fence around three sides, because the fourth side will be the wall of the house. He wants the length of the fenced yard (parallel to the house wall) to be 5 feet more than four times as long as the width. Find the length and the width.

Solution

Step 1: **Read the problem.**

Step 2: **Identify what you are looking for.**

We are looking for the length and width.

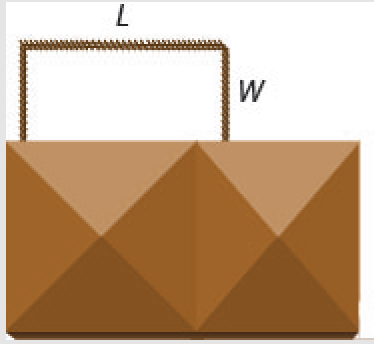


Figure 4.4.7

Step 3: **Name what we are looking for.**

Let L = the length of the fenced yard.

Let W =the width of the fenced yard

Step 4: **Translate into a system of equations.**

One length and two widths equal 125.

$$L+2W=125$$

The length will be 5 feet more than four times the width.

$$L=4W+5$$

The system is:

$$\begin{cases} L+2W=125 \\ L=4W+5 \end{cases}$$

Step 5: **Solve the system of equations by substitution.**

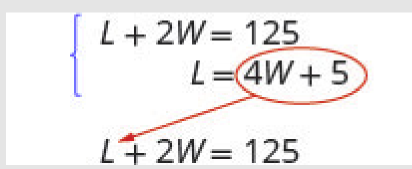


Figure 4.4.8

Substitute $L=4W+5$ into the first equation, then solve for W.

$$\begin{aligned} rgb]1.0,0.0,0.04rgb]1.0,0.0,0.0W rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.05+2W &= 125 \\ 6W+5 &= 125 \\ 6W &= 120 \end{aligned}$$

Substitute 20 for W in the second equation, then solve for L.

$$\begin{array}{rl} \text{Substitute W=20.} & L = 4rgb]1.0, 0.0, 0.0W + 5 \\ \text{Solve for L.} & L = 4 \times rgb]1.0, 0.0, 0.020 + 5 \\ & L = 80 + 5 \\ & L = 85 \end{array}$$

Step 6: **Check the answer in the problem.**

$$20 + 28 + 20 = 125\checkmark$$
$$85 = 4 \times 20 + 5\checkmark$$

Step 7: **Answer the equation.**

The length is 85 feet and the width is 20 feet.

Try It

16) *Translate to a system of equations and then solve:* Mario wants to put a rectangular fence around the pool in his backyard. Since one side is adjacent to the house, he will only need to fence three sides. There are two long sides and the one shorter side is parallel to the house. He needs 155 feet of fencing to enclose the pool. The length of the long side is 10 feet less than twice the width. Find the length and width of the pool area to be enclosed.

Solution

The length is 60 feet and the width is 35 feet.

17) *Translate to a system of equations and then solve:* Alexis wants to build a rectangular dog run in her yard adjacent to her neighbour's fence. She will use 136 feet of fencing to completely enclose the rectangular dog run. The length of the dog run along the neighbour's fence will be 16 feet less than twice the width. Find the length and width of the dog run.

Solution

The length is 60 feet and the width is 38 feet.

Solve Uniform Motion Applications

We used a table to organize the information in uniform motion problems when we introduced them earlier. We'll continue using the table here. The basic equation was $D = rt$ where D is the distance travelled, r is the rate, and t is the time.

Our first example of a uniform motion application will be for a situation similar to some we have already seen, but now we can use two variables and two equations.

Example 4.4.8

Translate to a system of equations and then solve:

Joni left St. Louis on the interstate, driving west towards Denver at a speed of 65 miles per hour. Half an hour later, Kelly left St. Louis on the same route as Joni, driving 78 miles per hour. How long will it take Kelly to catch up to Joni?

Solution

A diagram is useful in helping us visualize the situation.

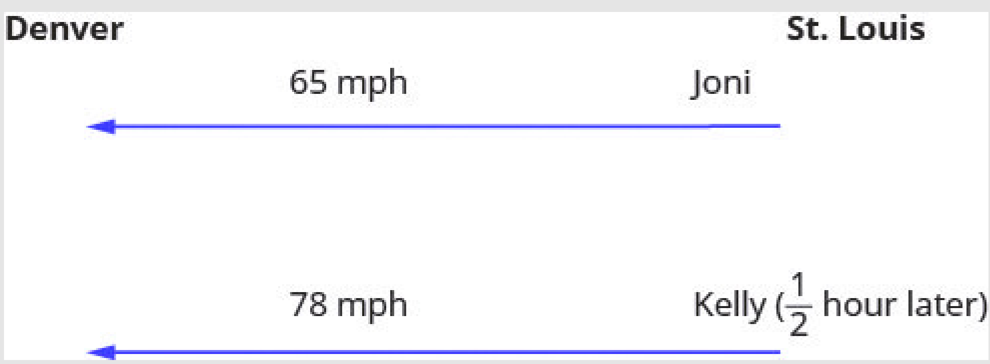


Figure 4.4.9

Step 1: **Identify and name what we are looking for.**

A chart will help us organize the data. We know the rates of both Joni and Kelly, and so we enter them in the chart.

We are looking for the length of time Kelly, k, and Joni, j, will each drive. Since $D = r \cdot t$ we can fill in the Distance column.

Type	Rate x Time = Distance		
Joni	65	j	65j
Kelly	78	k	78k

Step 2: **Translate into a system of equations.**

To make the system of equations, we must recognize that Kelly and Joni will drive the same distance. So, $65j = 78k$.

Also, since Kelly left later, her time will be $\frac{1}{2}$ hour less than Joni's time.

So, $k = j - \frac{1}{2}$

Now we have the system.

$$\begin{cases} k = j - \frac{1}{2} \\ 65j = 78k \end{cases}$$

Step 3: **Solve the system of equations by substitution.**

$$65j = 78k$$

Substitute $k = j - \frac{1}{2}$ into the second equation, then solve for j.

$$65j = 78(j - \frac{1}{2})$$
$$65j = 78j - 39$$
$$-13j = -39$$
$$rgb]1.0, 0.0, 0.0j = rgb]1.0, 0.0, 0.03$$

To find Kelly's time, substitute $j = 3$ into the first equation, then solve for k.

\begin{eqnarray*}\text{Solve for k.}\operatorname{ }\;k&=&{\color[rgb]{1.0, 0.0, 0.0}j}-\frac12\\k&=&{\color[rgb]{1.0, 0.0, 0.0}3}-\frac12\\k&=&\frac52\;or\;k&=&2\frac12\\\\\end{eqnarray*}

Step 4: **Check the answer in the problem.**

Joni 3 hours (65mph) = 195 miles.

Kelly $2\frac{1}{2}$ hours (78 mph) = 195 miles.

Yes, they will have travelled the same distance when they meet.

***Step 5:* Answer the question.**

Kelly will catch up to Joni in $2\frac{1}{2}$ hours. By then, Joni will have travelled 3 hours.

Try It

18) *Translate to a system of equations and then solve:* Mitchell left Detroit on the interstate driving south towards Orlando at a speed of 60 miles per hour. Clark left Detroit 1 hour later travelling at a speed of 75 miles per hour, following the same route as Mitchell. How long will it take Clark to catch Mitchell?

Solution

It will take Clark 4 hours to catch Mitchell.

19) *Translate to a system of equations and then solve:* Charlie left his mother's house travelling at an average speed of 36 miles per hour. His sister Sally left 15 minutes ($\frac{1}{4}$ hour) later travelling the same route at an average speed of 42 miles per hour. How long before Sally catches up to Charlie?

Solution

It will take Sally $1\frac{1}{2}$ hours to catch up to Charlie.

Many real-world applications of uniform motion arise because of the effects of currents—of water or air—on the actual speed of a vehicle. Cross-country airplane flights in the United States generally take longer going west than going east because of the prevailing wind currents.

Let's take a look at a boat travelling on a river. Depending on which way the boat is going, the current of the water is either slowing it down or speeding it up.

Figures 4.4.1 and 4.4.2 show how a river current affects the speed at which a boat is actually travelling. We'll call the speed of the boat in still water b and the speed of the river current c.

In Figure 4.4.1 boat is going downstream, in the same direction as the river current. The current helps push the boat, so the boat's actual speed is faster than its speed in still water. The actual speed at which the boat is moving is $b + c$.

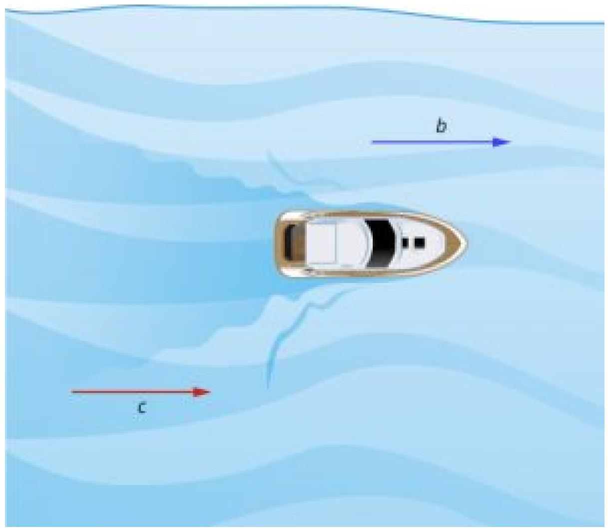


Figure 4.4.10

In Figure 4.4.2 the boat is going upstream, opposite to the river current. The current is going against the boat, so the boat's actual speed is slower than its speed in still water. The actual speed of the boat is $b - c$.

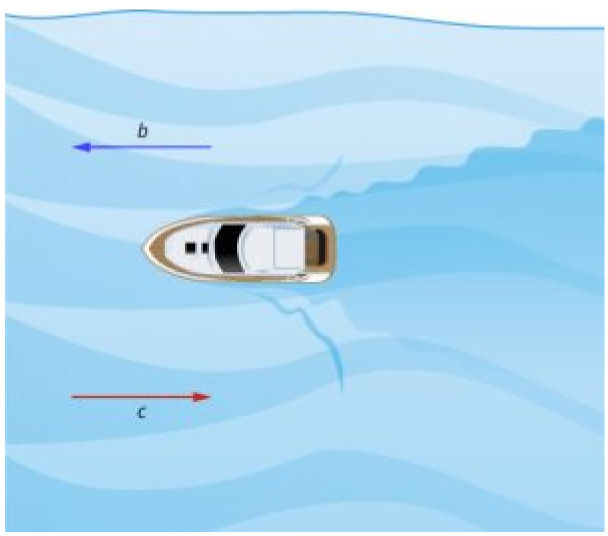


Figure 4.4.11

We'll put some numbers to this situation in Example 4.4.9

Example 4.4.9

Translate to a system of equations and then solve:

A river cruise ship sailed 60 miles downstream for 4 hours and then took 5 hours sailing upstream to return to the dock. Find the speed of the ship in still water and the speed of the river current.

Solution

Read the problem.

This is a uniform motion problem and a picture will help us visualize the situation.

Figure 4.4.12

Step 1: **Identify what we are looking for.**

We are looking for the speed of the ship in still water and the speed of the current.

Step 2: **Name what we are looking for.**

Let s = the rate of the ship in still water.

Let c = the rate of the current

A chart will help us organize the information. The ship goes downstream and then upstream. Going downstream, the current helps the ship; therefore, the ship's actual rate is $s + c$. Going upstream, the current slows the ship; therefore, the actual rate is $s - c$.

Type	Rate x Time = Distance		
Downstream	$s + c$	4	60
Upstream	$s - c$	5	60

Downstream it takes 4 hours. Upstream it takes 5 hours. Each way the distance is 60 miles.

Step 3: **Translate into a system of equations.**

Since rate times time is distance, we can write the system of equations.

$$\begin{cases} 4(s+c)=60 \\ 5(s-c)=60 \end{cases}$$

Step 4: **Solve the system of equations.**

Distribute to put both equations in standard form, then solve by elimination.

$$\begin{cases} 4s+4c=60 \\ 5s-5c=60 \end{cases}$$

Multiply the top equation by 5 and the bottom equation by 4. Add the equations, then solve for s.

$$\begin{array}{r} \begin{cases} 20s+20c=300 \\ 20s-20c=240 \end{cases} \\ \hline 40s=540 \end{array}$$

Substitute $s=13.5$ into one of the original equations.

$$\begin{aligned} \text{Substitute s=13.5.} \quad 4(rgb]1.0,0.0,0.0s+c) &= 60 \\ 4(rgb]1.0,0.0,0.013rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.05+c) &= 60 \\ \text{Solve for c.} \quad 54+4c &= 60 \\ 4c &= 6 \\ c &= 1.5 \end{aligned}$$

Step 5: **Check the answer in the problem.**

The downstream rate would be $13.5+1.5=15\ mph$.

In 4 hours the ship would travel $15\cdot 4=60\ miles$.

The upstream rate would be $13.5-1.5=12\ mph$.

In 5 hours the ship would travel $12\cdot 5=60\ miles$.

Step 6: **Answer the question**.

The rate of the ship is 13.5 mph and the rate of the current is 1.5 mph.

Try It

20) *Translate to a system of equations and then solve:* A Mississippi river boat cruise sailed 120 miles upstream for 12 hours and then took 10 hours to return to the dock. Find the speed of the river boat in still water and the speed of the river current.

Solution

The rate of the boat is 11 mph and the rate of the current is 1 mph.

21) *Translate to a system of equations and then solve:* Jason paddled his canoe 24 miles upstream for 4 hours. It took him 3 hours to paddle back. Find the speed of the canoe in still water and the speed of the river current.

Solution

The speed of the canoe is 7 mph and the speed of the current is 1 mph.

Wind currents affect airplane speeds in the same way as water currents affect boat speeds. We'll see this in Example 4.4.10. A wind current in the same direction as the plane is flying is called a *tailwind*. A wind current blowing against the direction of the plane is called a ***headwind***.

Example 4.4.10

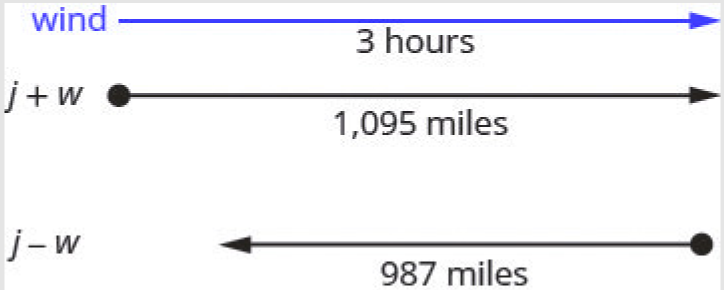
Translate to a system of equations and then solve:

A private jet can fly 1095 miles in three hours with a tailwind but only 987 miles in three hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Solution

Read the problem.

This is a uniform motion problem and a picture will help us visualize.



4.4.13

Step 1: **Identify what we are looking for.**

We are looking for the speed of the jet in still air and the speed of the wind.

Step 2: **Name what we are looking for.**

Let $j =$ the speed of the jet in still air.

Let $w =$ the speed of the wind

A chart will help us organize the information. The jet makes two trips-one in a tailwind and one in a headwind. In a tailwind, the wind helps the jet and so the rate is $j + w$. In a headwind, the wind slows the jet and so the rate is $j - w$.

Type	Rate x Time = Distance		
Tailwind	$j + w$	3	1095
Headwind	$j - w$	3	987

Each trip takes 3 hours. In a tailwind the jet flies 1095 miles. In a headwind the jet flies 987 miles.

Step 3: **Translate into a system of equations.**

Since rate times time is distance, we get the system of equations.

$$\begin{cases} 3(j+w) = 1095 \\ 3(j-w) = 987 \end{cases}$$

Step 4: **Solve the system of equations.**

Distribute, then solve by elimination.

$$\begin{cases} 3j + 3w = 1095 \\ 3j - 3w = 987 \end{cases}$$
$$6j = 2082$$

Add, and solve for j.

Substitute $j = 347$ into one of the original equations, then solve for w.

$$\begin{aligned} \text{Substitute j=347.} \quad 3(rgb]1.0,0.0,0.0j + w) &= 1095 \\ 3(rgb]1.0,0.0,0.0347 + w) &= 1095 \\ \text{Solve for w.} \quad 1041 + 3w &= 1095 \\ 3w &= 54 \\ w &= 18 \end{aligned}$$

Step 5: **Check the answer in the problem.**

With the tailwind, the actual rate of the jet would be $347 + 18 = 365\ mph.$

In 3 hours the jet would travel $365 \times 3 = 1095\ miles.$

Going into the headwind, the jet's actual rate would be $347 - 18 = 329\ mph.$

In 3 hours the jet would travel $329 \times 3 = 987\ miles.$

Step 6: **Answer the question.**

The rate of the jet is 347 mph and the rate of the wind is $18\ mph.$

Try It

22) Translate to a system of equations and then solve: A small jet can fly $1,325$ miles in 5 hours with a tailwind but only 1025 miles in 5 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Solution

The speed of the jet is 235 mph and the speed of the wind is 30 mph.

23) Translate to a system of equations and then solve: A commercial jet can fly 1728 miles in 4 hours with a tailwind but only 1536 miles in 4 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Solution

The speed of the jet is 408 mph and the speed of the wind is 24 mph.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=2450#h5p-11

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

complementary angles
: Two angles are complementary if the sum of the measures of their angles is 90 degrees.

supplementary angles
: Two angles are supplementary if the sum of the measures of their angles is 180 degrees.

4.5 SOLVE MIXTURE APPLICATIONS WITH SYSTEMS OF EQUATIONS

Learning Objectives

By the end of this section, you will be able to:

- Solve mixture applications
- Solve interest applications

Try It

Before you get started, take this readiness quiz:

1) Multiply $4.025(1,562)$.
2) Write 8.2% as a decimal.
3) Earl's dinner bill came to $\$32.50$ and he wanted to leave an 18% tip. How much should the tip be?

Solve Mixture Applications

When we solved mixture of applications with coins and tickets earlier, we started by creating a table so we could organize the information. For a coin example with nickels and dimes, the table looked like this:

Figure 4.5.1

Type	Number x Value ($) = Total Value ($)		
Nickels		0.05	
Dimes		0.10	

Using one variable meant that we had to relate the number of nickels and the number of dimes. We had to decide if we were going to let n be the number of nickels and then write the number of dimes in terms of n, or if we would let d be the number of dimes and write the number of nickels in terms of d.

Now that we know how to solve systems of equations with two variables, we'll just let n be the number of nickels and d be the number of dimes. We'll write one equation based on the total value column, like we did before, and the other equation will come from the number column.

For the first example, we'll do a ticket problem where the ticket prices are in whole dollars, so we won't need to use decimals just yet.

Example 4.5.1

Translate to a system of equations and solve:

The box office at a movie theatre sold 147 tickets for the evening show, and receipts totalled $\$1,302$. How many $\$11$ adult and how many $\$8$ child tickets were sold?

Solution

***Step 1:* Read the problem.**

We will create a table to organize the information.

***Step 2:* Identify what we are looking for.**

We are looking for the number of adult tickets and the number of child tickets sold.

***Step 3:* Name what we are looking for.**

Let a $=$ the number of adult tickets.
Let c $=$ the number of child tickets

A table will help us organize the data. We have two types of tickets: adult and child.

Write a and c for the number of tickets.

Write the total number of tickets sold at the bottom of the Number column.

Altogether 147 were sold.

Write the value of each type of ticket in the Value column.

The value of each adult ticket is $\$11$. The value of each child tickets is $\$8$.

The number times the value gives the total value, so the total value of adult tickets is $a \cdot 11 = 11a$, and the total value of child tickets is $c \cdot 8 = 8c$.

Type	Number $\times$ Value (\$) = Total Value (\$)		
Adult	a	11	$11a$
Child	c	8	$8c$
	147		1302

Altogether the total value of the tickets was $\$1,302$.

Fill in the Total Value column.

Step 4: **Translate into a system of equations.**

The Number column and the Total Value column give us the system of equations. We will use the elimination method to solve this system.

$$\begin{cases} a + c = 147 \\ 11a + 8c = 1302 \end{cases}$$

Multiply the first equation by -8.

$$\begin{cases} -8(a + c) = -8(147) \\ 11a + 8c = 1302 \end{cases}$$

Simplify and add, then solve for a.

$$\begin{cases} -8a + 8c = -1176 \\ 11a + 8c = 1302 \end{cases}$$
$$\overline{\qquad\qquad 3a = 126}$$

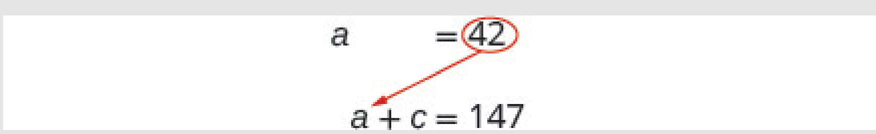


Figure 4.5.2

Substitute $a = 42$ into the first equation, then solve for c.

$$\begin{aligned} rgb]1.0,0.0,0.042 + c &= 147 \\ c &= 105 \end{aligned}$$

Step 5: **Check the answer in the problem.**

42 adult tickets at $\$11$ per ticket makes $\$462$.

105 child tickets at $\$8$ per ticket makes $\$840$.

The total receipts are $\$1,302\checkmark$.

Step 6: **Answer the question.**

The movie theatre sold 42 adult tickets and 105 child tickets.

Try It

4) *Translate to a system of equations and solve:* The ticket office at the zoo sold 553 tickets one day. The receipts totalled $3,936. How many $9 adult tickets and how many $6 child tickets were sold?

Solution

There were 206 adult tickets sold and 347 children tickets sold.

5) *Translate to a system of equations and solve:* A science centre sold 1,363 tickets on a busy weekend. The receipts totalled $12,146. How many $12 adult tickets and how many $7 child tickets were sold?

Solution

There were 521 adult tickets sold and 842 children tickets sold.

In Example 4.5.2 we'll solve a coin problem. Now that we know how to work with systems of two variables, naming the variables in the 'number' column will be easy.

Example 4.5.2

Translate to a system of equations and solve:

Priam has a collection of nickels and quarters, with a total value of $\$7.30$. The number of nickels is six less than three times the number of quarters. How many nickels and how many quarters does he have?

Solution

***Step 1:* Read the problem.**

We will create a table to organize the information.

***Step 2:* Identify what we are looking for.**

We are looking for the number of nickels and the number of quarters.

***Step 3:* Name what we are looking for.**

Let $n =$ the number of nickels.
Let $q =$ the number of quarters

A table will help us organize the data. We have two types of coins, nickels and quarters.

Write n and q for the number of each type of coin.

Fill in the Value column with the value of each type of coin.

The value of each nickel is $\$0.05$.

The value of each quarter is $\$0.25$.

The number times the value gives the total value, so, the total value of the nickels is $n(0.05) = 0.05n$ and the total value of quarters is $q(0.25) = 0.25q$. Altogether the total value of the coins is $\$7.30$.

Type	Number $\times$	Value ($) =	Total Value ($)
Nickels	n	0.05	$0.05n$
Quarters	q	0.25	$0.25q$
			7.30

Step 4: **Translate into a system of equations**.

The Total value column gives one equation.

$$0.05n + 0.25q = 7.30$$

We also know the number of nickels is six less than three times the number of quarters. Translate to get the second equation.

$$n = 3q - 6$$

Now we have the system to solve.

$$\begin{cases} 0.05n + 0.25q = 7.30 \\ n = 3q - 6 \end{cases}$$

Step 5: **Solve the system of equations**

We will use the substitution method. Substitute $n = 3q - 6$ into the first equation. Simplify and solve for q.

$$\begin{aligned} 0.05rgb]1.0, 0.0, 0.0n + 0.25q &= 7.30 \\ 0.05(rgb]1.0, 0.0, 0.03rgb]1.0, 0.0, 0.0qrgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.06) + 0.25q &= 7.3 \\ 0.15q - 0.3 + 0.25q &= 7.3 \\ 0.4q - 0.3 &= 7.3 \\ 0.4q &= 7.6 \\ q &= 19 \end{aligned}$$

To find the number of nickels, substitute $q = 19$ into the second equation.

$$\begin{aligned} n &= 3rgb]1.0, 0.0, 0.0q - 6 \\ n &= 3 \times rgb]1.0, 0.0, 0.019 - 6 \\ n &= 51 \end{aligned}$$

Step 6: **Check the answer in the problem.**

$$19 \text{ quarters at } \$0.25 = \$4.75$$

$$51 \text{ nickels at } \$0.05 = \$2.55$$

$$\begin{aligned} \text{Total} &= \$7.30 \\ 3.19 - 16 &= \$51 \checkmark \end{aligned}$$

Step 7: **Answer the question.**

Priam has 19 quarters and 51 nickels.

Try It

6) *Translate to a system of equations and solve:* Matilda has a handful of quarters and dimes, with a total value of $\$8.55$. The number of quarters is 3 more than twice the number of dimes. How many dimes and how many quarters does she have?

Solution

Matilda has 13 dimes and 29 quarters.

7) *Translate to a system of equations and solve:* Juan has a pocketful of nickels and dimes. The total value of the coins is $\$8.10$. The number of dimes is 9 less than twice the number of nickels. How many nickels and how many dimes does Juan have?

Solution

Juan has 36 nickels and 63 dimes.

Some mixture applications involve combining foods or drinks. Example situations might include combining raisins and nuts to make a trail mix or using two types of coffee beans to make a blend.

Example 4.5.3

Translate to a system of equations and solve:

Carson wants to make 20 pounds of trail mix using nuts and chocolate chips. His budget requires that the trail mix costs him $\$7.60$ per pound. Nuts cost $\$9.00$ per pound and chocolate chips cost $\$2.00$ per pound. How many pounds of nuts and how many pounds of chocolate chips should he use?

Solution

***Step 1:* Read the problem.**

We will create a table to organize the information.

Step 2: **Identify what we are looking for.**

We are looking for the number of pounds of nuts and the number of pounds of chocolate chips.

Step 3: **Name what we are looking for.**

Let $n =$ the number of pound of nuts.
Let $c =$ the number of pounds of chips

Carson will mix nuts and chocolate chips to get trail mix. Write in n and c for the number of pounds of nuts and chocolate chips.

There will be 20 pounds of trail mix. Put the price per pound of each item in the Value column. Fill in the last column using Number $\times$ Value = Total Value

Type	Number of Pounds $\times$ Value (\$) = Total Value (\$)		
Nuts	n	9.00	$9n$
Chocolate Chips	c	2.00	$2c$
Trail Mix	20	7.60	$7.60(20) = 152$

Step 4: **Translate into a system of equations. We get the equations from the Number and Total Value columns.**

$$\begin{cases} n + c = 20 \\ 9n + 2c = 152 \end{cases}$$

Step 5: **Solve the system of equations. We will use elimination to solve the system.**

Multiply the first equation by -2 to eliminate c.

$$\begin{cases} -2(n + c) = -2(20) \\ 9n + 2c = 152 \end{cases}$$

Simplify and add. Solve for n.

$$\begin{cases} -2n - 2c = -40 \\ 9n + 2c = 152 \end{cases}$$
$$7n = 112$$

$$n=16$$

To find the number of pounds of chocolate chips, substitute $n=16$ into the first equation, then solve for c.

$$\begin{aligned} rgb]1.0,0.0,0.0n+c&=20\\ rgb]1.0,0.0,0.016+c&=20\\ c&=4 \end{aligned}$$

Step 6: **Check the answer in the problem.**

$$\begin{aligned} 16+4&=20\checkmark\\ 9.16+2.4&=152\checkmark \end{aligned}$$

Step 7: **Answer the question.**

Carson should mix 16 pounds of nuts with 4 pounds of chocolate chips to create the trail mix.

Try It

8) *Translate to a system of equations and solve:* Greta wants to make 5 pounds of a nut mix using peanuts and cashews. Her budget requires the mixture to cost her $\$6$ per pound. Peanuts are $\$4$ per pound and cashews are $\$9$ per pound. How many pounds of peanuts and how many pounds of cashews should she use?

Solution
Greta should use 3 pounds of peanuts and 2 pounds of cashews.

9) *Translate to a system of equations and solve:* Sammy has most of the ingredients he needs to make a large batch of chili. The only items he lacks are beans and ground beef. He needs a total of 20 pounds combined of beans and ground beef and has a budget of $\$3$ per pound. The price of beans is $\$1$ per pound and the price of ground beef is $\$5$ per pound. How many pounds of beans and how many pounds of ground beef should he purchase?

Solution

Sammy should purchase 10 pounds of beans and 10 pounds of ground beef.

Another application of mixture problems relates to concentrated cleaning supplies, other chemicals, and mixed drinks. The concentration is given as a percent. For example, a 20% concentrated household cleanser means that 20 of the total amount is cleanser, and the rest is water. To make 35 ounces of a 20 concentration, you mix 7 ounces $(20$ of $35)$ of the cleanser with 28 ounces of water.

For these kinds of mixture problems, we'll use percent instead of value for one of the columns in our table.

Example 4.5.4

Translate to a system of equations and solve: Sasheena is a lab assistant at her community college. She needs to make 200 millilitres of a 40 solution of sulphuric acid for a lab experiment. The lab has only 25 and 50 solutions in the storeroom. How much should she mix of the 25 and the 50 solutions to make the 40 solution?

Solution

***Step 1:* Read the problem.**

A figure may help us visualize the situation, then we will create a table to organize the information.

Sasheena must mix some of the 25% solution and some of the 50% solution together to get 200 ml of the 40% solution.

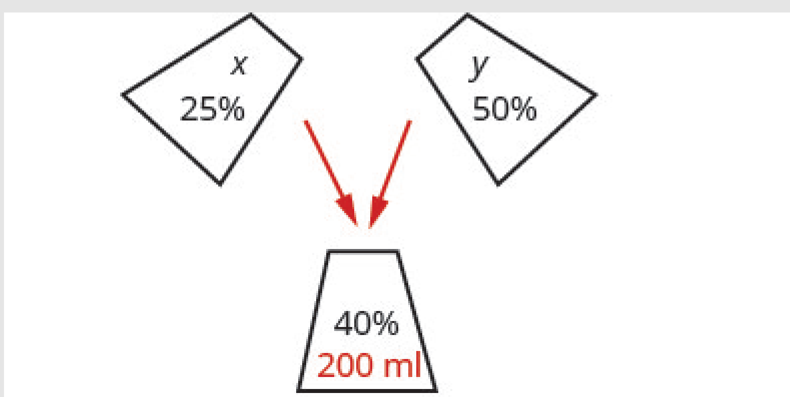


Figure 4.5.3

Step 2: **Identify what we are looking for.**

We are looking for how much of each solution she needs.

Step 3: **Name what we are looking for.**

Let x = number of ml of 25% solution.
Let y = number of ml of 50% solution.

A table will help us organize the data.

She will mix x ml of 25 with y ml of 50 to get 200 ml of 40 solution.

We write the percents as decimals in the chart.

We multiply the number of units times the concentration to get the total amount of sulphuric acid in each solution.

Type	Number of Units $\times$ Concentration % = Amount		
25%	x	0.25	$0.25x$
50%	y	0.50	$0.50y$
40%	200	0.40	$0.40(200)$

Step 4: **Translate into a system of equations. We get the equations from the Number column and the Amount column.**

Now we have the system.

$$\begin{cases} x+y=200 \\ 0.25x+0.50y=0.40(200) \end{cases}$$

Step 5: **Solve the system of equations. We will solve the system by elimination. Multiply the first equation by -0.5 to eliminate y.**

$$\begin{cases} rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.00rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.05(x+y)=rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.00rgb]1.0,0.0,0.0.rgb]1.0,0.0,0.05(200) \\ 0.25x+0.50y=0.40(200) \end{cases}$$

Simplify and add to solve for x.

$$\begin{cases} -0.5x-0.5y=-100 \\ 0.25x+0.50y=0.40(200) \end{cases}$$
$$\begin{aligned} -0.25x &= -20 \\ x &= 80 \end{aligned}$$

To solve for y, substitute $x=80$ into the first equation.

$$rgb]1.0, 0.0, 0.0x + y = 200$$
$$rgb]1.0, 0.0, 0.080 + y = 200$$
$$y = 120$$

Step 6: **Check the answer in the problem.**

$$80 + 120 = 120✓$$
$$0.25(80) + 0.50(120) = 80✓$$

Step 7: **Answer the question.**

Sasheena should mix 80 ml of the 25 solution with 120 ml of the 50 solution to get the 200 ml of the 40 solution.

Try It

10) *Translate to a system of equations and solve:* LeBron needs 150 millilitres of a 30 solution of sulphuric acid for a lab experiment but only has access to a 25 and a 50 solution. How much of the 25 and how much of the 50 solution should he mix to make the 30 solution?

Solution

LeBron needs 120 ml of the 25 solution and 30 ml of the 50 solution.

11) *Translate to a system of equations and solve:* Anatole needs to make 250 millilitres of a 25 solution of hydrochloric acid for a lab experiment. The lab only has a 10 solution and a 40 solution in the storeroom. How much of the 10 and how much of the 40 solutions should he mix to make the 25 solution?

Solution

Anatole should mix 125 ml of the 10 solution and 125 ml of the 40 solution.

Solve Interest Applications

The formula to model interest applications is $I = Prt$. Interest, I, is the product of the principal, P, the rate, r, and the time, t. In our work here, we will calculate the interest earned in one year, so t will be 1.

We modify the column titles in the mixture table to show the formula for interest, as you'll see in Example 4.5.5.

Example 4.5.5

Translate to a system of equations and solve:

Adnan has $\$40,000$ to invest and hopes to earn 7.1 interest per year. He will put some of the money into a stock fund that earns 8 per year and the rest into bonds that earns 3 per year. How much money should he put into each fund?

Solution

Step 1: **Read the problem.**

A chart will help us organize the information.

Step 2: **Identify what we are looking for.**

We are looking for the amount to invest in each fund.

Step 3: **Name what we are looking for.**

Let s $=$ the amount invested in stocks.
Let b $=$ the amount invested in bonds.

Write the interest rate as a decimal for each fund. Multiply: Principal $\times$ Rate $\times$Time to get the Interest.

Account	Principal $\times$ Rate $\times$ Time = Interest			
Stock Fund	s	0.08	1	$0.08s$
Bonds	b	0.03	1	$0.03b$
Total	$40,000$	0.071		$0.071(40,000)$

Step 4: **Translate into a system of equations.**

We get our system of equations from the Principal column and the Interest column.

$$\begin{cases} s + b = 40,000 \\ 0.08s + 0.03b = 0.071(40,000) \end{cases}$$

Step 5: **Solve the system of equations.**

Solve by elimination. Multiply the top equation by -0.03.

$$\begin{cases} rgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.00rgb]1.0, 0.0, 0.0.rgb]1.0, 0.0, 0.003(s + b) = rgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.00rgb]1.0, 0.0, 0.0.rgb]1.0, 0.0, 0.003(40,000) \\ 0.08s + 0.03b = 2,840 \end{cases}$$

$$s = 32,800$$

Simplify and add to solve for s.

$$\begin{cases} -0.03s - 0.03b = -1,200 \\ 0.08s + 0.03b = 2,840 \end{cases}$$
$$0.05s = 1,640$$

To find b, substitute $s = 32,800$ into the first equation.

$$rgb]1.0, 0.0, 0.0s + b = 40,000$$
$$rgb]1.0, 0.0, 0.032rgb]1.0, 0.0, 0.0,rgb]1.0, 0.0, 0.0800 + b = 40,000$$
$$b = 7,200$$

Step 6: **Check the answer in the problem.**

We leave the check to you.

Step 7: **Answer the question.**

Adnan should invest $\$32,800$ in stock and $\$7,200$ in bonds.

Did you notice that the Principal column represents the total amount of money invested while the Interest column represents only the interest earned? Likewise, the first equation in our system, $s + b = 40,000$, represents the total amount of money invested and the second equation, $0.08s + 0.03b = 0.071(40,000)$, represents the interest earned.

Try It

12) *Translate to a system of equations and solve:* Leon had $\$50,000$ to invest and hopes to earn 6.2 interest per year. He will put some of the money into a stock fund that earns 7 per year and the rest in to a savings account that earns 2 per year. How much money should he put into each fund?

Solution

Leon should put $\$42,000$ in the stock fund and $\$8000$ in the savings account.

13) *Translate to a system of equations and solve:* Julius invested $\$7,000$ into two stock investments. One stock paid 11 interest and the other stock paid 13 interest. He earned 12.5 interest on the total investment. How much money did he put in each stock?

Solution

Julius invested $\$1,750$ at 11 and $\$5,250$ at 13.

Example 4.5.6

Translate to a system of equations and solve: Rosie owes $\$21,540$ on her two student loans. The interest rate on her bank loan is 10.5 and the interest rate on the federal loan is 5.9. The total amount of interest she paid last year was $\$1,669.68$. What was the principal for each loan?

Solution

Step 1: **Read the problem.**

A chart will help us organize the information.

Step 2: **Identify what we are looking for.**

We are looking for the principal of each loan.

Step 3: **Name what we are looking for.**

Let $b =$ the principal for the bank loan.
Let $f =$ the principal on the federal loan.

The total loans are $\$21,540$.

Record the interest rates as decimals in the chart.

Account	Principal × Rate × Time = Interest			
Bank	b	0.105	1	$0.105b$
Federal	f	0.059	1	$0.059f$
Total	$21,540$			$1,669.68$

Multiply using the formula $I = Prt$ to get the Interest.

Step 4: **Translate into a system of equations.**

The system of equations comes from the Principal column and the Interest column.

$$\begin{cases} b + f = 21,540 \\ 0.105b + 0.059f = 1,669.68 \end{cases}$$

Step 5: **Solve the system of equations.**

We will use substitution to solve. Solve the first equation for b.

$$\begin{aligned} b + f &= 21,540 \\ b &= -f + 21,540 \end{aligned}$$

Substitute $b = -f + 21,540$ into the second equation.

$$0.105 rgb]1.0,0.0,0.0b + 0.59f = 1,669.68$$
$$0.105(rgb]1.0,0.0,0.0 - rgb]1.0,0.0,0.0 f rgb]1.0,0.0,0.0 + rgb]1.0,0.0,0.021 rgb]1.0,0.0,0.0, rgb]1.0,0.0,0.0540) + 0.59f = 1,669.68$$

Simplify and solve for f.

$$\begin{aligned} -0.105f + 2,261.70 + 0.059f &= 1,669.68 \\ -0.46f + 2,261.70 &= 1,669.68 \\ -0.046f &= -592.02 \\ f &= 12,870 \end{aligned}$$

To find *b*, substitute $f = 12,870$ into the first equation.

$$\begin{aligned} rgb]1.0,0.0,0.0b + f &= 21,540 \\ rgb]1.0,0.0,0.012 rgb]1.0,0.0,0.0, rgb]1.0,0.0,0.0870 + f &= 21,540 \\ f &= 8,670 \end{aligned}$$

Step 6: **Check the answer in the problem.**

We leave the check to you.

Step 7: **Answer the question.**

The principal of the bank loan is $\$12,870$ and the principal for the federal loan is $\$8,670.$

Try It

14) *Translate to a system of equations and solve:* Laura owes $\$18,000$ on her student loans. The interest rate on the bank loan is 2.5 and the interest rate on the federal loan is 6.9. The total amount of interest she paid last year was $\$1,066$. What was the principal for each loan?

Solution

The principal amount for the bank loan was $\$4,000$. The principal amount for the federal loan was $\$14,000$.

15) *Translate to a system of equations and solve:* Jill's Sandwich Shoppe owes $\$65,200$ on two business loans, one at 4.5 interest and the other at 7.2 interest. The total amount of interest owed last year was $\$3,582$. What was the principal for each loan?

Solution

The principal amount for was $\$41,200$ at 4.5. The principal amount was, $\$24,000$ at 7.2.

Access these online resources for additional instruction and practice with solving application problems with systems of linear equations.

- Cost and Mixture Word Problems

- Mixture Problems

Key Concepts

Table for coin and mixture applications

Type	Number •	Value($) =	Total Value($)
Total			

Figure 4.5.4

Table for concentration applications

Type	Number of units •	Concentration % =	Amount
Total			

Figure 4.5.5

Table for interest applications

Account	Principal •	Rate •	Time =	Interest
			1	
			1	
Total				

Figure 4.5.6

Self Check

After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=2511#h5p-14

After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

4.6 UNIT SOURCES

Unit 4 Sources

4.1 "Solve Systems of Equations by Graphing" from Elementary Algebra 2e. by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

4.2 "Solve Systems of Equations by Substitution" from Elementary Algebra 2e. by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

4.3 "Solve Systems of Equations by Elimination" from Elementary Algebra 2e. by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

4.4 "Solve Applications with Systems of Equations" from Elementary Algebra 2e. by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

4.5 "Solve Mixture Applications with Systems of Equations" from Elementary Algebra 2e. by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

UNIT 5: INTRODUCTION TO POLYNOMIALS

Chapter Outline

5.0 INTRODUCTION

Architects use polynomials to design curved shapes such as this bridge, the Blues Point Bridge, in Sydney NSW, Australia.

Figure 5.0.1: Blues Point Bridge, Sydney NSW, Australia Photo by Robert Cutts CC-BY-SA 2.0

We have seen that the graphs of linear equations are straight lines. Graphs of other types of equations, called polynomial equations, are curves, like the outline of this suspension bridge. Architects use polynomials to design the shape of a bridge like this and to draw the blueprints for it. Engineers use polynomials to calculate the stress on the bridge's supports to ensure they are strong enough for the intended load. In this chapter, you will explore operations with and properties of polynomials.

5.1 ADD AND SUBTRACT POLYNOMIALS

Learning Objectives

By the end of this section, you will be able to:

- Identify polynomials, monomials, binomials, and trinomials
- Determine the degree of polynomials
- Add and subtract monomials
- Add and subtract polynomials
- Evaluate a polynomial for a given value

Try it

Before you get started, take this readiness quiz:

1) Simplify: $8x+3x$.
2) Subtract: $(5n+8)-(2n-1)$
3) Write in expanded form: a^5.

Identify Polynomials, Monomials, binomial and Trinomials

You have learned that a **term** is a constant or the product of a constant and one or more variables. When it is of the form ax^m, where a is a constant and m is a whole number, it is called a **monomial**. Some examples of monomial are 8, $-2x^2$, $4y^3$, and $11z^7$.

A monomial, or two or more monomials combined by addition or subtraction, is a **polynomial**. Some polynomials have special names, based on the number of terms. A monomial is a polynomial with exactly one term. A **binomial** has exactly two terms, and a **trinomial** has exactly three terms. There are no special names for polynomials with more than three terms.

Polynomials

Monomial—A monomial is a term of the form ax^m, where a is a constant and m is a positive whole number.
Polynomial—A monomial, or two or more monomials combined by addition or subtraction, is a polynomial.

- **monomial**—A polynomial with exactly one term is called a monomial.
- **binomial**—A polynomial with exactly two terms is called a binomial.
- **trinomial**—A polynomial with exactly three terms is called a trinomial.

Here are some examples of polynomials.

Polynomial	$b+1$	$4y^2-7y+2$	$4x^4+x^3+8x^2-9x+1$	
Monomial	14	$8y^2$	$-9x^3y^5$	-13
Binomial	$a+7$	$4b-5$	y^2-16	$3x^3-9x^2$
Trinomial	$x^2-7x+12$	$9y^2+2y-8$	$6m^4-m^3+8m$	z^4+3z^2-1

Notice that every monomial, binomial, and trinomial is also a polynomial. They are just special members of the "family" of polynomials and so they have special names. We use the words *monomial*, *binomial*, and *trinomial* when referring to these special polynomials and just call all the rest *polynomials*.

Example 5.1.1

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial.

a. $4y^2 - 8y - 6$
b. $-5a^4b^2$
c. $2x^5 - 5x^3 - 9x^2 + 3x + 4$
d. $13 - 5m^3$
e. q

Solution

	Polynomial	Number of terms	Type
a.	$4y^2 - 8y - 6$	3	Trinomial
b.	$-5a^4b^2$	1	Monomial
c.	$2x^5 - 5x^3 - 9x^2 + 3x + 4$	5	Polynomial
d.	$13 - 5m^3$	2	Binomial
e.	q	1	Monomial

Try It

4) Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial:

a. $5b$
b. $8y^3 - 7y^2 - y - 3$
c. $-3x^2 - 5x + 9$

d. $81 - 4a^2$
e. $-5x^6$

Solution

a. monomial
b. polynomial
c. trinomial
d. binomial
e. monomial

5) Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial:

a. $27z^3 - 8$
b. $12m^3 - 5m^2 - 2m$
c. $\frac{5}{6}$
d. $8x^4 - 7x^2 - 6x - 5$
e. $-n^4$

Solution

a. binomial
b. trinomial
c. monomial
d. polynomial
e. monomial

Determine the Degree of Polynomials

The **degree of a polynomial** and the degree of its terms are determined by the exponents of the variable.

A monomial that has no variable, just a constant, is a special case. The **degree of a constant** is 0—it has no variable.

Degree of a Polynomial

The **degree of a term** is the sum of the exponents of its variables.

The degree of a constant is 0.

The degree of a polynomial is the highest degree of all its terms.

Let's see how this works by looking at several polynomials. We'll take it step by step, starting with monomials, and then progressing to polynomials with more terms.

Monomial	14	$8y^2$	$-9x^3y^5$	$-13a$
Degree	$rgb]0.0, 0.0, 1.00$	$rgb]0.0, 0.0, 1.02$	$rgb]0.0, 0.0, 1.08$	$rgb]0.0, 0.0, 1.01$
Binomial	$a+7$	$4b^2-5b$	x^2y^2-16	$3n^3-9n^2$
Degree of each term	$rgb]0.0, 0.0, 1.01 \quad 0$	$rgb]0.0, 0.0, 1.02 \quad 1$	$rgb]0.0, 0.0, 1.04 \quad 0$	$rgb]0.0, 0.0, 1.03 \quad 2$
Degree of polynomial	$rgb]1.0, 0.0, 0.01$	$rgb]1.0, 0.0, 0.02$	$rgb]1.0, 0.0, 0.04$	$rgb]1.0, 0.0, 0.03$
Trinomial	$x^2-7x+12$	$9a^2+6ab+b^2$	$6m^4-m^3n^2+8mn^5$	z^4+3z^2-1
Degree of each term	$rgb]0.0, 0.0, 1.02 \quad 1 \quad 0$	$rgb]0.0, 0.0, 1.02 \quad 2 \quad 2$	$rgb]0.0, 0.0, 1.04 \quad 5 \quad 6$	$rgb]0.0, 0.0, 1.04 \quad 2 \quad 0$
Degree of polynomial	$rgb]1.0, 0.0, 0.02$	$rgb]1.0, 0.0, 0.02$	$rgb]1.0, 0.0, 0.06$	$rgb]1.0, 0.0, 0.04$
Polynomial	$b+1$	$4y^2-7y+2$	$4x^4+x^3+8x^2-9x+1$	
Degree of each term	$rgb]0.0, 0.0, 1.01 \quad 0$	$rgb]0.0, 0.0, 1.02 \quad 1 \quad 0$	$rgb]0.0, 0.0, 1.04 \quad 3 \quad 2 \quad 1 \quad 0$	
Degree of polynomial	$rgb]1.0, 0.0, 0.01$	$rgb]1.0, 0.0, 0.02$	$rgb]1.0, 0.0, 0.04$	

A polynomial is in **standard form** when the terms of a polynomial are written in descending order of degrees. Get in the habit of writing the term with the highest degree first.

Example 5.1.2

Find the degree of the following polynomials.

a. $10y$
b. $4x^3-7x+5$
c. -15
d. $-8b^2+9b-2$
e. $8xy^2+2y$

Solution

a. The exponent of y is one. $y = y^1$

$$10y$$

The degree is 1.

---

b. The highest degree of all the terms is 3.

$$4x^3 - 7x + 5$$

The degree is 3.

---

c. The degree of a constant is 0.

$$-15$$

The degree is 0.

---

d. The highest degree of all the terms is 2.

$$-8b^2 + 9b - 2$$

The degree is 2.

---

e. The highest degree of all the terms is 3.

$$6m^4 - m^3 + 8m$$

The degree is 3.

Try It

6) Find the degree of the following polynomials:

a. $-15b$
b. $10z^4 + 4z^2 - 5$
c. $12c^5d^4 + 9c^3d^9 - 7$
d. $3x^2y - 4x$
e. -9

Solution

a. 1
b. 4
c. 12
d. 3
e. 0

7) Find the degree of the following polynomials:

a. 52
b. $a^4b - 17a^4$
c. $5x + 6y + 2z$
d. $3x^2 - 5x + 7$
e. $-a^3$

Solution

a. 0
b. 5
c. 1
d. 2
e. 3

Add and Subtract Monomials

You have learned how to simplify expressions by combining like terms. Remember, like terms must have the same variables with the same exponent. Since monomials are terms, adding and subtracting monomials is the same as combining like terms. If the monomials are like terms, we just combine them by adding or subtracting the coefficient.

Example 5.1.3

Add: $25y^2 + 15y^2$.

Solution

Step 1: **Combine like terms.**

$$40y^2$$

Try It

8) Add: $12q^2 + 9q^2$.

Solution

$21q^2$

9) Add: $-15c^2 + 8c^2$.

Solution

$-7c^2$

Example 5.1.4

Subtract: $16p - (-7p)$

Solution

Step 1: **Combine like terms.**

$$23p$$

Try It

10) Subtract: $8m-(-5m)$

Solution

$13m$

11) Subtract: $-15z^3-(-5z^3)$

Solution

$-10z^3$

Remember that like terms must have the same variables with the same exponents.

Example 5.1.5

Simplify: $c^2+7d^2-6c^2$

Solution

Step 1: **Combine like terms.**

$$-5c^2+7d^2$$

Try It

12) Add: $8y^2 + 3z^2 - 3y^2$.

Solution

$5y^2 + 3z^2$

13) Add: $3m^2 + n^2 - 7m^2$

Solution

$-4m^2 + n^2$

Example 5.1.6

Simplify: $u^2v + 5u^2 - 3v^2$

Solution

Step 1: **There are no like terms to combine.**

$$u^2v + 5u^2 - 3v^2$$

Try It

14) Simplify: $m^2n^2 - 8m^2 + 4n^2$

Solution
There are no like terms to combine.

15) Simplify: $pq^2 - 6p - 5q^2$

Solution
There are no like terms to combine.

Add and Subtract Polynomials

We can think of adding and subtracting polynomials as just adding and subtracting a series of monomials. Look for the like terms—those with the same variables and the same exponent. The Commutative Property allows us to rearrange the terms to put like terms together.

Example 5.1.7

Find the sum: $(5y^2 - 3y + 15) + (3y^2 - 4y - 11)$

Solution
Step 1: **Identify like terms.**

$$\left(rgb]1.0, 0.0, 0.05rgb]1.0, 0.0, 0.0y^{rgb]1.0,0.0,0.02} - rgb]0.0, 0.0, 1.03rgb]0.0, 0.0, 1.0y + rgb]0.3, 0.3, 0.315\right) + \left(rgb]1.0, 0.0, 0.03rgb]1.0, 0.0, 0.0y^{rgb]1.0,0.0,0.02} - rgb]0.0, 0.0, 1.04rgb]0.0, 0.0, 1.0y - rgb]0.3, 0.3, 0.311\right)$$

Step 2: **Rearrange to get the like terms together.**

$$rgb]0.0, 0.0, 1.05rgb]0.0, 0.0, 1.0y^{rgb]0.0,0.0,1.02}rgb]0.0, 0.0, 1.0+rgb]0.0, 0.0, 1.03rgb]0.0, 0.0, 1.0y^{rgb]0.0,0.0,1.02} - rgb]1.0, 0.0, 0.03rgb]1.0, 0.0, 0.0yrgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.04rgb]1.0, 0.0, 0.0y + rgb]0.5, 0.0, 0.515 - rgb]0.5, 0.0, 0.511$$

Step 3: **Combine like terms.**

$$8y^2 - 7y + 4$$

Try It

16) Find the sum: $(7x^2 - 4x + 5) + (x^2 - 7x + 3)$

Solution

$8x^2 - 11x + 1$

17) Find the sum: $(14y^2 + 6y - 4) + (3y^2 + 8y + 5)$

Solution

$17y^2 + 14y + 1$

Example 5.1.8

Find the difference: $(9w^2 - 7w + 5) - (2w^2 - 4)$

Solution

Step 1: **Distribute and identify like terms.**

$rgb]0.0,0.0,1.09rgb]0.0,0.0,1.0w^{rgb]0.0,0.0,1.02} - rgb]1.0,0.0,0.07rgb]1.0,0.0,0.0w + rgb]0.5,0.0,0.55 - rgb]0.0,0.0,1.02rgb]0.0,0.0,1.0w^{rgb]0.0,0.0,1.02} + rgb]0.5,0.0,0.54$

Step 2: **Rearrange the terms.**

$rgb]0.0,0.0,1.09rgb]0.0,0.0,1.0w^{rgb]0.0,0.0,1.02}rgb]0.0,0.0,1.0 - rgb]0.0,0.0,1.02rgb]0.0,0.0,1.0w^{rgb]0.0,0.0,1.02} - rgb]1.0,0.0,0.07rgb]1.0,0.0,0.0w + rgb]0.5,0.0,0.55 + rgb]0.5,0.0,0.54$

Step 3: **Combine like terms.**

$$7w^2 - 7w + 9$$

Try It

18) Find the difference: $(8x^2 + 3x - 19) - (7x^2 - 14)$

Solution

$15x^2 + 3x - 5$

19) Find the difference: $(9b^2 - 5b - 4) - (3b^2 - 5b - 7)$

Solution

$6b^2 + 3$

Example 5.1.9

Subtract: $(c^2 - 4c + 7)$ from $(7c^2 - 5c + 3)$.

Solution

***Step 1:* Write the equation.**

$$(7c^2 - 5c + 3) - (c^2 - 4c + 7)$$

***Step 2:* Distribute and identify like terms.**

$rgb]0.0, 0.0, 1.07rgb]0.0, 0.0, 1.0c^{rgb]0.0,0.0,1.02} - rgb]1.0, 0.0, 0.05rgb]1.0, 0.0, 0.0c + rgb]0.5, 0.0, 0.53 - rgb]0.0, 0.0, 1.0c^{rgb]0.0,0.0,1.02} + rgb]1.0, 0.0, 0.04rgb]1.0, 0.0, 0.0c - rgb]0.5, 0.0, 0.57$

***Step 2:* Rearrange the terms.**

$rgb]0.0, 0.0, 1.07rgb]0.0, 0.0, 1.0c^{rgb]0.0,0.0,1.02}rgb]0.0, 0.0, 1.0-rgb]0.0, 0.0, 1.0c^{rgb]0.0,0.0,1.02} - rgb]1.0, 0.0, 0.05rgb]1.0, 0.0, 0.0crgb]1.0, 0.0, 0.0+rgb]1.0, 0.0, 0.04rgb]1.0, 0.0, 0.0c + rgb]0.5, 0.0, 0.53rgb]0.5, 0.0, 0.5+rgb]0.5, 0.0, 0.57$

***Step 3:* Combine like terms.**

$$6c^2 - c - 4$$

Try It

20) Subtract: $(5z^2 - 6z - 2)$ from $(7z^2 + 6z - 4)$.

Solution

$2z^2 + 12z - 2$

21) Subtract: $(x^2 - 5x - 8)$ from $(6x^2 + 9x - 1)$.

Solution

$5x^2 + 14x + 7$

Example 5.1.10

Find the sum: $(u^2 - 6uv + 5v^2) + (3u^2 + 2uv)$

Solution

Step 1: **Distribute.**

$$u^2 - 6uv + 5v^2 + 3u^2 + 2uv$$

Step 2: **Rearrange the terms, to put like terms together.**

$$u^2 + 3u^2 - 6uv + 2uv + 5v^2$$

Step 3: **Combine like terms.**

$$4u^2 - 4uv + 5v^2$$

Try It

22) Find the sum: $(3x^2-4xy+5y^2)+(2x^2-xy)$.

Solution

$5x^2-5xy+5y^2$

23) Find the sum: $(2x^2-3xy-2y^2)+(5x^2-3xy)$.

Solution

$7x^2-6xy-2y^2$

Example 5.1.11

Find the difference: $(p^2+q^2)-(p^2+10pq-2q^2)$.

Solution

Step 1: **Distribute.**

$$p^2+q^2-p^2-10pq+2q^2$$

Step 2: **Rearrange the terms, to put like terms together.**

$$p^2-p^2-10pq+q^2+2q^2$$

Step 3: **Combine like terms.**

$$-10pq^2+3q^2$$

Try It

24) Find the difference: $(a^2+b^2)-(a^2+5ab-6b^2)$.

Solution

$-5ab-5b^2$

25) Find the difference: $(m^2+n^2)-(m^2-7mn-3n^2)$.

Solution

$4n^2+7mn$

Example 5.1.12

Simplify: $(a^3-a^2b)-(ab^2+b^3)+(a^2b+ab^2)$.

Solution

Step 1: **Distribute.**

$$a^3-a^2b-ab^2-b^3+a^2b+ab^2$$

Step 2: **Rearrange the terms, to put like terms together.**

$$a^3-a^2b+a^2b-ab^2+ab^2-b^3$$

Step 3: **Combine like terms.**

$$a^3-b^3$$

Try It

26) Simplify: $(x^3 - x^2 y) - (xy^2 + y^3) + (x^2 y + xy^2)$.

Solution

$x^3 - y^3$

27) Simplify: $(p^3 - p^2 q) + (pq^2 + q^3) - (p^2 q + pq^2)$.

Solution

$p^3 - 2p^2 q + q^3$

Evaluate a Polynomial for a Given Value

We have already learned how to evaluate expressions. Since polynomials are expressions, we'll follow the same procedures to evaluate a polynomial. We will substitute the given value for the variable and then simplify using the order of operations.

Example 5.1.13

Evaluate $5x^2 - 8x + 4$ when

a. $x = 4$
b. $x = -2$
c. $x = 0$

Solution

a.

Step 1: **Substitute 4 for x.**

$$5(rgb]1.0,0.0,0.04)^2 - 8\,(rgb]1.0,0.0,0.04) + 4$$

Step 2: **Simplify the exponents.**

$$5 \cdot 16 - 8\,(4) + 4$$

Step 3: **Multiply.**

$$80 - 32 + 4$$

Step 4: **Simplify.**

$$52$$

---

b.

Step 1: **Substitute -2 for x.**

$$5(rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.02)^2 - 8\,(rgb]1.0,0.0,0.0{-}rgb]1.0,0.0,0.02) + 4$$

Step 2: **Simplify the exponents.**

$$5 \cdot 4 - 8(-2) + 4$$

Step 3: **Multiply.**

$$20 + 16 + 4$$

Step 4: **Simplify.**

40

---

c.

Step 1: **Substitute 0 for x.**

$$5(rgb]1.0,0.0,0.00)^2 - 8\,(rgb]1.0,0.0,0.00) + 4$$

Step 2: **Simplify the exponents.**

$$5 \cdot 0 - 8(0) + 4$$

Step 3: **Multiply.**

$$0 + 0 + 4$$

Step 4: **Simplify.**

$$4$$

Try It

28) Evaluate: $3x^2 + 2x - 15$ when

a. $x = 3$
b. $x = -5$
c. $x = 0$

Solution

a. 18
b. 50
c. -15

29) Evaluate: $5z^2 - z - 4$ when

a. $z = -2$
b. $z = 0$
c. $z = 2$

Solution

a. 18
b. -4
c. 14

Example 5.1.14

The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250 foot tall building. Find the height after $t = 2$ seconds.

Solution

Step 1: **Substitute** $t = 2$**.**

$$-16(2)^2 + 250$$

***Step 2:* Simplify.**

$$\begin{aligned} & -16 \times 4 + 250 \\ \text{Simplify.} \quad & -64 + 250 \\ & = 186 \end{aligned}$$

After 2 seconds the height of the ball is 186 feet.

Try It

30) The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall building. Find the height after $t = 0$ seconds.

Solution

250

31) The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall building. Find the height after $t = 3$ seconds.

Solution

106

Example 5.1.15

The polynomial $6x^2 + 15xy$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side x feet and sides of height y feet. Find the cost of producing a box with $x = 4$ feet and $y = 6$ feet.

Solution

Step 1: **Substitute** $x = 4, y = 6$.

$$6(rgb]1.0,0.0,0.04)^2 + 15\,(rgb]1.0,0.0,0.04)\,rgb]0.0,0.0,1.0\,(6)$$

Step 2: **Simplify.**

$$\begin{array}{ll} & 6 \cdot 16 + 15\,(rgb]1.0,0.0,0.04)\,rgb]0.0,0.0,1.0\,(6) \\ \text{Simplify.} & 96 + 360 \\ & = 456 \end{array}$$

The cost of producing the box is $\$456$.

Try It

32) The polynomial $6x^2 + 15xy$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side x feet and sides of height y feet. Find the cost of producing a box with $x = 6$ feet and $y = 4$ feet.

Solution

$576

33) The polynomial $6x^2 + 15xy$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side x feet and sides of height y feet. Find the cost of producing a box with $x = 5$ feet and $y = 8$ feet.

Solution

$750

Access these online resources for additional instruction and practice with adding and subtracting polynomials.

- Add and Subtract Polynomials 1
- Add and Subtract Polynomial 2
- Add and Subtract Polynomial 3

Key Concepts

- **Monomials**
  - A monomial is a term of the form ax^m, where a is a constant and m is a whole number.
- **Polynomials**
  - **polynomial**—A monomial, or two or more monomials combined by addition or subtraction is a polynomial.
  - **monomial**—A polynomial with exactly one term is called a monomial.
  - **binomial**—A polynomial with exactly two terms is called a binomial.
  - **trinomial**—A polynomial with exactly three terms is called a trinomial.
- **Degree of a Polynomial**
  - The **degree of a term** is the sum of the exponents of its variables.
  - The **degree of a constant** is 0.
  - The **degree of a polynomial** is the highest degree of all its terms.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

b. If most of your checks were:

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=2618#h5p-26

Glossary

binomial
A binomial is a polynomial with exactly two terms.

degree of a constant
The degree of any constant is 0.

degree of a polynomial
The degree of a polynomial is the highest degree of all its terms.

degree of a term
The degree of a term is the exponent of its variable.

monomial
A monomial is a term of the form ax^m, where a is a constant and m is a whole number; a monomial has exactly one term.

polynomial
A polynomial is a monomial, or two or more monomials combined by addition or subtraction.

standard form

A polynomial is in standard form when the terms of a polynomial are written in descending order of degrees.

trinomial

A trinomial is a polynomial with exactly three terms.

5.2 USE MULTIPLICATION PROPERTIES OF EXPONENTS

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with exponents
- Simplify expressions using the Product Property for Exponents
- Simplify expressions using the Power Property for Exponents
- Simplify expressions using the Product to a Power Property
- Simplify expressions by applying several properties
- Multiply monomials
- Use the definition of a negative exponent
- Simplify expressions with integer exponents

Try it

Before you get started, take this readiness quiz:

1) Simplify: $\frac{3}{4} \times \frac{3}{4}$
2) Simplify: $(-2)(-2)(-2)$.
3) What is the place value of the 6 in the number $64,891$?

4) Name the decimal: 0.0012.
5) Subtract: $5-(-3)$.

Simplify Expressions with Exponents

Remember that an exponent indicates repeated multiplication of the same quantity. For example, 2^4 means to multiply 2 by itself 4 times, so 2^4 means $2 \cdot 2 \cdot 2 \cdot 2$.

Let's review the vocabulary for expressions with exponents.

Exponential Notation

$$a^{m rgb]0.0,0.0,1.0 \leftarrow rgb]0.0,0.0,1.0 \text{exponent}}$$
$$rgb]1.0,0.0,0.0 \underset{base}{\uparrow}$$

$$a^m \text{ means multiply } m \text{ factors of } a$$
$$a^m = \underbrace{a \cdot a \cdot a \cdot \ldots \ldots \cdot a}_{rgb]0.0,0.0,1.0 m \text{ factors}}$$

This is read a to the m^{th} power.

In the expression a^m, the *exponent* m tells us how many times we use the *base* a as a factor.

$$4^3 \qquad \underbrace{4 \cdot 4 \cdot 4}_{rgb]0.0,0.0,1.0 3 rgb]0.0,0.0,1.0 \ rgb]0.0,0.0,1.0 \text{factors}}$$

$$(-9)^5 \qquad \underbrace{(-9)(-9)(-9)(-9)(-9)}_{rgb]0.0,0.0,1.0 5 rgb]0.0,0.0,1.0 \ rgb]0.0,0.0,1.0 \text{factors}}$$

Before we begin working with variable expressions containing exponents, let's simplify a few expressions involving only numbers.

Example 5.2.1

Simplify:

a. 4^3
b. 7^1

c. $(\frac{5}{6})^2$

d. $(0.63)^2$

Solution

a.

Step 1: **Multiply three factors of 4.**

$$\begin{array}{rl} & 4 \times 4 \times 4 \\ \text{Simplify.} & 64 \end{array}$$

---

b.

Step 1: **Multiply one factor of 7.**

$$7$$

---

c.

Step 1: **Multiply two factors.**

$$\begin{array}{rl} & \left(\frac{5}{6}\right) \times \left(\frac{5}{6}\right) \\ \text{Simplify.} & \frac{25}{36} \end{array}$$

---

d.

Step 1: **Multiply two factors.**

$$\begin{array}{rl} & (0.63) \times (0.63) \\ \text{Simplify.} & 0.3969 \end{array}$$

Try It

6) Simplify:

a. 6^3
b. 15^1
c. $(\frac{3}{7})^2$
d. $(0.43)^2$

Solution

a. 216
b. 15
c. $\frac{9}{49}$
d. 0.1849

7) Simplify:

a. 2^5
b. 21^1
c. $(\frac{2}{5})^3$
d. $(0.218)^2$

Solution

a. 32
b. 21
c. $\frac{8}{125}$
d. 0.047524

Example 5.2.2

Simplify:

a. $(-5)^4$
b. -5^4

Solution
a.

Step 1: **Multiply four factors of -5.**

$$\begin{array}{rl} & (-5)(-5)(-5)(-5) \\ \text{Simplify.} & 625 \end{array}$$

---

b.

Step 1: **Multiply four factors of 5.**

$$\begin{array}{rl} & -(5 \times 5 \times 5 \times 5) \\ \text{Simplify.} & 6-25 \end{array}$$

Try It

8) Simplify:

a. $(-3)^4$
b. -3^4

Solution

a. 81
b. -81

9) Simplify:

a. $(-13)^2$
b. -13^2

Solution
a. 169
b. -169

Notice the similarities and differences in the "Example 5.2.2" a. and b.! Why are the answers different? As we follow the order of operations in part a. the parentheses tell us to raise the -5 to the 4^{th} power. In part b. we raise just the 5 to the 4^{th} power and then take the opposite.

Simplify Expressions Using the Product Property for Exponents

You have seen that when you combine like terms by adding and subtracting, you need to have the same base with the same exponent. But when you multiply and divide, the exponents may be different, and sometimes the bases may be different, too.

We'll derive the properties of exponents by looking for patterns in several examples.

First, we will look at an example that leads to the Product Property.

	$x^2 \cdot x^3$
What does this mean? How many factors altogether?	$\underbrace{\underbrace{x \cdot x}_{rgb]0.0,0.0,1.02 rgb]0.0,0.0,1.0\ rgb]0.0,0.0,1.0\text{factors}} \cdot \quad rgb]0.0,0.0,1.03 rgb]0.0,0.0,1.0\ rgb]0.0,0.}_{rgb]0.0,0.0,1.05 rgb]0.0,0.0}$
So, we have	x^5
Notice that 5 is the sum of the exponents, 2 and 3.	$x^2 \cdot x^3$ is x^{2+3}, or x^5

We write:

$x^2 \cdot x^3$

x^{2+3}
x^{5}

The base stayed the same and we added the exponents. This leads to the Product Property for Exponents.

Product Property for Exponents

If a is a real number, and m and n are counting numbers, then

$$a^m \times a^n = a^{m+n}$$

To multiply with like bases, add the exponents.

An example with numbers helps to verify this property:

$$\begin{aligned} 2^2 \times 2^3 &\stackrel{?}{=} 2^{2+3} \\ 4 \times 8 &\stackrel{?}{=} 2^5 \\ 32 &= 32 \end{aligned}$$

Example 5.2.3

Simplify: $y^5 \times y^6$

Solution

Step 1: **Use the product property,** $a^m \cdot a^n = a^{m+n}$

$y^{rgb]1.0,0.0,0.05rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.06}$

Simplify. y^{11}

Try It

10) Simplify: $b^9 \times b^8$

Solution

b^{17}

11) Simplify: $x^{12} \times x^4$.

Solution

x^{16}

Example 5.2.4

Simplify:

a. $2^5 \times 2^9$
b. 3×3^4

Solution

a.

Step 1: **Use the product property,** $a^m \cdot a^n = a^{m+n}$

$$2^{rgb]1.0,0.0,0.05rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.09}$$

$$\text{Simplify.}\quad 2^{14}$$

---

b.

Step 1: **Use the product property,** $a^m \cdot a^n = a^{m+n}$

$3^{rgb]1.0,0.0,0.01 rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.04}$

Simplify. 3^5

Try It

12) Simplify:

a. 5×5^5
b. $4^9 \times 4^9$

Solution
a. 5^6
b. 4^{18}

13) Simplify:

a. {7}^{6}\times{7}^{8}
b. 10×10^{10}.

Solution
a. 7^{14}
b. 10^{11}

Example 5.2.5

Simplify:

a. $a^7 \times a$
b. $x^{27} \times x^{13}$

Solution

a.

***Step 1:* Rewrite,** $a = a^1$.

$$a^7 \times a^1$$

***Step 2:* Use the product property,** $a^m \cdot a^n = a^{m+n}$

$$3a^{rgb]1.0,0.0,0.07rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.01}$$

$$\text{Simplify.} \quad a^8$$

---

b.

***Step 1:* Notice, the bases are the same, so add the exponents.**

$$x^{rgb]1.0,0.0,0.027rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.013}$$

$$\text{Simplify.} \quad x^{40}$$

Try It

14) Simplify:

a. $p^5 \times p$
b. $y^{14} \times y^{29}$

Solution
a. p^6
b. y^{43}

15) Simplify:

a. $z \times z^7$
b. $b^{15} \times b^{34}$.

Solution
a. z^8
b. b^{49}

We can extend the Product Property for Exponents to more than two factors.

Example 5.2.6

Simplify: $d^4 \cdot d^5 \cdot d^2$

Solution
Step 1: **Add the exponents, since bases are the same.**

$$d^{rgb]1.0,0.0,0.04 + 5 + 2}rgb]1.0, 0.0, 0.0$$

$$\text{Simplify.} \quad d^{11}$$

Try It

16) Simplify: $x^6 \times x^4 \times x^8$

Solution

x^{18}

17) Simplify: $b^5 \times b^9 \times b^5$

Solution

b^{19}

Simplify Expressions Using the Power Property for Exponents

Now let's look at an exponential expression that contains a power raised to a power. See if you can discover a general property.

	$(x^2)^3$
What does this mean? How many factors altogether?	$x^2 \cdot x^2 \cdot x^2$ $\underbrace{\underbrace{x \cdot x}_{rgb]0.0,0.0,1.02rgb]0.0,0.0,1.0\ rgb]0.0,0.0,1.0\text{factors}} \cdot \underbrace{x \cdot x}_{rgb]0.0,0.0,1.02rgb]0.0,0.0,1.0\ rgb]0.0,0.0,1.0\text{factors}} \cdot \ldots rgb]0.0,0.0,1.02rgb]0.0,}_{rgb]0.0,0.0,1.06rgb]0.0,0.0,1.0\ rgb]0.0,0.0,1.0\text{factors}}$
So we have	x^6
Notice that 6 is the product of the exponents, 2 and 3.	$(x^2)^3$ is $x^{2 \cdot 3}$ or x^6

We write:

$$(x^2)^3$$
$$x^{(2 \times 3)}$$
$$x^6$$

We multiplied the exponents. This leads to the **Power Property for Exponents.**

Power Property for Exponents

If a is a real number, and m and n are whole numbers, then

$$(a^m)^n = a^{mn}$$

To raise a power to a power, multiply the exponents.

An example with numbers helps to verify this property:

$$\begin{aligned}(3^2)^3 &\stackrel{?}{=} 3^{(2\,\times\,3)}\\ (9)^3 &\stackrel{?}{=} 3^6\\ 729 &= 729\end{aligned}$$

Example 5.2.7

Simplify:

a. $(y^5)^9$

b. $(4^4)^7$

Solution

a.

Step 1: **Use the power property,** $(a^m)^n = a^{a \cdot n}$

$$y^{rgb]1.0,0.0,0.05\ \times\ 9}$$

$$\text{Simplify.}\quad y^{45}$$

---

b.

Step 1: **Use the power property.**

$$4^{rgb]1.0,0.0,0.04} \times 7$$

$$\text{Simplify.} \quad 4^{28}$$

Try It

18) Simplify:

a. $(b^7)^5$
b. $(5^4)^3$

Solution
a. b^{35}
b. 5^{12}

19) Simplify:

a. $(z^6)^9$
b. $(3^7)^7$

Solution
a. z^{54}
b. 3^{49}

Simplify Expressions Using the Product to a Power Property

We will now look at an expression containing a product that is raised to a power. Can you find this pattern?

	$(2x)^3$
What does this mean?	$2x \cdot 2x \cdot 2x$
We group the like factors together.	$2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x$
How many factors of 2 and of x?	$2^3 \times x^3$

Notice that each factor was raised to the power and $(2x)^3$ is $2^3 \times x^3$.

We write:

$$(2x)^3$$
$$2^3 \times x^3$$

The exponent applies to each of the factors! This leads to the Product to a Power Property for Exponents.

Product to a Power Property for Exponents

If a and b are real numbers and m is a whole number, then

$$(ab)^m = a^m b^m$$

To raise a product to a power, raise each factor to that power.

An example with numbers helps to verify this property:

$$\begin{aligned} 2(2 \times 3)^2 &\stackrel{?}{=} 2^2 3^2 \\ 6^2 &\stackrel{?}{=} (4)(9) \\ 36 &= 36 \end{aligned}$$

Example 5.2.8

Simplify:

a. $(-9d)^2$
b. $(3mn)^3$

Solution
a.

Step 1: **Use Power of a Product Property,** $(ab)^m = a^m b^m$

$$(-9)^{rgb]1.0,0.0,0.0,0.02} d^{rgb]1.0,0.0,0.0,0.09}$$

$$\text{Simplify.} \quad 81d^2$$

---

b.

Step 1: **Use Power of a Product Property,** $(ab)^m = a^m b^m$

$$(3)^{rgb]1.0,0.0,0.0,0.03} m^{rgb]1.0,0.0,0.0,0.03} n^{rgb]1.0,0.0,0.0,0.03}$$

$$\text{Simplify.} \quad 27m^3n^3$$

Try It

20) Simplify:

a. $(-12y)^2$
b. $(2wx)^5$

Solution
a. $144y^2$
b. $32w^5x^5$

21) Simplify:

a. $5wx^3$
b. $(-3y)^3$

Solution

a. $125w^3x^3$
b. $-27y^3$

Simplify Expressions by Applying Several Properties

We now have three properties for multiplying expressions with exponents. Let's summarize them and then we'll do some examples that use more than one of the properties.

Properties of Exponents

If a and b are real numbers, and m and n are whole numbers, then

Product Property	{a}^{m}\times {a}^{n} = {a}^{m+n}
Power Property	$(a^m)^n = a^{mn}$
Product to a Power	$(ab)^m = a^m b^m$

All exponent properties hold true for any real numbers m and n. Right now, we only use whole-number exponents.

Example 5.2.9

Simplify:

a. ({y}^{3})^{6}({y}^{5})^{4}
b. $(-6x^4y^5)^2$

Solution
a.

Step 1: **Use the Power Property.**

$$y^{18} \times y^{20}$$

Step 2: **Add the exponents.**

$$y^{38}$$

---

b.

Step 1: **Use the Product to a Power Property.**

$$(-6)^2(x^4)^2(y^5)^2$$

Step 2: **Use the Power Property.**

$$(-6)^2(x^4)^2(y^5)^2$$
$$\text{Simplify.} \quad 36x^8y^{10}$$

Try It

22) Simplify:

a. $(a^4)^5(a^7)^4$
b. $(-2c^4d^2)^3$

Solution

a. a^{48}
b. $(-8c^{12}d^6)$

23) Simplify:

a. $(-3x^6y^7)^4$
b. $(q^4)^5(q^3)^3$

Solution

a. $81x^{24}y^{28}$
b. q^{29}

Example 5.2.10

Simplify:

a.(5m)^{2}(3{m})^{3}
b. $(3x^2y)^4(2xy^2)^3$

Solution

a.

Step 1: **Raise $5m$ to the second power.**

$$5^2m^2 \times 3m^3$$
$$\text{Simplify.}\ \ 25m^2 \times 3m^3$$

Step 2: **Use the Commutative Property.**

$$25 \times 3 \times m^2 \times m^3$$

Step 3: **Multiply the constants and add the exponents.**

$$75m^5$$

---

b.

Step 1: **Use the Product to a Power Property.**

$$(3^4x^8y^4)(2^3x^3y^6)$$

$$\text{Simplify.} \quad (81x^8y^4)(8x^3y^6)$$

Step 2: **Use the Commutative Property.**

$$81 \times 8 \times x^8 \times x^3 \times y^4 \times y^6$$

Step 3: **Multiply the constants and add the exponents.**

$$648x^{11}y^{10}$$

Try It

24) Simplify:

a. $(5n)^2(3n^{10})$
b. $(c^4d^2)^5(3cd^5)^4$

Solution

a. $75n^{12}$
b. $81c^{24}d^{30}$

25) Simplify:

a. $(a^3b^2)^6(4ab^3)^4$
b. $(2x)^3(5x^7)$

Solution

a. $256a^{22}b^{24}$
b. $40x^{10}$

Multiply Monomials

Since a monomial is an algebraic expression, we can use the properties of exponents to multiply monomials.

Example 5.2.11

Multiply: $(3x^2)(-4x^3)$

Solution

Step 1: **Use the Commutative Property to rearrange the terms.**

$$3(-4)x^2x^3$$

$$\text{Multiply.} \quad -12x^5$$

Try It

26) Multiply: $(5y^7)(-7y^4)$

Solution

$-35y^{11}$

27) Multiply: $(-6b^4)(-9b^5)$

Solution

$54b^9$

Example 5.2.12

Multiply: $(\frac{5}{6}x^3y)(12xy^2)$

Solution

Step 1: **Use the Commutative Property to rearrange the terms.**

$$\frac{5}{6} \times 12x^3xyy^2$$

$$\text{Multiply.} \quad 10x^4y^3$$

Try It

28) Multiply: $(\frac{2}{5}a^4b^3)(15ab^3)$

Solution

$6a^5b^6$

29) Multiply: $(\frac{2}{3}r^5s)(12r^6s^7)$

Solution

$8r^{11}s^8$

Use the Definition of a Negative Exponent

We saw that the Quotient Property for Exponents introduced earlier in this text, has two forms depending on whether the exponent is larger in the numerator or the denominator.

Quotient Property for Exponents

If a is a real number, $a \neq 0$, and m and n are whole numbers, then:

$$\frac{a^m}{a^n} = a^{m-n}, \text{ for } m > n \text{ and } \frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \text{ for } n > m.$$

What if we just subtract exponents regardless of which is larger?

Let's consider $\frac{x^2}{x^5}$

We subtract the exponent in the denominator from the exponent in the numerator.

$$\frac{x^2}{x^5}$$

$$x^{2-5}$$

$$x^{-3}$$

We can also simplify $\frac{x^2}{x^5}$ by dividing out common factors:

$$\frac{\cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}$$

$$\frac{1}{x^3}$$

This implies that $x^{-3} = \frac{1}{x^3}$ and it leads us to the definition of a *negative exponent*.

Negative Exponents

If n is an integer and $a \neq 0$, then $a^{-n} = \frac{1}{a^n}$.

The negative exponent tells us we can re-write the expression by taking the reciprocal of the base and then changing the sign of the exponent.

Any expression that has negative exponents is not considered to be in simplest form. We will use the definition of a negative exponent and other properties of exponents to write the expression with only positive exponents.

For example, if after simplifying an expression we end up with the expression x^{-3}, we will take one more step and write $\frac{1}{x^3}$. The answer is considered to be in simplest form when it has only positive exponents.

Example 5.2.13

Simplify:

a. 4^{-2}
b. 10^{-3}

Solution

a.

Step 1: **Use the definition of a negative exponent,** $a^{-n} = \frac{1}{a^n}$

$$\frac{1}{4^2}$$

$$\text{Simplify.} \quad \frac{1}{16}$$

---

b.

Step 1: **Use the definition of a negative exponent,** $a^{-n} = \frac{1}{a^n}$

$$\frac{1}{10^{-3}}$$

$$\text{Simplify.} \quad \frac{1}{1000}$$

Try It

30) Simplify:

a. 2^{-3}
b. 10^{-7}

Solution
a. $\frac{1}{8}$
b. $\frac{1}{10^7}$

31) Simplify:

a. 3^{-2}
b. 10^{-4}

Solution
a. $\frac{1}{9}$
b. $\frac{1}{10,000}$

In example 5.2.13 we raised an integer to a negative exponent. What happens when we raise a fraction to a negative exponent? We'll start by looking at what happens to a fraction whose numerator is one and whose denominator is an integer raised to a negative exponent.

	$\frac{1}{a^{-}n}$
Step 1: **Use the definition of a negative exponent,** $a^{-n} = \frac{1}{a^n}$	$\frac{1}{\frac{1}{a^n}}$
Step 2: **Simplify the complex fraction.**	$1 \cdot \frac{a^n}{1}$
Step 3. **Multiply.**	a^n

This leads to the Property of Negative Exponents.

Property of Negative Exponents

If n is an integer and $a \neq 0$, then $\frac{1}{a^{-n}} = a^n$

Example 5.2.14

Simplify:

a. \frac{1}{{y}^{-4}}

b. $\frac{1}{3^{-2}}$

Solution

a.

Step 1: **Use the property of a negative exponent,** $\frac{1}{a^{-n}} = a^n$

y^4

b.

Step 1: **Use the property of a negative exponent,** $\frac{1}{a^{-n}} = a^n$

$$3^2$$

$$\text{Simplify.} \quad 9$$

Try It

32) Simplify:

a. \frac{1}{{p}^{-8}}

b. $\frac{1}{4^{-3}}$

Solution

a. p^8

b. 64

33) Simplify:

a. \frac{1}{{q}^{-7}}

b. $\frac{1}{2^{-4}}$

Solution

a. q^7

b. 16

Suppose now we have a fraction raised to a negative exponent. Let's use our definition of negative exponents to lead us to a new property.

	$(\frac{3}{4})^{-2}$
Step 1: **Use the definition of a negative exponent,** $a^{-n} = \frac{1}{a^n}$	$\frac{1}{(\frac{3}{4})^2}$
Step 2: **Simplify the denominator.**	$\frac{1}{\frac{9}{16}}$
Step 3: **Simplify the complex fraction.**	$\frac{16}{9}$
But we know that $\frac{16}{9}$ is $(\frac{4}{3})^2$	$\frac{4^2}{3^2}$
This tells us that:	$(\frac{3}{4})^{-2} = (\frac{4}{3})^2$

To get from the original fraction raised to a negative exponent to the final result, we took the reciprocal of the base—the fraction—and changed the sign of the exponent.

This leads us to the Quotient to a Negative Power Property.

Quotient to a Negative Exponent Property

If a and b are real numbers, $a \neq 0$, $b \neq 0$, and n is an integer, then $(\frac{a}{b})^{-n} = (\frac{b}{a})^n$

Example 5.2.15

Simplify:

a. $(\frac{5}{7})^{-2}$
b. $(-\frac{2x}{y})^{-3}$

Solution

a.

Step 1: **Use the Quotient to a Negative Exponent Property,** $(\frac{a}{b})^{-n} = (\frac{b}{a})^{n}$

Step 2: **Take the reciprocal of the fraction and change the sign of the exponent.**

$$(\frac{7}{5})^2$$

$$\text{Simplify.} \quad \frac{49}{25}$$

---

b.

Step 1: **Use the Quotient to a Negative Exponent Property,** $(\frac{a}{b})^{-n} = (\frac{b}{a})^{n}$

Step 2: **Take the reciprocal of the fraction and change the sign of the exponent.**

$$(-\frac{y}{2x})^3$$

$$\text{Simplify.} \quad -\frac{y^3}{8x^3}$$

Try It

34) Simplify:

a. $\left(\frac{2}{3}\right)^{-4}$

b. $(-\frac{6m}{n})^{-2}$

Solution

a. $\frac{81}{16}$

b. $\frac{n^2}{36m^2}$

35) Simplify:

a. $(\frac{3}{5})^{-3}$

b. $(-\frac{a}{2b})^{-4}$

Solution

a. $\frac{125}{27}$

b. $\frac{16b^4}{a^4}$

When simplifying an expression with exponents, we must be careful to correctly identify the base.

Example 5.2.16

Simplify:

a. $(-3)^{-2}$

b. -3^{-2}

c. $(-\frac{1}{3})^{-2}$

d. $-(\frac{1}{3})^{-2}$

Solution

a. Here the exponent applies to the base -3.

Step 1: **Take the reciprocal of the base and change the sign of the exponent.**

$$\frac{1}{(-3)^2}$$

$$\text{Simplify.} \quad \frac{1}{9}$$

---

b. The expression -3^{-2} means "find the opposite of 3^{-2}"

Step 1: **Rewrite as a product with -1.**

$$-1 \times 3^{-2}$$

Step 2: **Take the reciprocal of the base and change the sign of the exponent.**

$$-1 \cdot \frac{1}{3^2}$$

$$\text{Simplify.} \quad -\frac{1}{9}$$

---

c. Here the exponent applies to the base $-\frac{1}{3}$.

Step 1: **Take the reciprocal of the base and change the sign of the exponent.**

$$(-\frac{3}{1})^2$$

$$\text{Simplify.} \quad (-\frac{1}{3})^{-2}$$

---

d. The expression $-(\frac{1}{3})^{-2}$ means "find the opposite of $(\frac{1}{3})^{-2}$." Here the exponent applies to the base $\frac{1}{3}$.

Step 1: **Rewrite as a product with -1.**

$$-1 \cdot (\frac{3}{1})^2$$

Step 2: **Take the reciprocal of the base and change the sign of the exponent.**

$$-9$$

Try It

36) Simplify:

a. $(-5)^{-2}$

b. -5^{-2}

c. $(-\frac{1}{5})^{-2}$

d. $-(\frac{1}{5})^{-2}$

Solution

a. $\frac{1}{25}$

b. $-\frac{1}{25}$

c. 25

d. -25

37) Simplify:

a. $(-7)^{-2}$

b. -7^{-2}

c. (-\frac{1}{7})^{-2}

d. $(-\frac{1}{7})^{-2}$

Solution

a. $\frac{1}{49}$

b. $-\frac{1}{49}$

c. 49

d. -49

We must be careful to follow the Order of Operations. In the next example, parts (a) and (b) look similar, but the results are different.

Example 5.2.17

Simplify:

a. 4×2^{-1}

b. $(4 \times 2)^{-1}$

Solution

a.

Step 1: **Do exponents before multiplication.**

$$4 \times 2^{-1}$$

Step 2: **Use** $a^{-n} = \frac{1}{a^n}$

$$4 \times \frac{1}{2^1}$$

$$\text{Simplify.} \quad 2$$

b.

Step 1: **Simplify inside the parentheses first.**

$$(8)^{-1}$$

Step 2: **Use** $a^{-n} = \frac{1}{a^n}$

$$\frac{1}{8^1}$$

$$\text{Simplify.} \quad \frac{1}{8}$$

Try It

38) Simplify:

a. 6×3^{-1}
b. $(6 \times 3)^{-1}$

Solution

a. 2
b. $\frac{1}{18}$

39) Simplify:

a. 8×2^{-2}
b. 8×2^{-2}

Solution

a. 2
b. $\frac{1}{256}$

When a variable is raised to a negative exponent, we apply the definition the same way we did with numbers. We will assume all variables are non-zero.

Example 5.2.18

Simplify:

a. {x}^{-6}
b. $(u^4)^{-3}$

Solution
a.

Step 1: **Use the definition of a negative exponent** $a^{-n} = \frac{1}{a^n}$

$$\frac{1}{x^6}$$

---

b.

Step 1: **Use the definition of a negative exponent** $a^{-n} = \frac{1}{a^n}$

$$\frac{1}{(u^4)^3}$$

$$\text{Simplify.} \quad \frac{1}{u^{12}}$$

Try It

40) Simplify:

a. y^{-7}
b. $(z^3)^{-5}$

Solution

a. $\frac{1}{y^7}$
b. $\frac{1}{z^{15}}$

41) Simplify:

a. p^{-9}
b. $(q^4)^{-6}$

Solution

a. $\frac{1}{p}^{9}$
b. $\frac{1}{q}^{24}$

When there is a product and an exponent we have to be careful to apply the exponent to the correct quantity. According to the Order of Operations, we simplify expressions in parentheses before applying exponents. We'll see how this works in the next example.

Example 5.2.19

Simplify:

a. $5y^{-1}$

b. 5y^{-1}

c. $(-5y)^{-1}$

Solution

a.

Step 1: **Notice the exponent applies to just the base y.**

$$5\cdot\left(\frac{1}{y^1}\right)$$

Step 2: **Take the reciprocal of y and change the sign of the exponent.**

$$\frac{5}{y}$$

$$\text{Simplify.}\quad (5y)^{-1}$$

---

b.

Step 1: **Here the parentheses make the exponent apply to the base $5y$.**

Step 2: **Take the reciprocal of $5y$ and change the sign of the exponent.**

$$\frac{1}{5y^1}$$

$$\text{Simplify.}\quad \frac{1}{5y}$$

---

c.

Step 1: **The base here is $-5y$.**

Step 2: **Take the reciprocal of $-5y$ and change the sign of the exponent.**

$$\frac{1}{-5y}$$

$$\text{Simplify.}\quad \left(-\frac{1}{5y}\right)$$

Try It

42) Simplify:

a. (8{p}^{-1})
b. $(8p)^{-1}$
c. $(-8p)^{-1}$

Solution

a. $\frac{8}{p}$
b. $\frac{1}{8p}$
c. $-\frac{1}{8p}$

43) Simplify:

a. {11q}^{-1}
b. $(11q)^{-1}$
c. $(-11q)^{-1}$

Solution

a. $\frac{1}{11q}$
b. $\frac{1}{11q}$
c. $-\frac{1}{11q}$

With negative exponents, the Quotient Rule needs only one form $\frac{a^m}{a^n} = a^{m-n}$, for $a \neq 0$. When the exponent in the denominator is larger than the exponent in the numerator, the exponent of the quotient will be negative.

Simplify Expressions with Integer Exponents

All of the exponent properties we developed earlier in the chapter with whole number exponents apply to integer exponents, too. We restate them here for reference.

Summary of Exponent Properties

If a and b are real numbers, and m and n are integers, then

Product Property	$a^m \times a^n = a^{m+n}$
Power Property	$(a^m)^n = amn$
Product to a Power	$(ab)^m = a^m b^m$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}$
Zero Exponent Property	$a^0 = 1, a \neq 1$
Quotient to a Power Property	$(\frac{a}{b})^m = \frac{a^m}{b^m}, b \neq 0$
Properties of Negative Exponents	$a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$
Quotient to a Negative Exponent	(\frac{a}{b})^{-n} = (\frac{b}{a})^{n}

Example 5.2.20

Simplify:

a. $x^{-4} \times x^{6}$
b. $y^{-6} \times y^{4}$
c. $z^{-5} \times z^{-3}$

Solution

a.

Step 1: **Use the Product Property,** $a^m a^n = a^{m=n}$

$$x^{-4+6}$$

$$\text{Simplify.} \;\; x^2$$

---

b.

Step 1: **Notice the same bases, so add the exponents.**

$$y^{-6+4}$$

$$\text{Simplify.} \;\; y^{-2}$$

---

c.

Step 1: **Add the exponents, since the bases are the same.**

$$z^{-5-3}$$

$$\text{Simplify.} \;\; z^{-8}$$

Step 3: **Take the reciprocal and change the sign of the exponent, using the definition of a negative exponent.**

$$\frac{1}{z^8}$$

Try It

44) Simplify:

a. $x^{-3} \times x^{7}$
b. $y^{-7} \times y^{2}$
c. $z^{-4} \times z^{-5}$

Solution

a. x^{4}
b. $\frac{1}{y^{5}}$
c. $\frac{1}{z^{9}}$

45) Simplify:

a. $a^{-1} \times a^{6}$
b. b^{-8}\times b^{4}
c. $c^{-8} \times c^{-7}$

Solution

a. a^{5}
b. $\frac{1}{b^{4}}$
c. $\frac{1}{c^{15}}$

In the next two examples, we'll start by using the Commutative Property to group the same variables together. This makes it easier to identify the like bases before using the Product Property.

Example 5.2.21

Simplify: $(m^4n^{-3})(m^{-5}n^{-2})$

Solution

Step 1: **Use the Commutative Property to get like bases together.**

$$m^4m^{-5}n^{-2}n^{-3}$$

Step 2: **Add the exponents for each base.**

$$m^{-1}n^{-5}$$

Step 3: **Take reciprocals and change the signs of the exponents.**

$$\frac{1}{m^1}\frac{1}{n^5}$$

$$\text{Simplify.}\quad \frac{1}{mn^5}$$

Try It

46) Simplify: $(p^6q^{-2})(p^{-9}q^{-1})$

Solution

$$\frac{1}{p^3q^3}$$

47) Simplify: $(r^5s^{-3})(r^{-7}s^{-5})$

Solution

$$\frac{1}{r^2 s^8}$$

If the monomials have numerical coefficients, we multiply the coefficients, just like we did earlier.

Example 5.2.22

Simplify: $(2x^{-6}y^8)(-5x^5y^{-3})$

Solution

Step 1: **Rewrite with the like bases together.**

$$2(-5)(x^{-6}x^5)(y^8y^{-3})$$

Step 2: **Multiply the coefficients and add the exponents of each variable.**

$$-10x^{-1}y^5$$

Step 3: **Use the definition of a negative exponent,** $a^{-n} = \frac{1}{a^n}$

$$-10 \cdot \left(\frac{1}{x^1}\right) \cdot y^5$$

$$\text{Simplify.} \quad -\frac{10y^5}{x}$$

Try It

48) Simplify: $(3u^{-5}v^7)(-4u^4v^{-2})$

Solution

$$-\frac{12v^5}{u}$$

49) Simplify: $(-6c^{-6}d^4)(-5c^{-2}d^{-1})$

Solution

$$\frac{30d^3}{c^8}$$

In the next two examples, we'll use the Power Property and the Product to a Power Property.

Example 5.2.23

Simplify: $(6k^3)^{-2}$

Solution

Step 1: **Use the Product to a Power Property,** $(ab)^m = a^m b^m$

$$(6)^{(-2)}(k^3)^{-2}$$

Step 2: **Use the Power Property,** $(a^m)^n = a^{mn}$

$$6^{-2}k^{-6}$$

Step 3: **Use the Definition of a Negative Exponent,** $a^{-n} = \frac{1}{a^n}$

$$\frac{1}{6^2} \cdot \frac{1}{k^6}$$

$$\text{Simplify.} \quad \frac{1}{36k^6}$$

Try It

50) Simplify: $(-4x^4)^{-2}$

Solution

$\dfrac{1}{16x^8}$

51) Simplify: $\left(2b^3\right)^{-4}$

Solution

$\dfrac{1}{16b^{12}}$

Example 5.2.24

Simplify: $(5x^{-3})^2$

Solution

Step 1: **Use the Product to a Power property,** $(ab)^m = a^m b^m$

$$5^2(x^{-3})^2$$

Step 2: **Simplify** 5^2 **and multiply the exponents of** x **using the Power Property,** $(a^m)^n = a^{mn}$

$$25x^{-6}$$

Step 3: **Rewrite** x^{-6} **by using the Definition of a Negative Exponent,** $a^{-n} = \dfrac{1}{a^n}$

$$25 \cdot \left(\frac{1}{x^6}\right)$$

$$\text{Simplify.} \quad \frac{25}{x^6}$$

Try It

52) Simplify: $(8a^{-4})^2$

Solution

$\frac{64}{a^8}$

53) Simplify: $(2c^{-4})^3$

Solution

$\frac{8}{c^{12}}$

To simplify a fraction, we use the Quotient Property and subtract the exponents.

Example 5.2.25

Simplify: $\frac{r^5}{r^{-4}}$

Solution

Step 1: **Use the Quotient Property,** $\frac{a^m}{a^n} = a^{m-n}$

$$r^{5-(-4)}$$

$$\text{Simplify.} \quad r^9$$

Try It

54) Simplify: $\frac{x^8}{x^{-3}}$

Solution

x^{11}

55) Simplify: $\frac{y^8}{y^{-6}}$

Solution

y^{14}

Access these online resources for additional instruction and practice with using multiplication properties of exponents:

- Multiplication Properties of Exponents

Key Concepts

- **Exponential Notation**

  $a^{m rgb]0.0,0.0,1.0 \leftarrow rgb]0.0,0.0,1.0 \text{exponent}}$

  $rgb]1.0, 0.0, 0.0 \underset{base}{\uparrow}$

  a^m means multiply m factors of a

  $$a^m = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{rgb]0.0,0.0,1.0 m \text{ factors}}$$

- **Properties of Exponents**
  - If a, and b are real numbers and m, n are whole numbers, then
  - Product Property $a^m a^n = a^{m+n}$
  - Power Property $(a^m)^n = a^{mn}$
  - Product to a Power $(ab)^m = a^m b^m$
- **Property of Negative Exponents**
  - If n is a positive integer and $a \neq 0$, then $\frac{1}{a}^{-n} = a^n$
- **Quotient to a Negative Exponent**
  - If a, and b are real numbers, $b \neq 0$ and n is an integer, then $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here: https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=2663#h5p-27

b. After reviewing this checklist, what will you do to become confident for all goals?

5.3 MULTIPLY POLYNOMIALS

Learning Objectives

By the end of this section, you will be able to:

- Multiply a polynomial by a monomial
- Multiply a binomial by a binomial
- Multiply a trinomial by a binomial

Try it

Before you get started, take this readiness quiz:

1) Distribute: $2(x+3)$
2) Combine like terms: $x^2+9x+7x+63$

Multiply a Polynomial by a Monomial

We have used the Distributive Property to simplify expressions like $2(x-3)$. You multiplied both terms in the parentheses, x and 3 by 2, to get $2x-6$. With this chapter's new vocabulary, you can say you were multiplying a binomial, $x-3$, by a monomial, 2.

Multiplying a binomial by a monomial is nothing new for you! Here's an example:

Example 5.3.1

Multiply: $4(x+3)$

Solution

Figure 5.3.1

Step 1: **Distribute.**

$$4 \times x + 4 \times 3$$

$$\text{Simplify.} \quad 4x + 12$$

Try It

3) Multiply: $5(x+7)$

Solution

$5x+35$

4) Multiply: $3(y+13)$

Solution

$3y+39$

Example 5.3.2

Multiply: $y(y-2)$

Solution

Figure 5.3.2

Step 1: **Distribute.**

$$y \times y - y \times 2$$

$$\text{Simplify.} \;\; y^2 - 2y$$

Try It

5) Multiply: $x(x-7)$

Solution

$x^2 - 7x$

6) Multiply: $d(d-11)$

Solution

$d^2 - 11d$

Example 5.3.3

Multiply: $7x(2x+y)$

Solution

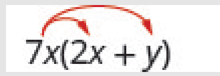


Figure 5.3.3

***Step 1:* Distribute.**

$$\begin{array}{r}7x \;\times\; 2x \;+\; 7x \;\times\; y \\ \text{Simplify.}\;\; 14x^2 \;+\; 7xy\end{array}$$

Try It

7) Multiply: $5x(x+4y)$

Solution

$5x^2+20xy$

8) Multiply: $2p(6p+r)$

Solution

$12p^2+2pr$

Example 5.3.4

Multiply: $-2y(4y^2+3y-5)$

Solution

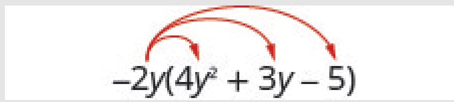


Figure 5.3.4

Step 1: **Distribute.**

$$\begin{array}{rl} & -2y \times 4y^2 + (-2y) \times 3y - (-2y) \times 5 \\ \text{Simplify.} & -8y^3 - 6y^2 + 10y \end{array}$$

Try It

9) Multiply: $-3y(5y^2+8y-7)$

Solution

$-15y^3-24y^2+21y$

10) Multiply: $4x^2(2x^2-3x+5)$

Solution

$8x^4-24x^3+20x^2$

Example 5.3.5

Multiply: $2x^3(x^2 - 8x + 1)$

Solution

Figure 5.3.5

Step 1: **Distribute.**

$$2x^3 \times x^2 + (2x^3) \times (-8x) + (2x^3) \times 1$$

$$\text{Simplify.} \quad 2x^5 - 16x^4 + 2x^3$$

Try It

11) Multiply: $4x(3x^2 - 5x + 3)$

Solution

$12x^3 - 20x^2 + 12x$

12) Multiply: $-6a^3(3a^2 - 2a + 6)$

Solution

$-18a^5 + 12a^4 - 36a^3$

Example 5.3.6

Multiply: $(x+3)p$.

Solution

Step 1: **The monomial is the second factor.**

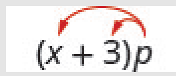


Figure 5.3.6

Step 2: **Distribute.**

$$x \times p + 3 \times p$$

$$\text{Simplify.} \quad xp + 3p$$

Try It

13) Multiply: $(x+8)p$

Solution

$xp+8p$

14) Multiply: $(a+4)p$

Solution

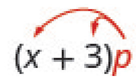
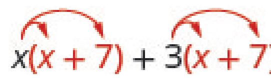
$ap+4p$

Multiply a Binomial by a Binomial

Just like there are different ways to represent multiplication of numbers, there are several methods that can be used to multiply a binomial times a binomial. We will start by using the Distributive Property.

Multiply a Binomial by a Binomial Using the Distributive Property

Look at below table, where we multiplied a binomial by a monomial.

	
We distributed the p to get:	$xrgb]1.0, 0.0, 0.0p + 3rgb]1.0, 0.0, 0.0p$
What if we have $(x+7)$ instead of p?	
Distribute $(x+7)$.	
Distribute again.	$x^2 + 7x + 3x + 21$
Combine like terms.	$x^2 + 10x + 21$

Notice that before combining like terms, you had four terms. You multiplied the two terms of the first binomial by the two terms of the second binomial—four multiplications.

Example 5.3.7

Multiply: $(y+5)(y+8)$

Solution

Figure 5.3.7

Step 1: **Distribute** $(y+8)$**.**

$yrgb]1.0, 0.0, 0.0(rgb]1.0, 0.0, 0.0yrgb]1.0, 0.0, 0.0\ rgb]1.0, 0.0, 0.0+rgb]1.0, 0.0, 0.0\ rgb]1.0, 0.0, 0.08rgb]1.0, 0.0, 0.0) + 5rgb]1.0, 0.0, 0.0(rgb]1.0, 0.0, 0.0yrgb]1.0, 0.0, 0.0\ rgb]1.0, 0.0, 0.0+rgb]1.0, 0.0, 0.0\ rgb]1.0, 0.0, 0.08rgb]1.0, 0.0, 0.0)$

Step 2: **Distribute again**

$$y^2 + 8y + 5y + 40$$

Step 3: **Combine like terms.**

$$y^2 + 13y + 40$$

Try It

15) Multiply: $(x+8)(x+9)$

Solution

$x^2 + 17x + 72$

16) Multiply: $(5x+9)(4x+3)$

Solution

$20x^2 + 51x + 27$

Example 5.3.8

Multiply: $(2y+5)(3y+4)$

Solution

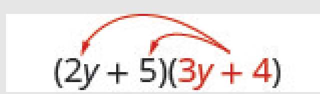


Figure 5.3.8

Step 1: **Distribute** $(3y+4)$**.**

$$2y rgb]1.0, 0.0, 0.0(rgb]1.0, 0.0, 0.03rgb]1.0, 0.0, 0.0y rgb]1.0, 0.0, 0.0\ rgb]1.0, 0.0, 0.0+rgb]1.0, 0.0, 0.0\ rgb]1.0, 0.0, 0.04rgb]1.0, 0.0, 0.0) \ + \ 5rgb]1.0, 0.0, 0.0(rgb]1.0, 0.0, 0.03rgb]1.0, 0.0, 0.0y rgb]1.0, 0.0, 0.0\ rgb]1.0, 0.0, 0.0+rgb]1.0, 0.0, 0.0\ rgb]1.0, 0.0, 0.04rgb]1.0, 0.0, 0.0)$$

Step 2: **Distribute again**

$$6y^2 \ + \ 8y \ + \ 15y \ + \ 20$$

Step 3: **Combine like terms.**

$$6y^2 \ + \ 23y \ + \ 20$$

Try It

17) Multiply: $(3b + 5)(4b + 6)$

Solution

$12b^2 + 38b + 30$

18) Multiply: $(a + 10)(a + 7)$

Solution

$a^2 + 17a + 70$

Example 5.3.9

Multiply: $(4y + 3)(2y - 5)$

Solution

Step 1: **Distribute.**

$$4y rgb]1.0, 0.0, 0.0(rgb]1.0, 0.0, 0.02rgb]1.0, 0.0, 0.0y rgb]1.0, 0.0, 0.0\ rgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.0\ rgb]1.0, 0.0, 0.05rgb]1.0, 0.0, 0.0) \ + \ 3rgb]1.0, 0.0, 0.0(rgb]1.0, 0.0, 0.02rgb]1.0, 0.0, 0.0y rgb]1.0, 0.0, 0.0\ rgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.0\ rgb]1.0, 0.0, 0.05rgb]1.0, 0.0, 0.0)$$

Step 2: **Distribute again.**

$$8y^2 \;-\; 20y \;+\; 6y \;-\; 15$$

***Step 3:* Combine like terms.**

$$8y^2 \;-\; 14y \;-\; 15$$

Try It

19) Multiply: $(5y+2)(6y-3)$

Solution

$30y^2-3y-6$

20) Multiply: $(3c+4)(5c-2)$

Solution

$15c^2+14c-8$

Example 5.3.10

Multiply: $(x-2)(x-y)$

Solution

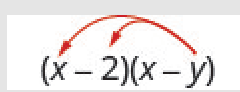


Figure 5.3.9

***Step 1:* Distribute.**

$x rgb]1.0, 0.0, 0.0(rgb]1.0, 0.0, 0.0x rgb]1.0, 0.0, 0.0\ rgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.0\ rgb]1.0, 0.0, 0.0y rgb]1.0, 0.0, 0.0)\ rgb]1.0, 0.0, 0.0-\ 2rgb]1.0, 0.0, 0.0(rgb]1.0, 0.0, 0.0x rgb]1.0, 0.0, 0.0\ rgb]1.0, 0.0, 0.0-rgb]1.0, 0.0, 0.0\ rgb]1.0, 0.0, 0.0y rgb]1.0, 0.0, 0.0)$

Step 2: **Distribute again.**

$$x^2\ -\ xy\ -\ 2x\ +\ 2y$$

Step 3: **There are no like terms to combine.**

Try It

21) Multiply: $(a+7)(a-b)$

Solution

$a^2-ab+7a-7b$

22) Multiply: $(x+5)(x-y)$

Solution

$x^2-xy+5x-5y$

Multiply a Binomial by a Binomial Using the FOIL Method

Remember that when you multiply a binomial by a binomial you get four terms. Sometimes you can combine like terms to get a trinomial, but sometimes, like in the Example 5.3.10 there are no like terms to combine.

Let's look at the last example again and pay particular attention to how we got the four terms.

$$(x-2)(x-y)$$
$$x^2-xy-2x+2y$$

Where did the first term, x^2, come from?

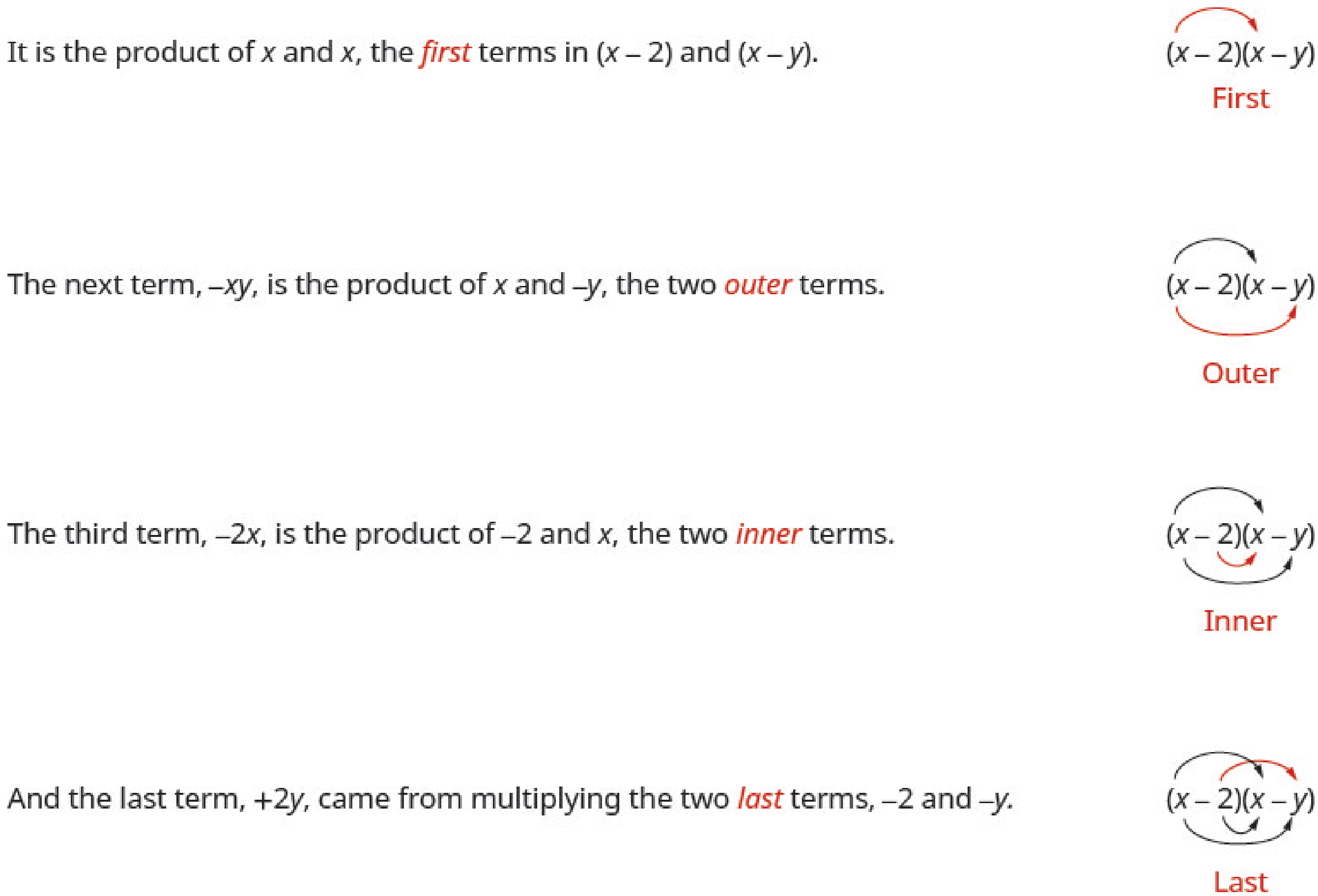
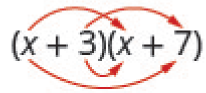


Figure 5.3.10

We abbreviate "First, Outer, Inner, Last" as FOIL. The letters stand for '**F**irst, **O**uter, **I**nner, **L**ast'. The word FOIL is easy to remember and ensures we find all four products.

$$(x-2)(x-y)$$
$$x^2-xy-2x+2y$$
F. O. I. L

Let's look at $(x+3)(x+7)$.

Distributive Property	FOIL
$(x+3)(x+7)$	 Figure 5.3.11
$x(x+7)+3(x+7)$	
$\begin{matrix} x^2 & + & 7x & + & 3x & + & 21 \\ F & & O & & I & & L \end{matrix}$	$\begin{matrix} x^2 & + & 7x & + & 3x & + & 21 \\ F & & O & & I & & L \end{matrix}$
$x^2+10x+21$	$x^2+10x+21$

Notice how the terms in third line fit the FOIL pattern.

Now we will do an example where we use the FOIL pattern to multiply two binomials.

Example 5.3.11

Multiply using the FOIL method: $(x+5)(x+9)$

Solution

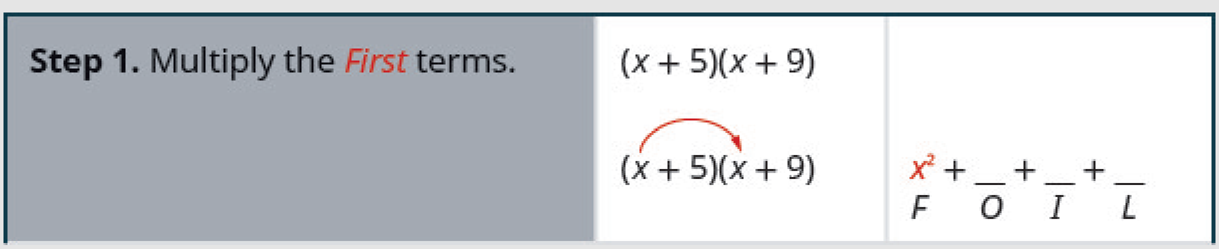


Figure 5.3.12

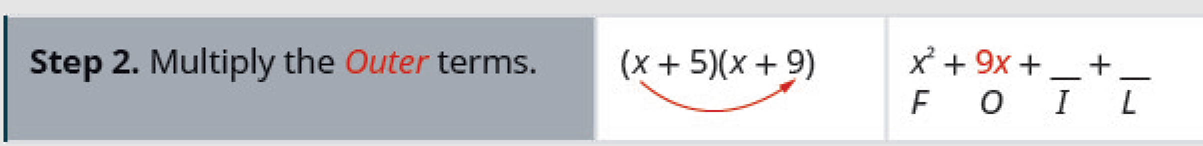


Figure 5.3.13

Step 3. Multiply the *Inner* terms.	$(x+5)(x+9)$	$x^2 + 9x + 5x + \_$ F O I L

Figure 5.3.14

Step 4. Multiply the *Last* terms.	$(x+5)(x+9)$	$x^2 + 9x + 5x + 45$ F O I L

Figure 5.3.15

Step 5. Combine like terms, when possible.		$x^2 + 14x + 45$

Figure 5.3.16

Try It

23) Multiply using the FOIL method: $(x+6)(x+8)$

Solution

$x^2 + 14x + 48$

24) Multiply using the FOIL method: $(y+17)(y+3)$

Solution

$y^2 + 20y + 51$

We summarize the steps of the FOIL method below. The FOIL method only applies to multiplying binomials, not other polynomials!

Multiply two binomials using the FOIL method

HOW TO

Step 1. Multiply the *First* terms.

Step 2. Multiply the *Outer* terms.

Step 3. Multiply the *Inner* terms.

Step 4. Multiply the *Last* terms.

Step 5. Combine like terms, when possible.

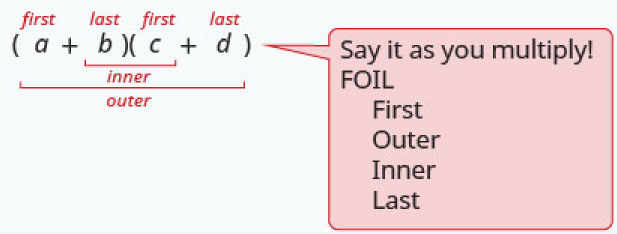


Figure 3.5.17

When you multiply by the FOIL method, drawing the lines will help your brain focus on the pattern and make it easier to apply.

Example 5.3.12

Multiply: $(y - 7)(y + 4)$

Solution

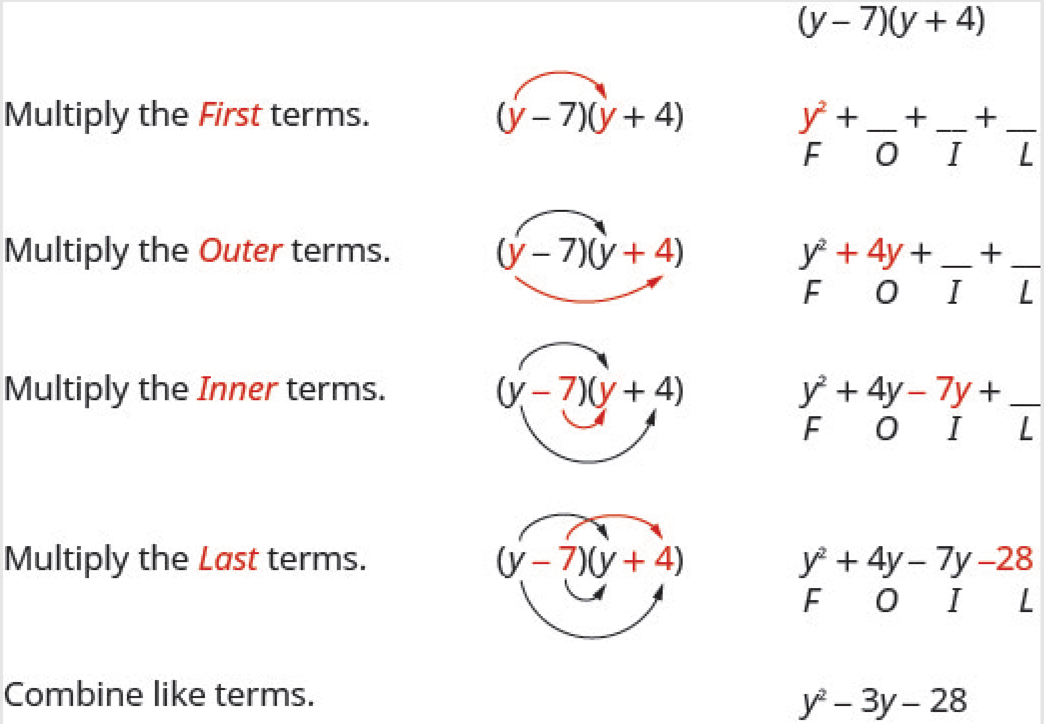


Figure 3.5.18

Try It

25) Multiply: $(x-7)(x+5)$

Solution

$x^2-2x-35$

26) Multiply: $(b-3)(b+6)$

Solution

$b^2+3b-18$

Example 5.3.13

Multiply: $(4x+3)(2x-5)$

Solution

	$(4x-3)(2x-5)$
	$(4x+3)(2x-5)$
Multiply the *First* terms, $4x \cdot 2x$.	$8x^2 + \underline{} + \underline{} + \underline{}$ F O I L
Multiply the *Outer* terms, $4x \cdot (-5)$.	$8x^2 - 20x + \underline{} + \underline{}$ F O I L
Multiply the *Inner* terms, $3 \cdot 2x$.	$8x^2 - 2x + 6x + \underline{}$ F O I L
Multiply the *Last* terms, $3 \cdot (-5)$.	$8x^2 - 20x + 6x - 15$ F O I L
Combine like terms.	$8x^2 - 14x - 15$

Figure 5.3.19

Try It

27) Multiply: $(3x+7)(5x-2)$

Solution

$15x^2 + 29x - 14$

28) Multiply: $(4y + 5)(4y - 10)$

Solution

$16y^2 - 20y - 50$

The final products in the last four examples were trinomials because we could combine the two middle terms. This is not always the case.

Example 5.3.14

Multiply: $(3x - y)(2x - 5)$

Solution

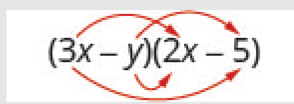


Figure 5.3.20

Step 1: **Multiply the *First.***

$$rgb]1.0, 0.0, 0.0 \underset{rgb]0.1,0.1,0.1F}{6x^2} + \underset{O}{\_\_} + \underset{I}{\_\_} + \underset{L}{\_\_}$$

Step 2: **Multiply the *Outer.***

$$rgb]1.0, 0.0, 0.0 \underset{rgb]0.1,0.1,0.1F}{6x^2} rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.0\ rgb]1.0, 0.0, 0.0 \underset{rgb]0.1,0.1,0.1O}{15x} + \underset{I}{\_\_} + \underset{L}{\_\_}$$

Step 3: **Multiply the *Inner.***

$$rgb]1.0, 0.0, 0.0 rgb]0.1, 0.1, 0.16 rgb]0.1, 0.1, 0.1x^{rgb]0.1,0.1,0.12}_{rgb]0.1,0.1,0.1F}\ rgb]0.1, 0.1, 0.1 - rgb]0.1, 0.1, 0.1\ rgb]0.1, 0.1, 0.1 \underset{O}{15x}\ rgb]1.0, 0.0, 0.0 \underset{rgb]0.1,0.1,0.1I}{-2xy} + \underset{L}{\_\_}$$

Step 4: **Multiply the *Last.***

$$rgb]1.0, 0.0, 0.0 rgb]0.1, 0.1, 0.16 rgb]0.1, 0.1, 0.1x^{rgb]0.1,0.1,0.12}_{rgb]0.1,0.1,0.1F}\ rgb]0.1, 0.1, 0.1 - rgb]0.1, 0.1, 0.1\ rgb]0.1, 0.1, 0.1 \underset{O}{15x}\ rgb]0.1, 0.1, 0.1 - \underset{I}{2xy}\ rgb]1.0, 0.0, 0.0 + rgb]1.0, 0.0, 0.0 \underset{rgb]0.1,0.1,0.1L}{5y}$$

Step 5: **Combine like terms—there are none.**

$$6x^2 - 15x - 2xy + 5y$$

Try It

29) Multiply: $(10c - d)(c - 6)$

Solution

$10c^2 - 60c - cd + 6d$

30) Multiply: $(7x - y)(2x - 5)$

Solution

$14x^2 - 35x - 2xy + 10y$

Be careful of the exponents in the next example.

Example 5.3.15

Multiply: $(n^2 + 4)(n - 1)$

Solution

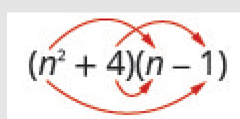


Figure 5.3.21

***Step 1:* Multiply the *First*.**

$$rgb]1.0, 0.0, 0.0 \underset{rgb]0.1,0.1,0.1F}{n^3} + \underset{O}{\_} + \underset{I}{\_} + \underset{L}{\_}$$

***Step 2:* Multiply the *Outer*.**

$$rgb]1.0, 0.0, 0.0 rgb]0.1, 0.1, 0.1 \underset{rgb]0.1,0.1,0.1F}{n^{rgb]0.1,0.1,0.13}} \; rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.0 \; rgb]1.0, 0.0, 0.0 \underset{rgb]0.1,0.1,0.1O}{n^2} + \underset{I}{\_} + \underset{L}{\_}$$

Step 3: **Multiply the *Inner*.**

$$rgb]1.0,0.0,0.0 rgb]0.1,0.1,0.1 \underset{rgb]0.1,0.1,0.1F}{n^{rgb]0.1,0.1,0.1 3}}\ rgb]0.1,0.1,0.1 - rgb]0.1,0.1,0.1\ rgb]0.1,0.1,0.1 \underset{O}{n^2}\ rgb]1.0,0.0,0.0 +\ rgb]1.0,0.0,0.0 \underset{rgb]0.1,0.1,0.1 I}{4n} + \underset{L}{\_}$$

Step 4: **Multiply the *Last*.**

$$rgb]1.0,0.0,0.0 rgb]0.1,0.1,0.1 \underset{rgb]0.1,0.1,0.1F}{n^{rgb]0.1,0.1,0.1 3}}\ rgb]0.1,0.1,0.1 - rgb]0.1,0.1,0.1\ rgb]0.1,0.1,0.1 \underset{O}{n^2}\ rgb]0.1,0.1,0.1 + rgb]0.1,0.1,0.1\ rgb]0.1,0.1,0.1 \underset{I}{4n}\ rgb]1.0,0.0,0.0 - rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0 \underset{rgb]0.1,0.1,0.1L}{4}$$

Step 5: **Combine like terms—there are none.**

$$n^3 \ - \ n^2 \ + \ 4n \ - \ 4$$

Try It

31) Multiply: $(x^2 + 6)(x - 8)$

Solution

$x^3 - 8x^2 + 6x - 48$

32) Multiply: $(y^2 + 7)(y - 9)$

Solution

$y^3 - 9y^2 + 7y - 63$

Example 5.3.16

Multiply: $(3pq + 5)(6pq - 11)$

Solution

Figure 5.3.22

Step 1: **Multiply the *First*.**

$$rgb]1.0,0.0,0.018rgb]1.0,0.0,0.0\underbrace{p^{rgb]1.0,0.0,0.02}rgb]1.0,0.0,0.0q^{rgb]1.0,0.0,0.02}}_{F} + \underline{\quad}_{O} + \underline{\quad}_{I} + \underline{\quad}_{L}$$

Step 2: **Multiply the *Outer*.**

$$rgb]0.1,0.1,0.118rgb]0.1,0.1,0.1p^{rgb]0.1,0.1,0.12}_{F}rgb]0.1,0.1,0.1q^{rgb]0.1,0.1,0.12}\ rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0\ \underbrace{33pq}_{rgb]0.1,0.1,0.1O} + \underline{\quad}_{I} + \underline{\quad}_{L}$$

Step 3: **Multiply the *Inner*.**

$$rgb]0.1,0.1,0.118rgb]0.1,0.1,0.1p^{rgb]0.1,0.1,0.12}_{F}rgb]0.1,0.1,0.1q^{rgb]0.1,0.1,0.12}\ rgb]0.1,0.1,0.1-rgb]0.1,0.1,0.1\ rgb]0.1,0.1,0.133\underset{O}{pq}\ rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0\ \underbrace{30pq}_{rgb]0.1,0.1,0.1I} + \underline{\quad}_{L}$$

Step 4: **Multiply the *Last*.**

$$rgb]0.1,0.1,0.118rgb]0.1,0.1,0.1p^{rgb]0.1,0.1,0.12}_{F}rgb]0.1,0.1,0.1q^{rgb]0.1,0.1,0.12}\ rgb]0.1,0.1,0.1-rgb]0.1,0.1,0.1\ rgb]0.1,0.1,0.133\underset{O}{pq}\ rgb]0.1,0.1,0.1+rgb]0.1,0.1,0.1\ rgb]0.1,0.1,0.130\underset{I}{pq}rgb]0.1,0.1,0.1\ rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0\ \underbrace{55}_{rgb]0.1,0.1,0.1L}\ rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0$$

Step 5: **Combine like terms—there are none.**

$$18p^2q^2\ -\ 3pq\ -\ 55$$

Try It

33) Multiply: $(2ab + 5)(4ab - 4)$

Solution

$8a^2b^2 + 12ab - 20$

34) Multiply: $(2xy + 3)(4xy - 5)$

Solution

$8x^2y^2 + 2xy - 15$

Multiply a Binomial by a Binomial Using the Vertical

Method

The FOIL method is usually the quickest method for multiplying two binomials, but it *only* works for binomials. You can use the Distributive Property to find the product of any two polynomials. Another method that works for all polynomials is the Vertical Method. It is very much like the method you use to multiply whole numbers. Look carefully at this example of multiplying two-digit numbers.

$$\begin{array}{rll} 23 & & \\ \underline{\times 46} & & \\ 138 & \text{partial product} & \text{Start by multiplying 23 by 6 to get 138.} \\ \underline{92\ \ } & \text{partial product} & \text{Next, multiply 23 by 4, lining up the partial product in the correct columns.} \\ 1058 & \text{product} & \text{Last you add the partial products.} \end{array}$$

Now we'll apply this same method to multiply two binomials.

Example 5.3.17

Multiply using the Vertical Method: $(3y-1)(2y-6)$

Solution

It does not matter which binomial goes on the top.

Step 1: **Multiply $3y-1$ by -6**

Step 2: **Multiply $3y-1$ by $2y$**

Step 3: Add like terms.

Notice the partial products are the same as the terms in the FOIL method.

$(3y-1)(2y-6)$ $rgb]0.0,0.0,1.0\underline{6y2^2-2y}rgb]1.0,0.0,0.0\underline{-18y+6}$ $6y^2-20y+6$	$3y-1$ $\underline{\times 2y-6}$ $rgb]1.0,0.0,0.0-18y+6$ $rgb]0.0,0.0,1.06y^2-2y$ $\overline{6y^2-20x+6}$

Try It

35) Multiply using the Vertical Method: $(5m-7)(3m-6)$

Solution

$15m^2-51m+42$

36) Multiply using the Vertical Method: $(6b-5)(7b-3)$

Solution

$42b^2-53b+15$

We have now used three methods for multiplying binomials. Be sure to practice each method, and try to decide which one you prefer. The methods are listed here all together, to help you remember them.

Multiplying Two Binomials

HOW TO

To multiply binomials, use the:

- Distributive Property
- FOIL Method
- Vertical Method

Remember, FOIL only works when multiplying two binomials.

Multiply a Trinomial by a Binomial

We have multiplied monomials by monomials, monomials by polynomials, and binomials by binomials. Now we're ready to multiply a trinomial by a binomial. Remember, FOIL will not work in this case, but we can use either the Distributive Property or the Vertical Method. We first look at an example using the Distributive Property.

Example 5.3.18

Multiply using the Distributive Property: $(b+3)(2b^2-5b+8)$

Solution

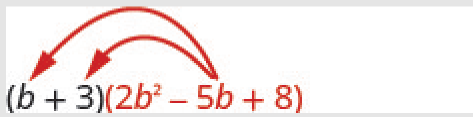


Figure 5.3.23

Step 1: **Distribute.**

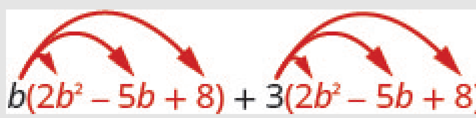


Figure 5.3.24

Step 2: **Multiply.**

$$2b^3 \ - \ 5b^2 \ + \ 8b \ + \ 6b^2 \ - \ 15b \ + \ 24$$

Step 3: **Combine like terms.**

$$2b^3 \ + \ b^2 \ - \ 7b \ + \ 24$$

Try It

37) Multiply using the Distributive Property: $(y-3)(y^2-5y+2)$

Solution

$y^3-8y^2+17y-6$

38) Multiply using the Distributive Property: $(x+4)(2x^2-3x+5)$

Solution

$2x^3+5x^2-7x+20$

Now let's do this same multiplication using the Vertical Method.

Example 5.3.19

Multiply using the Vertical Method: $(b+3)(2b^2-5b+8)$

Solution

Step 1: **Multiply** $(2b^2-5b+8)$ **by** 3**.**

$$\begin{array}{r} 2b^2-5b+8 \\ \times \qquad b+3 \\ \hline 6b^2-15b+24 \end{array}$$

Step 2: **Multiply** $(2b^2-5b+8)$ **by** b**.**

$$2b^3 \;-\; 5b^2 \;+\; 8b$$

Step 3: **Add like terms.**

$$2b^3 \;+\; b^2 \;-\; 7b \;+\; 24$$

It is easier to put the polynomial with fewer terms on the bottom because we get fewer partial products this way.

Try It

39) Multiply using the Vertical Method: $(y-3)(y^2-5y+2)$

Solution

$y^3-8y^2+17y-6$

40) Multiply using the Vertical Method: $(x+4)(2x^2-3x+5)$

Solution

$2x^3+5x^2-7x+20$

We have now seen two methods you can use to multiply a trinomial by a binomial. After you practice each method, you'll probably find you prefer one way over the other. We list both methods are listed here, for easy reference.

Multiplying a Trinomial by a Binomial

HOW TO

To multiply a trinomial by a binomial, use the:

- Distributive Property
- Vertical Method

Access these online resources for additional instruction and practice with multiplying polynomials:

- Multiplying Exponents 1
- Multiplying Exponents 2
- Multiplying Exponents 3

Key Concepts

- **FOIL Method for Multiplying Two Binomials**—To multiply two binomials:
  1. Multiply the **First** terms.
  2. Multiply the **Outer** terms.
  3. Multiply the **Inner** terms.
  4. Multiply the **Last** terms.
- **Multiplying Two Binomials**—To multiply binomials, use the:
  - Distributive Property (Example 5.3.7)
  - FOIL Method (Example 5.3.11)
  - Vertical Method (Example 5.3.17)
- **Multiplying a Trinomial by a Binomial**—To multiply a trinomial by a binomial, use the:
  - Distributive Property (Example 5.3.18)
  - Vertical Method (Example 5.3.19)

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=2751#h5p-28

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

5.4 SPECIAL PRODUCTS

Learning Objectives

By the end of this section, you will be able to:

- Square a binomial using the Binomial Squares Pattern
- Multiply conjugates using the Product of Conjugates Pattern
- Recognize and use the appropriate special product pattern

Try It

Before you get started, take this readiness quiz:

1) Simplify:

a. 9^2
b. $(-9)^2$
c. -9^2

Square a Binomial Using the Binomial Squares Pattern

Mathematicians like to look for patterns that will make their work easier. A good example of this is squaring binomials. While you can always get the product by writing the binomial twice and using the methods of the last section, there is less work to do if you learn to use a pattern.

Let's start by looking at $(x+9)^2$	
What does this mean?	$(x+9)^2$
It means to multiply $(x+9)$ by itself.	$(x+9)(x+9)$
Then, using FOIL, we get:	$x^2+9x+9x+81$
Combining like terms gives:	$x^2+18x+81$

Here's another one:	$(y-7)^2$
Multiply $(y-7)$ by itself.	$(y-7)(y-7)$
Using FOIL, we get:	$(y^2-7y-7y+49$
And combining like terms:	$y^2-14y+49$

And one more:	$(2x+3)^2$
Multiply.	$(2x+3)(2x+3)$
Use FOIL:	$4x^2+6x+6x+9$
Combine like terms.	$4x^2+12x+9$

Look at these results. Do you see any patterns?

What about the number of terms? In each example, we squared a binomial and the result was a trinomial.

$$(a+b)^2 = \_\_\_\_ + \_\_\_\_ + \_\_\_\_$$

Now look at the *first term* in each result. Where did it come from?

$(rgb]1.0, 0.0, 0.0x + rgb]0.0, 0.0, 1.09)^2$	$(rgb]1.0, 0.0, 0.0y - rgb]0.0, 0.0, 1.07)^2$	$(rgb]1.0, 0.0, 0.02x + rgb]0.0, 0.0, 1.03)^2$
$(rgb]1.0, 0.0, 0.0x + rgb]0.0, 0.0, 1.09)(rgb]1.0, 0.0, 0.0x + rgb]0.0, 0.0, 1.09)$	$(rgb]1.0, 0.0, 0.0y - rgb]0.0, 0.0, 1.07)(rgb]1.0, 0.0, 0.0y - rgb]0.0, 0.0, 1.07)$	$(rgb]1.0, 0.0, 0.02x + rgb]0.0, 0.0, 1.03)(rgb]1.0, 0.0, 0.02x + rgb]0.0, 0.0, 1.03)$
$rgb]1.0, 0.0, 0.0x^2 + 9x + 9x + rgb]0.0, 0.0, 1.081$	$rgb]1.0, 0.0, 0.0y^2 - 7y - 7y + rgb]0.0, 0.0, 1.049$	$rgb]1.0, 0.0, 0.04x^2 + 6x + 6x + rgb]0.0, 0.0, 1.09$
$rgb]1.0, 0.0, 0.0x^2 + 18x + rgb]0.0, 0.0, 1.081$	$rgb]1.0, 0.0, 0.0y^2 - 14y + rgb]0.0, 0.0, 1.049$	$rgb]1.0, 0.0, 0.04x^2 + 12x + rgb]0.0, 0.0, 1.09$

The first term is the product of the first terms of each binomial. Since the binomials are identical, it is just the square of the first term!

$$(a+b)^2 = a^2 + \_\_\_\_ + \_\_\_\_$$

To get the *first term* of the product, *square the first term*.

Where did the *last term* come from? Look at the examples and find the pattern.

The last term is the product of the last terms, which is the square of the last term.

$$(a+b)^2 = \_\_\_\_ + \_\_\_\_ + b^2$$

To get the last term of the product, square the last term.

Finally, look at the *middle term*. Notice it came from adding the "outer" and the "inner" terms—which are both the same! So the middle term is double the product of the two terms of the binomial.

$$(a+b)^2 = \_\_\_\_ + 2ab + \_\_\_\_$$
$$(a-b)^2 = \_\_\_\_ - 2ab + \_\_\_\_$$

To get the middle term of the product, multiply the terms and double their product.

Putting it all together:

Binomial Squares Pattern

if a and b are real numbers,

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$\underbrace{(a+b)^2}_{(binomial)^2} = \underbrace{a^2}_{(first\ term)} + \underbrace{2ab}_{2(product\ of\ terms)} + \underbrace{b^2}_{(last\ term)^2}$$

To square a binomial:

- square the first term
- square the last term
- double their product

A number example helps verify the pattern.

	$(10+4)^2$
Square the first term.	$10^2 + \_\_\_ +$
Square the last term.	$10^2 + \_\_\_ + 4^2$
Double their product.	$10^2 + 2 \times 10 \times 4 + 4^2$
Simplify.	$100 + 80 + 16$
Simplify.	196

To multiply $(10+4)^2$ usually you'd follow the Order of Operations.

$$
\begin{aligned}
&=(10+4)^2\\
&=14^2\\
&=196
\end{aligned}
$$

The pattern works!

Example 5.4.1

Multiply: $(x+5)^2$

Solution

$$\left(\begin{array}{c} rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0 \ rgb]1.0,0.0,0.0 + rgb]1.0,0.0,0.0 \ rgb]1.0,0.0,0.0b \\ x \ + \ 5 \end{array}\right)^2$$

Step 1: **Square the first term.**

$$
\begin{array}{l}
rgb]1.0,0.0,0.0a^2+2b+b^2\\
x^2+\_\_+\_\_
\end{array}
$$

Step 2: **Square the last term.**

$$
\begin{array}{l}
rgb]1.0,0.0,0.0a^2+2b+b^2\\
x^2+\_\_+5^2
\end{array}
$$

Step 3: **Double the product.**

$$
\begin{array}{l}
rgb]1.0,0.0,0.0a^2+2\times a\times b+b^2\\
x^2+2\times x\times 5+5^2
\end{array}
$$

Step 4: **Simplify.**

$$x^2 \ + \ 10x \ + \ 25$$

Try It

2) Multiply: $(x+9)^2$

Solution

$x^2+18x+81$

3) Multiply: $(y+11)^2$

Solution

$y^2+22y+121$

Example 5.4.2

Multiply: $(y-3)^2$

Solution

$$\left(\overset{rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0 - rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0b}{y\ -\ 3}\right)^2$$

Step 1: **Square the first term.**

$$\begin{array}{l} rgb]1.0,0.0,0.0a^2-2b+b^2 \\ y^2-\_\_+\_\_ \end{array}$$

Step 2: **Square the last term.**

$$\begin{array}{l} rgb]1.0,0.0,0.0a^2-2b+b^2 \\ y^2-\_\_+3^2 \end{array}$$

Step 3: **Double the product.**

$$rgb]1.0,0.0,0.0a^2 - 2 \times a \times b + b^2$$
$$y^2 - 2 \times y \times 3 + 3^2$$

***Step 4:* Simplify.**

$$y^2 \; - \; 6y \; + \; 9$$

Try It

4) Multiply: $(x - 9)^2$

Solution

$x^2 - 18x + 81$

5) Multiply: $(p - 13)^2$

Solution

$p^2 - 26p + 169$

Example 5.4.3

Multiply: $(4x + 6)^2$

Solution

$$\left(\begin{array}{c} rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0 \; rgb]1.0,0.0,0.0 \; rgb]1.0,0.0,0.0 \; rgb]1.0,0.0,0.0 \; rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.0 \; rgb]1.0,0.0,0.0 \; rgb]1.0,0.0,0.0b \\ 4x \; + \; 6 \end{array}\right)^2$$

***Step 1:* Use the pattern.**

$$rgb]1.0,0.0,0.0\ a^2 \quad + 2 \times \ a \ \times b + b^2$$
$$(4x)^2 + 2 \times 4x \times 6 + 6^2$$

***Step 2:* Simplify.**

$$16x^2 \ + \ 48x \ + \ 36$$

Try It

6) Multiply: $(6x + 3)^2$

Solution

$36x^2 + 36x + 9$

7) Multiply: $(4x + 9)^2$

Solution

$16x^2 + 72x + 81$

Example 5.4.4

Multiply: $(2x - 3y)^2$

Solution

$$\left(\overset{rgb]1.0,0.0,0.0argb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0b}{2x \ - \ 3y}\right)^2$$

***Step 1:* Use the pattern.**

$$rgb]1.0,0.0,0.0\ a^2 \quad -2\times \ a \ \times \ b \ + \ \ b^2$$
$$(2x)^2 - 2\times 2x\times 3y + (3y)^2$$

Step 2: **Simplify.**

$$4x^2 \ - \ 12xy \ + \ 9y^2$$

Try It

8) Multiply: $(2c-d)^2$

Solution

$4c^2-4cd+d^2$

9) Multiply: $(4x-5y)^2$

Solution

$16x^2-40xy+25y^2$

Example 5.4.5

Multiply: $(4u^3+1)^2$

Solution

$$\left(\overset{rgb]1.0,0.0,0.0argb]1.0,0.0,0.0 \ rgb]1.0,0.0,0.0 \ rgb]1.0,0.0,0.0 \ rgb]1.0,0.0,0.0 \ rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.0 \ rgb]1.0,0.0,0.0 \ rgb]1.0,0.0,0.0b}{4u^3 \ + \ 1}\right)^2$$

Step 1: **Use the pattern.**

$$rgb]1.0, 0.0, 0.0\ a^2 \quad + 2 \times \quad a \quad \times \ b \ + \quad b^2$$
$$(4u)^2 + 2 \times 4u^3 \times 1 + (1)^2$$

***Step 2:* Simplify.**

$$16u^6 \ + \ 8u^3 \ + \ 1$$

Try It

10) Multiply: $(2x^2 + 1)^2$

Solution

$4x^4 + 4x^2 + 1$

11) Multiply: $(3y^3 + 2)^2$

Solution

$9y^6 + 12y^3 + 4$

We just saw a pattern for squaring binomials that we can use to make multiplying some binomials easier. Similarly, there is a pattern for another product of binomials. But before we get to it, we need to introduce some vocabulary.

What do you notice about these pairs of binomials?

$$(x - 9)(x + 9) \qquad (y - 8)(y + 8) \qquad (2x - 5)(2x + 5)$$

Look at the first term of each binomial in each pair.

$$(rgb]1.0, 0.0, 0.0x - 9)(rgb]1.0, 0.0, 0.0x + 9)$$
$$(rgb]1.0, 0.0, 0.0y - 8)(rgb]1.0, 0.0, 0.0y + 8)$$
$$(rgb]1.0, 0.0, 0.02x - 5)(rgb]1.0, 0.0, 0.02x + 5)$$

Notice the first terms are the same in each pair.

Look at the last terms of each binomial in each pair.

$$\underbrace{(a+b)^2}_{(binomial)^2} = \underbrace{a^2}_{(first\ term)} + \underbrace{2ab}_{2(product\ of\ terms)} + \underbrace{b^2}_{(last\ term)^2}$$

Notice the last terms are the same in each pair.

Notice how each pair has one sum and one difference.

$$\left(\underset{Difference}{\underset{rgb]0.0,0.0,1.0\uparrow}{x rgb]1.0,0.0,0.0{-9}}}\right)\left(\underset{Sum}{\underset{rgb]0.0,0.0,1.0\uparrow}{x rgb]1.0,0.0,0.0{+9}}}\right) \quad \left(\underset{Difference}{\underset{rgb]0.0,0.0,1.0\uparrow}{y rgb]1.0,0.0,0.0{-8}}}\right)\left(\underset{Sum}{\underset{rgb]0.0,0.0,1.0\uparrow}{y rgb]1.0,0.0,0.0{+8}}}\right) \quad \left(\underset{Difference}{\underset{rgb]0.0,0.0,1.0\uparrow}{2x rgb]1.0,0.0,0.0{-5}}}\right)\left(\underset{Sum}{\underset{rgb]0.0,0.0,1.0\uparrow}{2x rgb]1.0,0.0,0.0{+5}}}\right)$$

A pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference has a special name. It is called a **conjugate pair** and is of the form $(a-b), (a+b)$.

Conjugate Pair

A conjugate pair is two binomials of the form

$$(a-b), (a+b)$$

The pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

There is a nice pattern for finding the product of conjugates. You could, of course, simply FOIL to get the product, but using the pattern makes your work easier.

Let's look for the pattern by using FOIL to multiply some conjugate pairs.

$(x-9)(x+9)$	$(y-8)(y+8)$	$(2x-5)(2x+5)$
$x^2+9x-9x-81$	$y^2+8y-8y-64$	$4x^2+10x-10x-25$
x^2-81	y^2-64	$4x^2-25$

$(rgb]1.0,0.0,0.0x rgb]0.0,0.0,1.0+9)(rgb]1.0,0.0,0.0x rgb]0.0,0.0,1.0-9)$	$(rgb]1.0,0.0,0.0y rgb]0.0,0.0,1.0-8)(rgb]1.0,0.0,0.0y rgb]0.0,0.0,1.0+8)$	$(rgb]1.0,0.0,0.02x rgb]0.0,0.0,1.0-5)(rgb]1.0,0.0,0.02x rgb]0.0,0.0,1.0+5)$
$rgb]1.0,0.0,0.0x^2-9x+9x rgb]0.0,0.0,1.0-81$	$rgb]1.0,0.0,0.0y^2+8y-8y rgb]0.0,0.0,1.0-64$	$rgb]1.0,0.0,0.04x^2+10x-10x rgb]0.0,0.0,1.0-25$
$rgb]1.0,0.0,0.0x^2 rgb]0.0,0.0,1.0-81$	$rgb]1.0,0.0,0.0y^2 rgb]0.0,0.0,1.0-64$	$rgb]1.0,0.0,0.04x^2 rgb]0.0,0.0,1.0-25$

Each *first term* is the product of the first terms of the binomials, and since they are identical it is the square of the first term.

$$(a+b)(a-b) = a^2\_\_\_\_$$

To get the first term, square the first term.

The *last term* came from multiplying the last terms, the square of the last term.

$$(a+b)(a-b)=a^2-b^2$$

To get the last term, square the last term.

What do you observe about the products?

The product of the two binomials is also a binomial! Most of the products resulting from FOIL have been trinomials.

Why is there no middle term? Notice the two middle terms you get from FOIL combine to 0 in every case, the result of one addition and one subtraction.

The product of conjugates is always of the form $a^2 - b^2$. This is called a difference of squares.

This leads to the pattern:

Product of Conjugates Pattern

If a and b are real numbers,

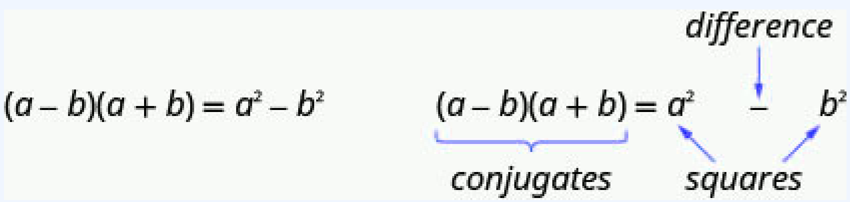


Figure 5.4.1

The product is called a difference of squares.

To multiply conjugates, square the first term, square the last term, and write the product as a difference of squares.

Let's test this pattern with a numerical example.

	$(10-2)(10+2)$
It is the product of conjugates, so the result will be the difference of two squares.	$\_\_\_\_ - \_\_\_\_$
Square the first term.	$10^2 - \_\_\_\_$
Square the last term.	$10^2 - 2^2$
Simplify.	$100 - 4$ $= 96$
What do you get using the order of operations?	$(10-2)(10+2)$ $(8)(12)$ $= 96$

Notice, the result is the same!

Example 5.4.6

Multiply: $(x-8)(x+8)$

Solution

First, recognize this as a product of conjugates. The binomials have the same first terms, and the same last terms, and one binomial is a sum and the other is a difference.

Step 1: **It fits the pattern.**

rgb]1.0,0.0,0,0.0argb]1.0,0.0,0,0.0 rgb]1.0,0.0,0,0.0−rgb]1.0,0.0,0,0.0 rgb]1.0,0.0,0,0.0b rgb]1.0,0.0,0,0.0argb]1.0,0.0,0,0.0 rgb]1.0,0.0,0,0.0+rgb]1.0,0.0,0,0.0 rgb]1.0,0.0,0,0.0b

$$(x - 8)(x + 8)$$

Step 2: **Square the first term, x.**

rgb]1.0,0.0,0,0.0a^rgb]1.0,0.0,0,0.02 rgb]1.0,0.0,0,0.0 rgb]1.0,0.0,0,0.0−rgb]1.0,0.0,0,0.0 rgb]1.0,0.0,0,0.0b^rgb]1.0,0.0,0,0.02

$$x^2 - \_\_$$

Step 3: **Square the last term, 8.**

rgb]1.0,0.0,0,0.0a^rgb]1.0,0.0,0,0.02 rgb]1.0,0.0,0,0.0 rgb]1.0,0.0,0,0.0−rgb]1.0,0.0,0,0.0 rgb]1.0,0.0,0,0.0b^rgb]1.0,0.0,0,0.02

$$x^2 - 8^2$$

Step 4: **The product is a difference of squares.**

rgb]1.0,0.0,0,0.0a^rgb]1.0,0.0,0,0.02 rgb]1.0,0.0,0,0.0 rgb]1.0,0.0,0,0.0−rgb]1.0,0.0,0,0.0 rgb]1.0,0.0,0,0.0b^rgb]1.0,0.0,0,0.02

$$x^2 - 64$$

Try It

12) Multiply: $(x-5)(x+5)$

Solution

x^2-25

13) Multiply: $(w-3)(w+3)$

Solution

w^2-9

Example 5.4.7

Multiply: $(2x+5)(2x-5)$

Solution

Are the binomials conjugates?

Step 1: **It is the product of conjugates.**

$$\left(\overset{rgb]1.0,0.0,0.0argb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0b}{2x\ +\ 5}\right)\left(\overset{rgb]1.0,0.0,0.0argb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0b}{2x\ -\ 5}\right)$$

Step 2: **Square the first term, $2x$.**

$$rgb]1.0,0.0,0.0a^{rgb]1.0,0.0,0.02}rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0b^{rgb]1.0,0.0,0.02}$$

$$(2x)^2\ -\ \_\_$$

Step 3: **Square the last term, 5.**

$$rgb]1.0,0.0,0.0a^{rgb]1.0,0.0,0.02}rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0b^{rgb]1.0,0.0,0.02}$$

$$(2x)^2\ -\ 5^2$$

Step 4: **Simplify. The product is a difference of squares.**

$$rgb]1.0,0.0,0.0a^{rgb]1.0,0.0,0.02} rgb]1.0,0.0,0.0 \; rgb]1.0,0.0,0.0 - rgb]1.0,0.0,0.0 \; rgb]1.0,0.0,0.0b^{rgb]1.0,0.0,0.02}$$
$$4x^2 \; - \; 25$$

Try It

14) Multiply: $(6x+5)(6x-5)$.

Solution

$36x^2 - 25$

15) Multiply: $(2x+7)(2x-7)$

Solution

$4x^2 - 49$

The binomials in the next example may look backwards – the variable is in the second term. But the two binomials are still conjugates, so we use the same pattern to multiply them.

Example 5.4.8

Find the product: $(3+5x)(3-5x)$

Solution

Step 1: **It is the product of conjugates.**

$$\left(\begin{matrix} rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0 \; rgb]1.0,0.0,0.0 - rgb]1.0,0.0,0.0 \; rgb]1.0,0.0,0.0b \\ 3 \; + \; 5x \end{matrix}\right) \left(\begin{matrix} rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0 \; rgb]1.0,0.0,0.0 + rgb]1.0,0.0,0.0 \; rgb]1.0,0.0,0.0b \\ 3 \; - \; 5x \end{matrix}\right)$$

Step 2: **Use the pattern.**

$$rgb]1.0,0.0,0.0a^{rgb]1.0,0.0,0.02} rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0b^{rgb]1.0,0.0,0.02}$$

$$3^2 \ - \ (5x)^2$$

***Step 3:* Simplify.**

$$9 \ - \ 25x^2$$

Try It

16) Multiply: $(7+4x)(7-4x)$

Solution

$49-16x^2$

17) Multiply: $(9-2y)(9+2y)$

Solution

$81-4y^2$

Now we'll multiply conjugates that have two variables.

Example 5.4.9

Find the product: $(5m-9n)(5m+9n)$

Solution

***Step 1:* This fits the pattern.**

$$\left(\begin{array}{c} rgb]1.0,0.0,0.0argb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0b \\ 5m \ - \ 9n \end{array}\right)\left(\begin{array}{c} rgb]1.0,0.0,0.0argb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0b \\ 5m \ + \ 9n \end{array}\right)$$

***Step 2:* Use the pattern.**

$$rgb]1.0,0.0,0.0a^{rgb]1.0,0.0,0.02} rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0b^{rgb]1.0,0.0,0.02}$$
$$(5m)^2 \ - \ (9n)^2$$

***Step 3:* Simplify.**

$$25m^2 \ - \ 81n^2$$

Try It

18) Find the product: $(4p-7q)(4p+7q)$

Solution

$16p^2-49q^2$

19) Find the product: $(3x-y)(3x+y)$

Solution

$9x^2-y^2$

Example 5.4.10

Find the product: $(cd-8)(cd+8)$

Solution

***Step 1:* This fits the pattern.**

$$\left(\overset{rgb]1.0,0.0,0.0argb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0b}{cd \ - \ 8}\right)\left(\overset{rgb]1.0,0.0,0.0argb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0b}{cd \ + \ 8}\right)$$

Step 2: **Use the pattern.**

$$rgb]1.0,0.0,0.0a^{rgb]1.0,0.0,0.02} rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0b^{rgb]1.0,0.0,0.02}$$
$$(cd)^2 \; - \; (8)^2$$

Step 3: **Simplify.**

$$c^2d^2 \; - \; 64$$

Try It

20) Find the product: $(xy - 6)(xy + 6)$

Solution

$x^2y^2 - 36$

21) Find the product: $(ab - 9)(ab + 9)$

Solution

$a^2b^2 - 81$

Example 5.4.11

Find the product: $(6u^2 - 11v^5)(6u^2 + 11v^5)$

Solution

Step 1: **This fits the pattern.**

$$\left(\begin{matrix} rgb]1.0,0.0,0.0argb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0b \\ 6u^2 \; - \; 11v^5 \end{matrix}\right)\left(\begin{matrix} rgb]1.0,0.0,0.0argb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0+rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0b \\ 6u^2 \; + \; 11v^5 \end{matrix}\right)$$

Step 2: **Use the pattern.**

$$rgb]1.0,0.0,0.0a^{rgb]1.0,0.0,0.02} rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0b^{rgb]1.0,0.0,0.02}$$

$$\left(6u^2\right)^2 \ - \ \left(11v^5\right)^2$$

Step 3: **Simplify.**

$$36u^4 \ - \ 121v^{10}$$

Try It

22) Find the product: $(3x^2 - 4y^3)(3x^2 + 4y^3)$

Solution

$9x^4 - 16y^6$

23) Find the product: $(2m^2 - 5n^3)(2m^2 + 5n^3)$

Solution

$4m^4 - 25n^6$

Recognize and Use the Appropriate Special Product Pattern

We just developed special product patterns for Binomial Squares and for the Product of Conjugates. The products look similar, so it is important to recognize when it is appropriate to use each of these patterns and to notice how they differ. Look at the two patterns together and note their similarities and differences.

Comparing the Special Product Patterns

(a-b)(a+b)={a}^{2}-{b}^{2}

Binomial Squares	**Product of Conjugates**
$(a+b)^2 = a^2 + 2ab + b^2$	$(a-b)^2 = a^2 - 2ab + b^2$
– Squaring a binomial	– Multiplying conjugates
– Product is a trinomial	– Product is a binomial
– Inner and outer terms with FOIL are the same.	– Inner and outer terms with FOIL are opposites.
– Middle term is double the product of the terms.	– There is no middle term.

Example 5.4.12

Choose the appropriate pattern and use it to find the product:

a. $(2x-3)(2x+3)$
b. $(8x-5)^2$
c. $(6m+7)^2$
d. $(5x-6)(6x+5)$

Solution

a. $(2x-3)(2x+3)$ These are conjugates. They have the same first numbers, and the same last numbers, and one binomial is a sum and the other is a difference. It fits the Product of Conjugates pattern.

Step 1: **This fits the pattern.**

$$\binom{rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0- rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0b}{2x\ -\ 3}\binom{rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0+ rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0b}{2x\ +\ 3}$$

Step 2: **Use the pattern.**

$$rgb]1.0,0.0,0.0\ a^2 \quad - b^2$$
$$(2x)^2 - 3^2$$

$$\text{Simplify.}\ \ 4x^2\ -\ 9$$

---

b. $(8x-5)^2$ We are asked to square a binomial. It fits the *binomial squares* pattern.

$$\left(\begin{array}{c} rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0 - rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0b \\ 8x\ -\ 5 \end{array}\right)^2$$

Step 1: **Use the pattern.**

$$\begin{array}{c} rgb]1.0,0.0,0.0a^{rgb]1.0,0.0,0.02} rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0 - rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.02 rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0b rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0 + rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0b^{rgb]1.0,0.0,0.02} \\ (8x)^2\ -\ 2\ \times\ 8x\ \times\ 5\ +\ 5^2 \end{array}$$

$\text{Simplify.}\ 64x^2\ -\ 80x\ +\ 25$

---

c. $(6m+7)^2$ Again, we will square a binomial so we use the binomial squares pattern.

$$\left(\begin{array}{c} rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0 + rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0b \\ 6m\ +\ 7 \end{array}\right)^2$$

Step 1: **Use the pattern.**

$$\begin{array}{c} rgb]1.0,0.0,0.0a^{rgb]1.0,0.0,0.02} rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0 + rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.02 rgb]1.0,0.0,0.0a rgb]1.0,0.0,0.0b rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0 + rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0b^{rgb]1.0,0.0,0.02} \\ (6m)^2\ +\ 2\ \times\ 6m\ \times\ 7\ + 7^2 \end{array}$$

$\text{Simplify.}\ 36m^2\ +\ 84m\ +\ 49$

---

d. $(5x-6)(6x+5)$ This product does not fit the patterns, so we will use FOIL.

Step 1: **Use FOIL.**

$$\begin{array}{rl} & 30x^2+25x-36x-30 \\ \text{Simplify.} & 30x^2-11x-30 \end{array}$$

Try It

24) Choose the appropriate pattern and use it to find the product:

a. $(9b-2)(2b+9)$
b. $(9p-4)^2$

c. $(7y+1)^2$
d. $(4r-3)(4r+3)$

Solution
a. FOIL; $18b^2+77b-18$
b. Binomial Squares; $81p^2-72p+16$
c. Binomial Squares; $49y^2+14y+1$
d. Product of Conjugates; $16r^2-9$

25) Choose the appropriate pattern and use it to find the product:

a. $(6x+7)^2$
b. $(3x-4)(3x+4)$
c. $(2x-5)(5x-2)$
d. $(6n-1)^2$

Solution
a. Binomial Squares; $36x^2+84x+49$
b. Product of Conjugates; $9x^2-16$
c. FOIL; $10x^2-29x+10$
d. Binomial Squares; $36n^2-12n+1$

Access these online resources for additional instruction and practice with special products:

- Special Products

Key Concepts

- **Binomial Squares Pattern**
  - If a and b are real numbers,
    $$\underbrace{(a\ +\ b)^2}_{(binomial)^2} = \underbrace{a^2}_{(first\ term)^2} + \underbrace{2ab}_{2(product\ of\ terms)} + \underbrace{b^2}_{(last\ term)^2}$$
  - $(a+b)^2 = a^2 + 2ab + b^2$
  - $(a-b)^2 = a^2 - 2ab + b^2$
  - To square a binomial: square the first term, square the last term, double their product.

- **Product of Conjugates Pattern**
  - If a and b are real numbers
    $$\underbrace{(a\ -\ b)(a\ +\ b)}_{conjugates} = \underbrace{a^2}\ \underset{squares}{\overset{\substack{difference\\ \downarrow}}{-}}\ \underbrace{b^2}$$
  - $(a-b)(a+b) = a^2 - b^2$
  - The product is called a difference of squares.

- **To multiply conjugates:**
  - square the first term square the last term write it as a difference of squares

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=2811#h5p-29

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

conjugate pair
: A conjugate pair is two binomials of the form $(a - b)$, $(a + b)$; the pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

5.5 DIVIDE MONOMIALS

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions using the Quotient Property for Exponents
- Simplify expressions with zero exponents
- Simplify expressions using the Quotient to a Power Property
- Simplify expressions by applying several properties
- Use the definition of a negative exponent
- Simplify expressions with integer exponents
- Simplify expressions by applying several properties with negative exponents
- Divide monomials

Try It

Before you get started, take this readiness quiz:

1) Simplify: $\frac{8}{24}$

2) Simplify: $(2m^3)^5$

3) Simplify: $\frac{12x}{12y}$

Simplify Expressions Using the Quotient Property for Exponents

Earlier in this chapter, we developed the properties of exponents for multiplication. We summarize these properties below.

Summary of Exponent Properties for Multiplication

If a and b are real numbers, and m and n are whole numbers, then

Product Property	$a^m \times a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \times n}$
Product to a Power	$(ab)^m = a^m b^m$

Now we will look at the exponent properties for division. A quick memory refresher may help before we get started. You have learned to simplify fractions by dividing out common factors from the numerator and denominator using the *Equivalent Fractions Property*. This property will also help you work with algebraic fractions—which are also quotients.

Equivalent Fractions Property

If a, b, and c are whole numbers where $b \neq 0, c \neq 0$ then

$$\frac{a}{b} = \frac{a \times c}{b \times c} \text{ and } \frac{a \times c}{b \times c} = \frac{a}{b}$$

As before, we'll try to discover a property by looking at some examples.

	Example 1	Example 2
Consider	$\frac{x^5}{x^2}$	$\frac{x^2}{x^3}$
What do they mean?	$\frac{x\cdot x\cdot x\cdot x\cdot x}{x\cdot x}$	$\frac{x\cdot x}{x\cdot x\cdot x}$
Use the Equivalent Fractions Property.	$\frac{\cancel{x}\cdot\cancel{x}\cdot x\cdot x\cdot x}{\cancel{x}\cdot\cancel{x}}$	$\frac{\cancel{x}\cdot\cancel{x}\cdot 1}{\cancel{x}\cdot\cancel{x}\cdot x}$
Simplify	x^3	$\frac{1}{x}$

Notice, in each case the bases were the same and we subtracted exponents.

When the larger exponent was in the numerator, we were left with factors in the numerator.

When the larger exponent was in the denominator, we were left with factors in the denominator—notice the numerator of 1.

We write:

$\frac{x^5}{x^2}$	$\frac{x^2}{x^3}$
x^{5-2}	$\frac{1}{x^{3-2}}$
x^3	$\frac{1}{x}$

This leads to the *Quotient Property for Exponents*.

Quotient Property for Exponents

If a is a real number, $a \neq 0$, and n are whole numbers, then

$$\frac{a^m}{a^n} = a^{m-n}, m > n \text{ and } \frac{a^m}{a^n} = \frac{1}{a^{m-n}}, n > m$$

A couple of examples with numbers may help to verify this property.

$\frac{3^4}{3^2} = 3^{4-2}$	$\frac{5^2}{5^3} = \frac{1}{5^{3-2}}$
$\frac{81}{9} = 3^2$	$\frac{25}{125} = \frac{1}{5^1}$
$9 = 9 \checkmark$	$\frac{1}{5} = \frac{1}{5} \checkmark$

Example 5.5.1

Simplify:

a. $\frac{x^9}{x^7}$

b. $\frac{3^{10}}{3^2}$

Solution

To simplify an expression with a quotient, we need to first compare the exponents in the numerator and denominator.

a.

Step 1: **Since $9 > 7$, there are more factors of x in the numerator.**

$$\frac{x^9}{x^7}$$

Step 2: **Use the Quotient Property,** $\dfrac{a^m}{a^n} = a^{m-n}$

$$= x^{rgb]1.0,0.0,0.0,0.09rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0,0.07}$$

$$\text{Simplify.} \quad = x^2$$

---

b.

Step 1: **Since $10 > 2$, there are more factors of x in the numerator.**

$$\frac{3^{10}}{3^2}$$

Step 2: **Use the Quotient Property,** $\dfrac{a^m}{a^n} = a^{m-n}$

$$= 3^{rgb]1.0,0.0,0.0,0.010\ -\ 2}$$

$$\text{Simplify.} \quad = 3^8$$

Notice that when the larger exponent is in the numerator, we are left with factors in the numerator.

Try It

4) Simplify:

a. $\dfrac{x^{15}}{x^{10}}$

b. $\dfrac{6^{14}}{6^5}$

Solution

a. x^5
b. 6^9

5) Simplify:

a. $\dfrac{y^{43}}{y^{37}}$
b. $\dfrac{10^{15}}{10^7}$

Solution
a. y^6
b. 10^8

Example 5.5.2

Simplify:

a. $\dfrac{b^8}{b^{12}}$
b. $\dfrac{7^3}{7^5}$

Solution
To simplify an expression with a quotient, we need to first compare the exponents in the numerator and denominator.

a.

Step 1: **Since $12 > 8$, there are more factors of b in the denominator.**

$$\frac{b^8}{b^{12}}$$

Step 2: **Use the Quotient Property,** $\dfrac{a^m}{a^n} = \dfrac{1}{a^{n-m}}$

$$= \frac{rgb]1.0, 0.0, 0.01}{b^{rgb]1.0,0.0,0.012\ -\ 8}}$$

$$\text{Simplify.} \quad = \frac{rgb]0.1, 0.1, 0.11}{b^4}$$

---

b.

Step 1: **Since $5 > 3$, there are more factors of 3 in the denominator.**

$$\frac{7^3}{7^5}$$

Step 2: **Use the Quotient Property,** $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$

$$= \frac{rgb]1.0, 0.0, 0.01}{7^{rgb]1.0,0.0,0.05-3}}$$

$$\text{Simplify.} \quad = \frac{rgb]0.1, 0.1, 0.11}{7^2}$$

$$\text{Simplify.} \quad = \frac{rgb]0.1, 0.1, 0.11}{49}$$

Notice that when the larger exponent is in the denominator, we are left with factors in the denominator.

Try It

6) Simplify:

a. $\frac{x^{18}}{x^{22}}$

b. $\frac{12^{15}}{12^{30}}$

Solution

a. $\dfrac{1}{x^4}$

b. $\dfrac{1}{12^{15}}$

7) Simplify:

a. $\dfrac{m^7}{m^{15}}$

b. $\dfrac{9^8}{9^{19}}$

Solution

a. $\dfrac{1}{m^8}$

b. $\dfrac{1}{9^{11}}$

Notice the difference in the two previous examples:

- If we start with more factors in the numerator, we will end up with factors in the numerator.
- If we start with more factors in the denominator, we will end up with factors in the denominator.

The first step in simplifying an expression using the *Quotient Property for Exponents* is to determine whether the exponent is larger in the numerator or the denominator. Later in this chapter, we will explore this further while using negative exponents.

Example 5.5.3

Simplify:

a. $\dfrac{a^5}{a^9}$

b. $\dfrac{x^{11}}{x^7}$

Solution

a. Is the exponent of a larger in the numerator or denominator? Since $9 > 5$, there are more a's in the denominator and so we will end up with factors in the denominator.

Step 1: **Use the Quotient Property,** $\dfrac{a^m}{a^n} = \dfrac{1}{a^{n-m}}$

$$= \frac{rgb]1.0,0.0,0.01}{a^{rgb]1.0,0.0,0.09\ -\ 5}}$$

$$\text{Simplify.} \quad = \frac{rgb]0.1,0.1,0.11}{a^4}$$

---

b. Notice there are more factors of x in the numerator, since $11 > 7$. So we will end up with factors in the numerator.

Step 1: **Use the Quotient Property,** $\dfrac{a^m}{a^n} = \dfrac{1}{a^{n-m}}$

$$= x^{rgb]1.0,0.0,0.011rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.0-rgb]1.0,0.0,0.0\ rgb]1.0,0.0,0.07}$$

$$\text{Simplify.} \quad = x^4$$

Try It

8) Simplify:

a. $\frac{b^{19}}{b^{11}}$
b. $\frac{z^5}{z^{11}}$

Solution

a. b^8
b. $\frac{1}{z^6}$

9) Simplify:

a. $\frac{p^9}{p^{17}}$
b. $\frac{w^{13}}{w^9}$

Solution

a. $\frac{1}{p^8}$
b. w^4

Simplify Expressions with an Exponent of Zero

A special case of the Quotient Property is when the exponents of the numerator and denominator are equal, such as an expression like $\frac{a^m}{a^m}$. From your earlier work with fractions, you know that:

$\frac{2}{2}=1$ $\frac{17}{17}=1$ \frac{-43}{-43}=1

In words, a number divided by itself is 1. So, $\frac{x}{x}=1$, for any x $(x \neq 0)$, since any number divided by itself is 1.

The *Quotient Property for Exponents* shows us how to simplify $\frac{a^m}{a^n}$ when $m > n$ and when $n < m$ by subtracting exponents. What if $m = n$?

Consider $\frac{8}{8}$, which we know is 1.

$$\frac{8}{8} = 1$$

$$\text{Write 8 as } 2^3 \qquad \frac{2^3}{2^3} = 1$$

$$\text{Subtract exponents} \qquad 2^{3-3} = 1$$

$$\text{Simplify} \qquad 2^0 = 1$$

Now we will simplify $\frac{a^m}{a^m}$ in two ways to lead us to the definition of the zero exponent. In general, for $a \neq 0$:

$$\frac{a^m}{a^m} \qquad \frac{a^m}{a^m}$$

$$a^{m-m} \qquad \frac{\overbrace{\cancel{a} \cdot \cancel{a} \cdot \ldots \cdot \cancel{a}}^{rgb]0.0,0.0,1.0m \text{ factors}}}{\underbrace{\cancel{a} \cdot \cancel{a} \cdot \ldots \cdot \cancel{a}}_{rgb]0.0,0.0,1.0m \text{ factors}}}$$

$$a^0 \qquad 1$$

We see $\frac{a^m}{a^m}$ simplifies to a^0 and to 1. So $a^0 = 1$

Zero Exponent

If a is a non-zero number, then $a^0 = 1$.

Any nonzero number raised to the zero power is 1.

In this text, we assume any variable that we raise to the zero power is not zero.

Example 5.5.4

Simplify:

a. 9^0
b. n^0

Solution
The definition says any non-zero number raised to the zero power is 1.

a.

Step 1: **Use the definition of the zero exponent.**

$$= 9^0$$
$$= 1$$

---

b.

Step 1: **Use the definition of the zero exponent.**

$$= n^0$$
$$= 1$$

Try It

10) Simplify:

a. 15^0
b. m^0

Solution

a. 1

b. 1

11) Simplify:

a. k^0

b. 29^0

Solution

a. 1

b. 1

Now that we have defined the zero exponent, we can expand all the *Properties of Exponents* to include whole number exponents.

What about raising an expression to the zero power? Let's look at $(2x)^0$. We can use the product to a power rule to rewrite this expression.

	$(2x)^0$
Use the product to a power rule.	$2^0 x^0$
Use the zero exponent property.	1×1
Simplify	1

This tells us that any nonzero expression raised to the zero power is one.

Example 5.5.5

Simplify:

a. $(5b)^0$

b. $(-4a^2b)^0$

Solution

a.

Step 1: **Use the definition of the zero exponent.**

$$= (5b)^0$$
$$= 1$$

---

b.

Step 1: **Use the definition of the zero exponent.**

$$= (-4a^{2b})^0$$
$$= 1$$

Try It

12) Simplify:

a. $(11z)^0$
b. $(-11pq^3)^0$

Solution

a. 1
b. 1

13) Simplify:

a. $(-6d)^0$
b. $(-8m^2n^3)^0$

Solution

a. 1
b. 1

Simplify Expressions Using the Quotient to a Power

Property

Now we will look at an example that will lead us to the *Quotient to a Power Property*.

$$\left(\frac{x}{y}\right)^3$$

This means: $\frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y}$

Multiply the fractions. $\frac{x \cdot x \cdot x}{y \cdot y \cdot y}$

Write with exponents. $\frac{x^3}{y^3}$

Notice that the exponent applies to both the numerator and the denominator.

We write: $\left(\frac{x}{y}\right)^3$

$$\frac{x^3}{y^3}$$

This leads to the *Quotient to a Power Property for Exponents*.

Quotient to a Power Property for Exponents

If a and b are real numbers, $b \neq 0$, and m is a counting number, then

$$(\frac{a}{b})^m = \frac{a^m}{b^m}$$

To raise a fraction to a power, raise the numerator and denominator to that power.

An example with numbers may help you understand this property:

$$(\frac{2}{3})^3 = \frac{2^3}{3^3}$$
$$\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{8}{27}$$
$$\frac{8}{27} = \frac{8}{27} \checkmark$$

Example 5.5.6

Simplify:

a. $(\frac{3}{7})^2$

b. (\frac{b}{3})^{4}

c. $\left(\frac{k}{j}\right)^3$

Solution

a.

Step 1: **Use the Quotient Property,** $(\frac{a}{b})^m = \frac{a^m}{b^m}$

$$= b^0$$

$$\text{Simplify.} \quad = 1$$

---

b.

Step 1: **Use the Quotient Property,** $(\frac{a}{b})^m = \frac{a^m}{b^m}$

$$= \frac{b^{rgb]1.0,0.0,0.04}}{3^{rgb]1.0,0.0,0.04}}$$

$$\text{Simplify.} \quad = \frac{b^{rgb]0.1,0.1,0.14}}{81}$$

---

c.

Step 1: **Raise the numerator and denominator to the third power.**

$$\frac{k^{rgb]1.0,0.0,0.03}}{j^{rgb]1.0,0.0,0.03}}$$

Try It

14) Simplify:

a. $(\frac{5}{8})^2$

b. $(\frac{p}{10})^4$

c. $(\frac{m}{n})^7$

Solution

a. $\frac{25}{64}$

b. $\frac{p^4}{10,000}$

c. $\frac{m^7}{n^7}$

15) Simplify:

a. $(\frac{1}{3})^3$

b. $(\frac{-2}{q})^3$

c. $(\frac{w}{x})^4$

Solution

a. $\frac{1}{27}$

b. $\frac{-8}{q^3}$

c. $\frac{w^4}{x^4}$

Simplify Expressions by Applying Several Properties

We'll now summarize all the properties of exponents so they are all together to refer to as we simplify expressions using several properties. Notice that they are now defined for whole number exponents.

Summary of Exponent Properties

If a and b are real numbers, and m and n are whole numbers, then

Product Property	$a^m \times a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \times n}$
Product to a Power	$(ab)^m = a^m b^m$
Quotient Property	$\frac{a^m}{b^m} = a^{m-n}, a \neq 0, m > n$
	$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, a \neq 0, n > m$
Zero Exponent Definition	$a^0 = 1, a \neq 0$
Quotient to a Power Property	$(\frac{a}{b})^m = \frac{a^m}{b^m}, b \neq 0$

Example 5.5.7

Simplify: $\frac{(y^4)^2}{y^6}$

Solution

Step 1: **Multiply the exponents in the numerator.**

$$\frac{y^8}{y^6}$$

Step 2: **Subtract the exponents.**

$$y^2$$

Try It

16) Simplify: $\dfrac{(m^5)^4}{m^7}$

Solution

m^{13}

17) Simplify: $\dfrac{(k^2)^6}{k^7}$

Solution

k^5

Example 5.5.8

Simplify: $\dfrac{b^{12}}{(b^2)^6}$

Solution

Step 1: **Multiply the exponents in the numerator.**

$$\frac{b^{12}}{b^{12}}$$

Step 2: **Subtract the exponents.**

$$= b^0$$

$$\text{Simplify.} \quad = 1$$

Try It

18) Simplify: $\dfrac{n^{12}}{(n^3)^4}$

Solution

1

19) Simplify: $\dfrac{x^{15}}{(x^3)^5}$

Solution

1

Example 5.5.9

Simplify: $\left(\dfrac{y^9}{y^4}\right)^2$

Solution

Step 1: **Remember parentheses come before exponents.**

Notice the bases are the same, so we can simplify inside the parentheses.

Subtract the exponents.

$$(y^5)^2$$

***Step 2:* Multiply the exponents.**

$$y^{10}$$

Try It

20) Simplify: $\left(\dfrac{r^5}{r^3}\right)^4$

Solution

r^8

21) Simplify: $\left(\dfrac{v^6}{v^4}\right)^3$

Solution

v^6

Example 5.5.10

Simplify: $\left(\dfrac{j^2}{k^3}\right)^4$

Solution

Here we cannot simplify inside the parentheses first, since the bases are not the same.

Step 1: **Raise the numerator and denominator to the third power using the Quotient to a Power Property,** $(\frac{a}{b})^m = \frac{a^m}{b^m}$

$$\left(\frac{j^2}{k^3}\right)^4$$

Step 2: **Use the Power Property and simplify.**

$$\frac{j^8}{k^{12}}$$

Try It

22) Simplify: $\left(\frac{a^3}{b^2}\right)^4$

Solution

$\frac{a^{12}}{b^8}$

23) Simplify: $\left(\frac{q^7}{r^5}\right)^3$

Solution

$\frac{q^{21}}{r^{15}}$

Example 5.5.11

Simplify: $\left(\frac{2m^2}{5n}\right)^4$

Solution

Step 1: **Raise the numerator and denominator to the fourth power, using the Quotient to a Power Property,** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

$$\frac{(2m^2)^4}{(5n)^4}$$

Step 2: **Raise each factor to the fourth power.**

$$\frac{(2m^2)^4}{(5n)^4}$$

Step 3: **Use the Power Property and simplify.**

$$\frac{16m^8}{625n^4}$$

Try It

24) Simplify: $\left(\frac{7x^3}{9y}\right)^2$

Solution

$\frac{49x^6}{81y^2}$

25) Simplify: $\left(\frac{3x^4}{7y}\right)^2$

Solution

$$\frac{9x^8}{49y^2}$$

Example 5.5.12

Simplify: $\frac{\left(x^3\right)^4 \times \left(x^2\right)^5}{\left(x^6\right)^5}$

Solution

Step 1: **Use the Power Property,** $\left(a^m\right)^n = a^{mn}$

$$\frac{\left(x^{12}\right) \times \left(x^{10}\right)}{\left(x^{30}\right)}$$

Step 2: **Add the exponents in the numerator.**

$$\frac{x^{22}}{x^{30}}$$

Step 3: **Use the Quotient Property,** $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$

$$\frac{1}{x^8}$$

Try It

26) Simplify: $\dfrac{\left(a^2\right)^3\left(a^2\right)^4}{\left(a^4\right)^5}$

Solution

$\dfrac{1}{a^6}$

27) Simplify: $\dfrac{\left(p^3\right)^4\left(p^5\right)^3}{\left(p^7\right)^6}$

Solution

$\dfrac{1}{p^{15}}$

Example 5.5.13

Simplify: $\dfrac{(10p^3)^2}{(5p)^3\cdot\left(2p^5\right)^4}$

Solution

Step 1: **Use the Product to a Power Property,** $(ab)^m = a^m b^m$

$$\frac{(10)^2\left(p^3\right)^2}{(5)^3(p)^3\cdot(2)^4\left(p^5\right)^4}$$

Step 2: **Use the Power Property,** $a^{mn} = a^{m\times n}$

$$\frac{100p^6}{125p^3 \cdot 16p^{20}}$$

Step 3: **Add the exponents in the denominator.**

$$\frac{100p^6}{125 \times 16p^{23}}$$

Step 4: **Use the Quotient Property,** $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$

$$= \frac{100}{125 \times 16p^{17}}$$

$$\text{Simplify.} \quad = \frac{1}{20p^{17}}$$

Try It

28) Simplify: $\frac{(3r^{3}{}^{2}(r^{3})^{7}}{(r^{3})^{3}}$

Solution

$9r^{18}$

29) Simplify: $\frac{(2x^{4})^{5}}{(4x^{3})^{2}(x^{3})^{5}}$

Solution

$\frac{2}{x}$

Use the Definition of a Negative Exponent

Up until this point, we have been careful to minimize our use of negative exponents. Before taking on division of polynomials, we will revisit some of the properties we've seen in the previous sections but, this time, with negative exponents.

Quotient Property for Exponents

If a is a real number, $a \neq 0$, and m and n are whole numbers, then

$$\frac{a^m}{a^n} = a^{m-n}\text{, for } m > n \text{ and } \frac{a^m}{a^n} = \frac{1}{a^{n-m}}\text{, for } n > m.$$

What if we just subtract exponents regardless of which is larger?

Let's consider $\frac{x^2}{x^5}$.

We subtract the exponent in the denominator from the exponent in the numerator.

$$\begin{aligned} &= \frac{x^2}{x^5} \\ &= x^{2-5} \\ &= x^{-3} \end{aligned}$$

We can also simplify $\frac{x^2}{x^5}$ by dividing out common factors:

$$\begin{aligned} &= \frac{rgb]1.0, 0.0, 0.0\cancel{x} \cdot rgb]1.0, 0.0, 0.0\cancel{x}}{rgb]1.0, 0.0, 0.0\cancel{x} \cdot rgb]1.0, 0.0, 0.0\cancel{x} \cdot x \cdot x \cdot x} \\ &= \frac{1}{x^3} \end{aligned}$$

This implies that $x^{-3} = \frac{1}{x^3}$ and it leads us to the definition of a *negative exponent*.

Negative Exponents

If n is an integer and $a \neq 0$, then $a^{-n} = \frac{1}{a^n}$.

The negative exponent tells us we can re-write the expression by taking the reciprocal of the base and then changing the sign of the exponent.

Any expression that has negative exponents is not considered to be in simplest form. We will use the definition of a negative exponent and other properties of exponents to write the expression with only positive exponents.

For example, if after simplifying an expression we end up with the expression x^{-3}, we will take one more step and write $\frac{1}{x^3}$. The answer is considered to be in simplest form when it has only positive exponents.

Example 5.5.14

Simplify:

a. 4^{-2}
b. 10^{-3}

Solution

a.

Step 1: **Use the definition of a negative exponent,** $a^{-n} = \frac{1}{a^n}$.

$$= \frac{1}{4^2}$$

$$\text{Simplify.} \quad = \frac{1}{16}$$

---

b.

Step 1: **Use the definition of a negative exponent,** $a^{-n} = \frac{1}{a^n}$.

$$= \frac{1}{10^3}$$

$$\text{Simplify.} \quad = \frac{1}{1000}$$

Try It

30) Simplify:

a. 2^{-3}
b. 10^{-7}

Solution
a. $\frac{1}{8}$
b. $\frac{1}{10^7}$

31) Simplify:

a. 3^{-2}
b. 10^{-4}

Solution
a. $\frac{1}{9}$
b. $\frac{1}{10,000}$

In example 5.2.13 we raised an integer to a negative exponent. What happens when we raise a fraction to a negative exponent? We'll start by looking at what happens to a fraction whose numerator is one and whose denominator is an integer raised to a negative exponent.

$$\frac{1}{a^{-n}}$$

Use the definition of a negative exponent, $a^{-n}=\frac{1}{a^n}$ $\qquad \frac{1}{\frac{1}{a^n}}$

Simplify the complex fraction. $\qquad 1\cdot\frac{a^n}{1}$

Multiply. $\qquad a^n$

This leads to the *Property of Negative Exponents*.

Property of Negative Exponents

If n is an integer and $a\neq 0$, then $\frac{1}{a^{-n}}=a^n$.

Example 5.5.15

Simplify:

a. \frac{1}{{y}^{-4}}

b. $\frac{1}{3^{-2}}$

Solution

a.

Step 1: **Use the property of a negative exponent,** $\frac{1}{a^{-n}}=a^n$**.**

$$y^4$$

b.

Step 1: **Use the property of a negative exponent,** $\dfrac{1}{a^{-n}} = a^n$.

$$= 3^2$$

$$\text{Simplify.} \quad = 9$$

Try It

32) Simplify:

a. $\dfrac{1}{p^{-8}}$

b. $\dfrac{1}{4^{-3}}$

Solution

a. p^8

b. 64

33) Simplify:

a. $\dfrac{1}{q^{-7}}$

b. $\dfrac{1}{2^{-4}}$

Solution

a. q^7

b. 16

Suppose now we have a fraction raised to a negative exponent. Let's use our definition of negative exponents to lead us to a new property.

	$(\frac{3}{4})^{-2}$
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{(\frac{3}{4})^2}$
Simplify the denominator.	$\frac{1}{\frac{9}{16}}$
Simplify the complex fraction.	$\frac{16}{9}$
But we know that $\frac{16}{9}$ is $(\frac{4}{3})^2$.	$\frac{4^2}{3^2}$
This tells us that:	$(\frac{3}{4})^{-2} = (\frac{4}{3})^2$

To get from the original fraction raised to a negative exponent to the final result, we took the reciprocal of the base—the fraction—and changed the sign of the exponent.

This leads us to the *Quotient to a Negative Power Property*.

Quotient to a Negative Exponent Property

If a and b are real numbers, $a \neq 0$, $b \neq 0$, and n is an integer, then $(\frac{a}{b})^{-n} = (\frac{b}{a})^n$.

Example 5.5.16

Simplify:

a. $\left(\frac{5}{7}\right)^{-2}$

b. $\left(-\frac{2x}{y}\right)^{-3}$

Solution

a.

Step 1: **Use the Quotient to a Negative Exponent Property, $(\frac{a}{b})^{-n} = (\frac{b}{a})^{n}$.**

Step 2: **Take the reciprocal of the fraction and change the sign of the exponent.**

$$= \left(\frac{7}{5}\right)^{2}$$

$$\text{Simplify.} \quad = \frac{49}{25}$$

---

b.

Step 1: **Use the Quotient to a Negative Exponent Property, $(\frac{a}{b})^{-n} = (\frac{b}{a})^{n}$**

Step 2: **Take the reciprocal of the fraction and change the sign of the exponent.**

$$= \left(-\frac{y}{2x}\right)^{3}$$

$$\text{Simplify.} \quad = -\frac{y^3}{8x^3}$$

Try It

34) Simplify:

a. $\left(\frac{2}{3}\right)^{-4}$
b. $(-\frac{6m}{n})^{-2}$

Solution
a. $\frac{81}{16}$
b. $\frac{n^2}{36m^2}$

35) Simplify:

a. $\left(\frac{3}{5}\right)^{-3}$
b. $\left(-\frac{a}{2b}\right)^{-4}$

Solution
a. $\frac{125}{27}$
b. $\frac{16b^4}{a^4}$

When simplifying an expression with exponents, we must be careful to correctly identify the base.

Example 5.5.17

Simplify:

a. $(-3)^{-2}$
b. -3^{-2}

c. $\left(-\frac{1}{3}\right)^{-2}$

d. $-\left(\frac{1}{3}\right)^{-2}$

Solution

a. Here the exponent applies to the base -3.

Step 1: **Take the reciprocal of the base and change the sign of the exponent.**

$$= \frac{1}{(-3)^2}$$

$$\text{Simplify.} \quad = \frac{1}{9}$$

---

b. The expression -3^{-2} means "find the opposite of 3^{-2}."

Step 1: **Rewrite as a product with -1.**

$$-1 \times 3^{-2}$$

Step 2: **Take the reciprocal of the base and change the sign of the exponent.**

$$= -1 \cdot \frac{1}{3^2}$$

$$\text{Simplify.} \quad = -\frac{1}{9}$$

---

c. Here the exponent applies to the base $-\frac{1}{3}$.

Step 1: **Take the reciprocal of the base and change the sign of the exponent.**

$$= \left(-\frac{3}{1}\right)^2$$

$$\text{Simplify.} \quad = 9$$

---

d. The expression $-(\frac{1}{3})^{-2}$ means "find the opposite of $(\frac{1}{3})^{-2}$." Here the exponent applies to the base $\frac{1}{3}$.

***Step 1:* Rewrite as a product with -1.**

$$-1 \cdot \left(\frac{3}{1}\right)^2$$

***Step 2:* Take the reciprocal of the base and change the sign of the exponent.**

$$-9$$

Try It

36) Simplify:

a. $(-5)^{-2}$

b. -5^{-2}

c. $\left(-\frac{1}{5}\right)^{-2}$

d. $-\left(\frac{1}{5}\right)^{-2}$

Solution

a. $\frac{1}{25}$

b. $-\frac{1}{25}$

c. 25

d. -25

37) Simplify:

a. $(-7)^{-2}$

b. -7^{-2}

c. $\left(-\frac{1}{7}\right)^{-2}$
d. $\left(-\frac{1}{7}\right)^{-2}$

Solution

a. $\frac{1}{49}$
b. $-\frac{1}{49}$
c. 49
d. -49

We must be careful to follow the Order of Operations. In the next example, parts (a) and (b) look similar, but the results are different.

Example 5.5.18

Simplify:

a. 4×2^{-1}
b. $(4 \times 2)^{-1}$

Solution

a.

Step 1: **Do exponents before multiplication.**

$$4 \times 2^{-1}$$

Step 2: **Use** $a^{-n} = \frac{1}{a^n}$

$$= 4 \times \frac{1}{2^1}$$

$$\text{Simplify.} \quad = 2$$

---

b.

Step 1: **Simplify inside the parentheses first.**

$$(8)^{-1}$$

Step 2: **Use** $a^{-n} = \dfrac{1}{a^n}$

$$= \frac{1}{8^1}$$

$$\text{Simplify.} \quad = \frac{1}{8}$$

Try It

38) Simplify:

a. 6×3^{-1}
b. $(6 \times 3)^{-1}$

Solution

a. 2
b. $\dfrac{1}{18}$

39) Simplify:

a. 8×2^{-2}
b. 8×2^{-2}

Solution

a. 2
b. $\frac{1}{256}$

When a variable is raised to a negative exponent, we apply the definition the same way we did with numbers. We will assume all variables are non-zero.

Example 5.5.19

Simplify:

a. {x}^{-6}
b. $(u^4)^{-3}$

Solution
a.

Step 1: **Use the definition of a negative exponent** $a^{-n} = \frac{1}{a^n}$

$$\frac{1}{x^6}$$

---

b.

Step 1: **Use the definition of a negative exponent** $a^{-n} = \frac{1}{a^n}$.

$$= \frac{1}{(u^4)^3}$$

$$\text{Simplify.} \quad = \frac{1}{u^{12}}$$

Try It

40) Simplify:

a. y^{-7}
b. $(z^3)^{-5}$

Solution

a. $\frac{1}{y^7}$
b. $\frac{1}{z^{15}}$

41) Simplify:

a. p^{-9}
b. $(q^4)^{-6}$

Solution

a. $\frac{1}{p}^{9}$
b. $\frac{1}{q}^{24}$

When there is a product and an exponent we have to be careful to apply the exponent to the correct quantity. According to the Order of Operations, we simplify expressions in parentheses before applying exponents. We'll see how this works in the next example.

Example 5.5.20

Simplify:

a. $5y^{-1}$
b. $(5y)^{-1}$
c. $(-5y)^{-1}$

Solution

a.

Step 1: **Notice the exponent applies to just the base y.**

Step 2: **Take the reciprocal of y and change the sign of the exponent.**

$$= 5 \cdot \left(\frac{1}{y^1}\right)$$

$$\text{Simplify.} \quad = \frac{5}{y}$$

---

b.

Step 1: **Here the parentheses make the exponent apply to the base $5y$.**

Step 2: **Take the reciprocal of $5y$ and change the sign of the exponent.**

$$= \frac{1}{5y^1}$$

$$\text{Simplify.} \quad = \frac{1}{5y}$$

---

c.

Step 1: **The base here is $-5y$.**

Step 2: **Take the reciprocal of $-5y$ and change the sign of the exponent.**

$$= \frac{1}{-5y}$$

Simplify. $$= \left(-\frac{1}{5y}\right)$$

Try It

42) Simplify:

a. $(8p^{-1})$

b. $(8p)^{-1}$

c. $(-8p)^{-1}$

Solution

a. $\frac{8}{p}$

b. $\frac{1}{8p}$

c. $-\frac{1}{8p}$

43) Simplify:

a. $11q^{-1}$

b. $(11q)^{-1}$

c. $(-11q)^{-1}$

Solution

a. $\frac{1}{11q}$

b. $\frac{1}{11q}$

c. $-\frac{1}{11q}$

With negative exponents, the Quotient Rule needs only one form $\frac{a^m}{a^n} = a^{m-n}$, for $a \neq 0$. When the exponent in the denominator is larger than the exponent in the numerator, the exponent of the quotient will be negative.

Simplify Expressions with Integer Exponents

All of the exponent properties we developed earlier in the chapter with whole number exponents apply to integer exponents, too. We restate them here for reference.

Summary of Exponent Properties

If a and b are real numbers, and m and n are integers, then

Product Property	$a^m \times a^n = a^{m+n}$
Power Property	$(a^m)^n = amn$
Product to a Power	$(ab)^m = a^m b^m$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}$
Zero Exponent Property	$a^0 = 1, a \neq 1$
Quotient to a Power Property	$(\frac{a}{b})^m = \frac{a^m}{b^m}, b \neq 0$
Properties of Negative Exponents	$a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$
Quotient to a Negative Exponent	(\frac{a}{b})^{-n} = (\frac{b}{a})^{n}

Example 5.5.21

Simplify:

a. $x^{-4} \times x^{6}$
b. $y^{-6} \times y^{4}$
c. $z^{-5} \times z^{-3}$

Solution

a.

Step 1: **Use the Product Property,** $a^m \cdot a^n = a^{m+n}$

$$= x^{-4+6}$$

$$\text{Simplify.} \quad = x^2$$

---

b.

Step 1: **Notice the same bases, so add the exponents.**

$$= y^{-6+4}$$

$$\text{Simplify.} \quad = y^{-2}$$

Step 3: **Use the definition of a negative exponent,** $a^{-n} = \dfrac{1}{a^n}$

$$\frac{1}{y^2}$$

---

c.

Step 1: **Add the exponents, since the bases are the same.**

$$= z^{-5-3}$$

$$\text{Simplify.} \quad = z^{-8}$$

Step 3: **Take the reciprocal and change the sign of the exponent, using the definition of a negative exponent.**

$$\frac{1}{z^8}$$

Try It

44) Simplify:

a. $x^{-3} \times x^{7}$
b. $y^{-7} \times y^{2}$
c. $z^{-4} \times z^{-5}$

Solution

a. x^{4}
b. $\frac{1}{y^5}$
c. $\frac{1}{z^9}$

45) Simplify:

a. $a^{-1} \times a^{6}$
b. b^{-8}\times b^{4}
c. $c^{-8} \times c^{-7}$

Solution

a. a^{5}
b. $\frac{1}{b^4}$
c. $\frac{1}{c^{15}}$

In the next two examples, we'll start by using the Commutative Property to group the same variables together. This makes it easier to identify the like bases before using the Product Property.

Example 5.5.22

Simplify: $(m^4n^{-3})(m^{-5}n^{-2})$

Solution

Step 1: **Use the Commutative Property to get like bases together.**

$$m^4m^{-5}n^{-2}n^{-3}$$

Step 2: **Add the exponents for each base.**

$$m^{-1}n^{-5}$$

Step 3: **Take reciprocals and change the signs of the exponents.**

$$= \frac{1}{m^1}\frac{1}{n^5}$$

$$\text{Simplify.} \quad = \frac{1}{mn^5}$$

Try It

46) Simplify: $(p^6q^{-2})(p^{-9}q^{-1})$

Solution

$$\frac{1}{p^3q^3}$$

47) Simplify: $(r^5s^{-3})(r^{-7}s^{-5})$

Solution

$$\frac{1}{r^2s^8}$$

If the monomials have numerical coefficients, we multiply the coefficients, just like we did earlier.

Example 5.5.23

Simplify: $(2x^{-6}y^{8})(-5x^{5}y^{-3})$

Solution

Step 1: **Rewrite with the like bases together**

$$2(-5)(x^{-6}x^{5})(y^{8}y^{-3})$$

Step 2: **Multiply the coefficients and add the exponents of each variable.**

$$-10x^{-1}y^{5}$$

Step 3: **Use the definition of a negative exponent,** $a^{-n} = \dfrac{1}{a^n}$

$$= -10 \cdot \left(\frac{1}{x^1}\right) \cdot y^5$$

$$\text{Simplify.} \quad = -\frac{10y^5}{x}$$

Try It

48) Simplify: $(3u^{-5}v^{7})(-4u^{4}v^{-2})$

Solution

$$-\frac{12v^5}{u}$$

49) Simplify: $(-6c^{-6}d^{4})(-5c^{-2}d^{-1})$

Solution

$$\frac{30d^3}{c^8}$$

In the next two examples, we'll use the *Power Property* and the *Product to a Power Property*.

Example 5.5.24

Simplify: $(6k^3)^{-2}$

Solution

Step 1: **Use the Product to a Power Property,** $(ab)^m = a^m b^m$

$$(6)^{(-2)}(k^3)^{-2}$$

Step 2: **Use the Power Property,** $(a^m)^n = a^{mn}$

$$6^{-2}k^{-6}$$

Step 3: **Use the Definition of a Negative Exponent,** $a^{-n} = \frac{1}{a^n}$

$$= \frac{1}{6^2} \cdot \frac{1}{k^6}$$

$$\text{Simplify.} \quad = \frac{1}{36k^6}$$

Try It

50) Simplify: $(-4x^4)^{-2}$

Solution

$$\frac{1}{16x^8}$$

51) Simplify: $\left(2b^3\right)^{-4}$

Solution

$$\frac{1}{16b^{12}}$$

Example 5.5.25

Simplify: $(5x^{-3})^2$

Solution

Step 1: **Use the Product to a Power property,** $(ab)^m = a^m b^m$

$$5^2(x^{-3})^2$$

Step 2: **Simplify** 5^2 **and multiply the exponents of** x **using the Power Property,** $(a^m)^n = a^{mn}$

$$25x^{-6}$$

Step 3: **Rewrite** x^{-6} **by using the Definition of a Negative Exponent,** $a^{-n} = \dfrac{1}{a^n}$

$$= 25 \cdot \left(\frac{1}{x^6}\right)$$

$$\text{Simplify.} \quad = \frac{25}{x^6}$$

Try It

52) Simplify: $(8a^{-4})^2$

Solution

$$\frac{64}{a^8}$$

53) Simplify: $(2c^{-4})^3$

Solution

$$\frac{8}{c^{12}}$$

To simplify a fraction, we use the Quotient Property and subtract the exponents.

Example 5.5.26

Simplify: $\frac{r^5}{r^{-4}}$

Solution

Step 1: **Use the Quotient Property,** $\frac{a^m}{a^n} = a^{m-n}$ **c**

$$= r^{5-(-4)}$$

$$\text{Simplify.} \quad = r^9$$

Try It

54) Simplify: $\frac{x^8}{x^{-3}}$

Solution

x^{11}

55) Simplify: $\frac{y^8}{y^{-6}}$

Solution

y^{14}

Divide Monomials

You have now been introduced to all the properties of exponents and used them to simplify expressions. Next, you'll see how to use these properties to divide monomials. Later, you'll use them to divide polynomials.

Example 5.5.27

Find the quotient: $56x^7 \div 8x^3$

Solution

Step 1: **Rewrite as a fraction.**

$$\frac{56x^7}{8x^3}$$

Step 2: **Use fraction multiplication.**

$$\frac{56}{8} \times \frac{x^7}{x^3}$$

Step 3: **Simplify and use the Quotient Property.**

7x^4

Try It

56) Find the quotient: $42y^9 \div 6y^3$

Solution

$7y^6$

57) Find the quotient: $48z^8 \div 8z^2$

Solution

$6z^6$

Example 5.5.28

Find the quotient: $\dfrac{45a^2b^3}{-5ab^5}$

Solution

Step 1: **Use fraction multiplication.**

$$\frac{45}{-5}\frac{a^2}{a}\frac{b^3}{b^5}$$

Step 2: **Simplify and use the Quotient Property.**

$$= -9 \cdot a \cdot \frac{1}{b^2}$$

$$\text{Multiply.} \quad = -\frac{9a}{b^2}$$

Try It

58) Find the quotient: $\dfrac{-72a^7b^3}{8a^{12}b^4}$

Solution

$$-\frac{9}{a^5b}$$

59) Find the quotient: $\dfrac{-63c^8d^3}{7c^{12}d^2}$

Solution

$$\frac{-9d}{c^4}$$

Example 5.5.29

Find the quotient: $\dfrac{24a^5b^3}{48ab^4}$

Solution

Step 1: **Use fraction multiplication.**

$$\frac{24}{48}\cdot\frac{a^5}{a}\cdot\frac{b^3}{b^4}$$

Step 2: **Simplify and use the Quotient Property.**

$$=\frac{1}{2}\cdot a^4\cdot\frac{1}{b}$$

$$\text{Multiply.}\quad=\frac{a^4}{2b}$$

Try It

60) Find the quotient: $\dfrac{16a^7b^6}{24ab^8}$

Solution

$$\frac{2a^6}{3b^2}$$

61) Find the quotient: $\dfrac{27p^4q^7}{-45p^{12}q}$

Solution

$$-\frac{3q^6}{5p^8}$$

Once you become familiar with the process and have practised it step by step several times, you may be able to simplify a fraction in one step.

Example 5.5.30

Find the quotient: $\dfrac{14x^7y^{12}}{21x^{11}y^6}$

Solution

Be very careful to simplify $\dfrac{14}{21}$ by dividing out a common factor, and to simplify the variables by subtracting their exponents.

Step 1: **Simplify and use the Quotient Property.**

$$\frac{2y^6}{3x^4}$$

Try It

62) Find the quotient: $\dfrac{28x^5y^{14}}{49x^9y^{12}}$

Solution

$\dfrac{4y^2}{7x^4}$

63) Find the quotient: $\dfrac{30m^5n^{11}}{48m^{10}n^{14}}$

Solution

$\dfrac{5}{8m^5n^3}$

In all examples so far, there was no work to do in the numerator or denominator before simplifying the fraction. In the next example, we'll first find the product of two monomials in the numerator before we simplify the fraction. This follows the order of operations. Remember, a fraction bar is a grouping symbol.

Example 5.5.31

Find the quotient: $\dfrac{(6x^2y^3)(5x^3y^2)}{(3x^4y^5)}$

Solution

Step 1: **Simplify the numerator.**

$$= \frac{30x^5y^5}{3x^4y^5}$$

Simplify. $= 10x$

Try It

64) Find the quotient: $\dfrac{(6a^4b^5)(4a^2b^5)}{12a^5b^8}$

Solution

$2ab^2$

65) Find the quotient: $\dfrac{(-12x^6y^9)(-4x^5y^8)}{-12x^{10}y^{12}}$

Solution

$-4xy^5$

Access these online resources for additional instruction and practice with dividing monomials:

- Rational Expressions
- Dividing Monomials
- Dividing Monomials 2

Key Concepts

- **Quotient Property for Exponents:**
    - If a is a real number, $a \neq 0$, and m, n are whole numbers, then: $\frac{a^m}{a^n} = a^{m-n}, m > n$ and $\frac{a^m}{a^n} = \frac{1}{a^{m-n}}, n > m$
- **Zero Exponent**
    - If a is a non-zero number, then $a^0 = 1$.
- **Quotient to a Power Property for Exponents**:
    - If a and b are real numbers, $b \neq 0$ and m is a counting number, then: $(\frac{a}{b})^m = \frac{a^m}{b^m}$
    - To raise a fraction to a power, raise the numerator and denominator to that power.
- **Summary of Exponent Properties**
    - If a, b are real numbers and m, n are whole numbers, then

Product Property	$a^m a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{mn}$
Product to a Power	$(ab)^m = a^m b^m$
Quotient Property	$\frac{a^m}{b^m} = a^{m-n}, a \neq 0, m > n$
	$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, a \neq 0, n > n = m$
Zero Exponent Definition	$a^0 = 1, a \neq 0$
Quotient to a Power Property	$(\frac{a}{b})^m = \frac{a^m}{b^m}, b \neq 0$

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=2842#h5p-30

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

5.6 DIVIDE POLYNOMIALS

Learning Objectives

By the end of this section, you will be able to:

- Divide a polynomial by a monomial
- Divide a polynomial by a binomial

Try It

Before you get started, take this readiness quiz:

1) Add: $\frac{3}{d}+\frac{x}{d}$

2) Simplify: $\frac{30xy^3}{5xy}$

3) Combine like terms: $8a^2+12a+1+3a^2-5a+4$

Divide a Polynomial by a Monomial

In the last section, you learned how to divide a monomial by a monomial. As you continue to build up your knowledge of polynomials the next procedure is to divide a polynomial of two or more terms by a monomial.

The method we'll use to divide a polynomial by a monomial is based on the properties of fraction addition. So we'll start with an example to review fraction addition.

\frac{y}{5}+\frac{2}{5}

The sum simplifies to

$$\frac{y+2}{5}$$

Now we will do this in reverse to split a single fraction into separate fractions.

We'll state the fraction addition property here just as you learned it and in reverse

Fraction Addition

If a, b and c are numbers where $c \neq 0$, then

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c} \text{ and } \frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$$

We use the form on the left to add fractions and we use the form on the right to divide a polynomial by a monomial.

For example, $\frac{y+2}{5}$ can be written $\frac{y}{5}+\frac{2}{5}$

We use this form of fraction addition to divide polynomials by monomials.

Division of a Polynomial by a Monomial

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

Example 5.6.1

Find the quotient: $\dfrac{7y^2+21}{7}$

Solution

Step 1: **Divide each term of the numerator by the denominator.**

$$\frac{7y^2}{7}+\frac{21}{7}$$

Step 2: **Simplify each fraction.**

$$y^2+3$$

Try It

4) Find the quotient: $\dfrac{8z^2+24}{4}$

Solution

$2z^2+6$

5) Find the quotient: $\dfrac{18z^2-27}{9}$

Solution

$2z^2-3$

Remember that division can be represented as a fraction. When you are asked to divide a polynomial by a monomial and it is not already in fraction form, write a fraction with the polynomial in the numerator and the monomial in the denominator.

Example 5.6.2

Find the quotient: $(18x^3 - 36x^2) \div 6x$

Solution

***Step 1:* Rewrite as a fraction.**

$$\frac{18x^3 - 36x^2}{6x}$$

***Step 2:* Divide each term of the numerator by the denominator.**

$$= \frac{18x^3}{6x} - \frac{36x^2}{6x}$$

$$\text{Simplify.} \quad = 3x^2 - 6x$$

Try It

6) Find the quotient: $(27b^3 - 33b^2) \div 3b$

Solution

$9b^2 - 11b$

7) Find the quotient: $(25y^3 - 55y^2) \div 5y$

Solution

$5y^2 - 11y$

When we divide by a negative, we must be extra careful with the signs.

Example 5.6.3

Find the quotient: $\dfrac{12d^2 - 16d}{-4}$

Solution

***Step 1:* Divide each term of the numerator by the denominator.**

$$\frac{12d^2}{-4} - \frac{16d}{-4}$$

***Step 2:* Simplify. Remember, subtracting a negative is like adding a positive!**

$$-3d^2 + 4d$$

Try It

8) Find the quotient: $\dfrac{25y^2 - 15y}{-5}$

Solution

$-5y^2 + 3y$

9) Find the quotient: $\dfrac{42b^2 - 18b}{-6}$

Solution

$-7b^2 + 3b$

Example 5.6.4

Find the quotient: $\dfrac{105y^5 + 75y^3}{5y^2}$

Solution

***Step 1:* Separate the terms.**

$$= \frac{105y^5}{5y^2} + \frac{75y^3}{5y^2}$$

$$\text{Simplify.} \quad = 21y^3 + 15y$$

Try It

10) Find the quotient: $\dfrac{60d^7 + 24d^5}{4d^3}$

Solution

$15d^4 + 6d^2$

11) Find the quotient: $\dfrac{216p^7 - 48p^5}{6p^3}$

Solution

$36p^4 - 8p^2$

Example 5.6.5

Find the quotient: $(15x^3y - 35xy^2) \div (-5xy)$

Solution

Step 1: **Rewrite as a fraction.**

$$\frac{15x^3y - 35xy^2}{-5xy}$$

Step 2: **Separate the terms.**

$$= \frac{15x^3y}{-5xy} - \frac{35xy^2}{-5xy}$$

$$\text{Simplify.} \quad = -3x^2 + 7y$$

Try It

12) Find the quotient: $(32a^2b - 16ab^2) \div (-8ab)$

Solution

$-4a + 2b$

13) Find the quotient: $(-48a^8b^4 - 36a^6b^5) \div (-6a^3b^3)$

Solution

$8a^5b + 6a^3b^2$

Example 5.6.6

Find the quotient: $\dfrac{36x^3y^2+27x^2y^2-9x^2y^3}{9x^2y}$

Solution

Step 1: **Separate the terms.**

$$=\frac{36x^3y^2}{9x^2y}+\frac{27x^2y^2}{9x^2y}-\frac{9x^2y^3}{9x^2y}$$

$$\text{Simplify.}\quad =4xy+3y-y^2$$

Try It

14) Find the quotient: $\dfrac{40x^3y^2+24x^2y^2-16x^2y^3}{8x^2y}$.

Solution

$5xy+3y-2y^2$

15) Find the quotient: $\dfrac{35a^4b^2+14a^4b^3-42a^2b^4}{7a^2b^2}$

Solution

$5a^2+2a^2b-6b^2$

Example 5.6.7

Find the quotient: $\dfrac{10x^2+5x-20}{5x}$

Solution

***Step 1:* Separate the terms.**

$$=\frac{10x^2}{5x}+\frac{5x}{5x}-\frac{20}{5x}$$

$$\text{Simplify.}\quad =2x+1+\frac{4}{x}$$

Try It

16) Find the quotient: $\dfrac{18c^2+6c-9}{6c}$

Solution

$3c+1-\dfrac{3}{2c}$

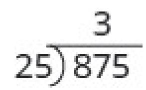
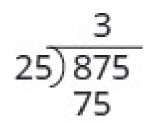
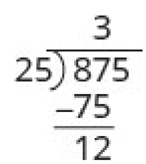
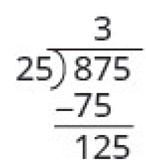
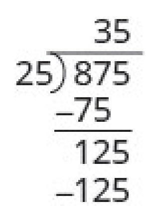
17) Find the quotient: $\dfrac{10d^2-5d-2}{5d}$

Solution

$2d-1-\dfrac{2}{5d}$

Divide a Polynomial by a Binomial

To divide a polynomial by a binomial, we follow a procedure very similar to long division of numbers. So let's look carefully the steps we take when we divide a 3-digit number, 875, by a 2-digit number, 25.

We write the long division		Figure
We divide the first two digits, 87, by 25.		Figure 5.6.2
We multiply 3 times 25 and write the product under the 87.		Figure 5.6.3
Now we subtract 75 from 87.		Figure 5.6.4
Then we bring down the third digit of the dividend, 5.		Figure 5.6.5
Repeat the process, dividing 25 into 125.		Figure 5.6.6

We check division by multiplying the quotient by the divisor.

If we did the division correctly, the product should equal the dividend.

$$35 \times 25$$
$$875 \checkmark$$

Now we will divide a trinomial by a binomial. As you read through the example, notice how similar the steps are to the numerical example above.

Example 5.6.8

Find the quotient: $(x^2 + 9x + 20) \div (x + 5)$

Solution

Step 1: **Write it as a long division problem.**

Step 2: **Be sure the dividend is in standard form.**

Figure 5.6.7

Step 3: **Divide x^2 by x.**

It may help to ask yourself, "What do I need to multiply x by to get x^2?"

Step 4: **Put the answer, x, in the quotient over the x term.**

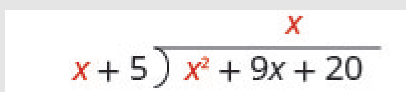


Figure 5.6.8

Step 5: **Multiply x times $x + 5$.**

Line up the like terms under the dividend.

Step 6: **Subtract $x^2 + 5x$ from $x^2 + 9x$.**

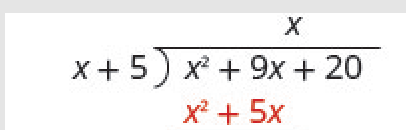


Figure 5.6.9

Step 7: **Add, changing the signs might make it easier to do this.**

Step 8: **Then bring down the last term, 20.**

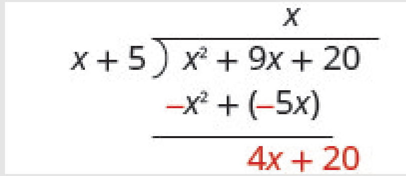


Figure 5.6.10

Step 9: **Divide $4x$ by x.**

It may help to ask yourself, “What do I need to multiply x by to get $4x$?”

Step 10: **Put the answer, 4, in the quotient over the constant term.**

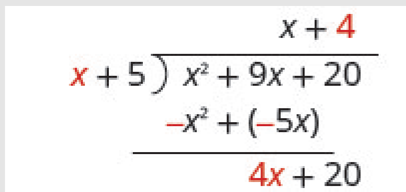


Figure 5.6.11

Step 11: **Multiply 4 times $x + 5$.**

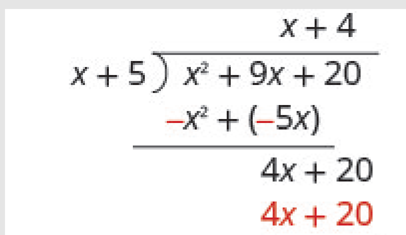


Figure 5.6.12

Step 12: **Subtract $4x + 20$ from $4x + 20$.**

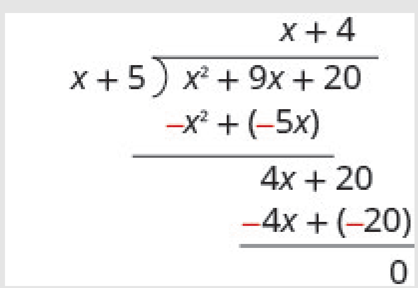


Figure 5.6.13

Step 13: **Check:**

Multiply the quotient by the divisor.
You should get the dividend.

$$(x+4)(x+5)$$
$$x^2+9x+20\checkmark$$

Try It

18) Find the quotient: $(y^2+10y+21)\div(y+3)$

Solution

$y+7$

19) Find the quotient: $(m^2+9m+20)\div(m+4)$

Solution

$m+5$

When the divisor has a subtraction sign, we must be extra careful when we multiply the partial quotient and then subtract. It may be safer to show that we change the signs and then add.

Example 5.6.9

Find the quotient: $(2x^2-5x-3)\div(x-3)$

Solution

Step 1: **Write it as a long division problem.**

Step 2: **Be sure the dividend is in standard form.**

$$x-3\overline{)2x^2-5x-3}$$

Figure 5.6.14

Step 3: **Divide $2x^2$ by x.**
Step 4: **Put the answer, $2x$, in the quotient over the x term.**

$$\begin{array}{r} 2x \\ x-3\overline{)2x^2-5x-3} \end{array}$$

Figure 5.6.15

Step 5: **Multiply $2x$ times $x-3$.**

Line up the like terms under the dividend.

$$\begin{array}{r} 2x \\ x-3\overline{)2x^2-5x-3} \\ \underline{2x^2-6x} \end{array}$$

Figure 5.6.16

Step 6: **Subtract $2x^2-6x$ from $2x^2-5x$.**
Step 7: **Change the signs and then add.**
Step 8: **Then bring down the last term.**

$$\begin{array}{r} 2x \\ x-3\overline{)2x^2-5x-3} \\ \underline{-2x^2+6x} \\ x-3 \end{array}$$

Figure 5.6.17

Step 9: **Divide x by x.**
Step 10: **Put the answer, 1, in the quotient over the constant term.**

$$\begin{array}{r} 2x+1 \\ x-3\overline{)2x^2-5x-3} \\ \underline{-2x^2-(-6x)} \\ x-3 \end{array}$$

Figure 5.6.18

Step 11: **Multiply 1 times $x-3$.**

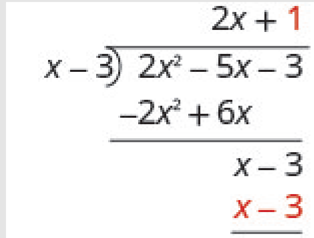


Figure 5.6.19

Step 12: **Subtract $x-3$ from $x-3$ by changing the signs and adding.**

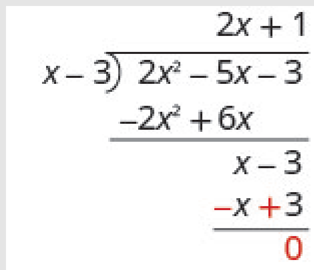


Figure 5.6.20

Step 13: **To check, multiply $(x-3)(2x+1)$.**

The result should be $2x^2-5x-3$.

Try It

20) Find the quotient: $(2x^2-3x-20)\div(x-4)$

Solution

$2x+5$

21) Find the quotient: $(3x^2-16x-12)\div(x-6)$

Solution

$3x+2$

When we divided 875 by 25, we had no remainder. But sometimes division of numbers does leave a remainder. The same is true when we divide polynomials. In example 5.6.10 we'll have a division that leaves a remainder. We write the remainder as a fraction with the divisor as the denominator.

Example 5.6.10

Find the quotient: $(x^3 - x^2 + x + 4) \div (x + 1)$

Solution

Step 1: **Write it as a long division problem.**

Step 2: **Be sure the dividend is in standard form.**

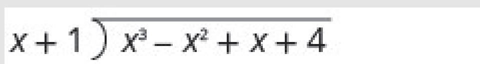


Figure 5.6.21

Step 3: **Divide** x^3 **by** x**.**

Step 4: **Put the answer,** x^2**, in the quotient over the** x^2 **term.**

Step 5: **Multiply** x^2 **times** $x + 1$**.**

Line up the like terms under the dividend.

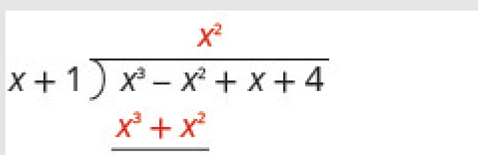


Figure 5.6.22

Step 6: **Subtract** $x^3 + x^2$ **from** $x^3 - x^2$ **by changing the signs and adding.**

Step 7: **Then bring down the next term.**

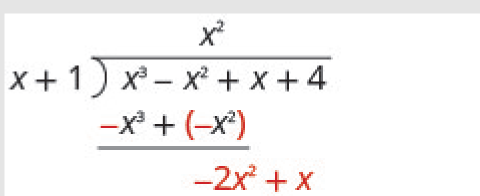


Figure 5.6.23

Step 9: **Divide** $-2x^2$ **by** x**.**

Step 10: **Put the answer, $-2x$, in the quotient over the x term.**

Step 11: **Multiply $-2x$ times $x+1$.**

Line up the like terms under the dividend.

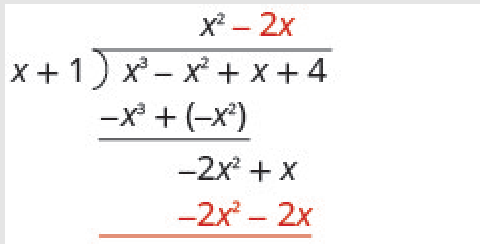


Figure 5.6.24

Step 12: **Subtract $-2x^2 - 2x$ from $-2x^2 + x$ by changing the signs and adding.**

Step 13: **Then bring down the last term.**

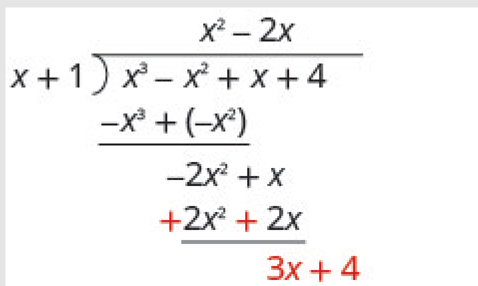


Figure 5.6.25

Step 14: **Divide $3x$ by x.**

Step 15: **Put the answer, 3, in the quotient over the constant term.**

Step 16: **Multiply 3 times $x+1$. Line up the like terms under the dividend.**

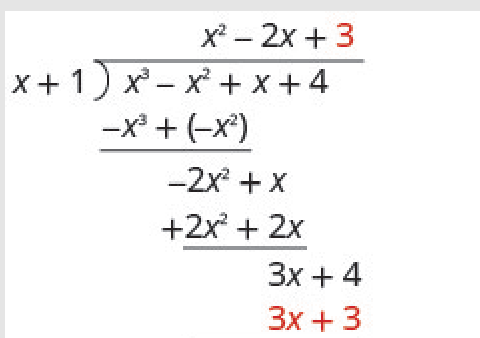


Figure 5.6.26

Step 17: **Subtract $3x+3$ from $3x+4$ by changing the signs and adding.**

Step 18: **Write the remainder as a fraction with the divisor as the denominator.**

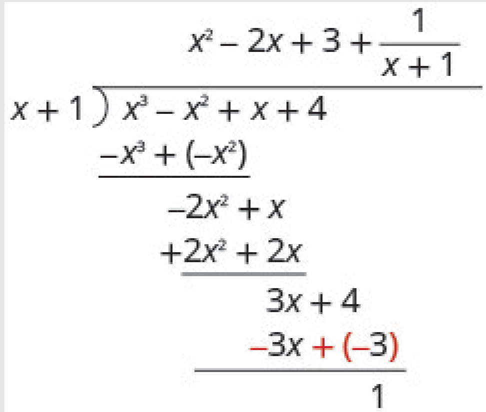


Figure 5.6.27

Step 19: **To check, multiply** $(x+1)(x^2-2x+3+\dfrac{1}{x+1})$.

The result should be x^3-x^2+x+4

Try It

22) Find the quotient: ({x}^{3}+5{x}^{2}+8x+6)\div(x+2)

Solution

$$\left(x^2+3x+2+\frac{2}{x+2}\right)$$

23) Find the quotient: $(2x^3+8x^2+x-8)\div(x+1)$

Solution

$$\left(2x^2+6x-5-\frac{3}{x+1}\right)$$

Look back at the dividends in Examples 5.6.8, 5.6.9, and 5.6.10 The terms were written in descending order of degrees, and there were no missing degrees. The dividend in Example 5.6.11 will be $x^4 - x^2 + 5x - 2$. It is missing an x^3 term. We will add in $0x^3$ as a placeholder.

Example 5.6.11

Find the quotient: $(x^4 - x^2 + 5x - 2) \div (x + 2)$

Solution

Notice that there is no x^3 term in the dividend. We will add $0x^3$ as a placeholder.

Step 1: **Write it as a long division problem.**

Be sure the dividend is in standard form with placeholders for missing terms.

$$x + 2 \overline{\smash{)}\, x^4 - 0x^3 - x^2 + 5x - 2}$$

Figure 5.6.28

Step 2: **Divide x^4 by x.**

Step 3: **Put the answer, x^3, in the quotient over the x^3 term.**

Step 4: **Multiply x^3 times $x + 2$.**

Line up the like terms.

Step 5: **Subtract and then bring down the next term.**

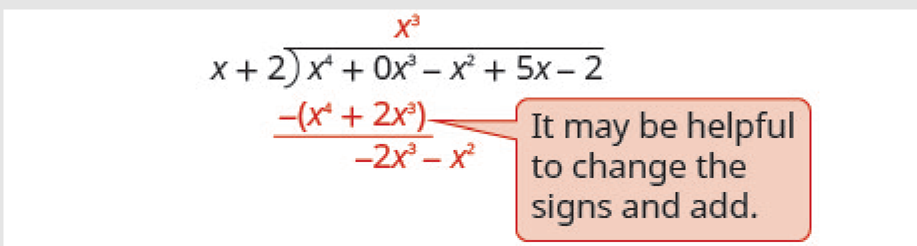


Figure 5.6.29

Step 6: **Divide $-2x^3$ by x.**

Step 7: **Put the answer, $-2x^2$, in the quotient over the x^2 term.**

Step 8: **Multiply $-2x^2$ times $x + 1$.**

Line up the like terms.

Step 9: **Subtract and bring down the next term.**

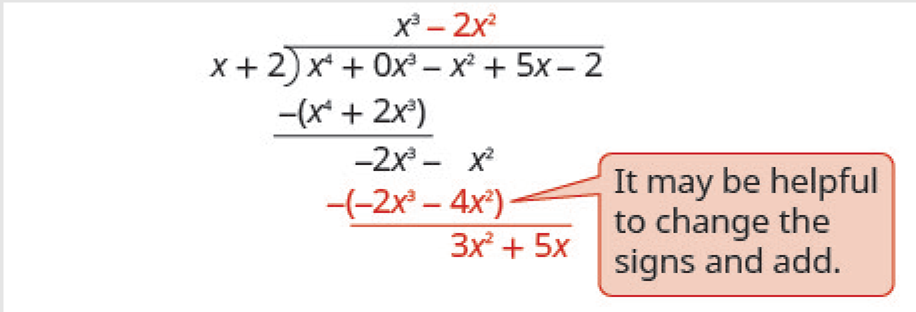


Figure 5.6.30

Step 10: **Divide $3x^2$ by x.**

Step 11: **Put the answer, $3x$, in the quotient over the x term.**

Step 12: **Multiply $3x$ times $x+1$.**

Line up the like terms.

Step 13: **Subtract and bring down the next term.**

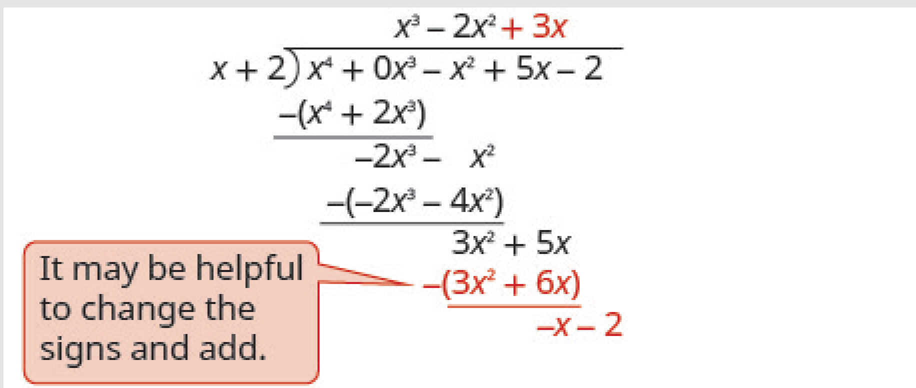


Figure 5.6.31

Step 14: **Divide $-x$ by x.**

Step 15: **Put the answer, -1, in the quotient over the constant term.**

Step 16: **Multiply -1 times $x+1$.**

Line up the like terms.

Step 17: **Change the signs, add.**

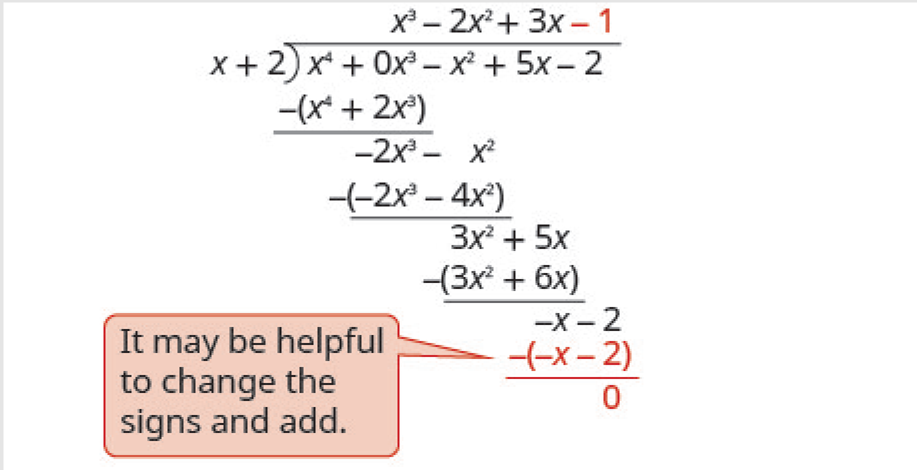


Figure 5.6.32

Step 18: **To check, multiply** $(x+2)(x^3-2x^2+3x-1)$.

The result should be (x^4-x^2+5x-2)

Try It

24) Find the quotient: $(x^3+3x+14)\div(x+2)$

Solution

x^2-2x+7

25) Find the quotient: $(x^4-3x^3-1000)\div(x+5)$

Solution

$x^3-8x^2+40x-200$

In Example 5.6.12, we will divide by $2a-3$. As we divide we will have to consider the constants as well as the variables.

Example 5.6.12

Find the quotient: $(8a^3 + 27) \div (2a + 3)$

Solution

This time we will show the division all in one step. We need to add two placeholders in order to divide.

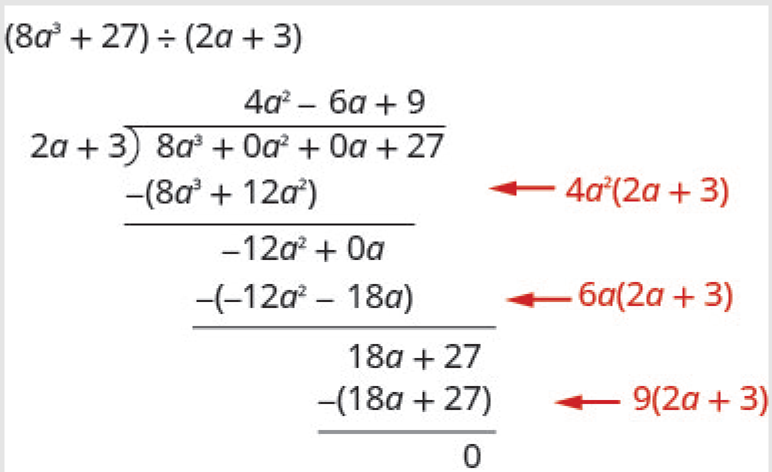


Figure 5.6.33

To check, multiply $(2a + 3)(4a^2 - 6a + 9)$

The result should be $8a^3 + 27$

Try It

26) Find the quotient: $(x^3 - 64) \div (x - 4)$

Solution

$x^2 + 4x + 16$

27) Find the quotient: $(125x^3 - 8) \div (5x - 2)$

Solution

$25x^2 + 10x + 4$

Access these online resources for additional instruction and practice with dividing polynomials:

- Divide a Polynomial by a Monomial
- Divide a Polynomial by a Monomial 2
- Divide Polynomial by Binomial

Key Concepts

- **Fraction Addition**
    - If a, b and c are numbers where $c \neq 0$, then $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ and $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$
- **Division of a Polynomial by a Monomial**
    - To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=2883#h5p-31

b. After reviewing this checklist, what will you do to become confident for all goals?

5.7 INTRODUCTION TO GRAPHING POLYNOMIALS

Learning Objectives

By the end of this section, you will be able to:

- Graph a basic polynomial function of the form $f(x) = ax^2 + k$.
- Graph a basic polynomial function of the form $f(x) = ax^3 + k$.

In this section, we will focus on graphing simple polynomial functions by identifying some important characteristics and creating a table of values. Polynomial functions can be represented graphically in the rectangular coordinate system, and these graphical representations can allow us to learn about the function visually and better understand the relationship between the variables. Polynomial functions can undergo many transformations to their form, but in this section, we will focus on vertical reflections and translations only.

Please note that the figures created for this section were created using the desmos graphing calculator. Feel free to use this calculator in your studies to better understand the graph of functions.

Graphs of Quadratic Equations

Let's examine the general shape of a graph of a general quadratic function $f(x) = ax^2$. The graph of a quadratic equation is called a parabola.

In the figure below (Figure 5.7.1), we see the graphs of the functions $f(x) = x^2$ and $f(x) = -x^2$. Notice that the only difference between these two equations is the sign in front of the x^2. This shows us that the sign of the leading coefficient, a, determines the direction of opening of a parabola. If the leading coefficient of a quadratic function is positive, $a > 0$, the parabola will open upwards, whereas if the leading coefficient of a quadratic function is negative, $a < 0$, the parabola will open downwards.

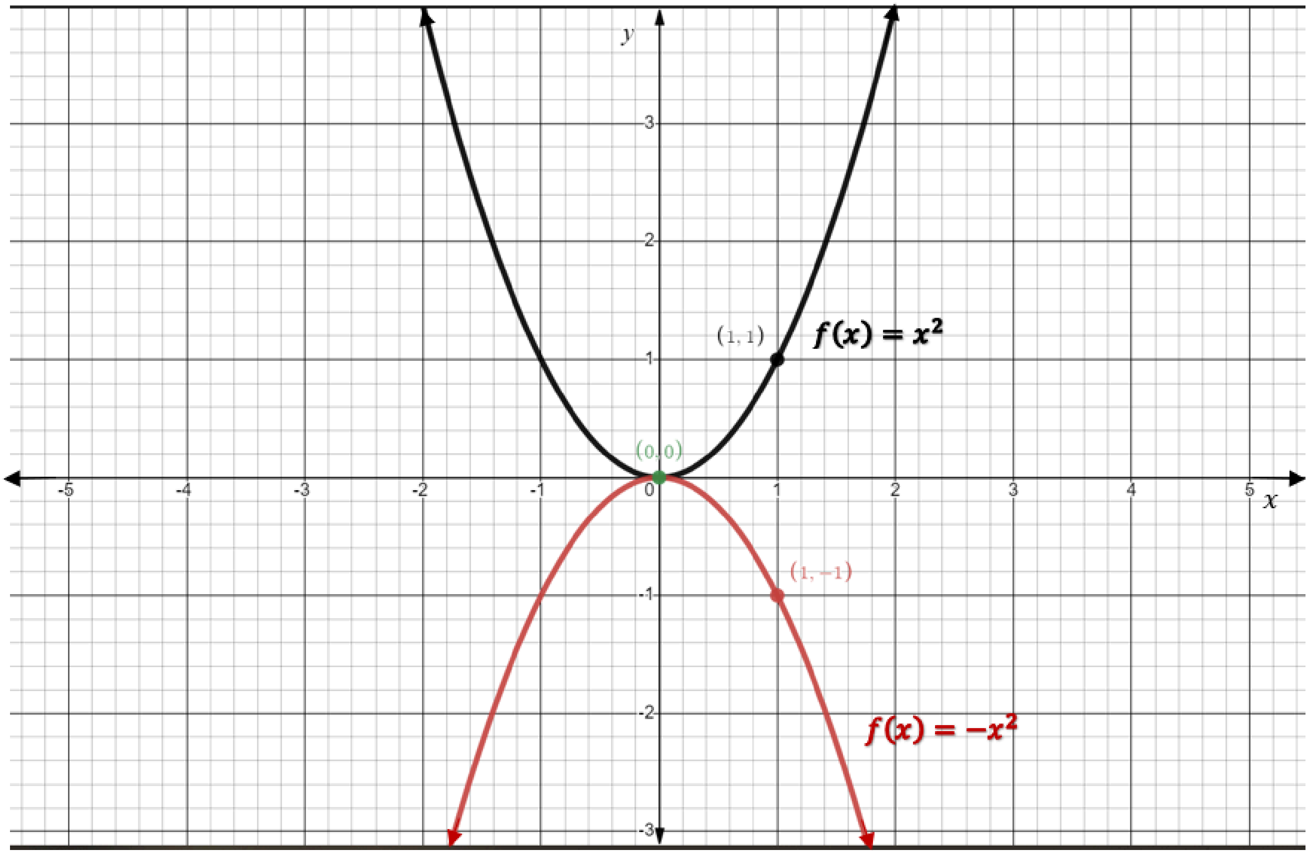


Figure 5.7.1 Comparing the graphs of equations that have undergone a vertical reflection.

Quadratic Equations Direction of Opening

For a function of the form $f(x) = ax^2$,

if $a > 0$, the parabola opens upwards.

if $a < 0$, the parabola opens downwards.

Another important characteristic of a parabola is its vertex. Here, we see that the vertex is the origin $(0, 0)$ regardless of the leading coefficient. In this case, the vertex is also the y-intercept. When graphing any function, it is always important to find the value of the y-intercept to use this point in the graphing process. To find the y-intercept of a function, set the x-value to zero and evaluate for y.

Let's examine a couple more graphs of quadratic functions.

This time, the figure below shows $f(x)=x^2$, $f(x)=2x^2$ and $f(x)=\frac{1}{2}x^2$ all in the same Cartesian plane for comparison purposes.

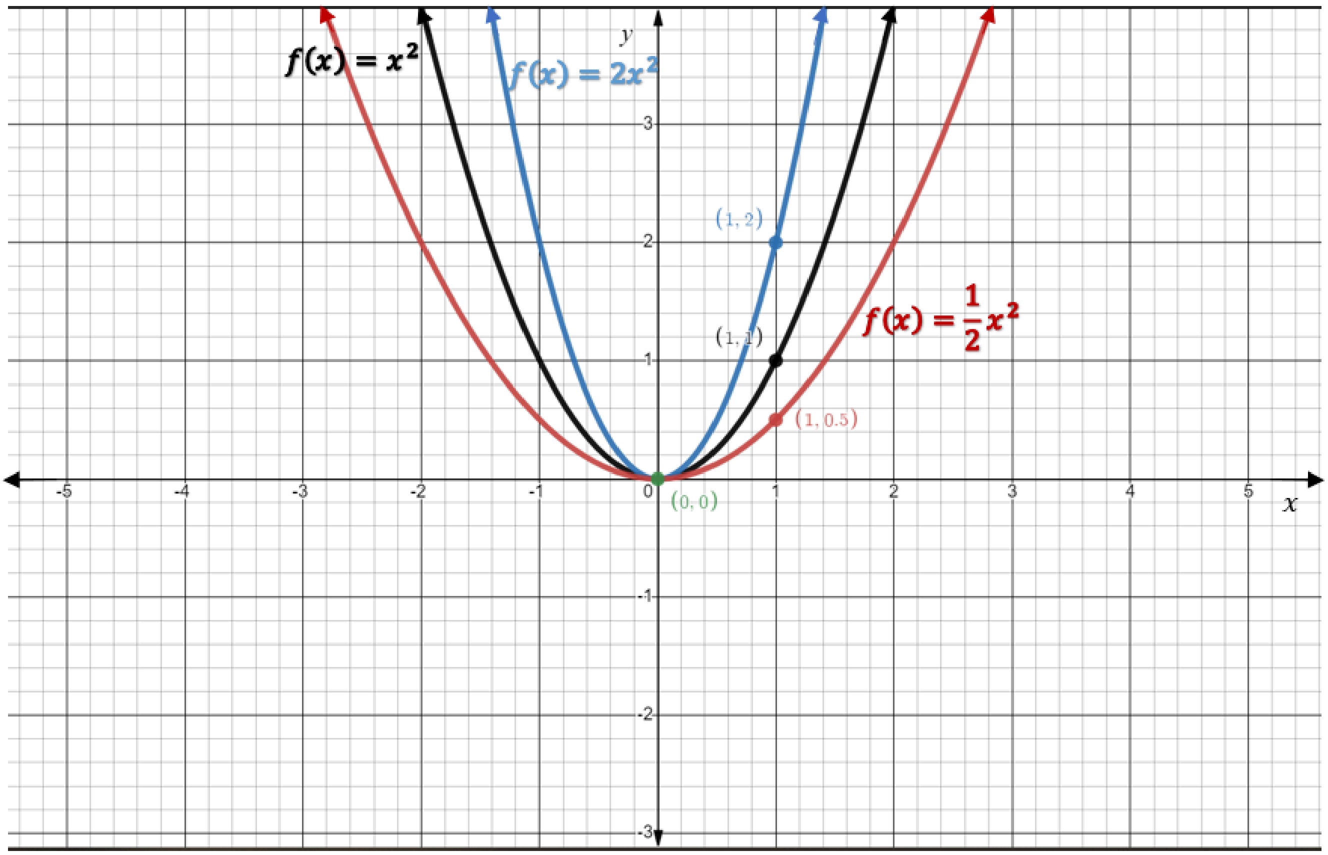


Figure 5.7.2 This figure compares the graphs of quadratic equations that have undergone vertical stretches or compressions.

Graphically, we can see that when the leading coefficient is 2, the parabola is stretched vertically, whereas when the leading coefficient is $\frac{1}{2}$, the parabola has been compressed vertically. This allows us to learn about the nature of a graph of a parabola when the leading coefficient is a value other than 1. In fact, the absolute value of a dictates the vertical stretch or compression of the graph of a quadratic equation.

Quadratic Equations Vertical Stretch or Compression

For a function of the form $f(x)=ax^2$,

if $|a| > 1$, the function is stretched by a factor of a.

if $|a| < 1$, the function is compressed by a factor of a.

Another way to observe the effects of vertical stretches and compressions is in the table of values associated with a particular function. For example, let's compare the table of values for the functions $f(x) = x^2$ and $f(x) = 2x^2$. You will see that the y-values of the graph of $f(x) = 2x^2$ have been multiplied by 2 and these values are reflected in Figure 5.7.2.

x	$f(x) = x^2$
-2	4
-1	1
0	0
1	1
2	4

x	$f(x) = 2x^2$
-2	8
-1	2
0	0
1	2
2	8

Finally, let's examine what happens when we add a vertical translation of k to the quadratic equation. In the figure below (Figure 5.7.3), we see graphs of the equations $f(x) = x^2$, $f(x) = x^2 + 2$, and $f(x) = x^2 - 2$.

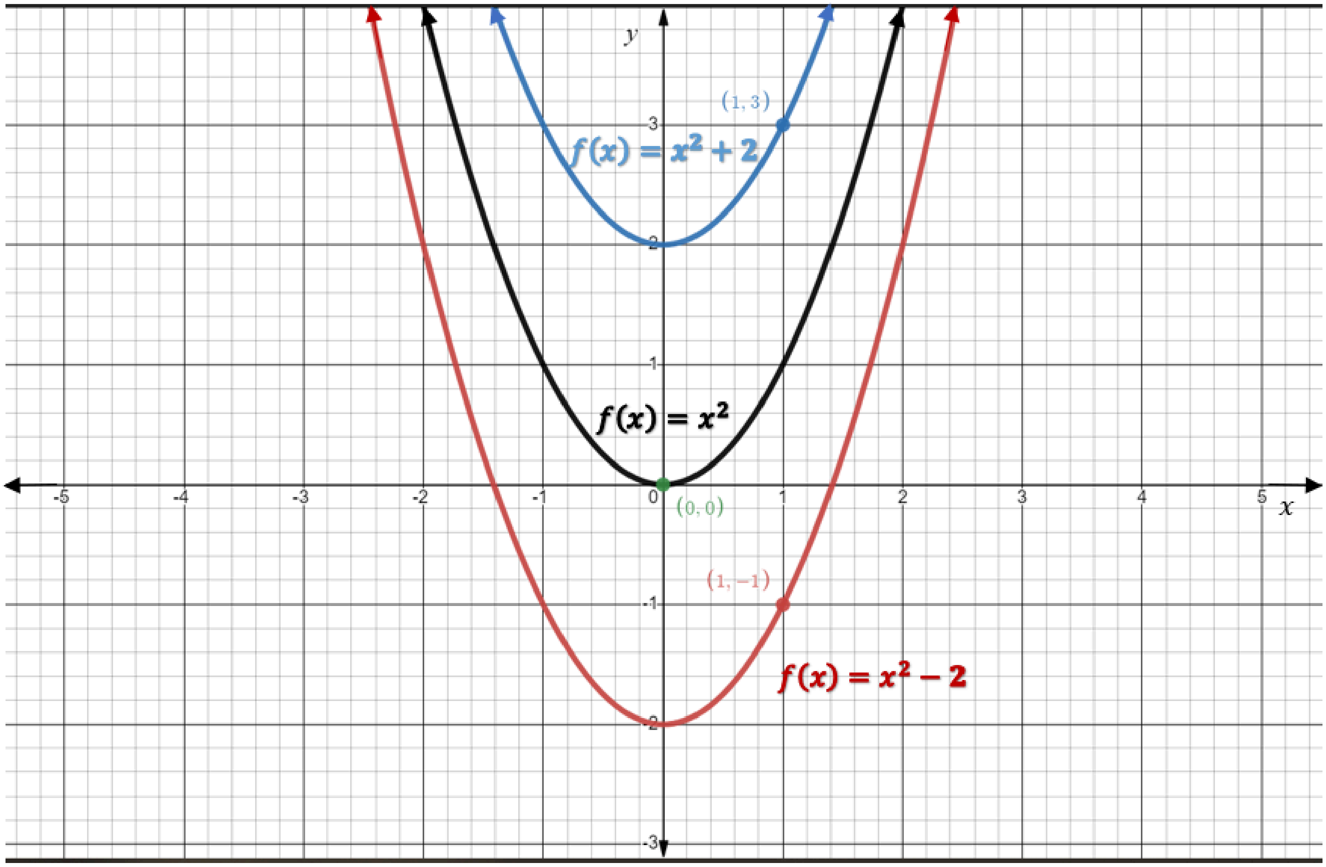


Figure 5.7.3 Comparing the graphs of equations with a vertical translation.

Graphically, we can see that a vertical translation either moves the parabola up by k units or down by $|k|$ units.

Once again, we can also see these transformations within the table of values associated with the given functions. Below, we compare $f(x) = x^2$ to $f(x) = x^2 - 2$. We see that the y-values have all been reduced by 2. It is also important to notice that the y-intercept (vertex) was also shifted down by 2 units.

x	$f(x) = x^2$
-2	4
-1	1
0	0
1	1
2	4

x	$f(x) = x^2 - 2$
-2	2
-1	-1
0	-2
1	-1
2	2

Quadratic Equations Vertical Translation

For a function of the form $f(x) = ax^2 + k$,

if $k > 0$, the function is translated up by k units.

if $k < 0$, the function is translated down by $|k|$ units.

HOW TO

Graph a polynomial function of the form $f(x) = ax^2 + k$.

1. Find the direction of opening by observing the sign of a.
2. Find the y-intercept, which is also the vertex, by setting $x = 0$ and evaluating for y.
3. Make a table of values for $-2 \leq x \leq 2$.
4. Plot the vertex and the ordered pairs from the table of values.
5. Join the points by a curve. Be sure to make the curve smooth with no sharp edges or points.
6. Remember the rules of good graphing (label axes, arrows on ends of axes and curves, at least 3 scale markings in each relevant, and original equations labelled).

Example 5.7.1

Graph the function $f(x) = -x^2 + 5$.

Solution

Step 1: **Note that the leading coefficient $a = -1$. This means that the parabola will open downwards.**

Step 2: **The y-intercept can be found by setting $x = 0$, and evaluating for the value of the function.**

$$\begin{aligned} f(x) &= -x^2 + 5 \\ f(0) &= -(0)^2 + 5 \\ f(0) &= 5 \end{aligned}$$

Therefore, the y-intercept is the point $(0, 5)$

Step 3: **Make a table of values for $-2 \leq x \leq 2$.**

x	$f(x) = -x^2 + 5$
-2	1
-1	4
0	5
1	4
2	1

Step 4: **Plot the vertex and the ordered pairs from the table of values.**

Step 5: **Join the points by a curve. Be sure to make the curve smooth with no sharp edges or points.**

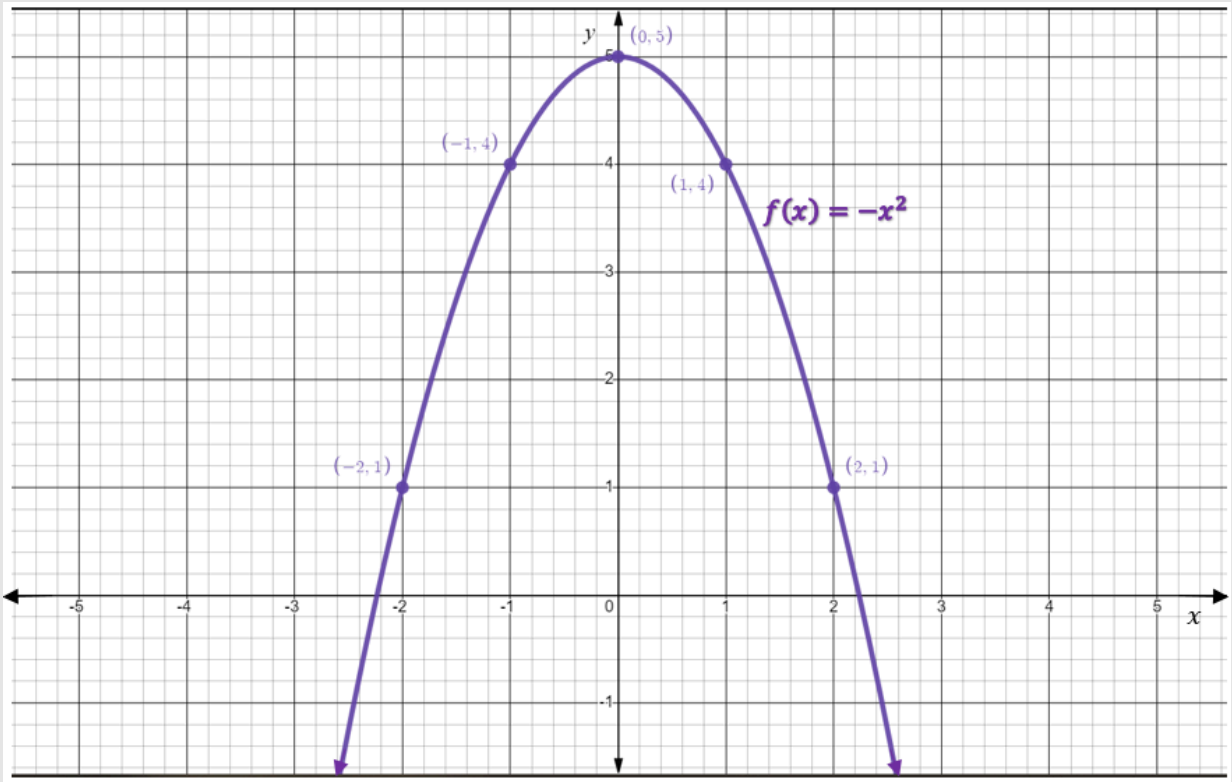


Figure 5.7.4

Step 6: **Remember the rules of good graphing (label axes, arrows on ends of axes and curves, at least 3 scale markings in each relevant, and original equations labelled).**

Example 5.7.2

Graph the function $f(x) = 2x^2 - 4$.

Solution

Step 1: **Note that the leading coefficient $a = 2$. This means that the parabola will open downwards.**

Step 2: **The y-intercept can be found by setting $x = 0$, and evaluating for the value of the function.**

$$f(x) = 2x^2 - 4$$
$$f(0) = 2(0)^2 - 4$$
$$f(0) = -4$$

Therefore, the y-intercept is the point $(0, -4)$

Step 3: **Make a table of values for $-2 \leq x \leq 2$.**

x	$f(x) = 2x^2 - 4$
-2	4
-1	-2
0	-4
1	-2
2	4

Step 4: **Plot the vertex and the ordered pairs from the table of values.**

Step 5: **Join the points by a curve. Be sure to make the curve smooth with no sharp edges or points.**

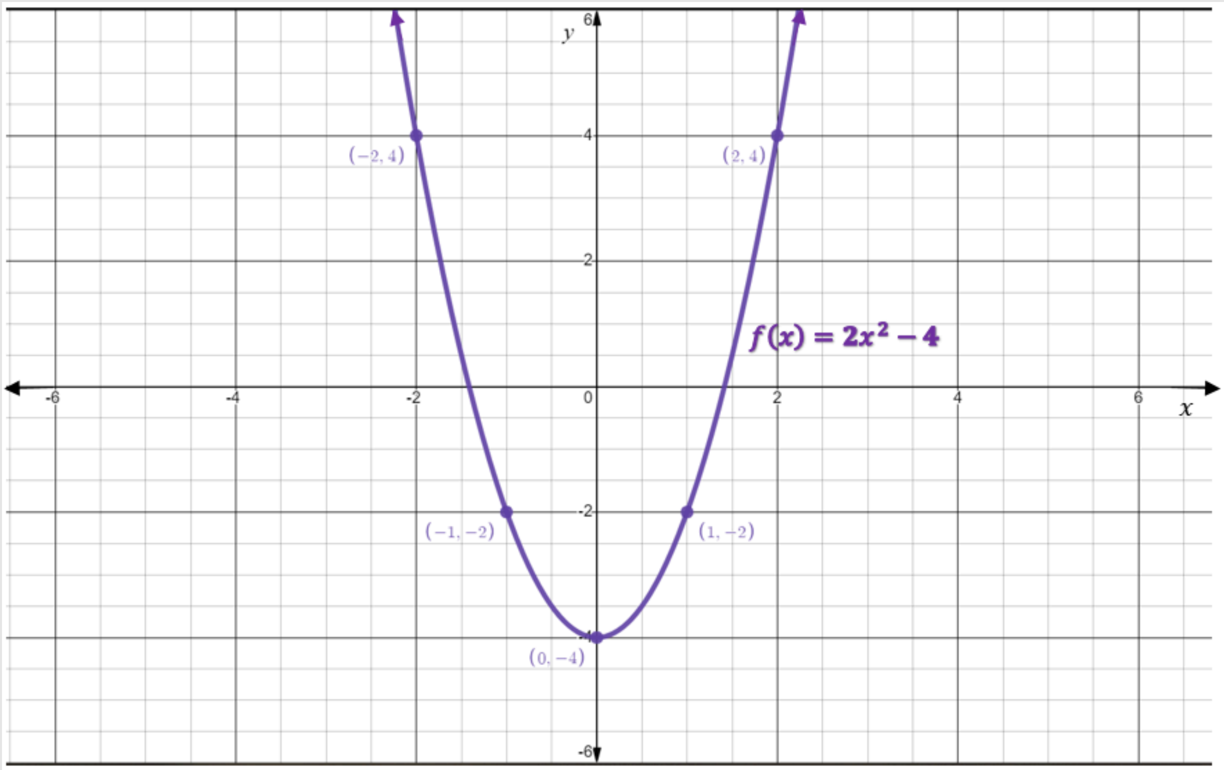


Figure 5.7.5

Step 6: **Remember the rules of good graphing (label axes, arrows on ends of axes and curves, at least 3 scale markings in each relevant, and original equations labelled).**

Try It

1) Graph the function $f(x) = -3x^2 + 6$.

Solution

Use the desmos graphing calculator to check your answer.

2) Graph the function $f(x) = x^2 - 10$.

Solution
Use the desmos graphing calculator to check your answer.

Graphs of Cubic Equations

A cubic equation is one that contains a cubed variable term. In this section, we will focus on the graphs of cubic equations of the form $f(x) = ax^3 + k$. It is important to note that cubic equations can have graphs that do not have this general shape, but we restrict our focus to cubic equations whose graphs have not undergone any further transformations other than vertical reflections and translations. We will see that changing the leading coefficient and the value of k will affect the graphs of cubic equations in the same (or similar) way as those transformations affected the graphs of quadratic equations.

Let's begin by comparing the graphs of the equations $f(x) = x^3$ and $f(x) = -x^3$.

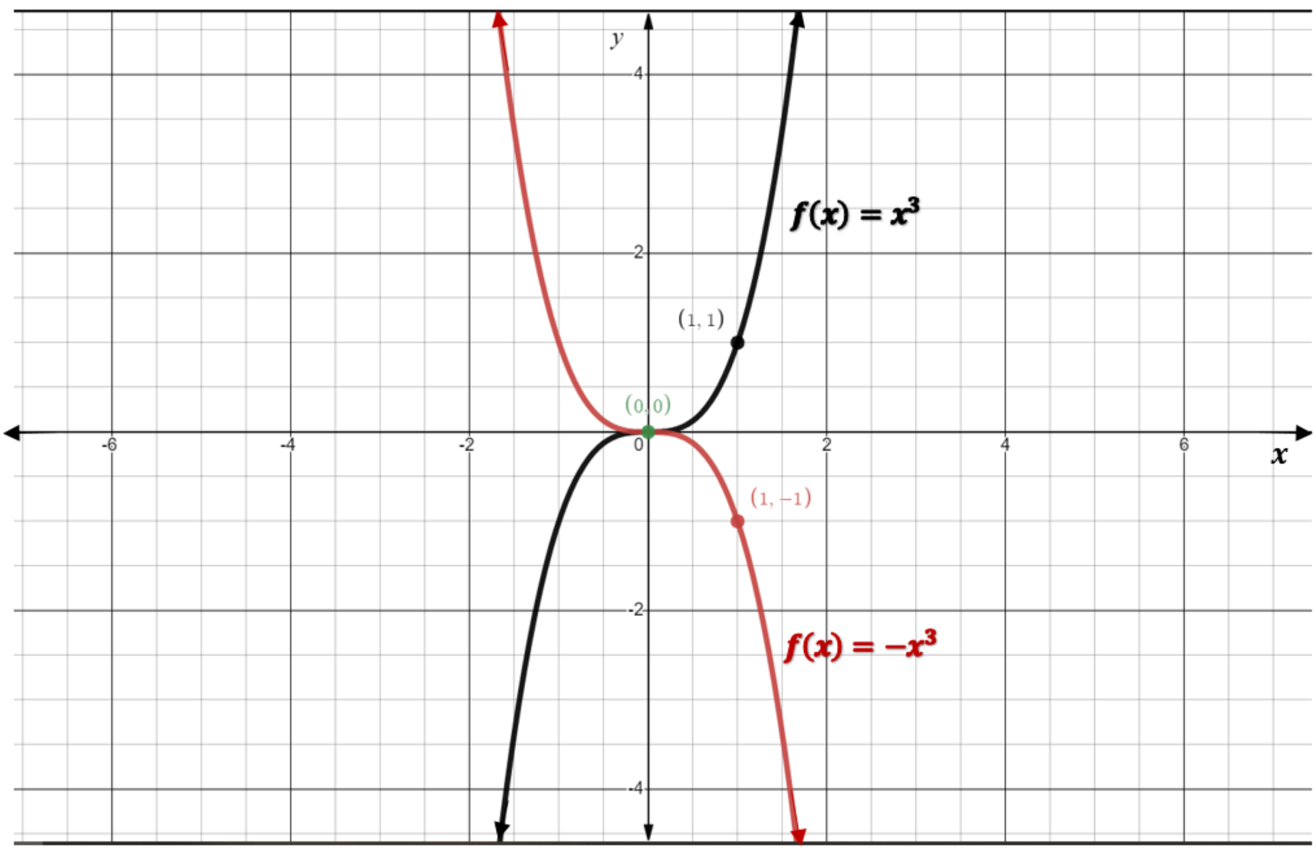


Figure 5.7.6 Comparing the graphs of cubic equations that have undergone vertical reflections.

As we can see above, the only difference between the two graphs is their orientation, and the only difference between the equations is the sign of the leading coefficient, a. When a is positive ($a > 0$), the graph increases initially and for all of the values of x. Contrarily, when a is negative ($a < 0$), the graph decreases initially and for all of the values of x. It is important to note that some cubic functions do have intervals of both increase and decrease, but since we are only studying cubic equations of the form $f(x) = ax^3 + k$, we will not investigate that case.

Cubic Equations Direction of Increase or Decrease

For cubic equations of the form $f(x) = ax^3 + k$,

If $a > 0$, the function is increasing for all values of x.

If $a < 0$, the function is decreasing for all values of x.

We can also make note that the y-intercept of these graphs is the origin, $(0, 0)$. In the case of cubic equations of this particular form, the y-intercept is also the point of inflection. An inflection point is a point where a curve changes its concavity.

Next, we'll examine the graphs $f(x) = 2x^3$ and $f(x) = \frac{1}{2}x^3$, alongside $f(x) = x^3$ for comparison. The figure below shows that cubic equations can undergo vertical stretches and compressions in the same way that quadratic equations could.

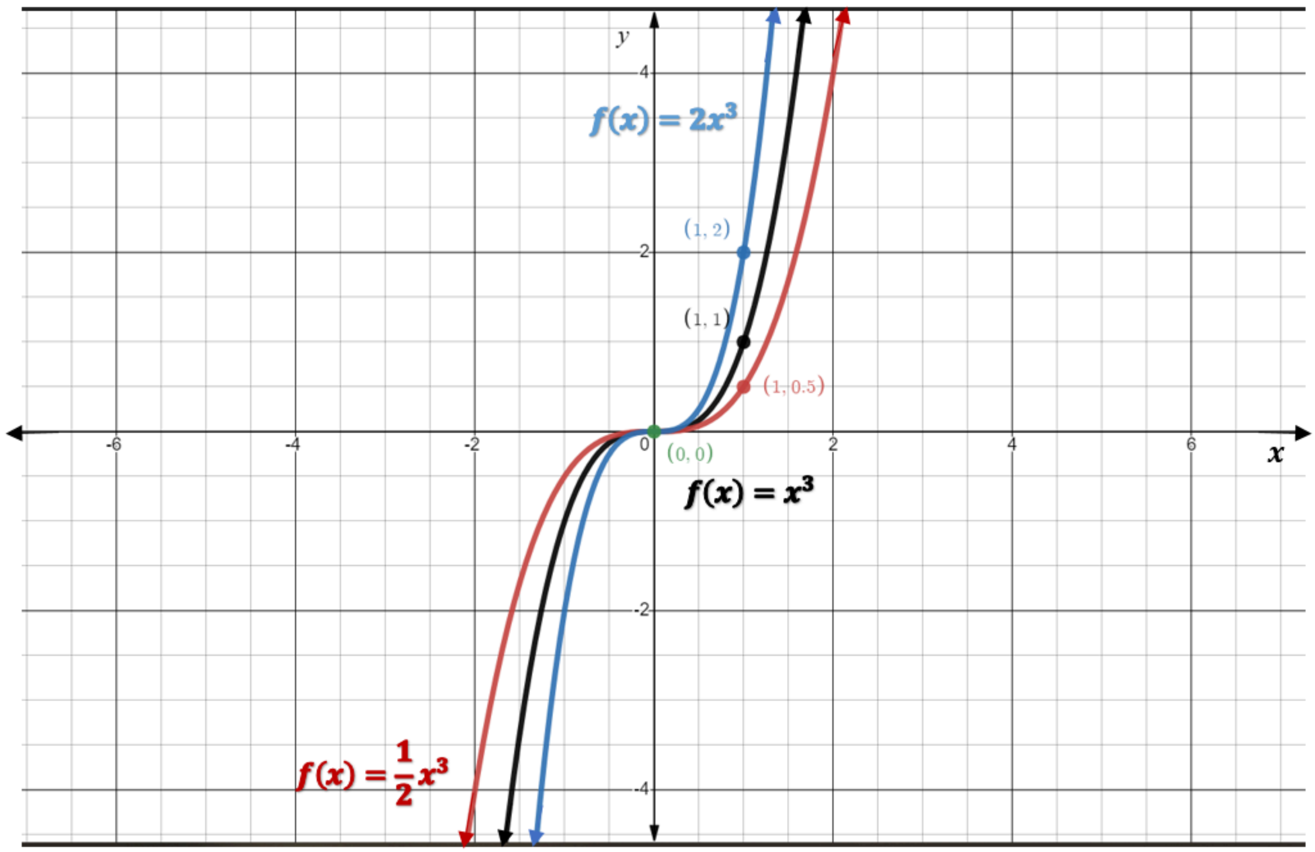


Figure 5.7.7 Comparing the graphs of cubic equations that have undergone vertical stretches and compressions.

Cubic Equations Vertical Stretch and Compression

For a function of the form $f(x) = ax^3$,

- if $|a| > 1$, the function is stretched by a factor of a.
- if $|a| < 1$, the function is compressed by a factor of a.

What do we think would happen if we added a vertical translation to the equation? What might that look like? As was the case with quadratic equations, the graphs of cubic equations behave in the same way when they undergo vertical translations. The following figure show the graphs of $f(x) = x^3$, $f(x) = x^3 + 2$ and $f(x) = x^3 - 2$.

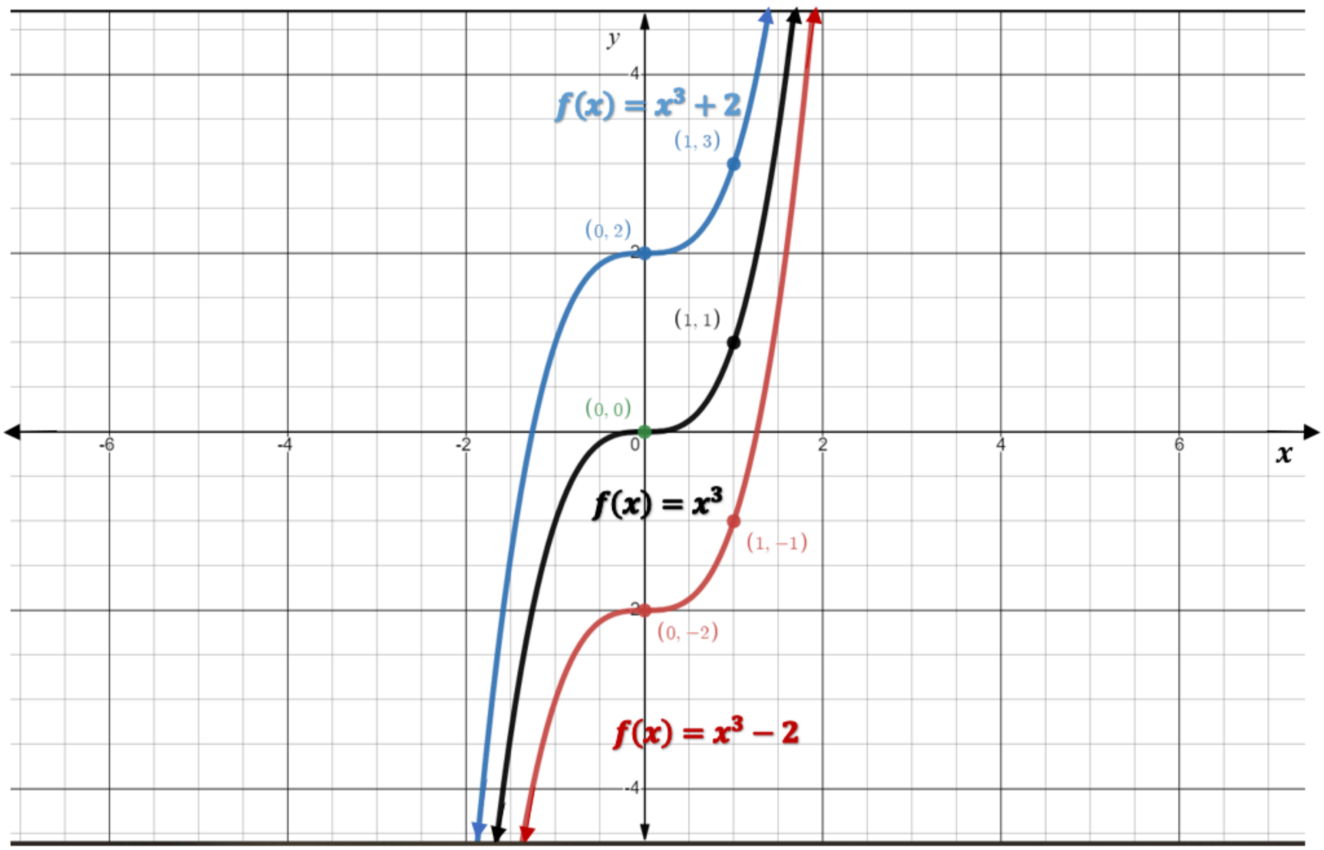


Figure 3.5.8 Comparing the graphs of cubic equations that have undergone vertical translations.

Cubic Equations Vertical Translation

For a function of the form $f(x) = ax^3 + k$,

if $k > 0$, the function is translated up by k units.

if $k < 0$, the function is translated down by $|k|$ units.

If we were to make a table of values for the cubic equations $f(x) = x^3$ and $f(x) = x^3 - 3$, we would see that the y-values have been reduced by 3.

x	$f(x) = x^3$
-2	-8
-1	-1
0	0
1	1
2	8

x	$f(x) = x^3 - 3$
-2	-11
-1	-4
0	-3
1	-2
2	5

Try It

3) Make a table of values for the functions $f(x) = x^3$ and $f(x) = x^3 + 7$ for $-2 \leq x \leq 2$. What transformation has occurred?

Solution

A translation of seven units upwards.

x	$f(x) = x^3$
-2	-8
-1	-1
0	0
1	1
2	8

x	$f(x) = x^3 + 7$
-2	-1
-1	6
0	7
1	8
2	15

HOW TO

Graph a polynomial function of the form $f(x) = ax^3 + k$.

1. Find the direction of the curve by observing the sign of a.
2. Find the y-intercept, which is also the inflection point, by setting $x = 0$ and evaluating for y.
3. Make a table of values for $-2 \leq x \leq 2$.
4. Plot the y-intercept and the ordered pairs from the table of values.
5. Join the points by a curve. Be sure to make the curve smooth with no sharp edges or points.
6. Remember the rules of good graphing (label axes, arrows on ends of axes and curves, at

least 3 scale markings in each relevant direction, and original equations labelled).

Example 5.7.3

Graph the function $f(x) = -x^3 + 2$.

Solution

Step 1: **Note that the leading coefficient $a = -1$. This means that the graph of the cubic will be decreasing.**

Step 2: **The y-intercept can be found by setting $x = 0$, and evaluating for the value of the function.**

$$f(x) = -x^3 + 2$$
$$f(0) = -(0)^3 + 2$$
$$f(0) = 2$$

Therefore, the y-intercept is the point $(0, 2)$

Step 3: **Make a table of values for $-2 \leq x \leq 2$.**

x	$f(x) = -x^3 + 2$
-2	10
-1	3
0	2
1	1
2	-6

Step 4: **Plot the y-intercept and the ordered pairs from the table of values.**

Step 5: **Join the points by a curve. Be sure to make the curve smooth with no sharp edges or points.**

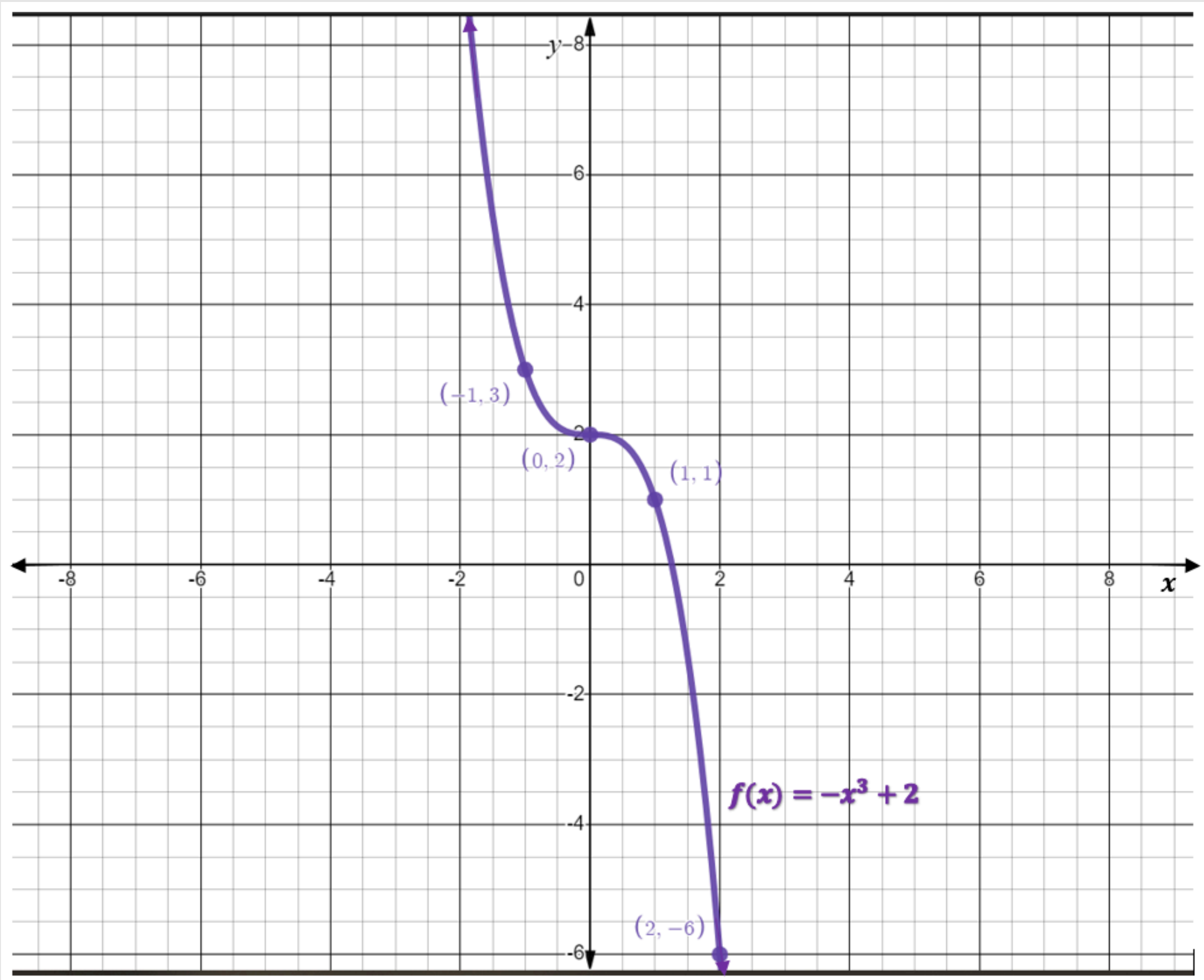


Figure 5.7.9

Step 6: **Remember the rules of good graphing (label axes, arrows on ends of axes and curves, at least 3 scale markings in each relevant, and original equations labelled).**

Example 5.7.4

Graph the function $f(x) = 3x^3 - 10$.

Solution

Step 1: Note that the leading coefficient $a = 3$. This means that the graph of the cubic will be increasing.

***Step 2:* The y-intercept can be found by setting $x = 0$, and evaluating for the value of the function.**

$$f(x) = 3x^3 - 10$$
$$f(0) = 3(0)^3 - 10$$
$$f(0) = -10$$

Therefore, the y-intercept is the point $(0, 2)$

***Step 3:* Make a table of values for $-2 \leq x \leq 2$.**

x	$f(x) = 3x^3 - 10$
-2	-34
-1	-13
0	-10
1	-7
2	14

***Step 4:* Plot the y-intercept and the ordered pairs from the table of values.**

***Step 5:* Join the points by a curve. Be sure to make the curve smooth with no sharp edges or points.**

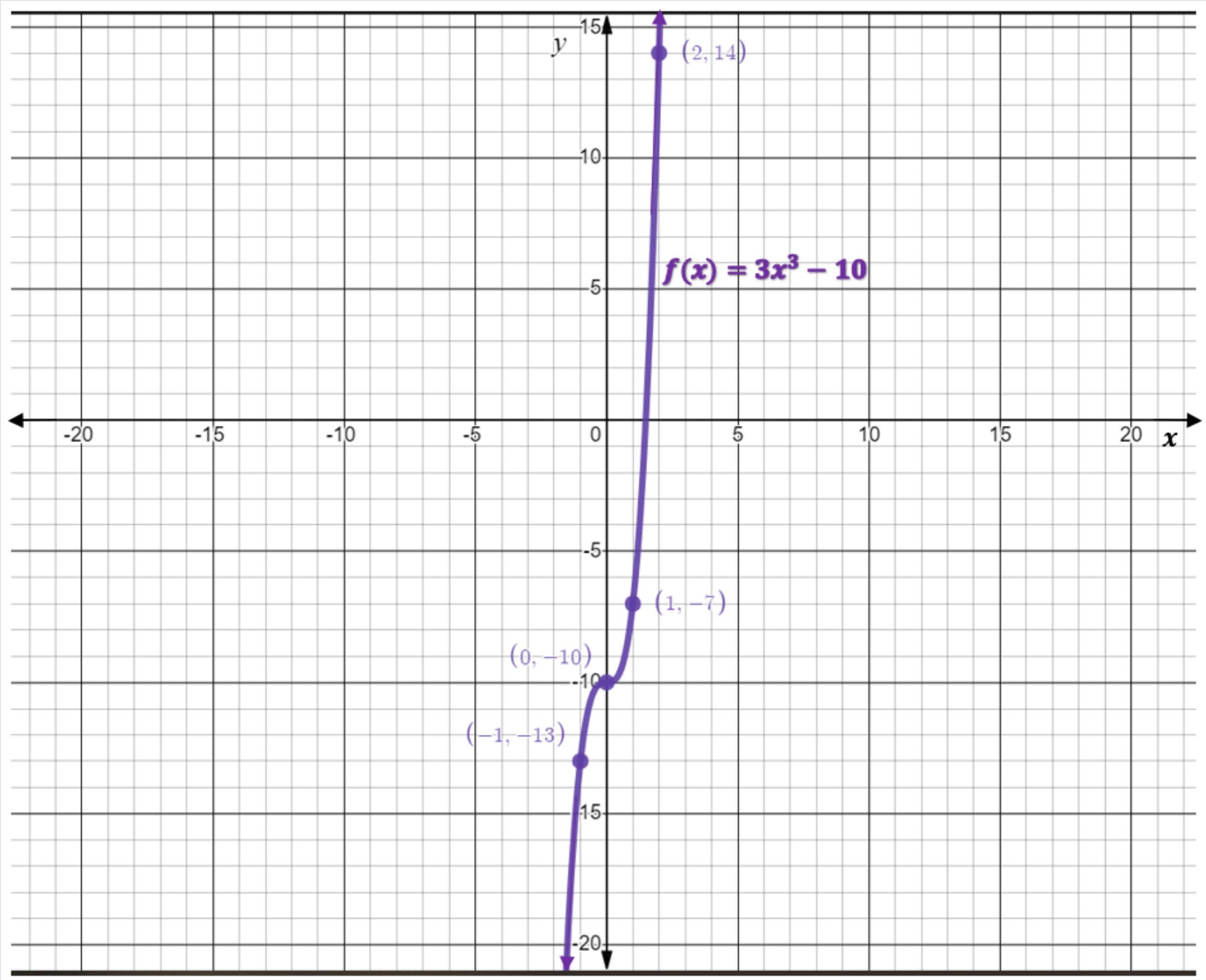


Figure 5.7.10

Step 6: **Remember the rules of good graphing (label axes, arrows on ends of axes and curves, at least 3 scale markings in each relevant, and original equations labelled).**

Try It

4) Graph the function $f(x) = -2x^3 + 1$.

Solution
Use the desmos graphing calculator to check your answer.

5) Graph the function $f(x) = x^3 - 4$.

Solution
Use the desmos graphing calculator to check your answer.

Key Concepts

1) Quadratic Equations Direction of Opening

For a function of the form $f(x) = ax^2$,

if $a > 0$, the parabola opens upwards;

if $a < 0$, the parabola opens downwards.

2) Quadratic Equations Vertical Stretch or Compression

For a function of the form $f(x) = ax^2$,

if $|a| > 1$, the function is stretched by a factor of a.

if $|a| < 1$, the function is compressed by a factor of a.

3) Quadratic Equations Vertical Translation

For a function of the form $f(x) = ax^2 + k$, if $k > 0$, the function is translated up by k units.

if $k < 0$, the function is translated down by $|k|$ units.

4) How To Graph a polynomial function of the form $f(x) = ax^2 + k$.

1. Find the direction of opening by observing the sign of a.
2. Find the y-intercept, which is also the vertex, by setting $x = 0$ and evaluating for y.
3. Make a table of values for $-2 \leq x \leq 2$.
4. Plot the vertex and the ordered pairs from the table of values.
5. Join the points by a curve. Be sure to make the curve smooth with no sharp edges or points.
6. Remember the rules of good graphing (label axes, arrows on ends of axes and curves, at least 3 scale markings in each relevant, and original equations labelled).

5) Cubic Equations Direction of Increase or Decrease

For cubic equations of the form $f(x) = ax^3 + k$,

If $a > 0$, the function is increasing for all values of x.

If $a < 0$, the function is decreasing for all values of x.

6) Cubic Equations Vertical Stretch and Compression

For a function of the form $f(x) = ax^3$,

if $|a| > 1$, the function is stretched by a factor of a.

if $|a| < 1$, the function is compressed by a factor of a.

7) Cubic Equations Vertical Translation

For a function of the form $f(x) = ax^3 + k$,

if $k > 0$, the function is translated up by k units.

if $k < 0$, the function is translated down by $|k|$ units.

8) How To Graph a polynomial function of the form $f(x) = ax^3 + k$.

1. Find the direction of the curve by observing the sign of a.
2. Find the y-intercept, which is also the inflection point, by setting $x = 0$ and evaluating for y.
3. Make a table of values for $-2 \leq x \leq 2$.
4. Plot the y-intercept and the ordered pairs from the table of values.

5. Join the points by a curve. Be sure to make the curve smooth with no sharp edges or points.
6. Remember the rules of good graphing (label axes, arrows on ends of axes and curves, at least 3 scale markings in each relevant direction, and original equations labelled).

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=4911#h5p-35

b. Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

5.8 UNIT SOURCES

Unit 5 Sources

5.1 "Add and Subtract Polynomials" from Elementary Algebra 2e. by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

5.2 "Use Multiplication Properties of Exponents" and "Integer Exponents and Scientific Notation" from Elementary Algebra 2e. by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

5.3 "Multiply Polynomials" from Elementary Algebra 2e. by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

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UNIT 6: GEOMETRY AND TRIGONOMETRY

Chapter Outline

6.0 INTRODUCTION

Figure 6.0.1 Note the many individual shapes in this building. Photo by Bert Kaufmann CC-BY 4.0

We are surrounded by all sorts of geometry. Architects use geometry to design buildings. Artists create vivid images out of colourful geometric shapes. Street signs, automobiles, and product packaging all take advantage of geometric properties. In this chapter, we will begin by considering a formal approach to solving problems and use it to solve a variety of common problems, including making decisions about money. Then we will explore geometry and relate it to everyday situations, using the problem-solving strategy we develop.

6.1 USE PROPERTIES OF ANGLES, TRIANGLES, AND THE PYTHAGOREAN THEOREM

Learning Objectives

By the end of this section, you will be able to:

- Use the properties of angles
- Use the properties of triangles
- Use the Pythagorean Theorem

Try It

Before you get started, take this readiness quiz:

1) Solve: $x + 3 + 6 = 11$
2) Solve: $\frac{a}{45} = \frac{4}{3}$
3) Simplify: $\sqrt{36 + 64}$

In this unit, we will study geometry and trigonometry in order to develop our understanding of how some of these tools can help us in a health career. For example, even having a simple understanding of angles will be helpful when performing injections.

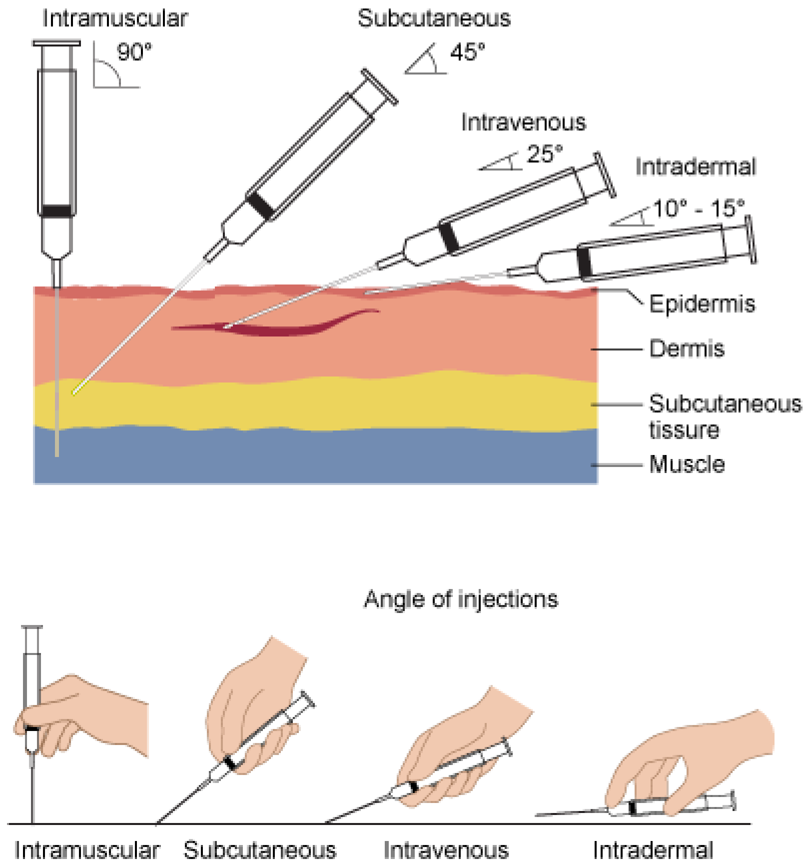


Figure 6.1.1

Use the Properties of Angles

Are you familiar with the phrase 'do a 180' It means to make a full turn so that you face the opposite direction. It comes from the fact that the measure of an angle that makes a straight line is 180 degrees. See Figure 6.1.2.

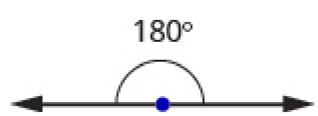


Figure 6.1.2

An **angle** is formed by two rays that share a common endpoint. Each ray is called a side of the angle and the common endpoint is called the vertex. An angle is named by its vertex. In Figure 6.1.3, A is the **angle with vertex** at point A. The measure of A is written $m\angle A$.

$m\angle A$ is the angle with vertex at point A.

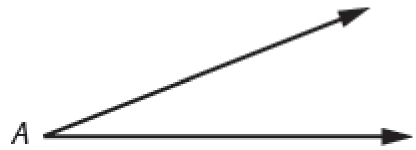


Figure 6.1.3

We measure angles in degrees, and use the symbol ○ to represent degrees. We use the abbreviation m to for the *measure* of an angle. So if A is $27°$, we would write $m\angle A = 27$.

If the sum of the measures of two angles is $180°$ then they are called **supplementary angles**. In Figure 6.1.4, each pair of angles is supplementary because their measures add to $180°$. Each angle is the *supplement* of the other.

The sum of the measures of supplementary angles is $180°$.

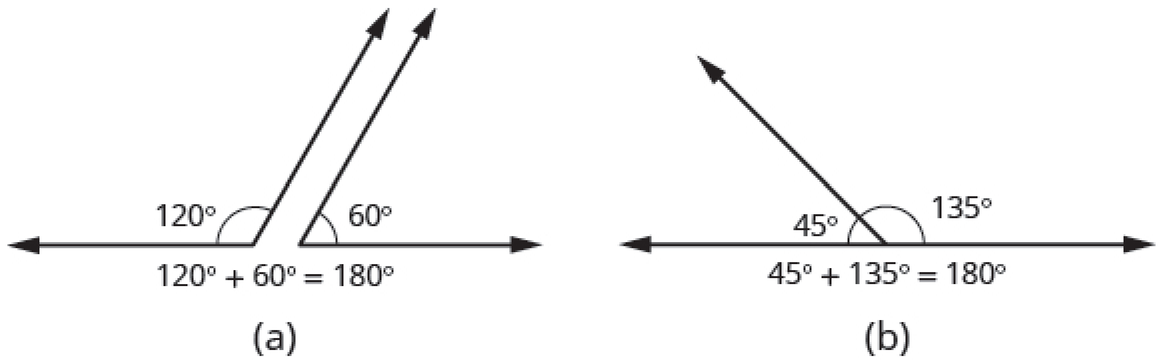


Figure 6.1.4

If the sum of the measures of two angles is $90°$, then the angles are complementary angles. In Figure 6.1.5, each pair of angles is complementary, because their measures add to $90°$. Each angle is the *complement* of the other.

The sum of the measures of complementary angles is $90°$.

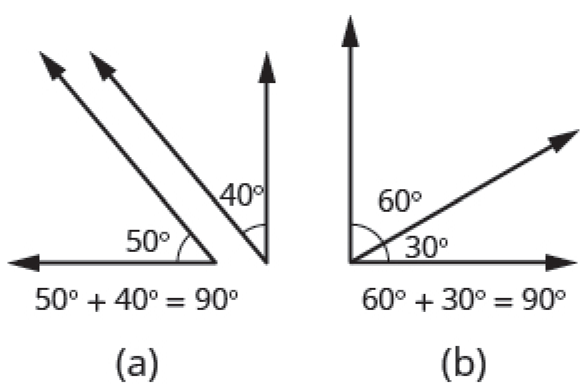


Figure 6.1.5

Supplementary and Complementary Angles

If the *sum* of the measures of two angles is $180°$ then the angles are *supplementary*.

If A and B are supplementary, then $m\angle A + m\angle B = 180$.

If the *sum* of the measures of two angles is $90°$, then the angles are *complementary*.

If A and B are complementary, then $m\angle A + m\angle B = 90$.

In this section and the next, you will be introduced to some common geometry formulas. We will adapt our Problem Solving Strategy for Geometry Applications. The geometry formula will name the variables and give us the equation to solve.

In addition, since these applications will all involve geometric shapes, it will be helpful to draw a figure and then label it with the information from the problem. We will include this step in the Problem Solving Strategy for Geometry Applications.

How To

Use a *Problem Solving Strategy for Geometry Applications*

1. Read the problem and make sure you understand all the words and ideas. Draw a figure and label it with the given information.
2. Identify what you are looking for.
3. Name what you are looking for and choose a variable to represent it.
4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. Solve the equation using good algebra techniques.
6. Check the answer in the problem and make sure it makes sense.
7. Answer the question with a complete sentence.

The next example will show how you can use the Problem Solving Strategy for Geometry Applications to answer questions about supplementary and complementary angles.

Example 6.1.1

An angle measures $40°$. Find:

a. its supplement
b. its complement

Solution

a. ***Step 1:*** **Read the problem. Draw the figure and label it with the given information.**

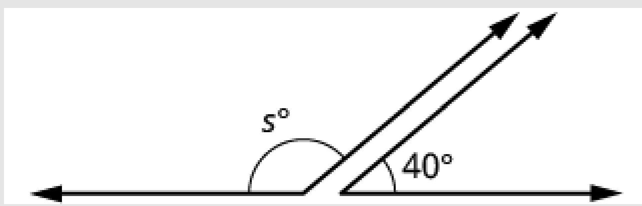


Figure 6.1.6

Step 2: **Identify what you are looking for.**

The supplement of a $40°$ angle.

Step 3: **Name. Choose a variable to represent it.**

Let $s =$ the measure of the supplement

Step 4: **Translate.**

Write the appropriate formula for the situation and substitute in the given information.

$$\begin{aligned} m\angle A + m\angle B &= 180 \\ s + 40 &= 180 \end{aligned}$$

Step 5: **Solve the equation.**

$$s = 140$$

Step 6: **Check.**

$$\begin{aligned} 140 + 40 &\stackrel{?}{=} 180 \\ 180 &= 180 \end{aligned}$$

Step 7: **Answer the question.**

The supplement of the $40°$ angle is $140°$.

---

b. ***Step 1:*** **Read the problem. Draw the figure and label it with the given information**

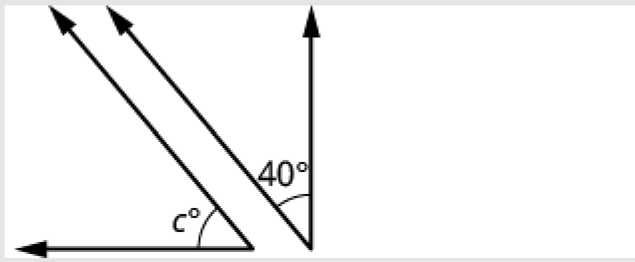


Figure 6.1.7

Step 2: **Identify what you are looking for.**

The compliment of a $40°$ angle.

Step 3: **Name. Choose a variable to represent it.**

Let $c =$ the measure of the complement.

Step 4: **Translate.**

Write the appropriate formula for the situation and substitute in given information.

$$m\angle A + m\angle B = 90$$

Step 5: **Solve the equation.**

$$\begin{aligned} c + 40 &= 90 \\ c &= 50 \end{aligned}$$

Step 6: **Check.**

$$\begin{aligned} 50 + 40 &\stackrel{?}{=} 90 \\ 90 &= 90 \end{aligned}$$

Step 7: **Answer the question.**

The complement of the $40°$ angle is $50°$.

Try it

4) An angle measures $25°$. Find its:

a. supplement
b. complement

Solution
a. 155°
b. 65°

Try it

5) An angle measures $77°$. Find its:

a. supplement
b. complement

Solution
a. 103°
b. 13°

Did you notice that the words complementary and supplementary are in alphabetical order just like 90 and 180 are in numerical order?

Example 6.1.2

Two angles are supplementary. The larger angle is $30°$ more than the smaller angle. Find the measure of both angles.

Solution

***Step 1:* Read the problem. Draw the figure and label it with the given information.**

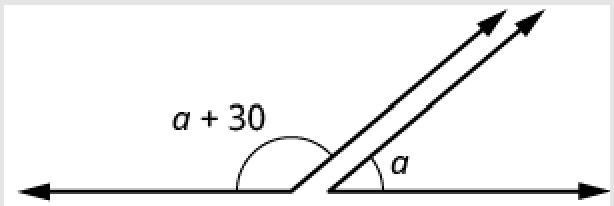


Figure 6.1.8

***Step 2:* Identify what you are looking for.**

The measures of both angles.

***Step 3:* Name. Choose a variable to represent it.**

Let a = measure of smaller angle
$a + 3$ = measure of larger angle

***Step 4:* Translate.**

Write the appropriate formula for the situation and substitute in the given information.

$$m\angle A + m\angle B = 180$$

***Step 5:* Solve the equation.**

$$\begin{aligned}(a + 30) + a &= 180\\ 2a + 30 &= 180\\ 2a &= 150\\ a &= 75\ measure\ of\ smaller\ angle\\ a &+ 30\ measure\ of\ larger\ angle\\ 75 + 30 &= 105\end{aligned}$$

Step 6: **Check.**

$$\begin{aligned} m\angle A + m\angle B &= 180 \\ 75 + 105 &\stackrel{?}{=} 180 \\ 180 &= 180 \end{aligned}$$

Step 7: **Answer the question.**

The measures of the angles are $75°$ and $105°$.

Try it

6) Two angles are supplementary. The larger angle is $100°$ more than the smaller angle. Find the measures of both angles.

Solution

$40°$ and $140°$

Try it

7) Two angles are complementary. The larger angle is $40°$ more than the smaller angle. Find the measures of both angles.

Solution

$25°$ and $65°$

Properties of Angles with Parallel Lines

It will be helpful to understand the relationship between parallel lines that are cut by a transversal line and their angles. When two lines are parallel, they will never intersect. We can use this property to find missing angles in various situations.

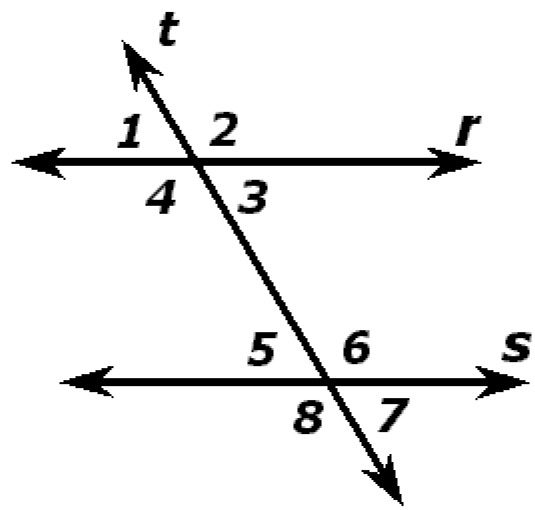


Figure 6.1.9

In the image above, the lines r and s are parallel. We can denote this mathematically using the following notation, $r \parallel s$. The line t is called a transversal line and no matter how it passes through the parallel lines, it will always result in the relationship above.

We notice that angles $\angle 1$, $\angle 3$, $\angle 5$, and $\angle 7$ will all be the same size. Similarly, angles $\angle 2$, $\angle 4$, $\angle 6$, and $\angle 8$ are all the same. Because of these properties, if we are missing certain angles we can find their values if given enough information. Before we do some examples, let's learn some terminology.

Corresponding angles: angles that occupy the same relative position. The corresponding angles are:

$\angle 1$ and $\angle 5$
$\angle 2$ and $\angle 6$
$\angle 3$ and $\angle 7$
$\angle 4$ and $\angle 8$

Consecutive angles: angles that are pairs on the same side and inside the two parallel lines. The consecutive angles are:

$\angle 3$ and $\angle 6$
$\angle 4$ and $\angle 5$

Alternate interior angles: angles that are pairs on opposite sides of the transversal and inside the two parallel lines. The alternate interior angles are:

$\angle 4$ and $\angle 6$

$\angle 3$ and $\angle 5$

Alternate exterior angles: angles that are pairs on opposite sides of the transversal and outside the two parallel lines. The alternate exterior angles are:

$\angle 1$ and $\angle 7$

$\angle 2$ and $\angle 8$

Example 6.1.3

If $r \parallel s$, and $\angle 8 = 110^\circ$, find the value of all of the missing angles.

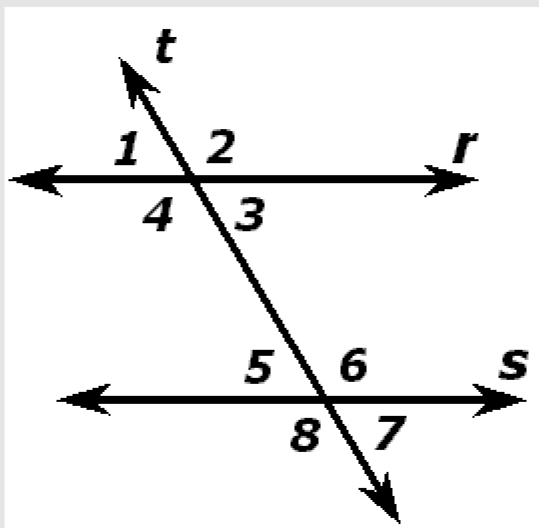


Figure 6.1.10

Solution

Since $\angle 8 = 110^\circ$ and $\angle 8$ and $\angle 7$ are supplementary angles, then

$\angle 8 + \angle 7 = 180^\circ$ we can solve this equation for the value of $\angle 7$.

$$110^\circ + \angle 7 = 180^\circ$$
$$\angle 7 = 180^\circ - 110^\circ$$
$$\angle 7 = 70^\circ$$

Now, since the lines are parallel, and we know the values of $\angle 8$ and $\angle 7$, we can find all the remaining angles. By our properties, $\angle 8 = \angle 6 = \angle 4 = \angle 2 = 110°$ and $\angle 7 = \angle 5 = \angle 3 = \angle 1 = 70°$

Use the Properties of Triangles

What do you already know about triangles? **Triangles** have three sides and three angles. Triangles are named by their vertices. The triangle in Figure 6.1.11 is called $\triangle\ ABC$, read 'triangle ABC'. We label each side with a lower case letter to match the upper case letter of the opposite vertex.

$\triangle\ ABC$ has vertices A, B, and C and sides a, b, and c.

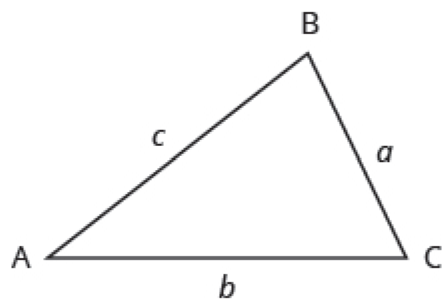


Figure 6.1.11

The three angles of a triangle are related in a special way. The sum of their measures is $180°$. $m\angle A\ +\ m\angle B\ +\ m\angle C\ =\ 180$.

Sum of the Measures of the Angles of a Triangle

For any $\triangle\ ABC$, the sum of the measures of the angles is $180°$. $m\angle A\ +\ m\angle B\ +\ m\angle C\ =\ 180$

Example 6.1.4

The measures of two angles of a triangle are $55°$ and $82°$. Find the measure of the third angle.

Solution

Step 1: **Read the problem. Draw the figure and label it with the given information.**

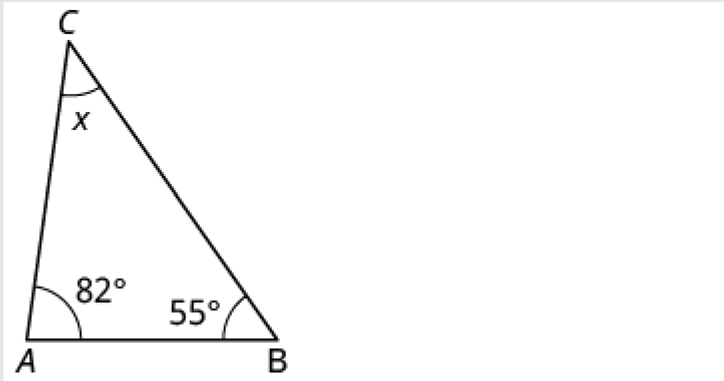


Figure 6.1.12

Step 2: **Identify what you are looking for.**

The measure of the third angle in a triangle.

Step 3: **Name. Choose a variable to represent it.**

Let x = measure of the angle.

Step 4: **Translate.**

Write the appropriate formula for the situation and substitute in the given information.

$$m\angle A + m\angle B + m\angle C = 180$$

Step 5: **Solve the equation.**

$$\begin{aligned} 24 &= 13 + x \\ 11 &= x \end{aligned}$$

Step 6: **Check.**

$$\begin{aligned} 55 + 82 + 43 &\stackrel{?}{=} 180 \\ 180 &= 180 \end{aligned}$$

Step 7: **Answer the question.**

The measures of the third angle is 43 degrees.

Try it

8) The measures of two angles of a triangle are $31°$ and $128°$. Find the measure of the third angle.

Solution

$21°$

Try it

9) A triangle has angles of $49°$ and $75°$. Find the measure of the third angle.

Solution

$56°$

Because the perimeter of a figure is the length of its boundary, the perimeter of $\triangle\ ABC$ is the sum of the lengths of its three sides.

$$P = a + b + c$$

To find the area of a triangle, we need to know its base and height. The height is a line that connects the base to the opposite vertex and makes a $90°$ angle with the base. We will draw $\triangle\ ABC$ again, and now show the height, h. See Figure 6.1.13.

The formula for the area of $\triangle\ ABC$ is $A = \frac{1}{2}bh$ where b is the base and h is the height.

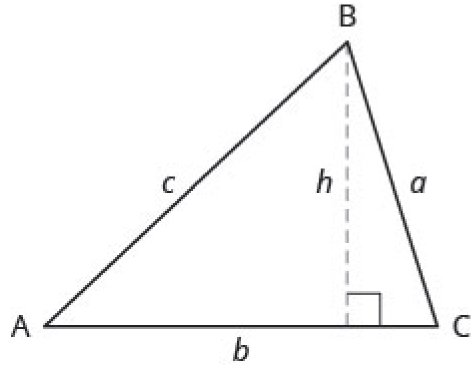


Figure 6.1.13

Triangle Properties

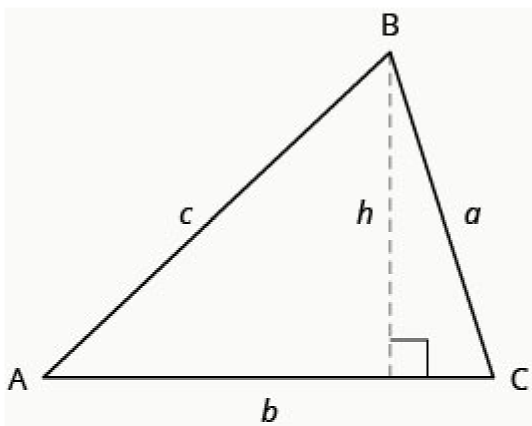


Figure 6.1.14

For $\triangle\ ABC$:

Angle measures:

$$m\angle A\ +\ m\angle B\ +\ m\angle C\ =\ 180$$

- The sum of the measures of the angles of a triangle is $180°$.

Perimeter:

$$P\ =\ a\ +\ b\ +\ c$$

- The perimeter is the sum of the lengths of the sides of the triangle.

Area:

$A = \frac{1}{2}bh$; b=base, h=height

- The area of a triangle is one-half the base times the height.

Example 6.1.5

The perimeter of a triangular garden is 24 feet. The lengths of two sides are four feet and nine feet. How long is the third side?

Solution

Step 1: **Read the problem. Draw the figure and label it with the given information.**

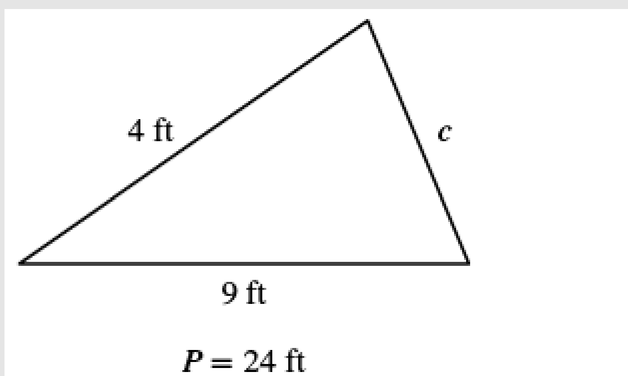


Figure 6.1.15

Step 2: **Identify what you are looking for.**

The length of the third side of a triangle.

Step 3: **Name. Choose a variable to represent it.**

Let $c =$ the third side.

Step 4: **Translate.**

Write the appropriate formula for the situation and substitute in the given information.

$$\begin{aligned} P &= a + b + c \\ P &= 4 + 9 + c \end{aligned}$$

Step 5: **Solve the equation.**

$$\begin{aligned} 24 &= 13 + c \\ 11 &= c \end{aligned}$$

Step 6: **Check.**

$$\begin{aligned} P &= a + b + c \\ 24 &\stackrel{?}{=} 4 + 9 + 11 \\ 24 &= 24 \end{aligned}$$

Step 7: **Answer the question.**

The third side is 11 feet long.

Try it

10) The perimeter of a triangular garden is 48 feet. The lengths of two sides are 18 feet and 22 feet. How long is the third side?

Solution

8 feet

Try it

11) The lengths of two sides of a triangular window are seven feet and five feet. The perimeter is 18 feet. How long is the third side?

Solution

6 feet

Example 6.1.6

The area of a triangular church window is 90 square meters. The base of the window is 15 meters. What is the window's height?

Solution

***Step 1:* Read the problem. Draw the figure and label it with the given information.**

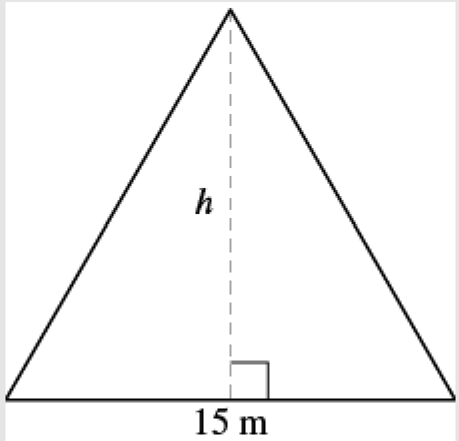


Figure 6.1.16

Step 2: **Identify what you are looking for.**

The height of a triangle.

Step 3: **Name. Choose a variable to represent it.**

Let $h =$ the height.

Step 4: **Translate.**

Write the appropriate formula for the situation and substitute in the given information.

$$A = \frac{1}{2} \times b \times h$$
$$90 = \frac{1}{2} \times 15 \times h$$

Step 5: **Solve the equation.**

$$90 = \frac{15}{2}h$$
$$12 = h$$

Step 6: **Check.**

$$A = \frac{1}{2}bh$$
$$90 \stackrel{?}{=} \frac{1}{2} \times 15 \times 12$$
$$90 = 90$$

Step 7: **Answer the question.**

The height of the triangle is 12 meters.

Try it

12) The area of a triangular painting is 126 square inches. The base is 18 inches. What is the height?

Solution

14 Inches

Try it

13) A triangular tent door has area 15 square feet. The height is five feet. What is the base?

Solution

6 feet

Right Triangles

The triangle properties we used so far apply to all triangles. Now we will look at one specific type of triangle—a **right triangle**. A right triangle has one $90°$ angle, which we usually mark with a small square in the corner, Figure 6.1.17.

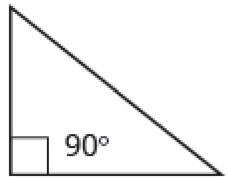


Figure 6.1.17

If we know that a triangle is a right triangle, we know that one angle measures $90°$ so we only need the measure of one of the other angles in order to determine the measure of the third angle.

Example 6.1.7

One angle of a right triangle measures $28°$. What is the measure of the third angle?

Solution

***Step 1:* Read the problem. Draw the figure and label it with the given information.**

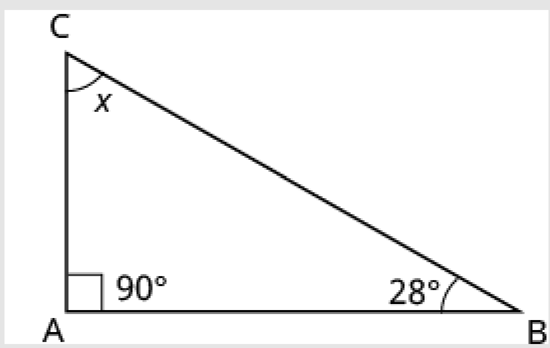


Figure 6.1.18

***Step 2:* Identify what you are looking for.**

The measure of an angle.

***Step 3:* Name. Choose a variable to represent it.**

Let $x =$ the measure of an angle.

Step 4: **Translate.**

Write the appropriate formula for the situation and substitute in the given information.

$$m\angle A + m\angle B + m\angle C = 180$$

Step 5: **Solve the equation.**

$$\begin{aligned} x + 90 + 28 &= 180 \\ x + 118 &= 180 \\ x &= 62 \end{aligned}$$

Step 6: **Check.**

$$\begin{aligned} 180 &\stackrel{?}{=} 90 + 28 + 62 \\ 180 &= 180 \end{aligned}$$

Step 7: **Answer the question.**

The measure of the third angle is $62°$.

Try it

14) One angle of a right triangle measures $56°$. What is the measure of the other angle?

Solution

$34°$

Try it

15) One angle of a right triangle measures $45°$. What is the measure of the other angle?

Solution

$45°$

In the examples so far, we could draw a figure and label it directly after reading the problem. In the next example, we will have to define one angle in terms of another. So we will wait to draw the figure until we write expressions for all the angles we are looking for.

Example 6.1.8

The measure of one angle of a right triangle is $20°$ more than the measure of the smallest angle. Find the measures of all three angles.

Solution

Step 1: **Read the problem.**

Step 2: **Identify what you are looking for.**

The measure of all three angles.

Step 3: **Name. Choose a variable to represent it.**

$$\begin{aligned} Let\ a &= the\ 1^{st}\ angle \\ a + 20 &= 2^{nd}\ angle \\ 90 &= 3^{rd}\ angle\ (the\ right\ angle) \end{aligned}$$

Now draw the figure and label it with the given information.

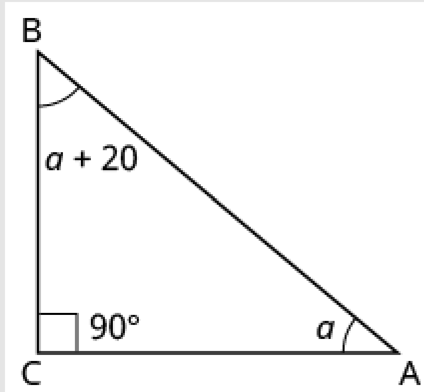


Figure 6.1.19

Step 4: **Translate.**

Write the appropriate formula for the situation and substitute in the given information.

$$\begin{aligned} m\angle A + m\angle B + m\angle C &= 180 \\ a + (a + 20) + 90 &= 180 \end{aligned}$$

Step 5: **Solve the equation.**

$$\begin{aligned} 2a + 110 &= 180 \\ 2a &= 70 \\ a &= 35\ (first\ angle) \\ a + & \ 20\ (second\ angle) \\ 35 + 20 &= 55 \\ 90 &= (third\ angle) \end{aligned}$$

Step 6: **Check.**

$$\begin{aligned} 35 + 55 + 90 &\stackrel{?}{=} 180 \\ 180 &= 180 \end{aligned}$$

Step 7: **Answer the question.**

The three angles measure $35°$, $55°$, and $90°$.

Try it

16) The measure of one angle of a right triangle is $50°$ more than the measure of the smallest angle. Find the measures of all three angles.

Solution

$20°$, $70°$, $90°$

Try it

17) The measure of one angle of a right triangle is $30°$ more than the measure of the smallest angle. Find the measures of all three angles.

Solution

$30°$, $60°$, $90°$

Similar Triangles

When we use a map to plan a trip, a sketch to build a bookcase, or a pattern to sew a dress, we are working with **similar figures**. In geometry, if two figures have exactly the same shape but different sizes, we say they are similar figures. One is a scale model of the other. The corresponding sides of the two figures have the same ratio, and all their corresponding angles are have the same measures.

The two triangles in Figure 6.1.20 are similar. Each side of $\triangle ABC$ is four times the length of the corresponding side of $\triangle XYZ$ and their corresponding angles have equal measures.

$\triangle ABC$ and $\triangle XYZ$ are similar triangles. Their corresponding sides have the same ratio and the corresponding angles have the same measure.

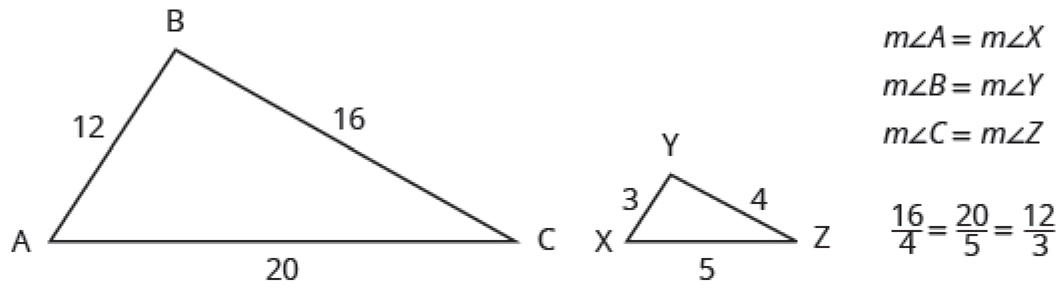


Figure 6.1.20

Properties of Similar Triangles

If two triangles are similar, then their corresponding angle measures are equal and their corresponding side lengths are in the same ratio.

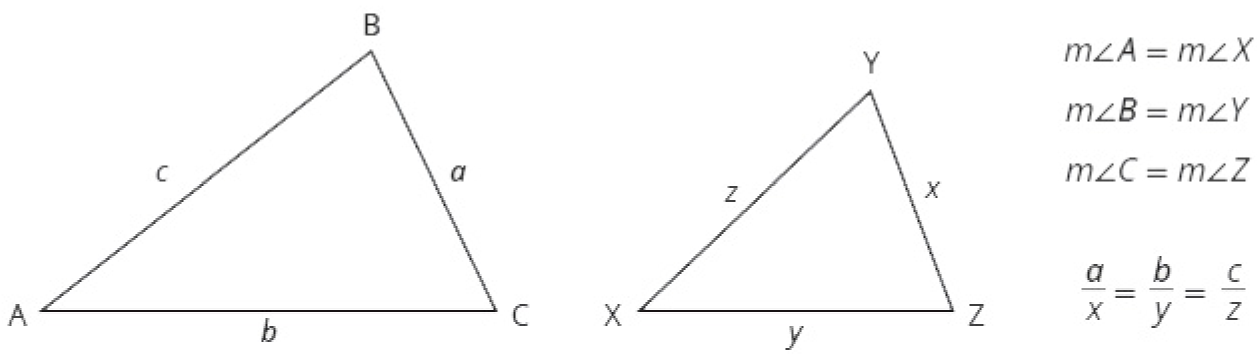


Figure 6.1.21

The length of a side of a triangle may be referred to by its endpoints, two vertices of the triangle. For example, in $\triangle ABC$:

The length a can also be written BC
The length b can also be written AC
The length c can also be written AB

We will often use this notation when we solve similar triangles because it will help us match up the corresponding side lengths.

Example 6.1.9

$\triangle\ ABC$ and $\triangle\ XYZ$ are similar triangles. The lengths of two sides of each triangle are shown. Find the lengths of the third side of each triangle.

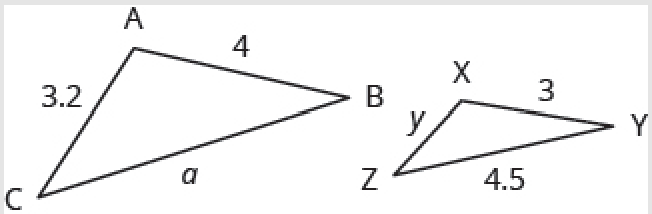


Figure 6.1.22

Solution

***Step 1:* Read the problem. Draw the figure and label it with the given information.**

The figure is provided.

***Step 2:* Identify what you are looking for.**

The length of the sides of similar triangles.

***Step 3:* Name. Choose a variable to represent it.**

Let $a =$ the length of the third side of $\triangle\ ABC$.
Let $y =$ the length of the third side of $\triangle\ XYZ$.

***Step 4:* Translate.**

Write the appropriate formula for the situation and substitute in the given information.
The triangles are similar, so the corresponding sides are in the same ratio. So:

$$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$$

Since the side $AB = 4$ corresponds to the side $XY = 3$, we will use the ratio $\frac{AB}{XY} = \frac{4}{3}$ to find the other sides.

Be careful to match up the corresponding sides correctly.

$$\text{Sides of large triangle (to find } a\text{):} \qquad \frac{AB}{XY} = \frac{BC}{YZ}$$

$$\text{Sides of large triangle (to find } y\text{):} \qquad \frac{AB}{XY} = \frac{AC}{XZ}$$

$$\text{Sides of small triangle (to find } a\text{):} \qquad \frac{4}{3} = \frac{a}{4.5}$$

$$\text{Sides of small triangle (to find } y\text{):} \qquad \frac{4}{3} = \frac{3.2}{y}$$

Step 5: **Solve the equation.**

$$\begin{aligned} 3a &= 4(4.5) \\ 3a &= 18 \\ a &= 6 \end{aligned}$$

$$\begin{aligned} 4y &= 3(3.2) \\ 4y &= 9.6 \\ y &= 2.4 \end{aligned}$$

Step 6: **Check.**

$$\begin{aligned} \frac{4}{3} &\stackrel{?}{=} \frac{6}{4.5} \\ 4(4.5) &\stackrel{?}{=} 6(3) \\ 18 &= 18 \end{aligned}$$

$$\begin{aligned} \frac{4}{3} &\stackrel{?}{=} \frac{3.2}{2.4} \\ 4(2.4) &\stackrel{?}{=} 3.2(3) \\ 9.6 &= 9.6 \end{aligned}$$

Step 7: **Answer the question.**

The third side of $\triangle\ ABC$ is 6 and the third side of $\triangle\ XYZ$ is 2.4.

Try it

18) $\triangle\ ABC$ is similar to $\triangle\ XYZ$. Find a.

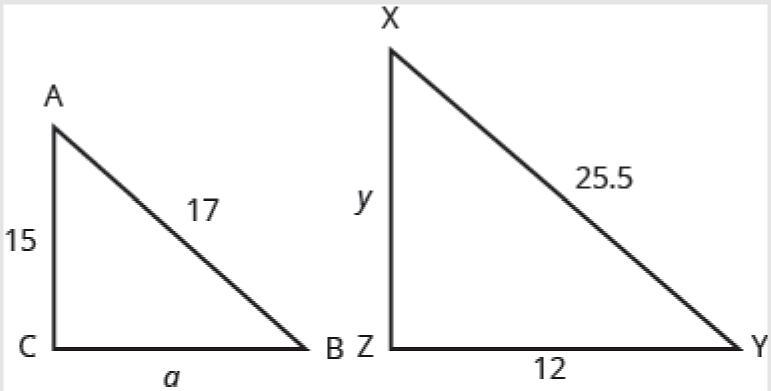


Figure 6.1.23

Solution

8

Try it

19) $\triangle\ ABC$ is similar to $\triangle\ XYZ$. Find y.

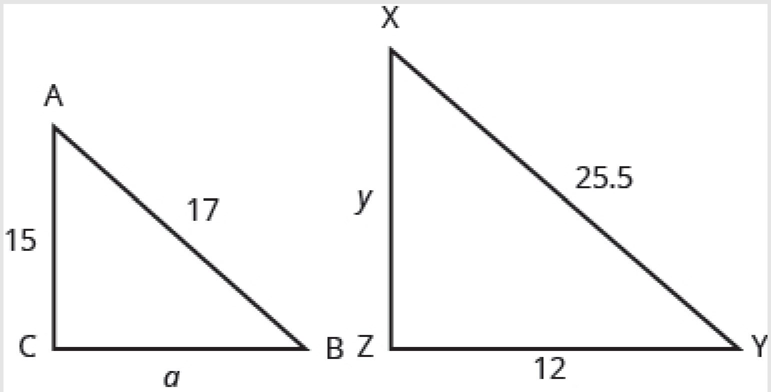


Figure 6.1.24

Solution

22.5

Use the Pythagorean Theorem

The Pythagorean Theorem is a special property of right triangles that has been used since ancient times. It is named after the Greek philosopher and mathematician Pythagoras who lived around 500 BCE.

Remember that a right triangle has a $90°$ angle, which we usually mark with a small square in the corner. The side of the triangle opposite the $90°$ angle is called the hypotenuse, and the other two sides are called the legs. See Figure 6.1.25.

In a right triangle, the side opposite the $90°$ angle is called the hypotenuse and each of the other sides is called a leg.

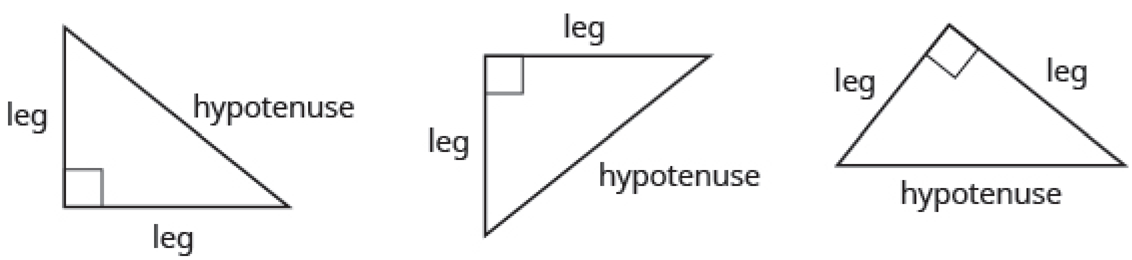


Figure 6.1.25

The Pythagorean Theorem tells how the lengths of the three sides of a right triangle relate to each other. It states that in any right triangle, the sum of the squares of the two legs equals the square of the hypotenuse.

The Pythagorean Theorem

In any right triangle ABC, $a^2 + b^2 = c^2$.

Where c is the length of the hypotenuse a and b are the lengths of the legs.

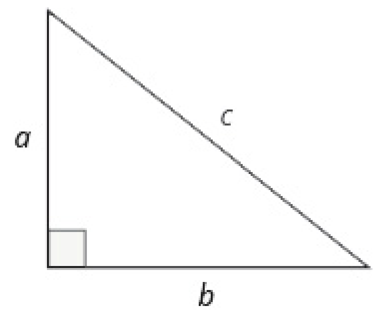


Figure 6.1.26

To solve problems that use the Pythagorean Theorem, we will need to find square roots. In 1.6 The Real Numbers we introduced the notation $\sqrt{m}$ and defined it in this way:

If $m = n^2$, then $\sqrt{m} = n$ for $n \geq 0$.

For example, we found that $\sqrt{25}$ is 5 because $5^2 = 25$.

We will use this definition of square roots to solve for the length of a side in a right triangle.

Example 6.1.10

Use the Pythagorean Theorem to find the length of the hypotenuse.

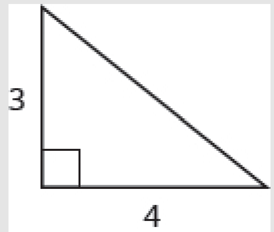


Figure 6.1.27

Solution

***Step 1:* Read the problem.**

***Step 2:* Identify what you are looking for.**

The length of the hypotenuse of the triangle.

***Step 3:* Name. Choose a variable to represent it.**

Let $c =$ the length of the hypotenuse.

***Step 4:* Translate.**

Write the appropriate formula for the situation and substitute in the given information.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 3^2 + 4^2 &= c^2 \end{aligned}$$

***Step 5:* Solve the equation.**

$$\begin{aligned} 3a &= 4(4.5) \\ 3a &= 18 \\ a &= 6 \end{aligned}$$

$$\begin{aligned} 4y &= 3(3.2) \\ 4y &= 9.6 \\ y &= 2.4 \end{aligned}$$

***Step 6:* Check.**

$$\begin{aligned} 3^2 + 4^2 &= 5^2 \\ 9 + 16 &\stackrel{?}{=} 25 \\ 25 &= 25 \end{aligned}$$

***Step 7:* Answer the question.**

The length of the hypotenuse is 5.

Try it

20) Use the Pythagorean Theorem to find the length of the hypotenuse.

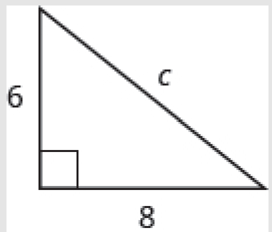


Figure 6.1.28

Solution

10

Try it

21) Use the Pythagorean Theorem to find the length of the hypotenuse.

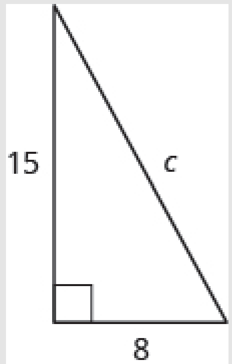


Figure 6.1.29

Solution

17

Example 6.1.11

Use the Pythagorean Theorem to find the length of the longer leg.

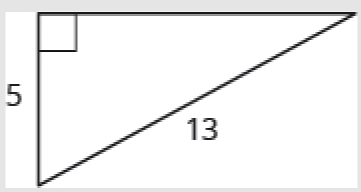


Figure 6.1.30

Solution

Step 1: **Read the problem.**

Step 2: **Identify what you are looking for.**

The length of the leg of the triangle.

Step 3: **Name. Choose a variable to represent it.**

Let $b =$ the leg of the triangle. Label side b.

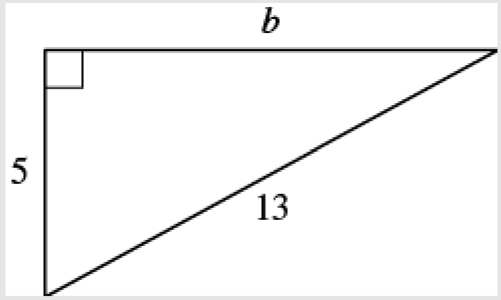


Figure 6.1.31

Step 4: **Translate.**

Write the appropriate formula for the situation and substitute in the given information.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 5^2 + b^2 &= 13^2 \end{aligned}$$

Step 5: **Solve the equation.**

$$\begin{aligned} 25 + b^2 &= 169 \\ b^2 &= 144 \\ b^2 &= \sqrt{144} \\ b &= 12 \end{aligned}$$

Step 6: **Check.**

$$\begin{aligned} 5^2 + 12^2 &\stackrel{?}{=} 13^2 \\ 25 + 144 &\stackrel{?}{=} 169 \\ 169 &= 169 \end{aligned}$$

Step 7: **Answer the question.**

The length of the leg is 12.

Try it

22) Use the Pythagorean Theorem to find the length of the leg.

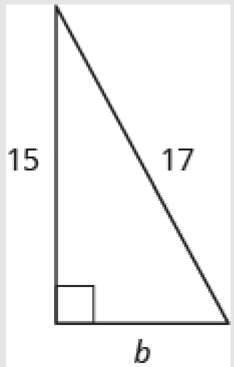


Figure 6.1.32

Solution

8

Try it

23) Use the Pythagorean Theorem to find the length of the leg.

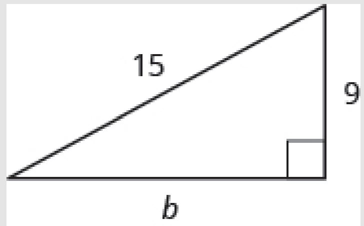


Figure 6.1.33

Solution

12

Example 6.1.12

Kelvin is building a gazebo and wants to brace each corner by placing a 10-inch wooden bracket diagonally as shown. How far below the corner should he fasten the bracket if he wants the distances from the corner to each end of the bracket to be equal? Approximate to the nearest tenth of an inch.

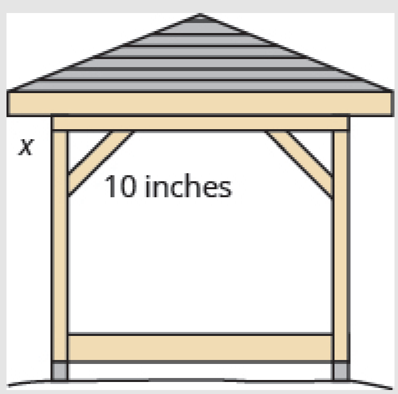


Figure 6.1.34

Solution

***Step 1:* Read the problem.**

Step 2: **Identify what you are looking for.**

The distance from the corner that the bracket should be attached.

Step 3: **Name. Choose a variable to represent it.**

Let x = the distance from the corner.

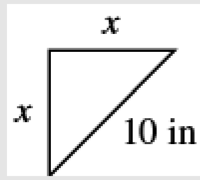


Figure 6.1.35

Step 4: **Translate.**

Write the appropriate formula for the situation and substitute in the given information.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ x^2 + x^2 &= 10^2 \end{aligned}$$

Step 5: **Solve the equation.**

	$2x^2 = 100$
Isolate the variable.	$x^2 = 50$
Use the definition of the square root. Simplify.	$x = \sqrt{50}$
Approximate to the nearest tenth.	$b \approx 7.1$

Step 6: **Check.**

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 7.1^2 + 7.1^2 &\stackrel{?}{\approx} 10^2 \\ &Yes. \end{aligned}$$

Step 7: **Answer the question.**

Kelvin should fasten each piece of wood approximately 7.1" from the corner.

Try it

24) John puts the base of a 13ft ladder 5 feet from the wall of his house. How far up the wall does the ladder reach?

Figure 6.1.36

Solution

12 feet

Try it

25) Randy wants to attach a 17 ft string of lights to the top of the 15 ft mast of his sailboat. How far from the base of the mast should he attach the end of the light string?

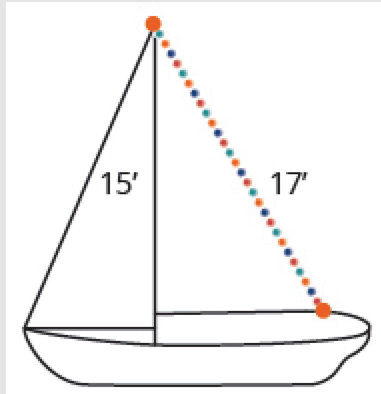


Figure 6.1.37

Solution

8 feet

ACCESS ADDITIONAL ONLINE RESOURCES

- Animation: The Sum of the Interior Angles of a Triangle
- Similar Polygons
- Example: Determine the Length of the Hypotenuse of a Right Triangle

Key Concepts

- **Supplementary and Complementary Angles**
  - If the sum of the measures of two angles is 180°, then the angles are supplementary.
  - If $\angle A$ and $\angle B$ are supplementary, then $m\angle A + m\angle B = 180$.
  - If the sum of the measures of two angles is $90°$, then the angles are complementary.

  - If $\angle A$ and $\angle B$ are complementary, then $m\angle A + m\angle B = 90$.

- **Solve Geometry Applications**

  1. Read the problem and make sure you understand all the words and ideas. Draw a figure and label it with the given information.
  2. Identify what you are looking for.
  3. Name what you are looking for and choose a variable to represent it.
  4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
  5. Solve the equation using good algebra techniques.
  6. Check the answer in the problem and make sure it makes sense.
  7. Answer the question with a complete sentence.

- **Sum of the Measures of the Angles of a Triangle**

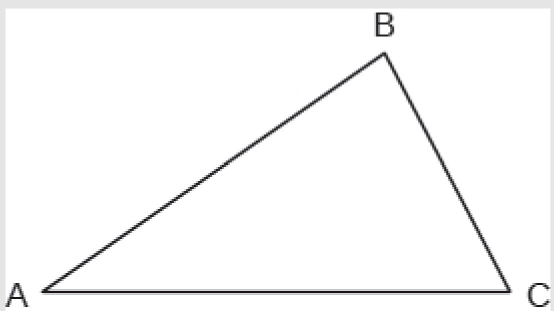


Figure 6.1.38

  - For any $\triangle ABC$, the sum of the measures is $180°$.
  - $mA + mB + mC = 180$.

- **Right Triangle**

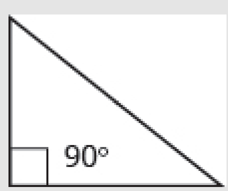


Figure 6.1.39

  - A right triangle is a triangle that has one $90°$ angle.

- **Properties of Similar Triangles**
  - If two triangles are similar, then their corresponding angle measures are equal and their corresponding side lengths have the same ratio.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=6531#h5p-37

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

Glossary

angle
An angle is formed by two rays that share a common endpoint. Each ray is called a side of the angle.

complementary angles
If the sum of the measures of two angles is 90°, then they are called complementary angles.

hypotenuse
The side of the triangle opposite the 90° angle is called the hypotenuse.

legs of a right triangle
The sides of a right triangle adjacent to the right angle are called the legs.

right triangle
A right triangle is a triangle that has one 90° angle.

similar figures
In geometry, if two figures have exactly the same shape but different sizes, we say they are similar figures.

supplementary angles
If the sum of the measures of two angles is 180°, then they are called supplementary angles.

triangle
A triangle is a geometric figure with three sides and three angles.

vertex of an angle
When two rays meet to form an angle, the common endpoint is called the vertex of the angle.

6.2 USE PROPERTIES OF RECTANGLES, TRIANGLES, AND TRAPEZOIDS

Learning Objectives

By the end of this section, you will be able to:

- Understand linear, square, and cubic measure
- Use properties of rectangles
- Use properties of triangles
- Use properties of trapezoids

Try It

Before you get started, take this readiness quiz:

1) The length of a rectangle is 3 less than the width. Let w represent the width. Write an expression for the length of the rectangle.

2) Simplify: $\frac{1}{2}(6h)$

3) Simplify: $\frac{5}{2}(10.3-7.9)$

In this section, we'll continue working with geometry applications. We will add some more properties of triangles, and we'll learn about the properties of rectangles and trapezoids.

Understand Linear, Square, and Cubic Measure

When you measure your height or the length of a garden hose, you use a ruler or tape measure Figure 6.2.1. A tape measure might remind you of a line—you use it for linear measure, which measures length. Inch, foot, yard, mile, centimetre and meter are units of linear measure.
This tape measure measures inches along the top and centimetres along the bottom.

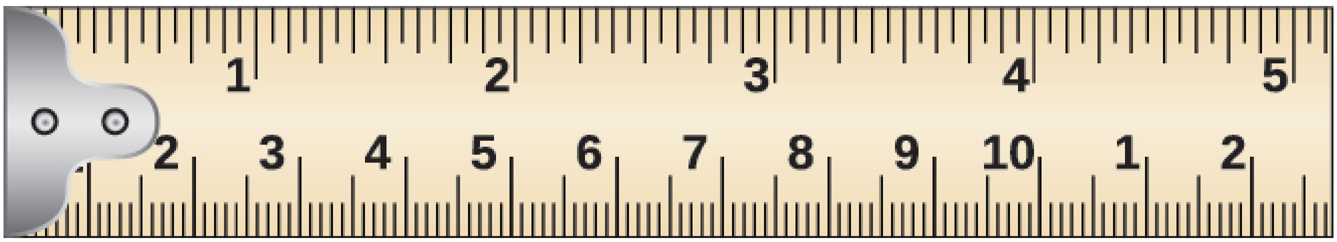


Figure 6.2.1

When you want to know how much tile is needed to cover a floor, or the size of a wall to be painted, you need to know the **area**, a measure of the region needed to cover a surface. Area is measured is square units. We often use square inches, square feet, square centimetres, or square miles to measure area. A square centimetre is a square that is one centimetre (cm) on each side. A square inch is a square that is one inch on each side Figure 6.2.2.

Square measures have sides that are each 1 unit in length.

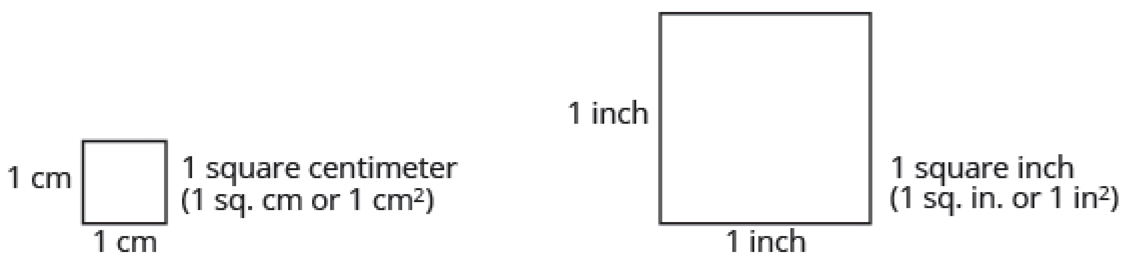


Figure 6.2.2

Figure 6.2.3 shows a rectangular rug that is 2 feet long by 3 feet wide. Each square is 1 foot wide by 1 foot long, or 1 square foot. The rug is made of 6 squares. The area of the rug is 6 square feet.

The rug contains six squares of 1 square foot each, so the total area of the rug is 6 square feet.

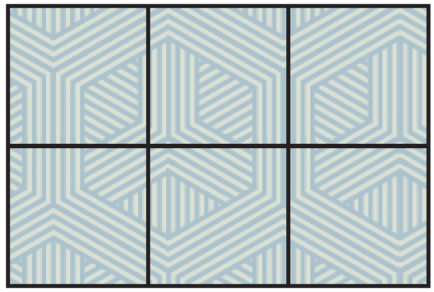


Figure 6.2.3

When you measure how much it takes to fill a container, such as the amount of gasoline that can fit in a tank, or the amount of medicine in a syringe, you are measuring volume. Volume is measured in cubic units such as cubic inches or cubic centimetres. When measuring the volume of a rectangular solid, you measure how many cubes fill the container. We often use cubic centimetres, cubic inches, and cubic feet. A cubic centimetre is a cube that measures one centimetre on each side, while a cubic inch is a cube that measures one inch on each side Figure 6.2.4.

Cubic measures have sides that are 1 unit in length.

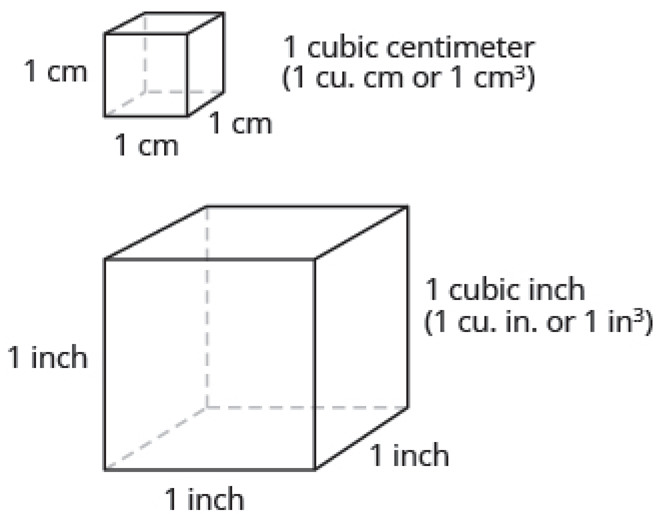


Figure 6.2.4

Suppose the cube in Figure 6.2.5 measures 3 inches on each side and is cut on the lines shown. How many little cubes does it contain? If we were to take the big cube apart, we would find 27 little cubes, with each one measuring one inch on all sides. So each little cube has a volume of 1 cubic inch, and the volume of the big cube is 27 cubic inches.

A cube that measures 3 inches on each side is made up of 27 one-inch cubes, or 27 cubic inches.

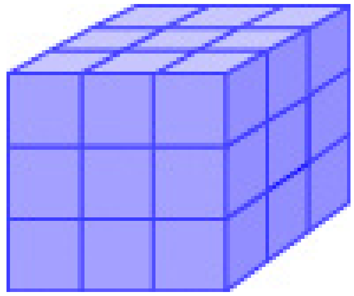


Figure 6.2.5

Example 6.2.1

For each item, state whether you would use linear, square, or cubic measure:

a. amount of carpeting needed in a room
b. extension cord length
c. amount of sand in a sandbox
d. length of a curtain rod
e. amount of flour in a canister
f. size of the roof of a doghouse.

Solution

a. You are measuring how much surface the carpet covers, which is the area.	square measure
b. You are measuring how long the extension cord is, which is the length.	linear measure
c. You are measuring the volume of the sand.	cubic measure
d. You are measuring the length of the curtain rod.	linear measure
e. You are measuring the volume of the flour.	cubic measure
f. You are measuring the area of the roof.	square measure

Try It

4) Determine whether you would use linear, square, or cubic measure for each item.

a. amount of paint in a can
b. height of a tree
c. floor of your bedroom
d. diameter of bike wheel
e. size of a piece of sod
f. amount of water in a swimming pool

Solution

a. cubic
b. linear
c. square
d. linear
e. square
f. cubic

5) Determine whether you would use linear, square, or cubic measure for each item.

a. volume of a packing box
b. size of patio
c. amount of medicine in a syringe
d. length of a piece of yarn
e. size of housing lot
f. height of a flagpole

Solution

a. cubic
b. square
c. cubic
d. linear
e. square
f. linear

Many geometry applications will involve finding the **perimeter** or the area of a figure. There are also many applications of perimeter and area in everyday life, so it is important to make sure you understand what they each mean.

Picture a room that needs new floor tiles. The tiles come in squares that are a foot on each side—one square foot. How many of those squares are needed to cover the floor? This is the area of the floor.

Next, think about putting new baseboard around the room, once the tiles have been laid. To figure out how many strips are needed, you must know the distance around the room. You would use a tape measure to measure the number of feet around the room. This distance is the perimeter.

Perimeter and Area

The perimeter is a measure of the distance around a figure.

The area is a measure of the surface covered by a figure

Figure 6.2.6 shows a square tile that is 1 inch on each side. If an ant walked around the edge of the tile, it would walk 4 inches. This distance is the perimeter of the tile.

Since the tile is a square that is 1 inch on each side, its area is one square inch. The area of a shape is measured by determining how many square units cover the shape.

$$\text{Perimeter} = 4\text{inches}$$
$$\text{Area} = 1\text{square inch}$$

When the ant walks completely around the tile on its edge, it is tracing the perimeter of the tile. The area of the tile is 1 square inch.

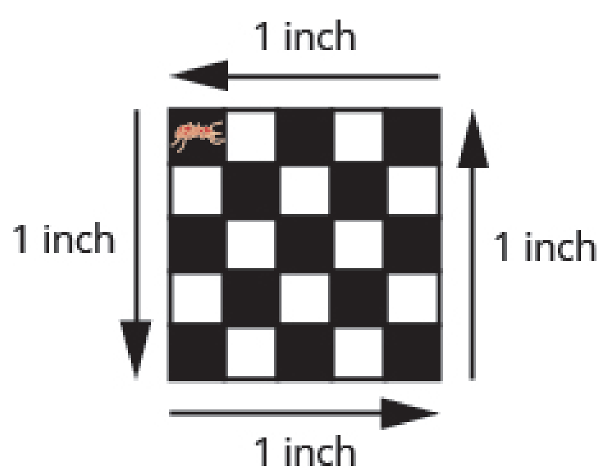


Figure 6.2.6

Example 6.2.2

Each of two square tiles is 1 square inch. Two tiles are shown together.

a. What is the perimeter of the figure?

b. What is the area?

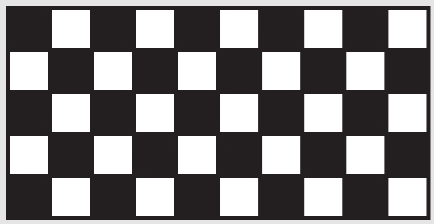


Figure 6.2.7

Solution

a. The perimeter is the distance around the figure. The perimeter is 6 inches.

b. The area is the surface covered by the figure. There are 2 square inch tiles so the area is 2 square inches.

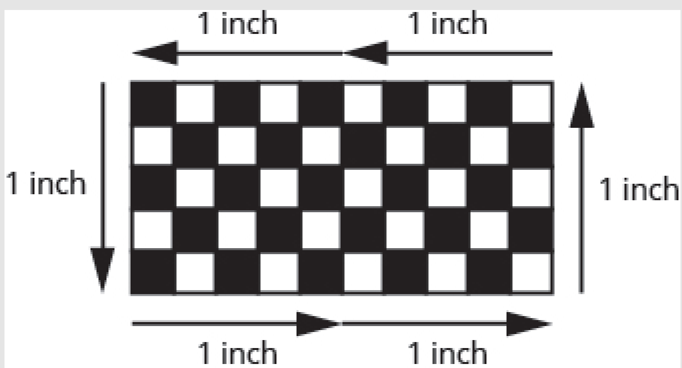


Figure 6.2.8

Try It

6) Find the: a. perimeter and b. area of the figure:

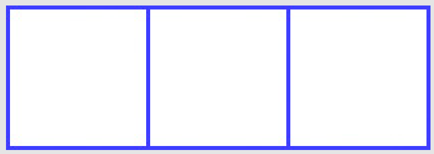


Figure 6.2.9

Solution

a. 8 inches

b. 3 sq. inches

7) Find the: a. perimeter b. area of the figure

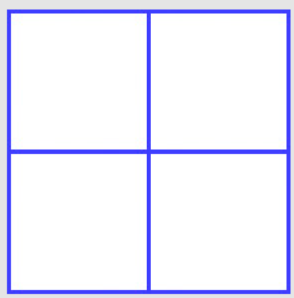


Figure 6.2.10

Solution

a. 8 centimetres

b. 4 sq. centimetres

Use the Properties of Rectangles

A **rectangle** has four sides and four right angles. The opposite sides of a rectangle are the same length. We refer to one side of the rectangle as the length, L, and the adjacent side as the width, W. See Figure 6.2.11.

A rectangle has four sides, and four right angles. The sides are labelled L for length and W for width.

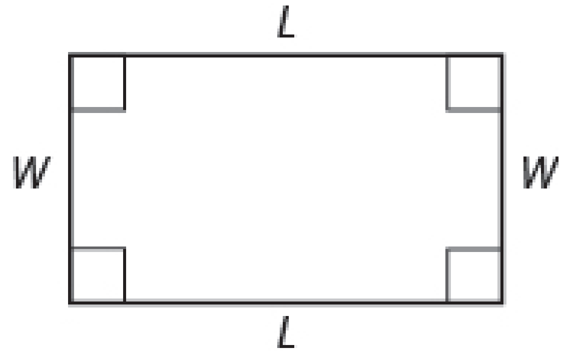


Figure 6.2.11

The perimeter, P, of the rectangle is the distance around the rectangle. If you started at one corner and walked around the rectangle, you would walk $L+W+L+W$ units, or two lengths and two widths. The perimeter then is

$P=L+W+L+W$ or
$P=2L+2W$

What about the area of a rectangle? Remember the rectangular rug from the beginning of this section. It was 2 feet long by 3 feet wide, and its area was 6 square feet. See Figure 6.2.13. Since $A=2\times\ 3$ we see that the area, A, is the length, L, times the width, W, so the area of a rectangle is $A=L\times\ W$

The area of this rectangular rug is 6 square feet, its length times its width.

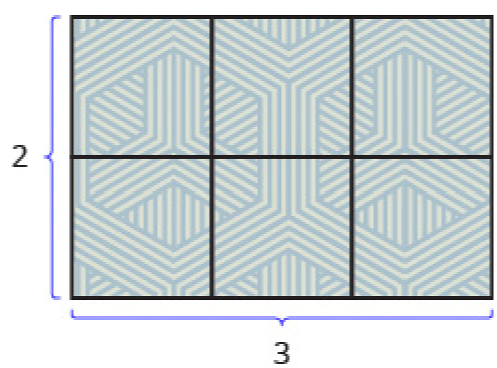


Figure 6.2.12

Properties of Rectangles

- Rectangles have four sides and four right $90°$ angles.
- The lengths of opposite sides are equal.
- The perimeter, P, of a rectangle is the sum of twice the length and twice the width. See Figure 6.2.11.
  $P = 2L + 2W$.
- The area, A, of a rectangle is the length times the width.
  $A = L \times W$

For easy reference as we work the examples in this section, we will restate the Problem Solving Strategy for Geometry Applications here.

HOW TO

Use a Problem Solving Strategy for Geometry Applications

1. Read the problem and make sure you understand all the words and ideas. Draw the figure and label it with the given information.
2. Identify what you are looking for.
3. Name what you are looking for. Choose a variable to represent that quantity.
4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. Solve the equation using good algebra techniques.
6. Check the answer in the problem and make sure it makes sense.
7. Answer the question with a complete sentence.

Example 6.2.3

The length of a rectangle is 32 meters and the width is 20 meters. Find a. the perimeter b. the area

Solution

a. ***Step 1:*** **Read the problem. Draw the figure and label it with the given information.**

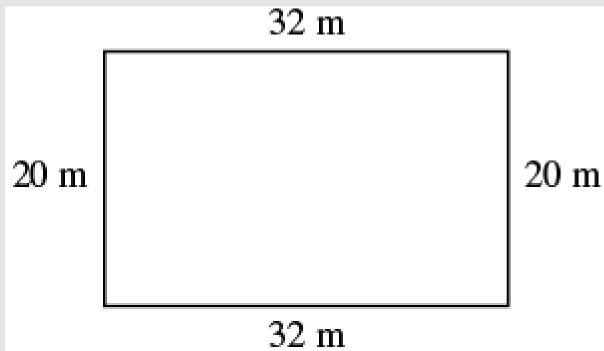


Figure 6.2.13

Step 2: **Identify what you are looking for.**

The perimeter of a rectangle.

Step 3: **Name. Choose a variable to represent it.**

Let P= the perimeter.

Step 4: **Translate.**

Write the appropriate formula. Substitute.

$$P = 2L + 2W$$
$$P = 2(32) + 2(20)$$

Step 5: **Solve the equation.**

$$P = 64 + 40$$
$$P = 104$$

Step 6: **Check.**

$$\begin{aligned} P &\stackrel{?}{=} 104 \\ 20 + 32 + 20 + 32 &\stackrel{?}{=} 104 \\ 104 &\stackrel{?}{=} 104 \end{aligned}$$

Step 7: **Answer the question.**

The perimeter of the rectangle is 104 meters.

---

b. ***Step 1:*** **Read the problem. Draw the figure and label it with the given information.**

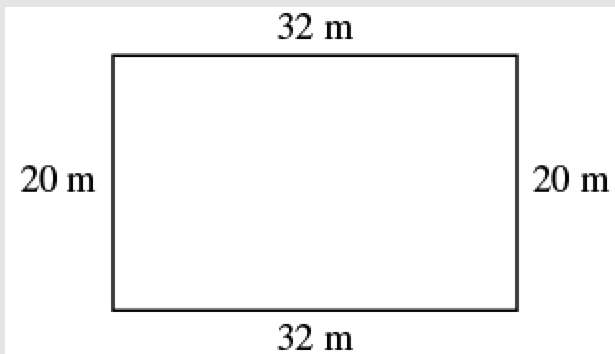


Figure 6.2.14

Step 2: **Identify what you are looking for.**

The area of a rectangle

Step 3: **Name. Choose a variable to represent it.**

Let A = the area.

Step 4: **Translate.**

Write the appropriate formula. Substitute.

$$\begin{aligned} A &= L \times W \\ A &= 32m \times 20m \end{aligned}$$

Step 5: **Solve the equation.**

$$A = 640$$

Step 6: **Check.**

$$A \stackrel{?}{=} 640$$
$$32 \times 20 \stackrel{?}{=} 640$$
$$640 \stackrel{?}{=} 640$$

***Step 7:* Answer the question.**

The area of the rectangle is 60 square meters.

Try It

8) The length of a rectangle is 120 yards and the width is 50 yards. Find a. the perimeter and b. the area.

Solution

a. 340 yd

b. 6000 sq. yd

9) The length of a rectangle is 62 feet and the width is 48 feet. Find a. the perimeter and b. the area.

Solution

a. 220 ft

b. 2976 sq. ft

Example 6.2.4

Find the length of a rectangle with perimeter 50 inches and width 10 inches.

Solution

***Step 1:* Read the problem. Draw the figure and label it with the given information.**

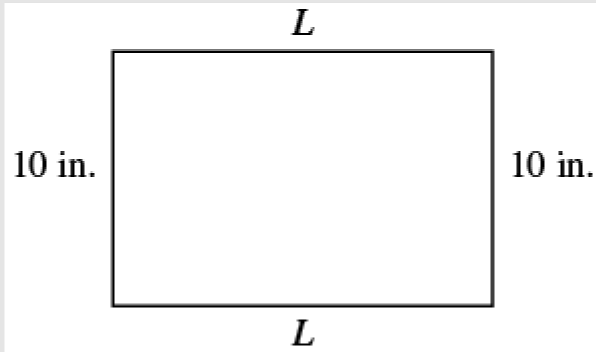


Figure 6.2.15

***Step 2:* Identify what you are looking for.**

The length of the rectangle.

***Step 3:* Name. Choose a variable to represent it.**

Let L = the length.

***Step 4:* Translate.**

Write the appropriate formula. Substitute.

$$P = 2L + 2W$$
$$50 = 2L + 2(10)$$

***Step 5:* Solve the equation.**

$$\begin{aligned} 50\ rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.0\ rgb]1.0, 0.0, 0.020 &= 2L\ +\ 20\ rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.020 \\ 30 &= 2L \\ \frac{30}{rgb]1.0, 0.0, 0.02} &= \frac{2L}{rgb]1.0, 0.0, 0.02} \\ 15 &= L \end{aligned}$$

***Step 6:* Check.**

$$P = 50$$
$$15 + 10 + 15 + 10 \stackrel{?}{=} 50$$
$$50 = 50 \checkmark$$

***Step 7:* Answer the question.**

The length is 15 inches.

Try It

10) Find the length of a rectangle with a perimeter of 80 inches and width of 25 inches.

Solution

15 in.

11) Find the length of a rectangle with a perimeter of 30 yards and width of 6 yards.

Solution

9 yd

In the next example, the width is defined in terms of the length. We'll wait to draw the figure until we write an expression for the width so that we can label one side with that expression.

Example 6.2.5

The width of a rectangle is two inches less than the length. The perimeter is 52 inches. Find the length and width.

Solution

***Step 1:* Read the problem.**

***Step 2:* Identify what you are looking for.**

The length of the rectangle.

***Step 3:* Name. Choose a variable to represent it.**

Since the width is defined in terms of the length, we let $L =$ length. The width is two feet less that the length, so we let width $(W) = L - 2$.

Now we can create a visual to represent it.

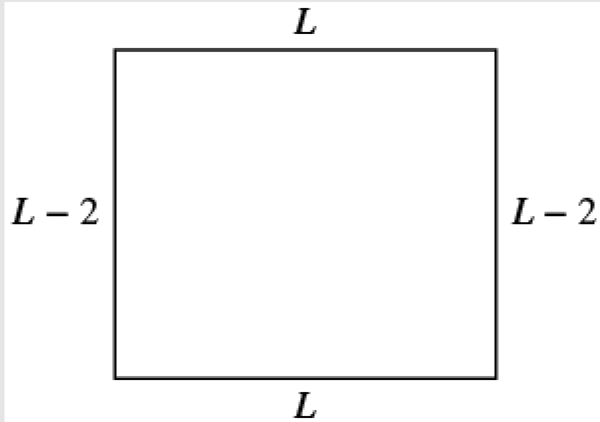


Figure 6.2.16

Step 4: **Translate.**

Write the appropriate formula. The formula for the perimeter of a rectangle relates all the information. Substitute in the given information.

$$\begin{aligned} P &= 2L + 2W \\ 52 &= 2L + 2(L-2) \end{aligned}$$

Step 5: **Solve the equation.**

Combine like terms.

$$52 = 2L + 2(L-2)$$
$$52 = rgb]1.0, 0.0, 0.02rgb]1.0, 0.0, 0.0Lrgb]1.0, 0.0, 0.0+rgb]1.0, 0.0, 0.02rgb]1.0, 0.0, 0.0L - 4$$
$$52 = rgb]1.0, 0.0, 0.04rgb]1.0, 0.0, 0.0L - 4$$

Add 4 to each side.

$$rgb]0.0, 0.0, 1.04rgb]0.0, 0.0, 1.0+52 = 4L - 4rgb]0.0, 0.0, 1.0+rgb]0.0, 0.0, 1.04$$
$$56 = 4L \cancel{-4+4}$$

Divide by 4 and solve.

$$\frac{56}{rgb]0.0, 0.0, 1.04} = \frac{4L}{rgb]0.0, 0.0, 1.04}$$
$$\frac{56}{rgb]0.0, 0.0, 1.04} = \frac{\cancel{4}L}{rgb]0.0, 0.0, 1.0\cancel{4}}$$
$$\frac{56}{4} = L$$
$$14 = L$$

Step 6: Find the width.

$$W = L - 2$$
$$W = rgb]1.0, 0.0, 0.014 - 2$$
$$W = 12$$

Step 7: **Check.**

$$P = 2rgb]1.0, 0.0, 0.0L + 2rgb]0.0, 0.0, 1.0W$$
$$52 = 2(rgb]1.0, 0.0, 0.014) + 2(rgb]0.0, 0.0, 1.012)$$
$$52 = rgb]1.0, 0.0, 0.028 + rgb]0.0, 0.0, 1.024$$
$$52 = 52\checkmark$$

Step 8: **Answer the question.**

The length is 14 feet and the width is 12 feet.

Try It

12) The width of a rectangle is seven meters less than the length. The perimeter is 58 meters. Find the length and width.

Solution

18 m, 11 m

13) The length of a rectangle is eight feet more than the width. The perimeter is 60 feet. Find the length and width.

Solution

11 ft , 19 ft

Example 6.2.6

The length of a rectangle is four centimetres more than twice the width. The perimeter is 32 centimetres. Find the length and width.

Solution

Step 1: **Read the problem.**

Step 2: **Identify what you are looking for.**

The length and width.

Step 3: **Name. Choose a variable to represent it. Then draw a diagram to show what we are solving for.**

Let $W =$ width. The length is four more than twice the width. $2w + 4 =$ length.

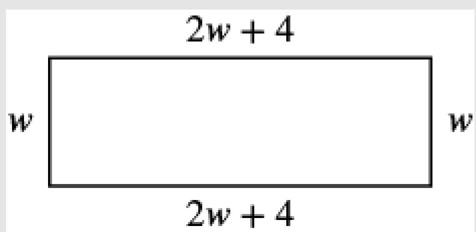


Figure 6.2.17

Step 4: **Translate.**

Write the appropriate formula and substitute in the given information.

$$\begin{aligned} P &= rgb]0.0,0.0,1.02rgb]0.0,0.0,1.0L + rgb]1.0,0.0,0.02rgb]1.0,0.0,0.0W \\ 32 &= rgb]0.0,0.0,1.02rgb]0.0,0.0,1.0(rgb]0.0,0.0,1.02rgb]0.0,0.0,1.0Wrgb]0.0,0.0,1.0\ rgb]0.0,0.0,1.0+rgb]0.0,0.0,1.0\ rgb]0.0,0.0,1.04rgb]0.0,0.0,1.0) + rgb]1.0,0.0,0.02rgb]1.0,0.0,0.0W \end{aligned}$$

Step 5: **Solve the equation.**

$$\begin{aligned} 32 &= rgb]0.0,0.0,1.02rgb]0.0,0.0,1.0(rgb]0.0,0.0,1.02rgb]0.0,0.0,1.0Wrgb]0.0,0.0,1.0+rgb]0.0,0.0,1.04rgb]0.0,0.0,1.0) + rgb]1.0,0.0,0.02rgb]1.0,0.0,0.0W \\ 32 &= rgb]0.0,0.0,1.04rgb]0.0,0.0,1.0Wrgb]0.0,0.0,1.0+rgb]0.0,0.0,1.08 + rgb]1.0,0.0,0.02rgb]1.0,0.0,0.0W \\ 32 &= 6W + 8 \\ 24 &= 6W \\ rgb]1.0,0.0,0.04 &= W \\ 2W + 4 &= length \\ 2(rgb]1.0,0.0,0.04) + 4 &= \ length \\ 12 &= \ legnth \end{aligned}$$

Step 6: **Check.**

$$P = 2L + 2W$$
$$32 \stackrel{?}{=} 2 \times 12 + 2 \times 4$$
$$32 = 32 \checkmark$$

Step 7: **Answer the question.**

The length is 12 cm and the width is 4 cm.

Try It

14) The length of a rectangle is eight more than twice the width. The perimeter is 64 feet. Find the length and width.

Solution

8 ft, 24 ft

15) The width of a rectangle is six less than twice the length. The perimeter is 18 centimetres. Find the length and width.

Solution

5 cm, 4 cm

Example 6.2.7

The area of a rectangular room is 168 square feet. The length is 14 feet. What is the width?

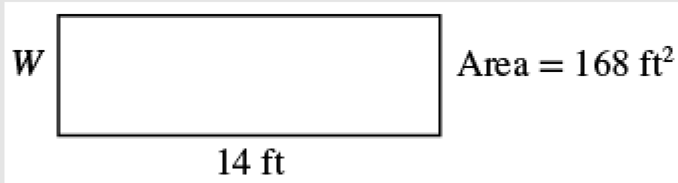


Figure 6.2.18

Solution

Step 1: **Read the problem.**

Step 2: **Identify what you are looking for.**

The width of a rectangular room

Step 3: **Name. Choose a variable to represent it.**

Let W = width.

Step 4: **Translate.**

Write the appropriate formula and substitute in the given information.

$$\begin{aligned} A &= LW \\ 168 &= 14W \end{aligned}$$

Step 5: **Solve the equation.**

$$\begin{aligned} \frac{168}{14} &= \frac{14W}{14} \\ 12 &= W \end{aligned}$$

Step 6: **Check.**

$$A = LW$$

$$168 \stackrel{?}{=} 14 \times 12$$

$$168 = 168 \checkmark$$

Step 7: **Answer the question.**

The width of the room is 12 feet.

Try It

16) The area of a rectangle is 598 square feet. The length is 23 feet. What is the width?

Solution

26 ft

17) The width of a rectangle is 21 meters. The area is 609 square meters. What is the length?

Solution

29 m

Example 6.2.8

Find the length of a rectangle with perimeter 50 inches and width 10 inches.

Solution

Step 1: **Read the problem. Draw the figure and label it with the given information.**

$$P = 50 \text{ inches.}$$

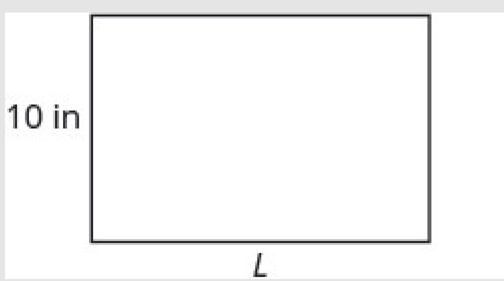


Figure 6.2.19

Step 2: **Identify what you are looking for.**

The length of the rectangle.

Step 3: **Name. Choose a variable to represent it.**

Let L = the length.

Step 4: **Translate.**

Write the appropriate formula and substitute in the given information.

$$\begin{aligned} P &= 2L + 2W \\ 50 &= 2L + 2(10) \end{aligned}$$

Step 5: **Solve the equation.**

$$50 rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.0 rgb]1.0, 0.0, 0.020 = 2L + 20 rgb]1.0, 0.0, 0.0 - rgb]1.0, 0.0, 0.020$$

$$30 = 2L$$

$$\frac{30}{2} = \frac{2L}{2}$$

$$15 = L$$

Step 6: **Check.**

$$15 + 10 + 15 + 10 \stackrel{?}{=} 50$$

$$50 = 50 \checkmark$$

Step 7: **Answer the question.**

The length is 15 inches.

Try It

18) Find the length of a rectangle with: perimeter 80 and width 25.

Solution

15

19) Find the length of a rectangle with: perimeter 30 and width 6.

Solution

9

Example 6.2.9

The perimeter of a rectangular swimming pool is 150 feet. The length is 15 feet more than the width. Find the length and width.

Solution

Step 1: **Read the problem. Draw the figure and label it with the given information.**

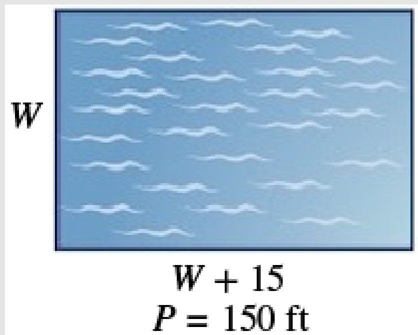


Figure 6.2.20

Step 2: **Identify what you are looking for.**

The length and width of the pool.

Step 3: **Name. Choose a variable to represent it.**

Let $W =$ width.
Let $W + 15 =$length.

Step 4: **Translate.**

Write the appropriate formula and substitute in the given information.

$$\begin{aligned} \mathrm{P} &= 2\mathrm{L} + 2\mathrm{W} \\ 150 &= 2(\mathrm{W} + 15) + 2\mathrm{W} \end{aligned}$$

Step 5: **Solve the equation.**

$$\begin{aligned} 150 &= 2\text{W} + 30 + 2\text{W} \\ 150 &= 4\text{W} + 30 \\ 120 &= 4\text{W} \\ rgb]1.0,0.0,0.030 &= \text{W (the width of the pool)} \end{aligned}$$

$$\begin{aligned} \text{W} + 15 &= \text{length of pool} \\ rgb]1.0,0.0,0.030 + 15 &= \text{L} \\ 45 &= \text{L} \end{aligned}$$

Step 6: **Check.**

$$\begin{aligned} \text{P} &= 2\text{L} + 2\text{W} \\ 150 &\stackrel{?}{=} 2(45) + 2(30) \\ 150 &= 150\checkmark \end{aligned}$$

Step 7: **Answer the question.**

The length of the pool is 45 feet and the width is 30 feet.

Try It

20) The perimeter of a rectangular swimming pool is 200 feet. The length is 40 feet more than the width. Find the length and width.

Solution

30 ft, 70 ft

21) The length of a rectangular garden is 30 yards more than the width. The perimeter is 300 yards. Find the length and width.

Solution

60 yd, 90 yd

Use the Properties of Triangles

We now know how to find the area of a rectangle. We can use this fact to help us visualize the formula for the area of a triangle. In the rectangle in Figure 6.2.21, we've labelled the length b and the width h, so it's area is bh.

The area of a rectangle is the base, b, times the height, h.

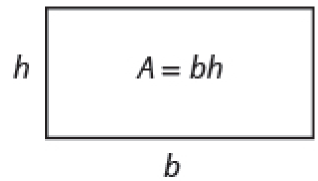


Figure 6.2.21

We can divide this rectangle into two congruent triangles Figure 6.2.22. Triangles that are congruent have identical side lengths and angles, and so their areas are equal. The area of each triangle is one-half the area of the rectangle, or $\frac{1}{2}bh$. This example helps us see why the formula for the area of a triangle is $A = \frac{1}{2}bh$.

A rectangle can be divided into two triangles of equal area. The area of each triangle is one-half the area of the rectangle.

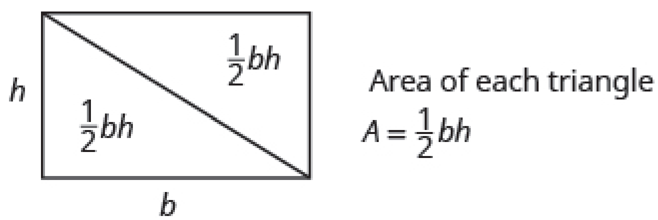


Figure 6.2.22

The formula for the area of a triangle is $A = \frac{1}{2}bh$, where b is the base and h is the height.

To find the area of the triangle, you need to know its base and height. The base is the length of one side of the triangle, usually the side at the bottom. The height is the length of the line that connects the base to the opposite vertex, and makes a $90°$ angle with the base. Figure 6.2.23 shows three triangles with the base and height of each marked.

The height h of a triangle is the length of a line segment that connects the the base to the opposite vertex and makes a $90°$ angle with the base.

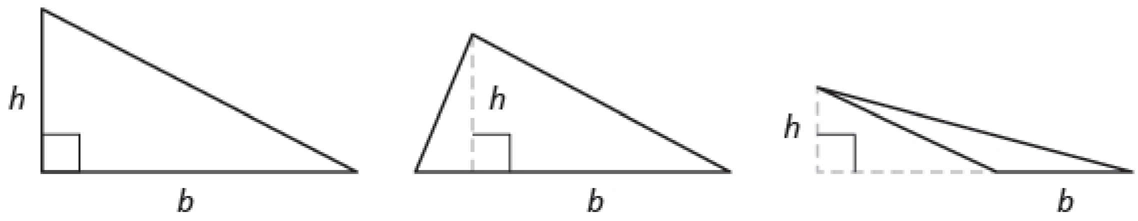


Figure 6.2.23

Triangle Properties

For any triangle $\triangle\ ABC$, the sum of the measures of the angles is $180°$.

$$m\angle A + m\angle B + m\angle C = 180°$$

The perimeter of a triangle is the sum of the lengths of the sides.

$$P = a + b + c$$

The area of a triangle is one-half the base, b, times the height, h.

$$A = \frac{1}{2}bh$$

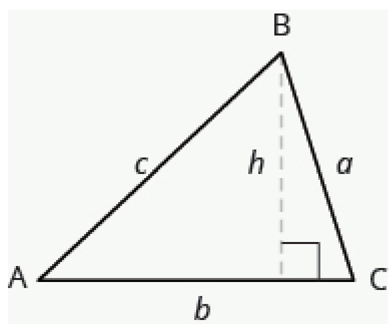


Figure 6.2.24

Example 6.2.10

Find the area of a triangle whose base is 11 inches and whose height is 8 inches.

Solution

Step 1: **Read the problem. Draw the figure and label it with the given information.**

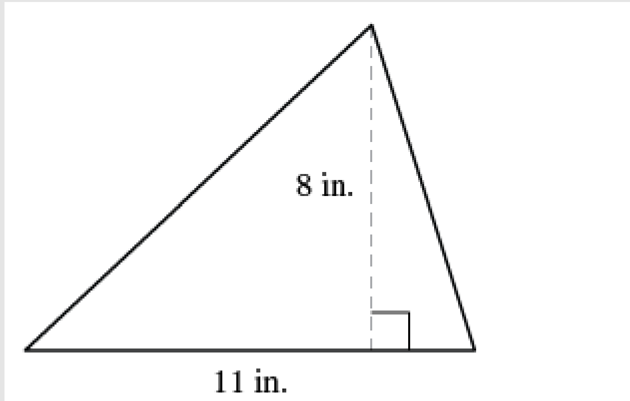


Figure 6.2.25

Step 2: **Identify what you are looking for.**

The area of the triangle.

Step 3: **Name. Choose a variable to represent it.**

Let $A =$ area of the triangle.

Step 4: **Translate.**

Write the appropriate formula and substitute in the given information.

$$A = \frac{1}{2} \times b \times h$$
$$A = \frac{1}{2} \times 11 \times 8$$

Step 5: **Solve the equation.**

$$A = 44 \text{ square inches}$$

Step 6: **Check.**

$$A = \frac{1}{2}bh$$
$$44 \stackrel{?}{=} \frac{1}{2}(11)8$$
$$44 = 44 \checkmark$$

Step 7: **Answer the question.**

The area is 44 square inches.

Try It

22) Find the area of a triangle with base 13 inches and height 2 inches.

Solution

13 sq. in.

23) Find the area of a triangle with base 14 inches and height 7 inches.

Solution

49 sq. in.

Example 6.2.11

The perimeter of a triangular garden is 24 feet. The lengths of two sides are 4 feet and 9 feet. How long is the third side?

Solution

Step 1: **Read the problem. Draw the figure and label it with the given information.**

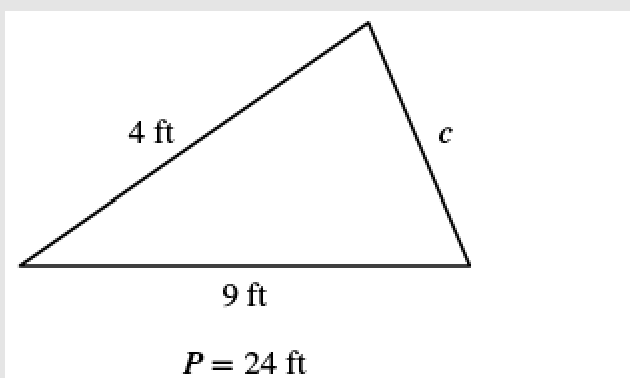


Figure 6.2.26

Step 2: **Identify what you are looking for.**

The length of the third side of a triangle.

Step 3: **Name. Choose a variable to represent it.**

Let c = the third side.

Step 4: **Translate.**

Write the appropriate formula and substitute in the given information.

$$\begin{aligned} P &= a + b + c \\ P &= 4 + 9 + c \end{aligned}$$

Step 5: **Solve the equation.**

$$\begin{aligned} 24 &= 13 + c \\ 11 &= c \end{aligned}$$

Step 6: **Check.**

$$\begin{aligned} P &= a + b + c \\ 24 &\stackrel{?}{=} 4 + 9 + 11 \\ 24 &= 24 \checkmark \end{aligned}$$

Step 7: **Answer the question.**

The third side is 11 feet long.

Try It

24) The perimeter of a triangular garden is 24 feet. The lengths of two sides are 18 feet and 22 feet. How long is the third side?

Solution

8 ft

25) The lengths of two sides of a triangular window are 7 feet and 5 feet. The perimeter is 18 feet. How long is the third side?

Solution

6 ft

Example 6.2.12

The area of a triangular church window is 90 square meters. The base of the window is 15 meters. What is the window's height?

Solution

Step 1: **Read the problem. Draw the figure and label it with the given information.**

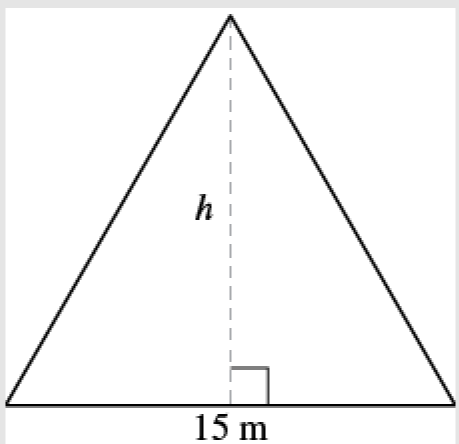


Figure 6.2.27

Step 2: **Identify what you are looking for.**

The height of a triangle.

Step 3: **Name. Choose a variable to represent it.**

Let $h =$ the height.

Step 4: **Translate.**

Write the appropriate formula and substitute in the given information.

$$A = \frac{1}{2} \times b \times h$$
$$90 = \frac{1}{2} \times 15 \times h$$

Step 5: **Solve the equation.**

$$\begin{aligned} 90 &= \frac{15}{2}h \\ 12 &= h \end{aligned}$$

Step 6: **Check.**

$$\begin{aligned} A &= \frac{1}{2}bh \\ 90 &\stackrel{?}{=} \frac{1}{2} \times 15 \times 12 \\ 90 &= 90 \checkmark \end{aligned}$$

Step 7: **Answer the question.**

The height of the triangle is 12 meters.

Try It

26) The area of a triangular painting is 126 square inches. The base is 18 inches. What is the height?

Solution

14 in.

27) A triangular tent door has an area of 15 square feet. The height is 5 feet. What is the base?

Solution

6 ft

Isosceles and Equilateral Triangles

Besides the right triangle, some other triangles have special names. A triangle with two sides of equal length is called an **isosceles triangle**. A triangle that has three sides of equal length is called an **equilateral triangle**. Figure 6.2.28 shows both types of triangles.

In an isosceles triangle, two sides have the same length, and the third side is the base. In an equilateral triangle, all three sides have the same length.

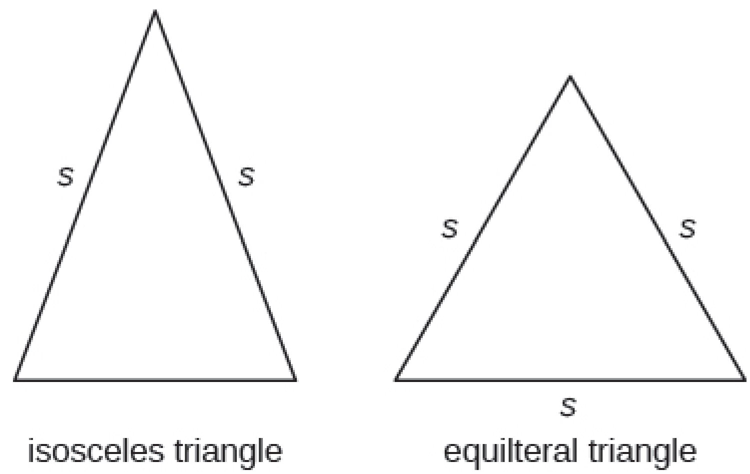


Figure 6.2.28

Isosceles and Equilateral Triangles

An isosceles triangle has two sides the same length.

An equilateral triangle has three sides of equal length.

Example 6.2.13

The perimeter of an equilateral triangle is 93 inches. Find the length of each side.

Solution

Step 1: **Read the problem. Draw the figure and label it with the given information.**

Perimeter = 93 in.

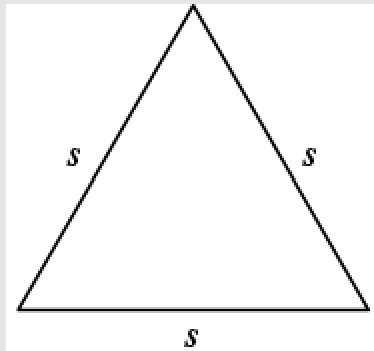


Figure 6.2.29

Step 2: **Identify what you are looking for.**

The length of the sides of an equilateral triangle.

Step 3: **Name. Choose a variable to represent it.**

Let s = length of each side.

Step 4: **Translate.**

Write the appropriate formula and substitute in the given information.

$$\begin{aligned} P &= a + b + c \\ 93 &= s + s + s \end{aligned}$$

Step 5: **Solve the equation.**

$$\begin{aligned} 93 &= 3s \\ 31 &= s \end{aligned}$$

Step 6: **Check.**

$$\begin{aligned} 93 &\stackrel{?}{=} 31 + 31 + 31 \\ 93 &= 93 \checkmark \end{aligned}$$

Step 7: **Answer the question.**

Each side is 31 inches.

Try It

28) Find the length of each side of an equilateral triangle with perimeter 39 inches.

Solution

13 in.

29) Find the length of each side of an equilateral triangle with perimeter 51 centimetres.

Solution

17 cm

Example 6.2.14

Arianna has 156 inches of beading to use as trim around a scarf. The scarf will be an isosceles triangle with a base of 60 inches. How long can she make the two equal sides?

Solution

Step 1: **Read the problem. Draw the figure and label it with the given information.**

P = 156 in

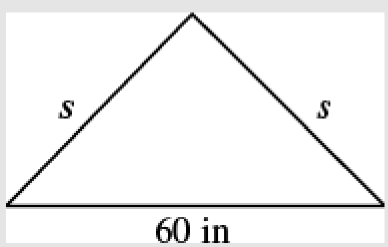


Figure 6.2.30

Step 2: **Identify what you are looking for.**

The lengths of the two equal sides.

Step 3: **Name. Choose a variable to represent it.**

Let s = the length of each side.

Step 4: **Translate.**

Write the appropriate formula and substitute in the given information.

$$\begin{aligned} P &= a + b + c \\ 156 &= s + 60 + s \end{aligned}$$

Step 5: **Solve the equation.**

$$\begin{aligned} 156 &= 2s + 60 \\ 90 &= 2s \\ 48 &= s \end{aligned}$$

Step 6: **Check.**

$$\begin{aligned} P &= a + b + c \\ 156 &\stackrel{?}{=} 48 + 60 + 48 \\ 156 &= 156 \checkmark \end{aligned}$$

Step 7: **Answer the question.**

Arianna can make each of the two equal sides 48 inches long.

Try It

30) A backyard deck is in the shape of an isosceles triangle with a base of 20 feet. The perimeter of the deck is 48 feet. How long is each of the equal sides of the deck?

Solution

14 ft

31) A boat's sail is an isosceles triangle with base of 8 meters. The perimeter is 22 meters. How long is each of the equal sides of the sail?

Solution

7 m

Use the Properties of Trapezoids

A **trapezoid** is four-sided figure, a *quadrilateral*, with two sides that are parallel and two sides that are not. The parallel sides are called the bases. We call the length of the smaller base b, and the length of the bigger base B. The height, h, of a trapezoid is he distance between the two bases as shown in Figure 6.2.31

A trapezoid has a larger base, B, and a smaller base, b. The height h is the distance between the bases.

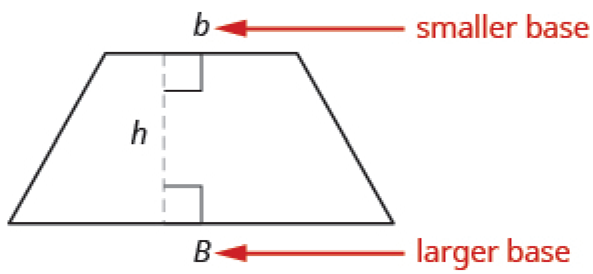


Figure 6.2.31

The formula for the area of a trapezoid is:

$$Area_{trapezoid} = \frac{1}{2}(b+B)$$

Splitting the trapezoid into two triangles may help us understand the formula. The area of the trapezoid is the sum of the areas of the two triangles. See Figure 6.2.32.

Splitting a trapezoid into two triangles may help you understand the formula for its area.

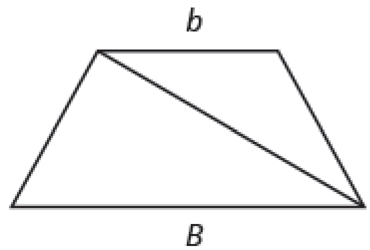


Figure 6.2.32

The height of the trapezoid is also the height of each of the two triangles. See Figure 6.2.33.

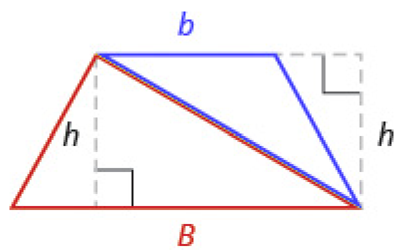


Figure 6.2.33

The formula for the area of a trapezoid is

$$\text{Area}_{trapezoid} = \frac{1}{2}bh + \frac{1}{2}Bh$$

$$\text{Area}_{trapezoid} = A_{blue\Delta} + A_{red\Delta}$$

Figure 6.2.34

If we distribute, we get,

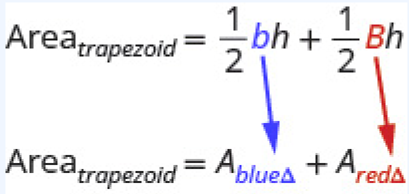


Figure 6.2.35

Properties of Trapezoids

- A trapezoid has four sides. See Figure 6.2.31
- Two of its sides are parallel and two sides are not.
- The area, A, of a trapezoid is $A = \frac{1}{2}h(b + B)$.

Example 6.2.15

Find the area of a trapezoid whose height is 6 inches and whose bases are 14 and 11 inches.

Solution

Step 1: **Read the problem. Draw the figure and label it with the given information.**

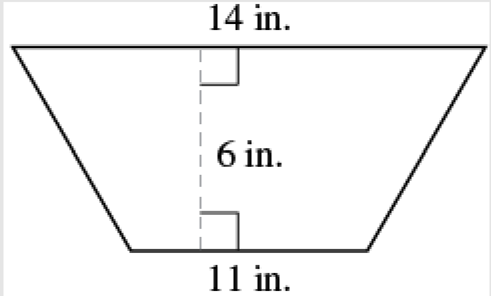


Figure 6.2.36

Step 2: **Identify what you are looking for.**

The area of the trapezoid.

Step 3: **Name. Choose a variable to represent it.**

Let $A =$the area.

Step 4: **Translate.**

Write the appropriate formula and substitute in the given information.

$$A = \frac{1}{2} \times b \times h$$
$$A = \frac{1}{2} \times 6 \times (11 + 14)$$

Step 5: **Solve the equation.**

$$1A = \frac{1}{2} \times 6(25)$$
$$A = 3(25)$$
$$A = 75 \; sq. \; inches$$

Step 6: **Check.**

Is this answer reasonable?

The area of the larger rectangle is 84 square inches and the area of the smaller rectangle is 66 square inches. So it makes sense that the area of the trapezoid is between 84 and 66 square inches

Step 7: **Answer the question.**

The area of the trapezoid is 75 square inches.

Try It

32) The height of a trapezoid is 14 yards and the bases are 7 and 16 yards. What is the area?

Solution

161 sq. yd

33) The height of a trapezoid is 18 centimetres and the bases are 17 and 8 centimetres. What is the area?

Solution

225 sq. cm

Example 6.2.16

Find the area of a trapezoid whose height is 5 feet and whose bases are 10.3 and 13.7 feet.

Solution

Step 1: **Read the problem. Draw the figure and label it with the given information.**

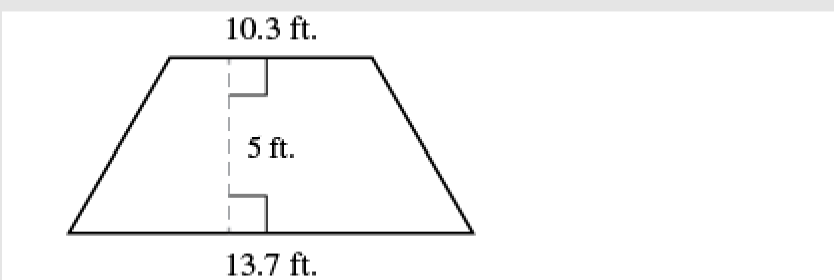


Figure 6.2.37

Step 2: **Identify what you are looking for.**

The area of the trapezoid.

Step 3: **Name. Choose a variable to represent it.**

Let $A=$ the area.

Step 4: **Translate.**

Write the appropriate formula and substitute in the given information.

$$A = \frac{1}{2} \times h \times (b+B)$$
$$A = \frac{1}{2} \times 5 \times (10.3 + 13.7)$$

Step 5: **Solve the equation.**

$$1A = \frac{1}{2} \times 5(24)$$
$$A = 12 \times 5$$
$$A = 60 \; sq. \; ft$$

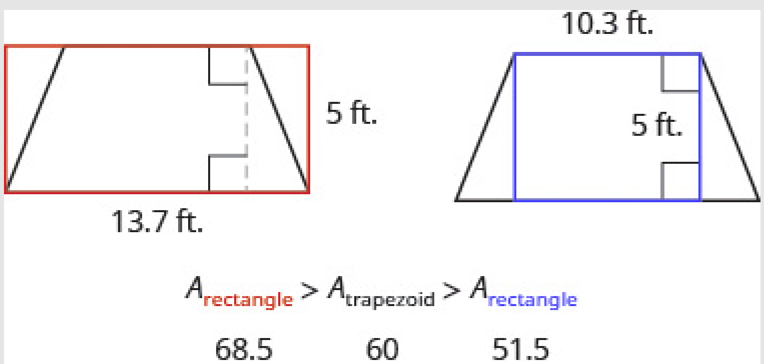


Figure 6.2.38

Step 6: **Check: Is this answer reasonable?**

The area of the trapezoid should be less than the area of a rectangle with base 13.7 and height 5, but more than the area of a rectangle with base 10.3 and height 5.

Step 7: **Answer the question.**

The area of the trapezoid is 60 square feet.

Try It

34) The height of a trapezoid is 7 centimetres and the bases are 4.6 and 7.4 centimetres. What is the area?

Solution

42 sq. cm

35) The height of a trapezoid is 9 meters and the bases are 6.2 and 7.8 meters. What is the area?

Solution

63 sq. m

Example 6.2.17

Vinny has a garden that is shaped like a trapezoid. The trapezoid has a height of 3.4 yards and the bases are 8.2 and 5.6 yards. How many square yards will be available to plant?

Solution

Step 1: **Read the problem. Draw the figure and label it with the given information.**

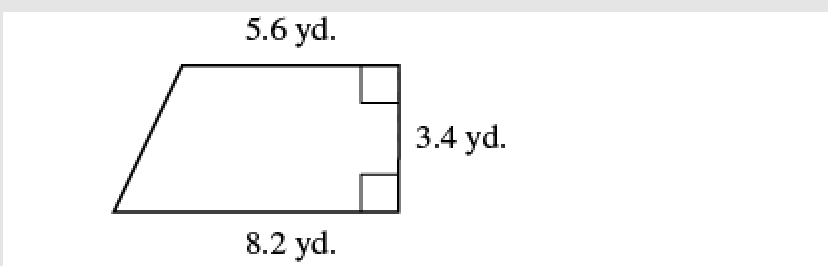


Figure 6.2.39

Step 2: **Identify what you are looking for.**

The area of a trapezoid.

Step 3: **Name. Choose a variable to represent it.**

Let $A =$ the area.

Step 4: **Translate.**

Write the appropriate formula and substitute in the given information.

$$A = \frac{1}{2} \times b \times (b + B)$$
$$A = \frac{1}{2} \times 3.4 \times (5.6 + 8.2)$$

Step 5: **Solve the equation.**

$$A = \frac{1}{2}(3.4)(13.8)$$
$$A = 23.46\ sq.\ yards$$

$A_{rgb]1.0,0.0,0.0\text{rectangle}} = Bh$ $= (8.2)(3.4)$ $= 27.88\ yd^2$	$A_{\text{trapezpoid}} = \frac{1}{2}(3.4\ yd)(5.6 = 8.2)$ $= 23.46\ yd^2$	$A_{rgb]0.0,0.0,1.0\text{rectangle}} = bh$ $= (5.6)(3.4)$ $= 19.04\ yd^2$
$A_{rgb]1.0,0.0,0.0\text{rectangle}}$ $>$ $rgb]1.0, 0.0, 0.027.88$	$A_{\text{trapezoid}}$ $>$ 23.46	$A_{rgb]0.0,0.0,1.0\text{rectangle}}$ $rgb]0.0, 0.0, 1.019.04$

Step 6: **Check: Is this answer reasonable?**

Yes. The area of the trapezoid is less than the area of a rectangle with a base of 8.2 yd and height 3.4 yd, but more than the area of a rectangle with base 5.6 yd and height 3.4 yd.

Step 7: **Answer the question.**

Vinny has 23.46 square yards in which he can plant.

Try It

36) Lin wants to sod his lawn, which is shaped like a trapezoid. The bases are 10.8 yards and 6.7 yards, and the height is 4.6 yards. How many square yards of sod does he need?

Solution

40.25 sq. yd

37) Kira wants cover his patio with concrete pavers. If the patio is shaped like a trapezoid whose bases are 18 feet and 14 feet and whose height is 15 feet, how many square feet of pavers will he need?

Solution

240 sq. ft

Additional resources listed. The Links to Literacy activity *Spaghetti and Meatballs for All* will provide you with another view of the topics covered in this section.

- Perimeter of a Rectangle
- Area of a Rectangle
- Perimeter and Area Formulas
- Area of a Triangle
- Area of a Triangle with Fractions
- Area of a Trapezoid

Key Concepts

- **Properties of Rectangles**
  - Rectangles have four sides and four right (90°) angles.
  - The lengths of opposite sides are equal.
  - The perimeter, P, of a rectangle is the sum of twice the length and twice the width.
    - $P = 2L + 2W$
  - The area, A, of a rectangle is the length times the width.
    - $A = L \times W$
- **Triangle Properties**
  - For any triangle $\triangle ABC$, the sum of the measures of the angles is 180°.
    - $m\angle A + m\angle B + m\angle C = 180^\circ$
  - The perimeter of a triangle is the sum of the lengths of the sides.
    - $P = a + b + c$
  - The area of a triangle is one-half the base, b, times the height, h.
    - $A = \frac{1}{2}bh$

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=6623#h5p-39

b. On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

Glossary

area
The area is a measure of the surface covered by a figure.

equilateral triangle
A triangle with all three sides of equal length is called an equilateral triangle.

isosceles triangle
A triangle with two sides of equal length is called an isosceles triangle.

perimeter
The perimeter is a measure of the distance around a figure.

rectangle
A rectangle is a geometric figure that has four sides and four right angles.

trapezoid
A trapezoid is four-sided figure, a quadrilateral, with two sides that are parallel and two sides that are not.

6.3 SOLVE GEOMETRY APPLICATIONS: CIRCLES AND IRREGULAR FIGURES

Learning Objectives

By the end of this section, you will be able to:

- Use the properties of circles
- Find the area of irregular figures

Try It

Before you get started, take this readiness quiz:

1) Evaluate x^2 when $x = 5$.
2) Using 3.14 for π, approximate the (a) circumference and (b) the area of a circle with radius 8 inches.
3) Simplify $\frac{22}{7}(0.25)^2$ and round to the nearest thousandth.

In this section, we'll continue working with geometry applications. We will add several new formulas to our collection of formulas. To help you as you do the examples and exercises in this section, we will show the Problem Solving Strategy for Geometry Applications here.

HOW TO

Problem Solving Strategy for Geometry Applications

1. Read the problem and make sure you understand all the words and ideas. Draw the figure and label it with the given information.
2. Identify what you are looking for.
3. Name what you are looking for. Choose a variable to represent that quantity.
4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. Solve the equation using good algebra techniques.
6. Check the answer in the problem and make sure it makes sense.
7. Answer the question with a complete sentence.

Use the Properties of Circles

Do you remember the properties of circles from Decimals and Fractions Together? We'll show them here again to refer to as we use them to solve applications.

Properties of Circles

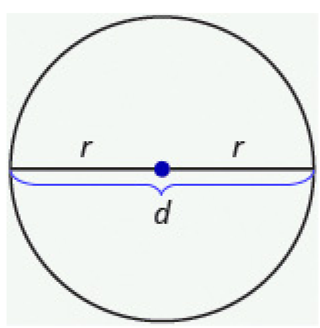


Figure 6.3.1

- r is the length of the radius
- d is the length of the diameter
- $d = 2r$

- Circumference is the perimeter of a circle. The formula for circumference is $C = 2\pi\mathrm{r}$
- The formula for area of a circle is $A = \pi r^2$

Remember, that we approximate π with 3.14 or $\frac{22}{7}$ depending on whether the radius of the circle is given as a decimal or a fraction. If you use the π key on your calculator to do the calculations in this section, your answers will be slightly different from the answers shown. That is because the π key uses more than two decimal places. On assessments, we will always give specific instructions on whether to use the approximations above or not.

Example 6.3.1

A circular sandbox has a radius of 2.5 feet. Find the:

a. circumference
b. area of the sandbox

Solution

a. ***Step 1:*** **Read the problem. Draw the figure and label it with the given information.**

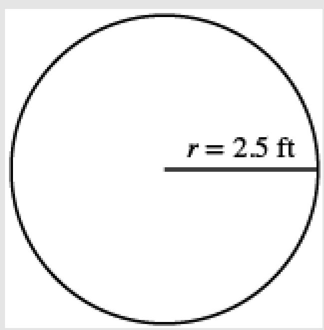


Figure 6.3.2

Step 2: **Identify what you are looking for.**

The circumference of the circle.

Step 3: **Name. Choose a variable to represent it.**

Let $c =$ the circumference of the circle.

Step 4: **Translate.**

Write the appropriate formula for the situation and substitute in the given information.

$$C = 2\pi r$$
$$\text{C} = 2\pi(2.5)$$

Step 5: **Solve the equation.**

$$C \approx 2(3.14)(2.5)$$
$$C \approx 15 ft$$

Step 6: **Check. Does this answer make sense?**

Yes. If we draw a square around the circle, its sides would be 5 ft (twice the radius), so its perimeter would be 20 ft. This is slightly more than the circle's circumference, 15.7 ft.

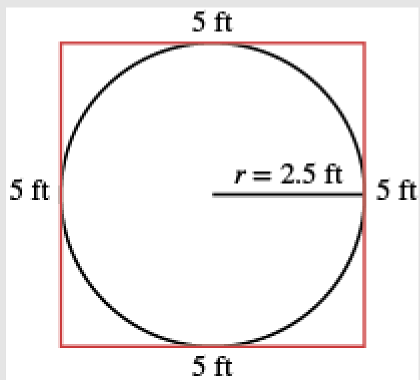


Figure 6.3.3

Step 7: **Answer the question.**

The circumference of the sandbox is 15.7 feet.

---

b. ***Step 1:*** **Read the problem. Draw the figure and label it with the given information**

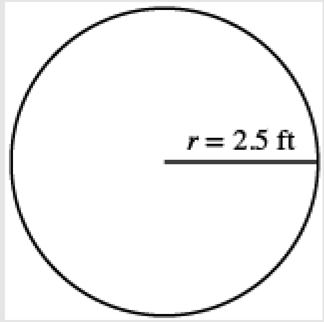


Figure 6.3.4

Step 2: **Identify what you are looking for.**

The area of the circle.

Step 3: **Name. Choose a variable to represent it.**

Let $A =$ the area of the circle.

Step 4: **Translate.**

Write the appropriate formula for the situation and substitute in the given information.

$$A = \pi \mathrm{r}^2$$
$$\mathrm{A} = \pi(2.5)^2$$

Step 5: **Solve the equation.**

$$A \approx (3.14)(2.5)^2$$
$$A \approx 19.625\ sq.\ ft$$

Step 6: **Check. Does this answer make sense?**

Yes. If we draw a square around the circle, its sides would be 5 ft, as shown in part a.. So the area of the square would be 25 sq. ft. This is slightly more than the circle's area, 19.625 sq. ft.

Step 7: **Answer the question.**

The area of the circle is 19.625 square feet.

Try It

4) A circular mirror has radius of 5 inches. Find the a. circumference and b. area of the mirror.

Solution

a. 31.4 in.
b. 78.5 sq. in.

Try It

5) A circular spa has radius of 4.5 feet. Find the a. circumference and b. area of the spa.

Solution

a. 28.26 ft
b. 63.585 sq. ft

We usually see the formula for circumference in terms of the radius r of the circle: $C = 2\pi\mathrm{r}$

But since the diameter of a circle is two times the radius, we could write the formula for the circumference in terms d.

Using the Communitive Property, we get: $$\begin{aligned} C &= 2\pi\mathrm{r} \\ \mathrm{C} &= \pi 2\mathrm{r} \end{aligned}$$

Then, substituting $d = 2r$: $C = \pi\mathrm{d}$

We will use this form of the circumference when we're given the length of the diameter instead of the radius.

Example 6.3.2

A circular table has a diameter of four feet. What is the circumference of the table?

Solution

Step 1: **Read the problem. Draw the figure and label it with the given information.**

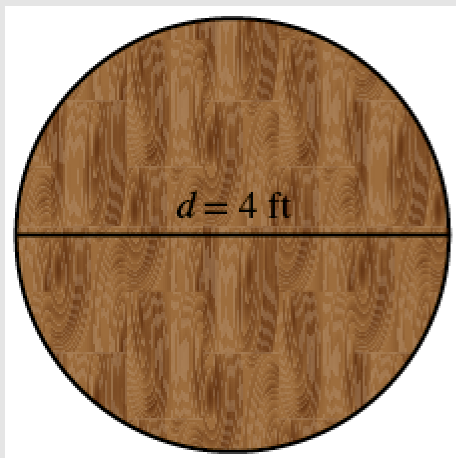


Figure 6.3.5

Step 2: **Identify what you are looking for.**

The circumference of the table.

Step 3: **Name. Choose a variable to represent it.**

Let $c =$ the circumference of the table.

Step 4: **Translate.**

Write the appropriate formula for the situation and substitute in the given information.

$$\begin{aligned} C &= \pi \text{d} \\ \text{C} &= \pi(4) \end{aligned}$$

Step 5: **Solve the equation, using 3.14 for π.**

$$\begin{aligned} C &\approx (3.14)(4) \\ C &\approx 12.56\ feet \end{aligned}$$

Step 6: **Check.**

If we put a square around the circle, its side would be 4. The perimeter would be 16. It makes sense that the circumference of the circle, 12.56, is a little less than 16.

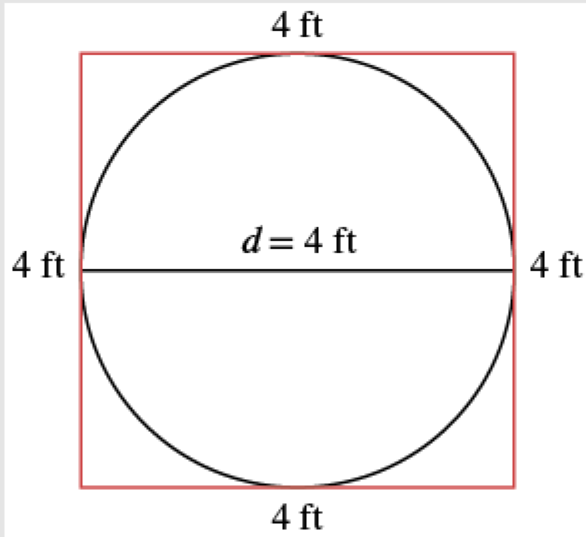


Figure 6.3.6

***Step 7:* Answer the question.**

The diameter of the table is 12.56 square feet.

Try It

6) Find the circumference of a circular fire pit whose diameter is 5.5 feet.

Solution

17.27 ft

Try It

7) If the diameter of a circular trampoline is 12 feet, what is its circumference?

Solution

37.68 ft

Example 6.3.3

Find the diameter of a circle with a circumference of 47.1 centimetres.

Solution

Step 1: **Read the problem. Draw the figure and label it with the given information.**

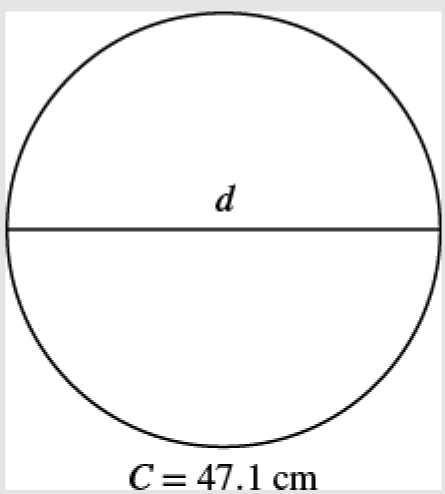


Figure 6.3.7

Step 2: **Identify what you are looking for.**

The diameter of the circle.

Step 3: **Name. Choose a variable to represent it.**

Let $d=$ the diameter of the circle.

Step 4: **Translate.**

Write the formula, using 3.14 for π.

$$\begin{aligned} C &= \pi \mathrm{d} \\ \mathrm{C} &= \pi(4) \end{aligned}$$

Step 5: **Solve the equation.**

$$\begin{aligned} C &\approx (3.14)(4) \\ C &\approx 12.56\ feet \end{aligned}$$

Step 6: **Check.**

$$\begin{aligned} C &= \pi \mathrm{d} \\ 47.1 &\stackrel{?}{=} (3.14)(15) \\ 47.1 &= 47.1 \end{aligned}$$

Step 7: **Answer the question.**

The diameter of the circle is approximately 15 centimetres.

Try It

8) Find the diameter of a circle with circumference of 94.2 centimetres.

Solution

30 cm

Try It

9) Find the diameter of a circle with circumference of 345.4 feet.

Solution

110 ft

Find the Area of Irregular Figures

So far, we have found area for rectangles, triangles, trapezoids, and circles. An **irregular figure** is a figure that is not a standard geometric shape. Its area cannot be calculated using any of the standard area formulas. But some irregular figures are made up of two or more standard geometric shapes. To find the area of one of these irregular figures, we can split it into figures whose formulas we know and then add the areas of the figures.

Example 6.3.4

Find the area of the shaded region.

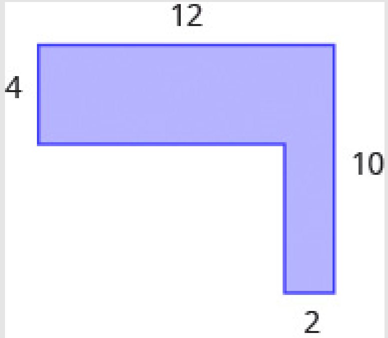


Figure 6.3.8

Solution

The given figure is irregular, but we can break it into two rectangles. The area of the shaded region will be the sum of the areas of both rectangles.

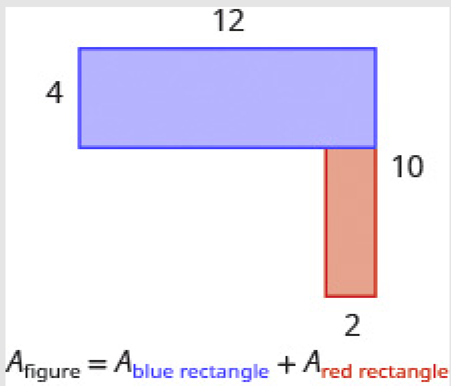


Figure 6.3.9

The blue rectangle has a width of 12 and a length of 4 The red rectangle has a width of 2, but its length is not labelled. The right side of the figure is the length of the red rectangle plus the length of the blue rectangle. Since the right side of the blue rectangle is 4 units long, the length of the red rectangle must be 6 units.

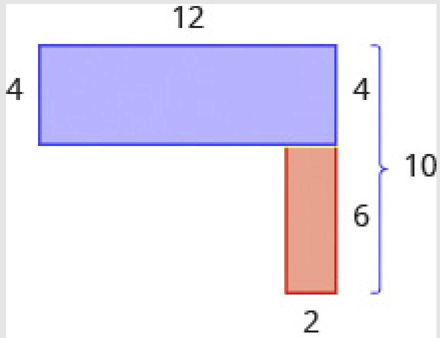


Figure 6.3.10

$$A_{figure} = A_{blue\ rectangle} + A_{red\ rectangle}$$
$$A_{figure} = bh + bh$$
$$A_{figure} = 12 \times 4 + 2 \times 6$$
$$A_{figure} = 48 + 12$$
$$A_{figure} = 60$$

The area of the figure is 60 square units.

Is there another way to split this figure into two rectangles? Try it, and make sure you get the same area.

Try It

10) Find the area of each shaded region:

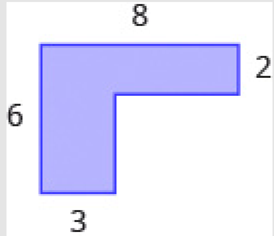


Figure 6.3.11

Solution

28 sq. units

Try It

11) Find the area of each shaded region:

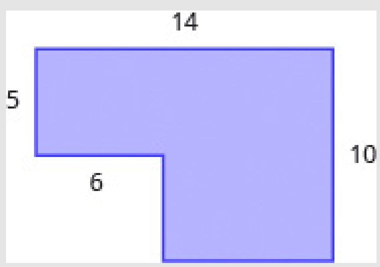


Figure 6.3.12

Solution

110 sq. units

Example 6.3.5

Find the area of the shaded region.

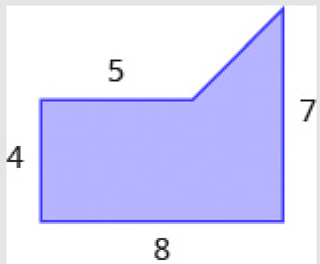


Figure 6.3.13

Solution

We can break this irregular figure into a triangle and rectangle. The area of the figure will be the sum of the areas of triangle and rectangle.

The rectangle has a length of 8 units and a width of 4 units.

We need to find the base and height of the triangle.

Since both sides of the rectangle are 4, the vertical side of the triangle is 3, which is $7-4$.

The length of the rectangle is 8, so the base of the triangle will be 3, which is $8-4$.

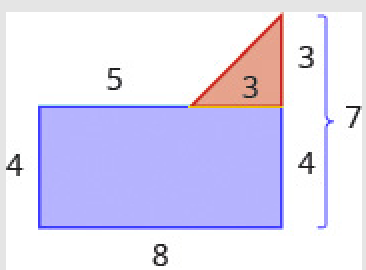


Figure 6.3.14

Now we can add the areas to find the area of the irregular figure.

$$A_{Figure} = A_{rectangle} + A_{triangle}$$
$$A_{Figure} = lw + \frac{1}{2}bh$$
$$A_{Figure} = 8 \times 4 + \frac{1}{2} \times 3 \times 3$$
$$A_{Figure} = 32 + 4.5$$
$$A_{Figure} = 36.4\ sq.\ units$$

The area of the figure is 36.5 square units.

Try It

12) Find the area of each shaded region.

Figure 6.3.15

Solution

36.5 sq. units

Try It

13) Find the area of each shaded region.

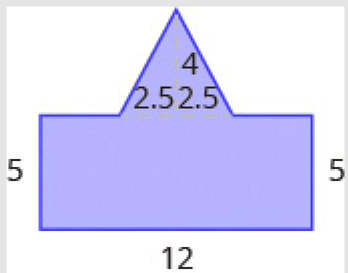


Figure 6.3.16

Solution

70 sq. units

Example 6.3.6

A high school track is shaped like a rectangle with a semi-circle (half a circle) on each end. The rectangle has length 105 meters and width 68 meters. Find the area enclosed by the track. Round your answer to the nearest hundredth.

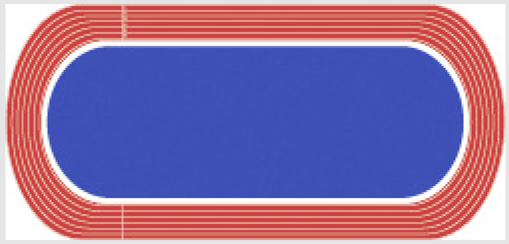


Figure 6.3.17

Solution

We will break the figure into a rectangle and two semi-circles. The area of the figure will be the sum of the areas of the rectangle and the semicircles.

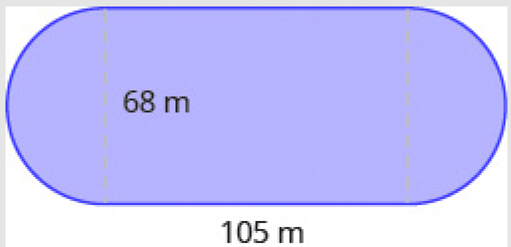


Figure 6.3.18

The rectangle has a length of 105m and a width of 68m. The semi-circles have a diameter of 68m, so each has a radius of 34m.

$$\begin{aligned} A_{Figure} &= A_{rectangle} + A_{semicircles} \\ A_{Figure} &= bh + 2(\frac{1}{2}\pi \times \text{r}^2) \\ A_{Figure} &\approx 105 \times 68 + \left(\frac{1}{2} \times 3.14 \times 34^2\right) \\ A_{Figure} &\approx 7140 + 3629.84 \\ A_{Figure} &\approx 10,769.84 \end{aligned}$$

The area enclosed is $10,769.84$ square meters.

Try It

14) Find the area:

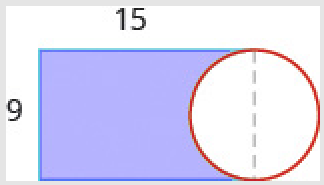


Figure 6.3.19

Solution

103.2 sq. units

Try It

15) Find the area:

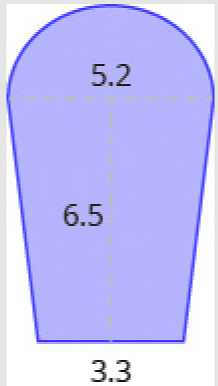


Figure 6.3.20

Solution

38.24 sq. units

Access Additional Online Resources

- Circumference of a Circle
- Area of a Circle
- Area of an L-shaped polygon
- Area of an L-shaped polygon with Decimals
- Perimeter Involving a Rectangle and Circle
- Area Involving a Rectangle and Circle

Key Concepts

- **Problem Solving Strategy for Geometry Applications**
  1. Read the problem and make sure you understand all the words and ideas. Draw the figure and label it with the given information.
  2. Identify what you are looking for.
  3. Name what you are looking for. Choose a variable to represent that quantity.
  4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
  5. Solve the equation using good algebra techniques.
  6. Check the answer in the problem and make sure it makes sense.
  7. Answer the question with a complete sentence.
- **Properties of Circles**

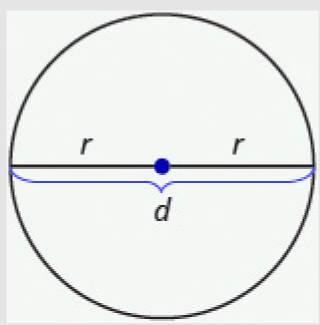


Figure 6.3.21

- $d = 2r$
- **Circumference:** $C = 2\pi \mathrm{r}$ or $C = \pi \mathrm{d}$
- **Area:** $\mathrm{A} = \pi \mathrm{r}^2$

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=6679#h5p-40

b. After looking at the checklist, do you think you are well prepared for the next section? Why or why not?

Glossary

irregular figure
An irregular figure is a figure that is not a standard geometric shape. Its area cannot be calculated using any of the standard area formulas.

6.4 SOLVE GEOMETRY APPLICATIONS: VOLUME AND SURFACE AREA

Learning Objectives

By the end of this section, you will be able to:

- Find volume and surface area of rectangular solids
- Find volume and surface area of spheres
- Find volume and surface area of cylinders
- Find volume of cones

Try It

Before you get started, take this readiness quiz:

1) Evaluate x^3 when $x = 5$.
2) Evaluate 2^x when $x = 5$.
3) Find the area of a circle with radius $\frac{7}{2}$.

In this section, we will finish our study of geometry applications. We find the volume and surface area of some three-dimensional figures. Since we will be solving applications, we will once again show our Problem-Solving Strategy for Geometry Applications.

HOW TO

Problem Solving Strategy for Geometry Applications

1. Read the problem and make sure you understand all the words and ideas. Draw the figure and label it with the given information.
2. Identify what you are looking for.
3. Name what you are looking for. Choose a variable to represent that quantity.
4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. Solve the equation using good algebra techniques.
6. Check the answer in the problem and make sure it makes sense.
7. Answer the question with a complete sentence.

Find Volume and Surface Area of Rectangular Solids

A cheerleading coach is having the squad paint wooden crates with the school colours to stand on at the games. (See Figure 6.4.1.). The amount of paint needed to cover the outside of each box is the surface area, a square measure of the total area of all the sides. The amount of space inside the crate is the volume, a cubic measure.

This wooden crate is in the shape of a rectangular solid. (Figure 6.4.1)

Each crate is in the shape of a rectangular solid. Its dimensions are the length, width, and height. The rectangular solid shown in (Figure 6.4.1) has length 4 units, width 2 units, and height 3 units. Can you tell how many cubic units there are altogether? Let's look layer by layer.

Breaking a rectangular solid into layers makes it easier to visualize the number of cubic units it contains. This 4 by 2 by 3 rectangular solid has 24 cubic units.

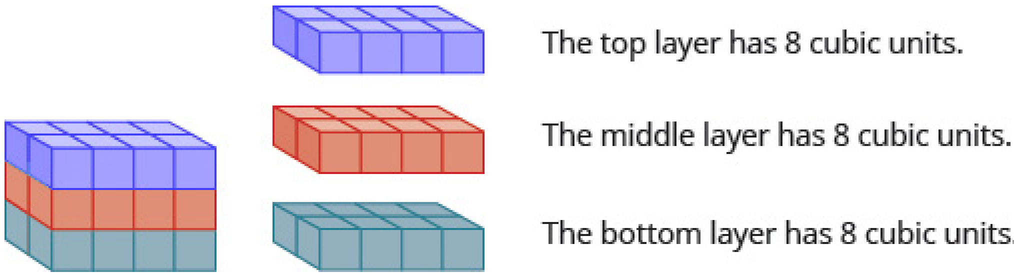


Figure 6.4.1

Altogether there are 24 cubic units. Notice that 24 is the $length \times width \times height$.

$$\begin{aligned} Volume &= L \times W \times H \\ 24 &= 4 \times 2 \times 3 \end{aligned}$$

The volume, V, of any rectangular solid is the product of the length, width, and height.

$$V = LWH$$

We could also write the formula for volume of a rectangular solid in terms of the area of the base. The area of the base, B, is equal to $length \times width$.

$$B = LW$$

We can substitute B for LW in the volume formula to get another form of the volume formula.

$$\begin{aligned} V &= L \times W \times H \\ V &= (L \times W) \times H \\ V &= Bh \end{aligned}$$

We now have another version of the volume formula for rectangular solids. Let's see how this works with the $4 \times 2 \times 3$ rectangular solid we started with. See Figure 6.4.2.

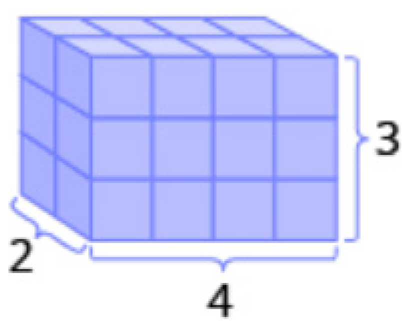


Figure 6.4.2

To find the *surface area* of a rectangular solid, think about finding the area of each of its faces. How many faces does the rectangular solid above have? You can see three of them.

$$\begin{array}{lll} A_{front} = LW & A_{side} = LW & A_{top} = LW \\ A_{front} = 4 \times 3 & A_{side} = 2 \times 3 & A_{top} = 4 \times 2 \\ A_{front} = 12 & A_{side} = 6 & A_{top} = 8 \end{array}$$

Notice for each of the three faces you see, there is an identical opposite face that does not show.

$$\begin{aligned} S &= (front + back) + (left\ side + right\ side) + (top + bottom) \\ S &= 2(front) + 2(left\ side) + 2(top) \\ S &= 2(12) + 2(6) + 2(8) \\ S &= 24 + 12 + 16 \\ S &= 52\ sq.\ units \end{aligned}$$

The surface area S of the rectangular solid shown in Figure 6.4.2 is 52 square units.

In general, to find the surface area of a rectangular solid, remember that each face is a rectangle, so its area is the product of its length and its width (see Figure 6.4.3). Find the area of each face that you see and then multiply each area by two to account for the face on the opposite side.

$$S = 2LH + 2LW + 2WH$$

For each face of the rectangular solid facing you, there is another face on the opposite side. There are 6 faces in all.

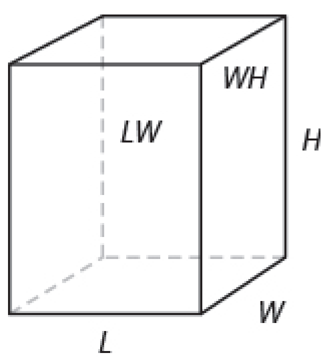


Figure 6.4.3

Volume and Surface Area of a Rectangular Solid

For a rectangular solid with length L, width W, and height H:

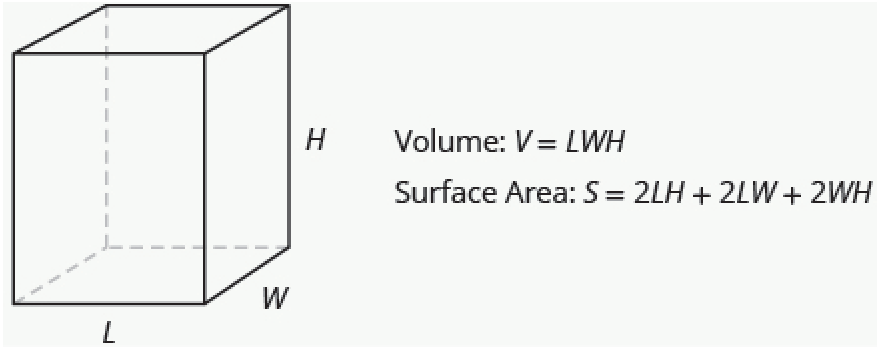


Figure 6.4.4

Example 6.4.1

For a rectangular solid with length 14 cm, height 17 cm, and width 9 cm, find:

a. volume
b. surface area

Solution

Step 1 is the same for both a. and b., so we will show it just once.

a. ***Step 1:*** **Read the problem. Draw the figure and label it with the given information.**

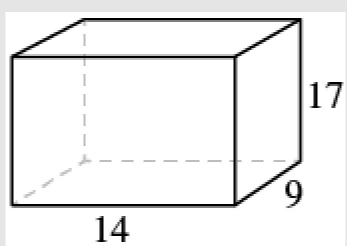


Figure 6.4.5

Step 2: **Identify what you are looking for.**

The volume of the rectangular solid.

Step 3: **Name. Choose a variable to represent it.**

Let V = volume.

Step 4: **Translate.**

Write the appropriate formula for the situation and substitute in the given information.

$$V = LWH$$
$$V = 14 \times 9 \times 17$$

Step 5: **Solve the equation.**

$$V = 2,142$$

Step 6: **Check. Does this answer make sense?**

We leave it to you to check your calculations.

Step 7: **Answer the question.**

The volume is $2,142$ cubic centimetres.

---

b. ***Step 2:*** **Identify what you are looking for.**

The surface area of the solid

Step 3: **Name. Choose a variable to represent it.**

Let S = surface area.

Step 4: **Translate.**

Write the appropriate formula for the situation and substitute in the given information.

$$S = 2LH + 2LW + 2WH$$
$$S = 2(14 \times 17) + 2(14 \times 9) + 2(9 \times 17)$$

Step 5: **Solve the equation.**

$$S = 1,034$$

Step 6: **Check. Does this answer make sense?**

Double-check with a calculator.

Step 7: **Answer the question.**

The surface area is $1,034$ square centimetres.

Try It

4) Find the a. volume and b. surface area of rectangular solid with the: length 8 feet, width 9 feet, and height 11 feet.

Solution

a. 792 cu. ft

b. 518 sq. ft

Try It

5) Find the a. volume and b. surface area of rectangular solid with the: length 15 feet, width 12 feet, and height 8 feet.

Solution

a. $1,440$ cu. ft

b. 792 sq. ft

Example 6.4.2

A rectangular crate has a length of 30 inches, width of 25 inches, and height of 20 inches. Find:

a. volume
b. surface area

Solution

Step 1 is the same for both a. and b., so we will show it just once.

a. ***Step 1:*** **Read the problem. Draw the figure and label it with the given information.**

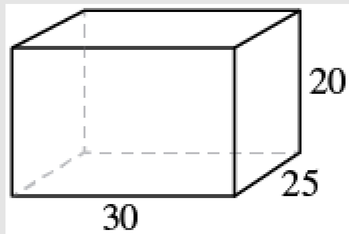


Figure 6.4.6

Step 2: **Identify what you are looking for.**

The volume of the crate.

Step 3: **Name. Choose a variable to represent it.**

Let V = volume.

Step 4: **Translate.**

Write the appropriate formula for the situation and substitute in the given information.

$$V = LWH$$
$$V = 30 \times 25 \times 20$$

Step 5: **Solve the equation.**

$$V = 15,000$$

Step 6: **Check. Does this answer make sense?**

Double check your math.

Step 7: **Answer the question.**

The volume is $15,000$ cubic inches.

---

b. ***Step 2:*** **Identify what you are looking for.**

The surface area of the crate.

Step 3: **Name. Choose a variable to represent it.**

Let $S =$ surface area.

Step 4: **Translate.**

Write the appropriate formula for the situation and substitute in the given information.

$$S = 2LH + 2LW + 2WH$$
$$S = 2(30 \times 20) + 2(30 \times 25) + 2(25 \times 20)$$

Step 5: **Solve the equation.**

$$S = 3,700$$

Step 6: **Check. Does this answer make sense?**

Check it yourself!

Step 7: **Answer the question.**

The surface area is $3,700$ square inches.

Try It

6) A rectangular box has length 9 feet, width 4 feet, and height 6 feet. Find its a. volume and b. surface area.

Solution

a. 216 cu. ft
b. 228 sq. ft

Try It

7) A rectangular suitcase has length 22 inches, width 14 inches, and height 9 inches. Find its a. volume and b. surface area.

Solution

a. $2,772$ cu. in.

b. $1,264$ sq. in.

Volume and Surface Area of a Cube

A **cube** is a rectangular solid whose length, width, and height are equal. See Volume and Surface Area of a Cube, below. Substituting, s for the length, width and height into the formulas for volume and surface area of a rectangular solid, we get:

$$V = LWH$$

$$S = 2LH + 2LW + 2WH$$

$$V = sss$$

$$S = 2ss + 2ss + 2ss$$

$$V = s^3$$

$$S = 2s^2 + 2s^2 + 2s^2$$

$$S = 6s^2$$

So for a cube, the formulas for volume and surface area are $V = s^3$ and $S = 6s^2$.

Volume and Surface Area of a Cube

For any cube with sides of length s,

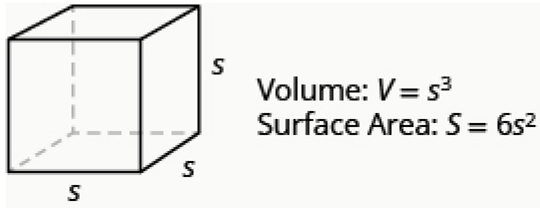


Figure 6.4.7

Example 6.4.3

A cube is 2.5 inches on each side. Find:

a. volume

b. surface area

Solution

Step 1 is the same for both a. and b., so we will show it just once.

a. ***Step 1:*** **Read the problem. Draw the figure and label it with the given information.**

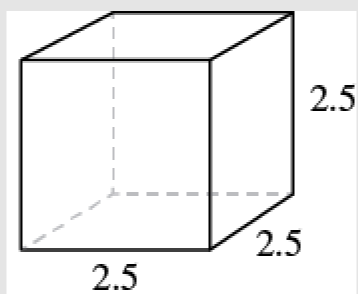


Figure 6.4.8

Step 2: **Identify what you are looking for.**

The volume of the cube.

Step 3: **Name. Choose a variable to represent it.**

Let V = volume.

Step 4: **Translate.**

Write the appropriate formula for the situation.

$$V = s^3$$

Step 5: **Solve. Substitute and Solve.**

$$\begin{aligned} V &= (2.5)^3 \\ V &= 15.625 \end{aligned}$$

Step 6: **Check — Check your work.**

Step 7: **Answer the question.**

The volume is 15.625 cubic inches.

---

b. ***Step 2:*** **Identify what you are looking for.**

The surface area of the cube.

Step 3: **Name. Choose a variable to represent it.**

Let S = surface area.

Step 4: **Translate.**

Write the appropriate formula for the situation.

$$S = 6s^2$$

Step 5: **Solve and substitute into the equation.**

$$\begin{aligned} S &= 6 \times (2.5)^2 \\ S &= 37.5 \end{aligned}$$

Step 6: **Check. Does this answer make sense?**

Check is left to you.

Step 7: **Answer the question.**

The surface area is 37.5 square inches.

Try It

8) For a cube with side 4.5 meters, find:

a. volume
b. surface area of the cube

Solution

a. 91.125 cu. m
b. 121.5 sq. m

Try It

9) For a cube with side 7.3 yards, find:

a. volume
b. surface area of the cube

Solution

a. 389.017 cu. yd.
b. 319.74 sq. yd.

Example 6.4.4

A notepad cube measures 2 inches on each side. Find:

a. volume

b. surface area

Solution

a. ***Step 1:*** **Read the problem. Draw the figure and label it with the given information.**

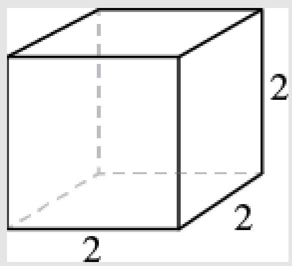


Figure 6.4.9

Step 2: **Identify what you are looking for.**

The volume of the cube.

Step 3: **Name. Choose a variable to represent it.**

Let V = volume.

Step 4: **Translate.**

Write the appropriate formula for the situation.

$$V = s^3$$

Step 5: **Solve. Substitute and Solve.**

$$V = 2^3$$
$$V = 8$$

Step 6: **Check**

Check that you did the calculations correctly.

Step 7: **Answer the question.**

The volume is 8 cubic inches.

---

b. ***Step 2:*** **Identify what you are looking for.**

The surface area of the cube

Step 3: **Name. Choose a variable to represent it.**

Let $S =$ surface area.

Step 4: **Translate.**

Write the appropriate formula for the situation.

$$S = 6s^2$$

Step 5: **Solve and substitute into the equation.**

$$\begin{aligned} S &= 6 \times (2)^2 \\ S &= 24 \end{aligned}$$

Step 6: **Check. Does this answer make sense?**

Check is left to you.

Step 7: **Answer the question.**

The surface area is 24 square inches.

Try It

10) A packing box is a cube measuring 4 feet on each side. Find:

a. volume

b. surface area.

Solution

a. 64 cu. ft

b. 96 sq. ft

Try It

11) A wall is made up of cube-shaped bricks. Each cube is 16 inches on each side. Find the a. volume and b. surface area of each cube.

Solution

a. $4,096$ cu. in.

b. 1536 sq. in.

Find the Volume and Surface Area of Spheres

A sphere is the shape of a basketball, like a three-dimensional circle. Just like a circle, the size of a sphere is determined by its radius, which is the distance from the centre of the sphere to any point on its surface. The formulas for the volume and surface area of a sphere are given below.

Showing where these formulas come from, like we did for a rectangular solid, is beyond the scope of this course. We will approximate π with 3.14.

Volume and Surface Area of a Sphere

For a sphere with radius r:

Figure 6.4.10

Example 6.4.5

A sphere has a radius 6 inches. Find:

a. volume

b. surface area

Solution

Step 1 is the same for both a. and b., so we will show it just once.

a. ***Step 1:*** **Read the problem. Draw the figure and label it with the given information.**

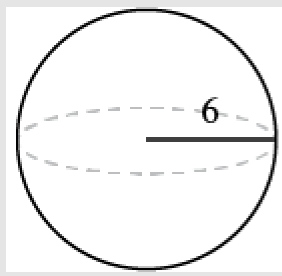


Figure 6.4.11

Step 2: **Identify what you are looking for.**

The volume of the sphere.

Step 3: **Name. Choose a variable to represent it.**

Let $V=$ volume.

Step 4: **Translate.**

Write the appropriate formula for the situation.

$$V = \frac{4}{3}\pi \mathrm{r}^3$$

Step 5: **Solve. Substitute and Solve.**

$$V \approx \frac{4}{3}(3.14)6^3$$
$$V \approx 904.32$$

Step 6: **Check. Double-check your math on a calculator.**

Step 7: **Answer the question.**

The volume is 904.32 cubic inches.

---

b. ***Step 2:*** **Identify what you are looking for.**

The surface area of the sphere.

Step 3: **Name. Choose a variable to represent it.**

Let $S =$ surface area.

Step 4: **Translate.**

Write the appropriate formula for the situation.

$$S = 4\pi \mathrm{r}^2$$

Step 5: **Solve and substitute into the equation.**

$$S \approx 4(3.14)6^2$$
$$S \approx 452.16$$

Step 6: **Check math on a calculator.**

Step 7: **Answer the question.**

The surface area is 452.16 square inches.

Try It

12) Find the a. volume and b. surface area of a sphere with radius 3 centimetres.

Solution

a. 113.04 cu. cm
b. 113.04 sq. cm

Try It

13) Find the a. volume and b. surface area of each sphere with a radius of 1 foot.

Solution

a. 4.19 cu. ft

b. 12.56 sq. ft

Example 6.4.6

A globe of Earth is in the shape of a sphere with radius 14 centimetres. Find its a. volume and b. surface area. Round the answer to the nearest hundredth.

Solution

a. ***Step 1:*** **Read the problem. Draw the figure and label it with the given information.**

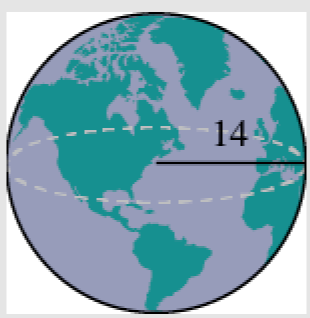


Figure 6.4.12

Step 2: **Identify what you are looking for.**

The volume of the sphere.

Step 3: **Name. Choose a variable to represent it.**

Let $V =$ volume.

Step 4: **Translate.**

Write the appropriate formula for the situation.

$$V = \frac{4}{3}\pi\text{r}^3$$

Step 5: **Solve. Substitute and Solve.**

$$\begin{aligned} V &\approx \frac{4}{3}(3.14)14^3 \\ V &\approx 11,488.21 \end{aligned}$$

Step 6: **Check. Double-check your math on a calculator.**

Step 7: **Answer the question.**

The volume is $11,488.21$ cubic inches.

---

b. ***Step 2:*** **Identify what you are looking for.**

The surface area of the sphere.

Step 3: **Name. Choose a variable to represent it.**

Let $S =$ surface area.

Step 4: **Translate.**

Write the appropriate formula for the situation.

$$S = 4\pi\text{r}^2$$

Step 5: **Solve and substitute into the equation.**

$$\begin{aligned} S &\approx 4(3.14)14^2 \\ S &\approx 2461.76 \end{aligned}$$

Step 6: **Check math on a calculator.**

Step 7: **Answer the question.**

The surface area is 2461.76 square inches.

Try It

14) A beach ball is in the shape of a sphere with radius of 9 inches. Find:

a. volume
b. surface area

Solution

a. 3052.08 cu. in.
b. 1017.36 sq. in.

Try It

15) A Roman statue depicts Atlas holding a globe with radius of 1.5 feet. Find:

a. volume
b. surface area of the globe

Solution

a. 14.13 cu. ft
b. 28.26 sq. ft

Find the Volume and Surface Area of a Cylinder

If you have ever seen a can of soda, you know what a **cylinder** looks like. A cylinder is a solid figure with two parallel circles of the same size at the top and bottom. The top and bottom of a cylinder are called the bases. The height h of a cylinder is the distance between the two bases. For all the cylinders we will work with here, the sides and the height, h , will be perpendicular to the bases.

A cylinder has two circular bases of equal size. The height is the distance between the bases.

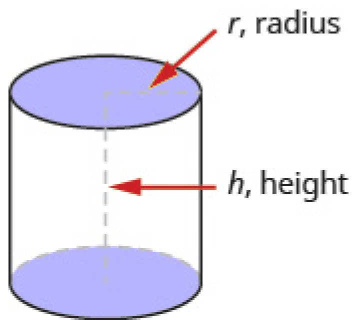


Figure 6.4.13

Rectangular solids and cylinders are somewhat similar because they both have two bases and a height. The formula for the volume of a rectangular solid, $V = Bh$, can also be used to find the volume of a cylinder.

For the rectangular solid, the area of the base, B, is the area of the rectangular base, $length \times width$. For a cylinder, the area of the base, B, is the area of its circular base, πr^2. Figure 6.4.14 compares how the formula $V = Bh$ is used for rectangular solids and cylinders.

Seeing how a cylinder is similar to a rectangular solid may make it easier to understand the formula for the volume of a cylinder.

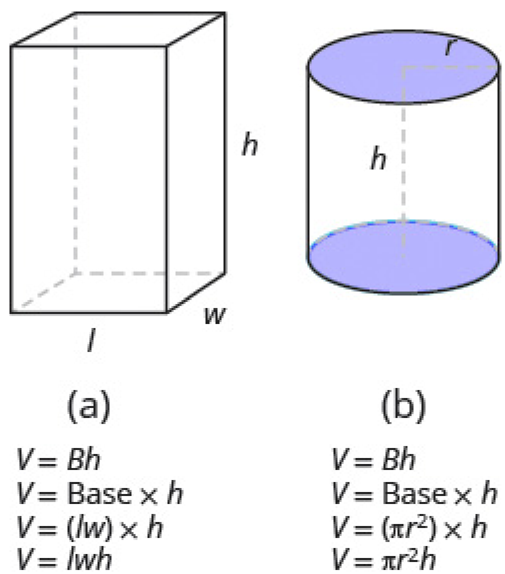


Figure 6.4.14

To understand the formula for the surface area of a cylinder, think of a can of vegetables. It has three surfaces: the top, the bottom, and the piece that forms the sides of the can. If you carefully cut the label off the side of the can and unroll it, you will see that it is a rectangle. See Figure 6.4.15.

By cutting and unrolling the label of a can of vegetables, we can see that the surface of a cylinder is a rectangle. The length of the rectangle is the circumference of the cylinder's base, and the width is the height of the cylinder.

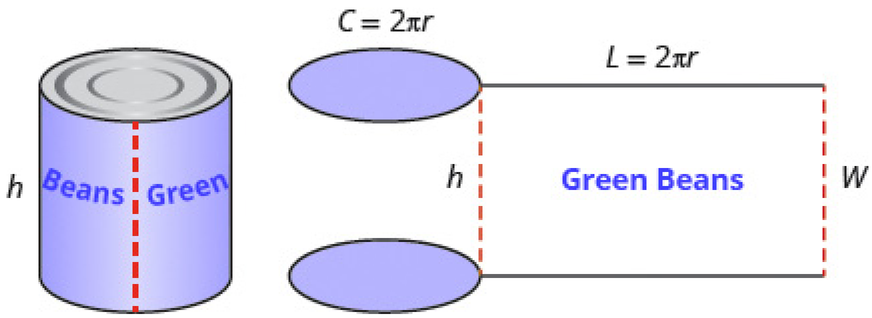
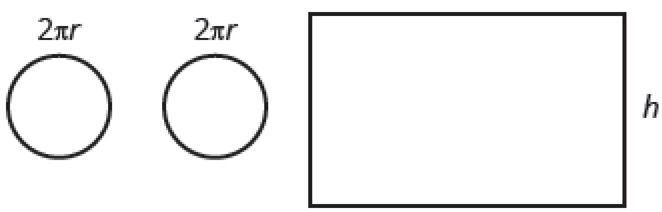


Figure 6.4.15

The distance around the edge of the can is the circumference of the cylinder's base it is also the length L of the rectangular label. The height of the cylinder is the width W of the rectangular label. So the area of the label can be represented as:

$$A = L \times W$$
$$A = (2\pi \mathrm{r}) \times (h)$$

To find the total surface area of the cylinder, we add the areas of the two circles to the area of the rectangle.



$$S = A_{\text{top circle}} + A_{\text{bottom circle}} + A_{\text{rectangle}}$$
$$S = \underbrace{\pi r^2 + \pi r^2} + 2\pi r \cdot h$$
$$S = 2 \cdot \pi r^2 + 2\pi rh$$
$$S = 2\pi r^2 + 2\pi rh$$

Figure 6.4.16

The surface area of a cylinder with radius r and height h, is:

$$S = 2\pi \mathrm{r}^2 + 2\pi \mathrm{rh}$$

Volume and Surface Area of a Cylinder

For a cylinder with radius r and height h:

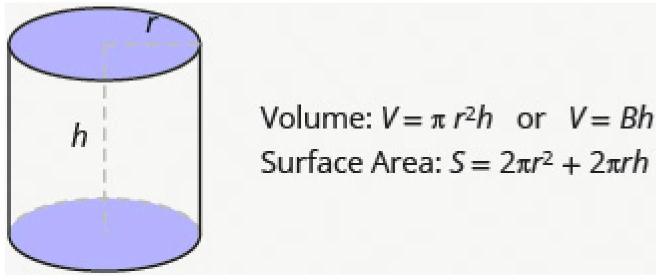


Figure 6.4.17

Example 6.4.7

A cylinder has height 5 centimetres and radius 3 centimetres. Find the a. volume and b. surface area.

Solution

a. ***Step 1:* Read the problem. Draw the figure and label it with the given information.**

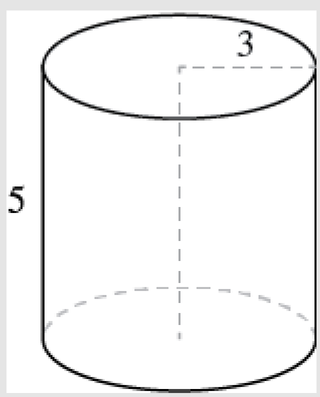


Figure 6.4.18

***Step 2:* Identify what you are looking for.**

The volume of the cylinder.

***Step 3:* Name. Choose a variable to represent it.**

Let $V =$ volume.

***Step 4:* Translate.**

Write the appropriate formula for the situation.
Use 3.14 for π.

$$V = \pi \mathrm{r}^2\mathrm{h}$$

Step 5: **Solve. Substitute and Solve.**

$$V \approx (3.14)3^2 \times 5$$
$$V \approx 141.3$$

Step 6: **Check. Double-check your math on a calculator.**

Step 7: **Answer the question.**

The volume is 141.3 cubic inches.

---

b. ***Step 2:*** **Identify what you are looking for.**

The surface area of the cylinder.

Step 3: **Name. Choose a variable to represent it.**

Let $S =$ surface area.

Step 4: **Translate.**

Write the appropriate formula for the situation.
Use 3.14 for π.

$$S = 2\pi \mathrm{r}^2 + 2\pi \mathrm{rh}$$

Step 5: **Solve and substitute into the equation.**

$$S \approx 2(3.14)3^2 + 2(3.14)(3)5$$
$$S \approx 150.72$$

Step 6: **Check math on a calculator.**

Step 7: **Answer the question.**

The surface area is approximately 150.72 square inches.

Try It

16) Find the a. volume and b. surface area of the cylinder with radius 4 cm and height 7 cm.

Solution

a. 351.68 cu. cm
b. 276.32 sq. cm

Try It

17) Find the a. volume and b. surface area of the cylinder with given radius 2 ft and height 8 ft.

Solution

a. 100.48 cu. ft
b. 125.6 sq. ft

Example 6.4.8

Find the a. volume and b. surface area of a can of soda. The radius of the base is 4 centimetres and the height is 13 centimetres. Assume the can is shaped exactly like a cylinder.

Solution

a. ***Step 1:*** **Read the problem. Draw the figure and label it with the given information.**

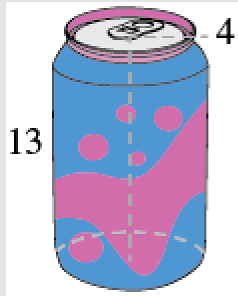


Figure 6.4.19

Step 2: **Identify what you are looking for.**

The volume of the cylinder.

Step 3: **Name. Choose a variable to represent it.**

Let $V =$ volume.

Step 4: **Translate.**

Write the appropriate formula for the situation.
Use 3.14 for π.

$$V = \pi r^2 h$$

Step 5: **Solve. Substitute and Solve.**

$$V \approx (3.14)4^2 \times 13$$
$$V \approx 653.12$$

Step 6: **Check. Double-check your math on a calculator.**

Step 7: **Answer the question.**

The volume is approximately 653.12 cubic inches.

---

b. ***Step 2:*** **Identify what you are looking for.**

The surface area of the cylinder.

Step 3: **Name. Choose a variable to represent it.**

Let $S =$ surface area.

Step 4: **Translate.**

Write the appropriate formula for the situation.

Use 3.14 for π.

$$S = 2\pi\mathrm{r}^2 + 2\pi\mathrm{rh}$$

Step 5: **Solve and substitute into the equation.**

$$S \approx 2(3.14)4^2 + 2(3.14)(4)13$$
$$S \approx 427.04$$

Step 6: **Check math on a calculator.**

Step 7: **Answer the question.**

The surface area is approximately 427.04 square inches.

Try It

18) Find the a. volume and b. surface area of a can of paint with radius 8 centimetres and height 19 centimetres. Assume the can is shaped exactly like a cylinder.

Solution

a. $3,818.24$ cu. cm

b. $1,356.48$ sq. cm

Try It

19) Find the a. volume and b. surface area of a cylindrical drum with radius 2.7 feet and height 4 feet. Assume the drum is shaped exactly like a cylinder.

Solution
a. 91.5624 cu. ft
b. 113.6052 sq. ft

Find the Volume of Cones

The first image that many of us have when we hear the word '**cone**' is an ice cream cone. There are many other applications of cones (but most are not as tasty as ice cream cones). In this section, we will see how to find the volume of a cone.

In geometry, a cone is a solid figure with one circular base and a vertex. The height of a cone is the distance between its base and the vertex. The cones that we will look at in this section will always have the height perpendicular to the base. See Figure 6.4.20. The height of a cone is the distance between its base and the vertex.

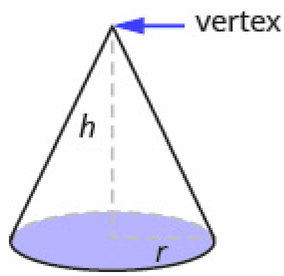


Figure 6.4.20

Earlier in this section, we saw that the volume of a cylinder is $V = \pi r^2 h$. We can think of a cone as part of a cylinder. Figure 6.4.21 shows a cone placed inside a cylinder with the same height and same base. If we compare the volume of the cone and the cylinder, we can see that the volume of the cone is less than that of the cylinder.

The volume of a cone is less than the volume of a cylinder with the same base and height.

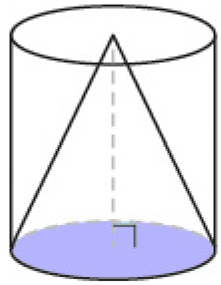


Figure 6.4.21

In fact, the volume of a cone is exactly one-third of the volume of a cylinder with the same base and height. The volume of a cone is:

$$V = \frac{1}{3}Bh$$

Since the base of a cone is a circle, we can substitute the formula of area of a circle, πr^2, for B to get the formula for volume of a cone.

$$V = \frac{1}{3}\pi r^2 h$$

In this book, we will only find the volume of a cone, and not its surface area.

Volume of a Cone

For a cone with radius r and height h.

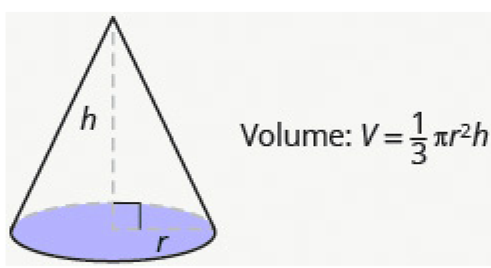


Figure 6.4.22

Example 6.4.9

Find the volume of a cone with height 6 inches and radius of its base 2 inches.

Solution

Step 1: **Read the problem. Draw the figure and label it with the given information.**

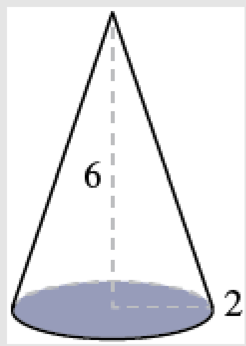


Figure 6.4.23

Step 2: **Identify what you are looking for.**

The volume of the cone.

Step 3: **Name. Choose a variable to represent it.**

Let $V =$ volume.

Step 4: **Translate.**

Write the appropriate formula for the situation.

Use 3.14 for π.

$$\mathrm{V} = \frac{1}{3}\pi \mathrm{r}^2 \mathrm{h}$$

Step 5: **Solve. Substitute and Solve.**

$$\mathrm{V} \approx \frac{1}{3} \times 3.14 \times (2)^2 \times (6)$$
$$\mathrm{V} \approx 25.12$$

Step 6: **Check. Double-check your math on a calculator.**

Step 7: **Answer the question.**

The volume is approximately 25.12 cubic inches.

Try It

20) Find the volume of a cone with height 7 inches and radius 3 inches

Solution

65.94 cu. in.

Try It

21) Find the volume of a cone with height 9 centimetres and radius 5 centimetres

Solution

235.5 cu. cm

Example 6.4.10

Marty's favourite gastro pub serves french fries in a paper wrap shaped like a cone. What is the volume of a conic wrap that is 8 inches tall and 5 inches in diameter? Round the answer to the nearest hundredth.

Solution

Step 1: **Read the problem. Draw the figure and label it with the given information.**

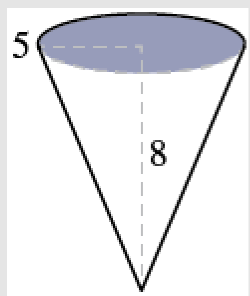


Figure 6.4.24

Step 2: **Identify what you are looking for.**

The volume of the cone.

Step 3: **Name. Choose a variable to represent it.**

Let V = volume.

Step 4: **Translate.**

Write the appropriate formula for the situation.
Use 3.14 for π.

$$\mathrm{V} = \frac{1}{3}\pi \mathrm{r}^2 \mathrm{h}$$

Step 5: **Solve. Substitute and Solve.**

$$\mathrm{V} \approx \frac{1}{3} \times 3.14 \times (2.5)^2 \times (8)$$
$$\mathrm{V} \approx 52.33$$

Step 6: **Check. Double-check your math on a calculator.**

***Step 7:* Answer the question.**

The volume of the wrap is approximately 52.33 cubic inches.

Try It

22) How many cubic inches of candy will fit in a cone-shaped piñata that is 18 inches long and 12 inches across its base? Round the answer to the nearest hundredth.

Solution

678.24 cu. in.

Try It

23) What is the volume of a cone-shaped party hat that is 10 inches tall and 7 inches across at the base? Round the answer to the nearest hundredth.

Solution

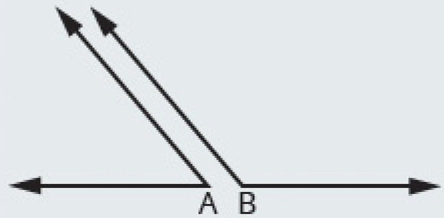
128.2 cu. in.

Summary of Geometry Formulas

The following charts summarize all of the formulas covered in this chapter.

Supplementary and Complementary Angles

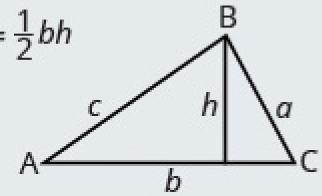
$m\angle A + m\angle B = 180°$ for supplementary angles A and B
$m\angle C + m\angle D = 90°$ for complementary angles C and D



Triangle

For ΔABC, angle measures.
$m\angle A + m\angle B + m\angle C = 180°$
Perimeter. $P = a + b + c$

$A = \frac{1}{2}bh$



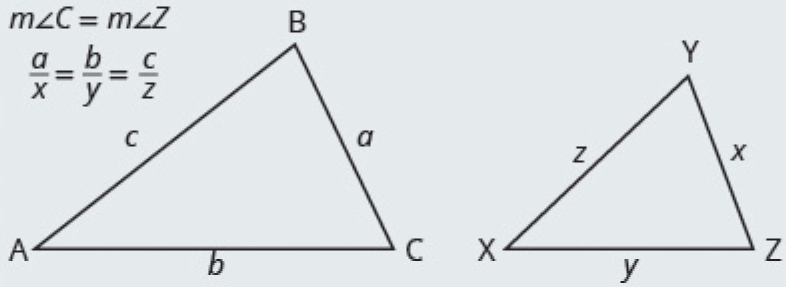
Similar Triangles

$m\angle A = m\angle X$

$m\angle B = m\angle Y$

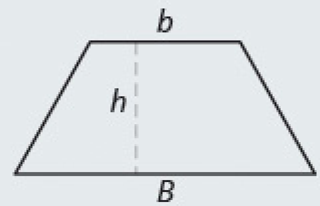
$m\angle C = m\angle Z$

$\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$



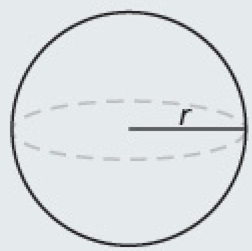
Trapezoid

Area. $A = \frac{1}{2}h(b + B)$



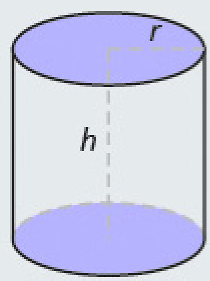
Sphere

Volume: $V = \frac{4}{3}\pi r^3$
Surface Area: $S = 4\pi r^2$



Cylinder

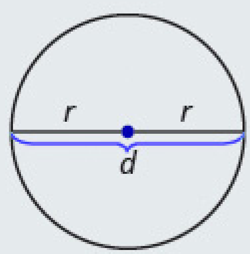
Volume: $V = \pi r^2 h$ or $V = Bh$
Surface Area: $S = 2\pi r^2 + 2\pi rh$



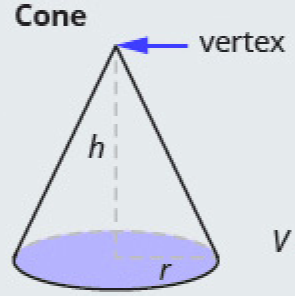
Circle

Circumference: $C = 2\pi r$
or
$C = \pi d$

Area: $A = \pi r^2$



Cone



$V = \frac{1}{3}\pi r^2 h$

Figure 6.4.25

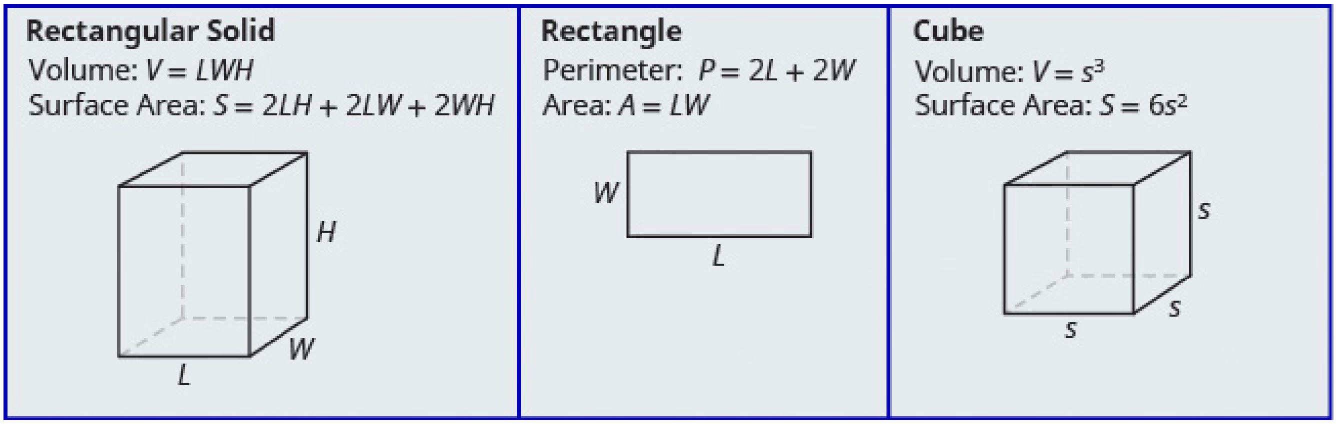


Figure 6.4.26

Access Additional Online Resources

- Volume of a Cone

Key Concepts

- **Volume and Surface Area of a Rectangular Solid**
  - $V = LWH$
  - $S = 2LH + 2LW + 2WH$
- **Volume and Surface Area of a Cube**
  - $V = s^3$
  - $S = 6s^2$
- **Volume and Surface Area of a Sphere**

    - $V = \frac{4}{3}\pi r^3$
    - $S = 4\pi r^2$
- **Volume and Surface Area of a Cylinder**
    - $V = \pi r^2 h$
    - $S = 2\pi r^2 2\pi rh$
- **Volume of a Cone**
    - For a cone with radius r and height h:
      Volume: $V = \frac{1}{3}\pi r^2 h$

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=6714#h5p-41

b. After reviewing this checklist, what will you do to become confident for all objectives?

Glossary

cone

A cone is a solid figure with one circular base and a vertex.

cube

A cube is a rectangular solid whose length, width, and height are equal.

cylinder

A cylinder is a solid figure with two parallel circles of the same size at the top and bottom.

6.5 SINE, COSINE AND TANGENT RATIOS AND APPLICATIONS OF TRIGONOMETRY

Learning Objectives

By the end of this section it is expected that you will be able to

- Find missing side of a right triangle using sine, cosine, or tangent ratios
- Find missing angle of a right triangle using sine, cosine, or tangent ratios
- Solve applications using right angle trigonometry

Now, that we know the fundamentals of algebra and geometry associated with a right triangle, we can start exploring trigonometry. Many real life problems can be represented and solved using right angle trigonometry.

Sine, Cosine, and Tangent Ratios

We know that any right triangle has three sides and a right angle. The side opposite to the right angle is called the hypotenuse. The other two angles in a right triangle are acute angles (with a measure less than 90 degrees). One of those angles we call reference angle and we use θ (theta) to represent it.

The hypotenuse is always the longest side of a right triangle. The other two sides are called opposite side and adjacent side. The names of those sides depends on which of the two acute angles is being used as a reference angle.

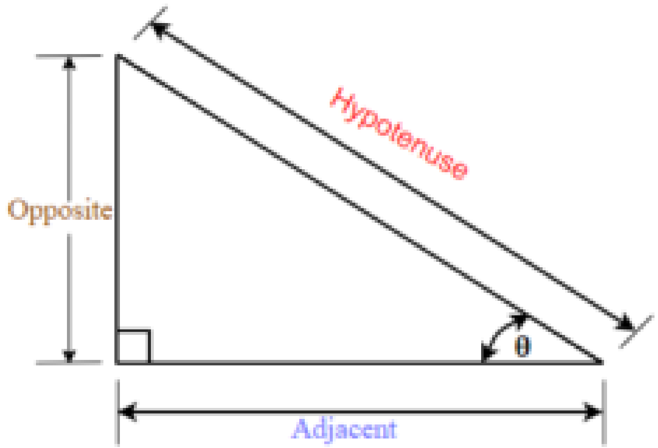


Figure 6.5.1

In the right triangle each side is labelled with a lowercase letter to match the uppercase letter of the opposite vertex.

Example 6.5.1

Label the sides of the triangle and find the hypotenuse, opposite, and adjacent.

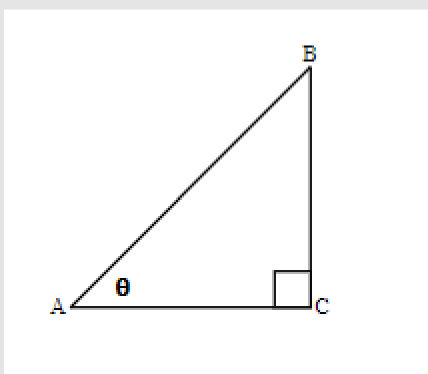


Figure 6.5.2

Solution

We labelled the sides with a lowercase letter to match the uppercase letter of the opposite vertex.

c is hypotenuse
a is opposite
b is adjacent

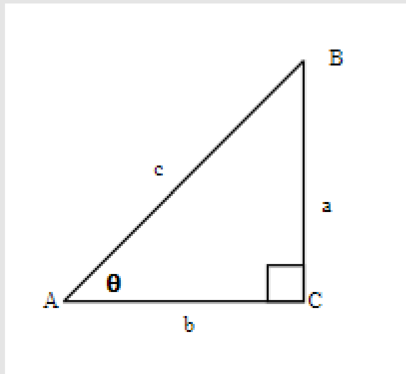


Figure 6.5.3

Try It

1) Label the sides of the triangle and find the hypotenuse, opposite and adjacent.

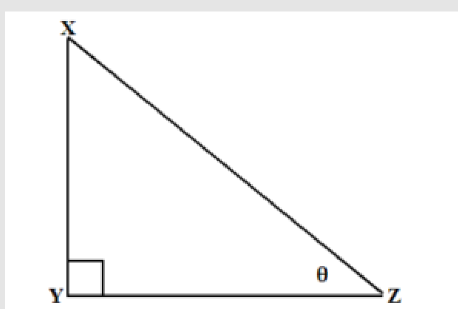


Figure 6.5.4

Solution

y is hypotenuse
z is opposite
x is adjacent

Trigonometric Ratios

Trigonometric ratios are the ratios of the sides in the right triangle. For any right triangle we can define three basic trigonometric ratios: sine, cosine, and tangent.

Let us refer to Figure 6.5.1 and define the three basic trigonometric ratios as:

Three Basic Trigonometric Ratios

- sine(\theta) = \frac{\text{the length of the opposite side}}{\text{the length of the hypotenuse side}}
- cosine(\theta) = \frac{\text{the length of the adjacent side}}{\text{the length of the hypotenuse side}}
- tangent(\theta) = \frac{\text{the length of the opposite side}}{\text{the length of the adjacent side}}

Where θ is the measure of a reference angle measured in degrees.

Very often we use the abbreviations for sine, cosine, and tangent ratios.

- $sin(\theta) = \frac{opp}{hyp}$
- $cos(\theta) = \frac{adj}{hyp}$
- $tan(\theta) = \frac{opp}{adj}$

Some people remember the definition of the trigonometric ratios as SOH CAH TOA.

Let's use the ΔDEF from Example 6.5.2 to find the three ratios.

Example 6.5.2

For the given triangle find the sine, cosine and tangent ratio.

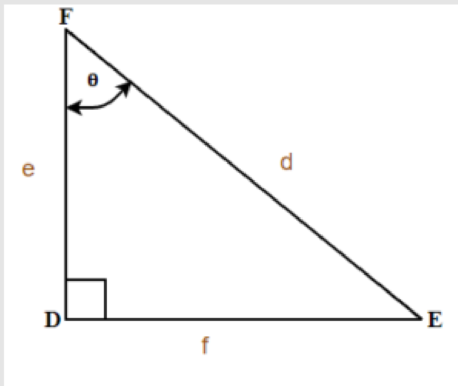


Figure 6.5.5

Solution

$$sin(\theta) = \frac{f}{d}$$

$$cos(\theta) = \frac{e}{d}$$

$$tan(\theta) = \frac{f}{e}$$

Try It

2) For the given triangle find the sine cosine and tangent ratio.

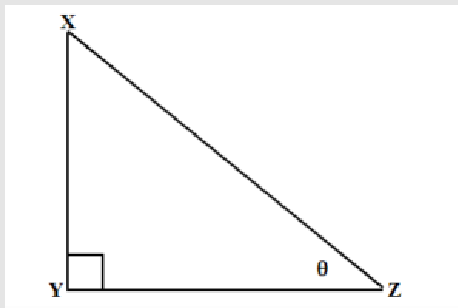


Figure 6.5.6

Solution

$$sin(\theta) = \frac{z}{y}$$

$$cos(\theta) = \frac{x}{y}$$

$$tan(\theta) = \frac{z}{x}$$

In Example 6.5.2, our reference angles can be angle E or angle F. Using the definition of trigonometric ratios, we can write $sin(E) = \frac{e}{d}$, $cos(E) = \frac{f}{d}$, and $tan(E) = \frac{e}{f}$.

When calculating we will usually round the ratios to four decimal places and at the end our final answer to one decimal place unless stated otherwise.

Example 6.5.3

For the given triangle find the sine, cosine and tangent ratios. If necessary round to four decimal places.

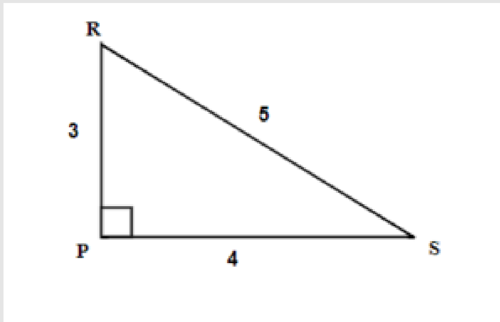


Figure 6.5.7

Solution

We have two possible reference angles: R and S.

Using the definitions, the trigonometric ratios for angle R are:

- $sin(R) = \frac{4}{5} = 0.8$
- $cos(R) = \frac{3}{5} = 0.6$
- $tan(R) = \frac{4}{3} = 1.3333...$

Using the definitions, the trigonometric ratios for angle S:

- $sin(S) = \frac{3}{5} = 0.6$
- $cos(S) = \frac{4}{5} = 0.8$
- $tan(S) = \frac{3}{4} = 0.75$

Try It

3) For the given triangle find the sine, cosine, and tangent ratios. If necessary round to four decimal places.

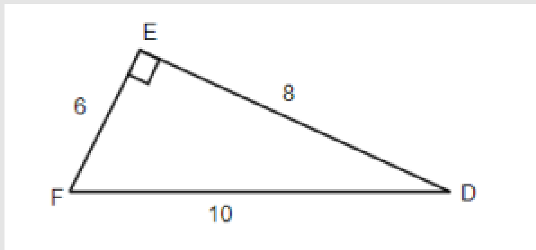


Figure 6.5.8

Solution

- $sin(F) = \frac{8}{10} = 0.8$
- $cos(F) = \frac{6}{10} = 0.6$
- $tan(F) = \frac{8}{6} = 1.3333...$
- $sin(D) = \frac{6}{10} = 0.6$
- $cos(D) = \frac{8}{10} = 0.8$
- $tan(D) = \frac{6}{8} = 0.75$

Now, let us use a scientific calculator to find the trigonometric ratios. Can you find the sin, cos, and tan buttons on your calculator? To find the trigonometric ratios make sure your calculator is in Degree Mode.

Example 6.5.4

Using a calculator find the trigonometric ratios. If necessary, round to 4 decimal places.

a) $sin(30^{\circ})$
b) $cos(45^{\circ})$
c) $tan(60^{\circ})$

Solution
Make sure your calculator is in Degree Mode.

a. Using a calculator find that $sin(30^{\circ}) = 0.5$
b. Using a calculator find that $cos(45^{\circ}) = 0.7071$ Rounded to 4 decimal places.
c. Using a calculator find that $tan(60^{\circ}) = 1.7321$ Rounded to 4 decimal places.

Try It

4) Find the trigonometric ratios. If necessary, round to 4 decimal places.

a. $sin(60^{\circ})$
b. $cos(30^{\circ})$
c. $tan(45^{\circ})$

Solution:
a. $sin(60^{\circ}) = 0.8660$
b. $cos(30^{\circ}) = 0.8660$
c. $tan(45^{\circ}) = 1$

Finding Missing Sides of a Right Triangle

In this section you will be using trigonometric ratios to solve right triangle problems. We will adapt our problem solving strategy for trigonometry applications. In addition, since those problems will involve the right triangle, it is helpful to draw it (if the drawing is not given) and label it with the given information. We will include this in the first step of the problem solving strategy for trigonometry applications.

HOW TO:

Solve Trigonometry Applications

1. Read the problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.
2. Identify what we are looking for.
3. Label what we are looking for by choosing a variable to represent it.
4. Find the required trigonometric ratio.
5. Solve the ratio using good algebra techniques.
6. Check the answer by substituting it back into the ratio in step 4 and by making sure it makes sense in the context of the problem.
7. Answer the question with a complete sentence

In the next few examples, having given the measure of one acute angle and the length of one side of the right triangle, we will solve the right triangle for the missing sides.

Example 6.5.5

Find the missing sides. Round your final answer to two decimal places

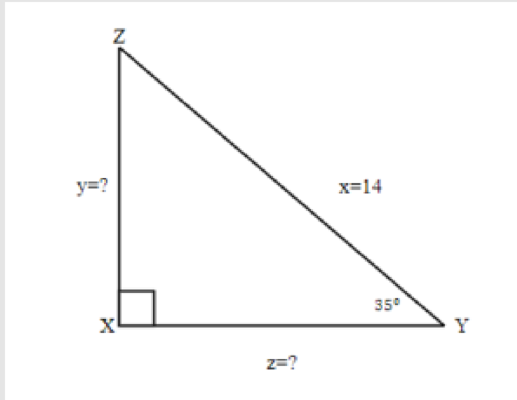


Figure 6.5.9

Solution

Step 1: **Read problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.**

A drawing is given. Angle Y is our reference angle, y is opposite side, z is adjacent side, and $x = 14$ is the hypotenuse.

Step 2: **Identify what you are looking for.**

a. The opposite side.

b. Adjacent side.

Step 3: **Label what we are looking for by choosing a variable to represent it.**

$$y = ?$$
$$z = ?$$

Step 4: **Find the required trigonometric ratio.**

$$\text{a. } sin(35^\circ) = \frac{y}{14}$$

$$\text{b. } cos(35^\circ) = \frac{z}{14}$$

Step 5: **Solve the ratio using good algebra techniques.**

$$\text{a. } 14\ sin(35^{\circ}) = y$$
$$8.03 = y$$

$$\text{b. } 14\ sin(35^{\circ}) = z$$
$$11.47 = z$$

Step 6: **Check the answer in the problem and by making sure it makes sense.**

$$\text{a. } 0.57 \stackrel{?}{=} 8.03 \div 14$$
$$0.57 = 0.57\checkmark$$

$$\text{b. } 0.82 \stackrel{?}{=} 11.47 \div 14$$
$$0.82 = 0.82\checkmark$$

Step 7: **Answer the question with a complete sentence.**

a. The opposite side is 8.03.
b. The adjacent side is 11.47.

Try It

5) Find the missing sides. Round your final answer to one decimal place.

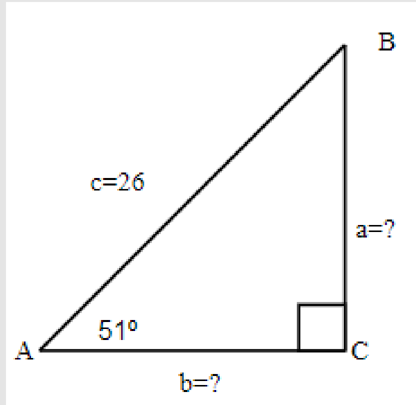


Figure 6.5.10

Solution

$a = 20.2$

$b = 16.4$

Example 6.5.6

Find the hypotenuse. Round your final answer to one decimal place.

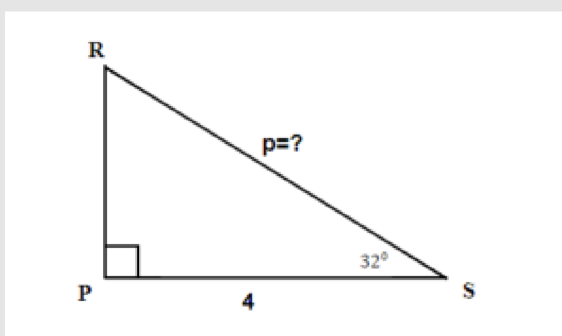


Figure 6.5.11

Solution

Step 1: **Read problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.**

A drawing is given. Angle S is our reference angle, R is opposite side, $r = 4$ is the adjacent side, and P is the hypotenuse.

Step 2: **Identify what you are looking for.**

The hypotenuse.

Step 3: **Label what we are looking for by choosing a variable to represent it.**

$$p = ?$$

Step 4: **Find the required trigonometric ratio.**

$$cos(32^\circ) = \frac{4}{p}$$

Step 5: **Solve the ratio using good algebra techniques.**

$$\begin{aligned} 0.8480 &= \frac{4}{p} \\ p &= 4.7170 \quad \text{Rounding the ratios to 4 decimal places} \end{aligned}$$

Step 6: **Check the answer in the problem and by making sure it makes sense.**

$$\begin{aligned} 0.8480 &\stackrel{?}{=} \frac{4}{4.7170} \\ 0.8480 &= 0.8480\checkmark \end{aligned}$$

Step 7: **Answer the question with a complete sentence.**

The hypotenuse is 4.7. *Round final answer to one decimal place.*

Try It

6) Find the hypotenuse. Round your final answer to one decimal place.

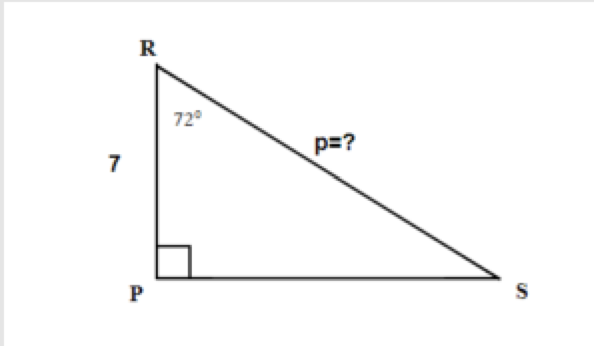


Figure 6.5.12

Solution

$p = 22.7$

Finding Missing Angles of a Right Triangle

Sometimes we have a right triangle with only the sides given. How can we find the missing angles? To find the missing angles, we use the inverse of the trigonometric ratios. The inverse buttons sin^{-1}, cos^{-1}, and tan^{-1} are on your scientific calculator.

Example 6.5.7

Find the angles. Round your final answer to one decimal place.

a. $sin(A) = 0.5$
b. $cos(B) = 0.9735$
c. $tan(C) = 2.89358$

Solution

Use your calculator and press the 2nd FUNCTION key and then press the SIN, COS, or TAN key

a. $$\begin{align*}A &= sin^{-1}(0.5)\\ \angle A &= 30^\circ\end{align*}$$

b. \begin{align*}B &= cos^{-1}(0.9735)\\\angle B\ &= 13.2^\circ\end{align*} *Rounded to one decimal place*

c. $$\begin{align*}C &= tan^{-1}(2.89358)\\ \angle C &= 70.9^\circ\end{align*}$$ *Rounded to one decimal place*

Try It

7) Find the angles. Round your final answer to one decimal place.

a. $sin(X) = 1$

b. $cos(Y) = 0.375$

c. $tan(Z) = 1.676767$

Solution

a. \angle X\ = 90^\circ

b. \angle Y\ = 68^\circ

c. \angle Z\ = 59.2^\circ

In the example below we have a right triangle with two sides given. Our acute angles are missing. Let us see what the steps are to find the missing angles.

Example 6.5.8

Find the missing $\angle T$. Round your final answer to one decimal place.

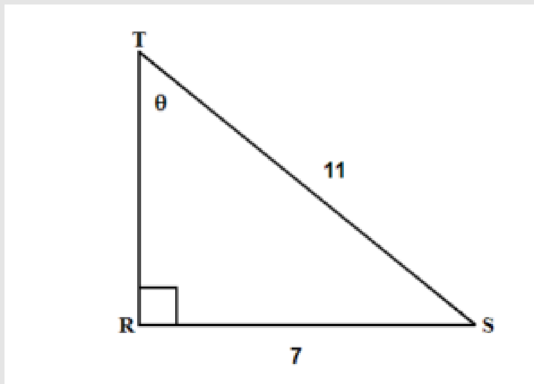


Figure 6.5.13

Solution

Step 1: **Read problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.**

A drawing is given. Angle T is our reference angle, $t = 7$ is the opposite side, s is adjacent side, and $r = 11$ is the hypotenuse.

Step 2: **Identify what you are looking for.**

Angle T.

Step 3: **Label what we are looking for by choosing a variable to represent it.**

$$\angle T = ?$$

Step 4: **Find the required trigonometric ratio.**

$$\sin(T) = \frac{7}{11}$$

Step 5: **Solve the ratio using good algebra techniques.**

$$\begin{aligned} sin(T) &= 0.6364 \\ T &= sin^{-1}(0.6364) \\ \angle T &= 39.5239^\circ \end{aligned}$$

Step 6: **Check the answer in the problem and by making sure it makes sense.**

$$\begin{aligned} sin\left(39.523^\circ\right) &\stackrel{?}{=} 0.6364 \\ 0.6364 &= 0.6364\checkmark \end{aligned}$$

Step 7: **Answer the question with a complete sentence.**

The missing angle T is 39.5°.

Try It

8) Find the missing angle X. Round your final answer to one decimal place.

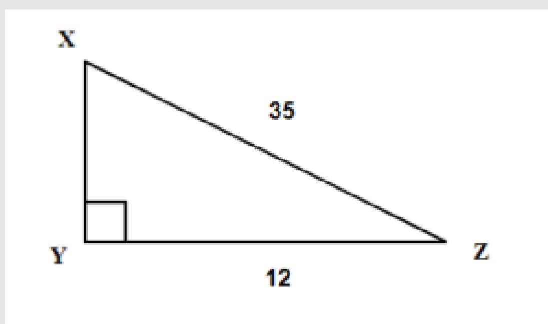


Figure 6.5.14

Solution

20.1°

Example 6.5.9

Find the missing angle A. Round your final answer to one decimal place.

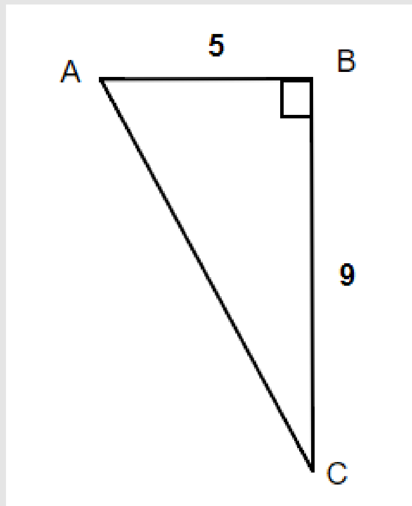


Figure 6.5.15

Solution

***Step 1:* Read problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.**

A drawing is given. Angle A is our reference angle, $a = 9$ is the opposite side, $c = 5$ is the adjacent side, and b is the hypotenuse.

***Step 2:* Identify what you are looking for.**

Angle A.

***Step 3:* Label what we are looking for by choosing a variable to represent it.**

$$\angle A = ?$$

***Step 4:* Find the required trigonometric ratio.**

$$tanA = \frac{9}{5}$$

***Step 5:* Solve the ratio using good algebra techniques.**

$$tan(A) = 1.8$$
$$A = tan^{-1}(1.8)$$
$$\angle A = 60.9^{\circ}$$

Step 6: **Check the answer in the problem and by making sure it makes sense.**

$$tan\ 60.9^{\circ} \stackrel{?}{=} 1.8$$
$$1.8 = 1.8\checkmark$$

Step 7: **Answer the question with a complete sentence.**

The missing angle A is 60.9°.

Try It

9) Find the missing angle C. Round your final answer to one decimal place.

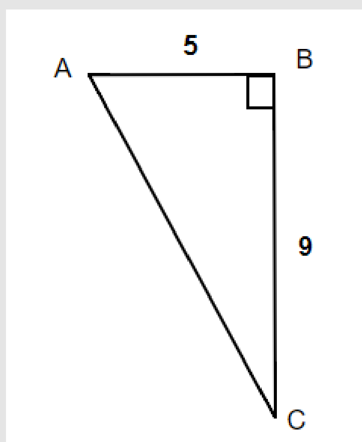


Figure 6.5.16

Solution

29.1°

Solving a Right Triangle

From the section before we know that any triangle has three sides and three interior angles. In a right triangle, when all six parts of the triangle are known, we say that the right triangle is solved.

Example 6.5.10

Solve the right triangle. Round your final answer to one decimal place.

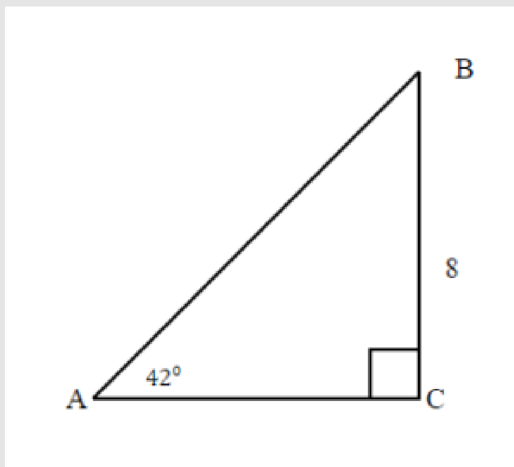


Figure 6.5.17

Solution

Since the sum of angles in any triangle is 180°, the measure of angle B can be easy calculated.

\begin{align*}\angle B\ &= 180^\circ - 90^\circ - 42^\circ\\\angle B\ &= 48^\circ\end{align*}

***Step 1:* Read problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.**

A drawing is given. Angle A is our reference angle, $a = 8$ is the opposite side, b is the adjacent side, and c is the hypotenuse.

Step 2: **Identify what you are looking for.**

a. The adjacent side.
b. The hypotenuse.

Step 3: **Label what we are looking for by choosing a variable to represent it.**

$$\text{a. } b = ?$$
$$\text{b. } c = ?$$

Step 4: **Find the required trigonometric ratio.**

$$\text{a. } tan(42^\circ) = \frac{8}{b}$$

$$\text{b. } sin(42^\circ) = \frac{8}{c}$$

Step 5: **Solve the ratio using good algebra techniques.**

$$\begin{aligned} \text{a. } 0.9004 &= \frac{8}{b} \\ 0.9004b &= 8 \\ b &= 8.8849 \end{aligned}$$

$$\begin{aligned} \text{b. } 0.6691 &= \frac{8}{c} \\ 0.6691c &= 8 \\ c &= 11.9563 \end{aligned}$$

Step 6: **Check the answer in the problem and by making sure it makes sense.**

$$\begin{aligned} \text{a. } tan(42^\circ) &\stackrel{?}{=} \frac{8}{8.8849} \\ 0.9 &= 0.9\checkmark \end{aligned}$$

$$\begin{aligned} \text{b. } sin(42^\circ) &\stackrel{?}{=} \frac{8}{11.9563} \\ 0.6691 &= 0.6691\checkmark \end{aligned}$$

Step 7: **Answer the question with a complete sentence.**

a. The adjacent side is 8.9. *Rounded to one decimal place.*
b. The hypotenuse is 12.

We solved the right triangle

$$\begin{aligned}\angle A &= 42^\circ\\ \angle B &= 48^\circ\\ \angle C &= 90^\circ\end{aligned}$$

$$\begin{aligned}a &= 8\\ b &= 8.9\\ c &= 12\end{aligned}$$

Try It

10) Solve the right triangle. Round your final answer to one decimal place.

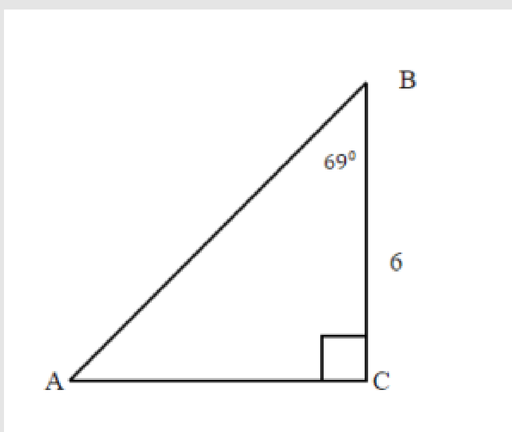


Figure 6.5.18

$$\begin{aligned}\angle A &= 21^\circ\\ \angle B &= 69^\circ\\ \angle C &= 90^\circ\end{aligned}$$

Solution

$a = 6$
$b = 15.6$
$c = 16.7$

Example 6.5.11

Solve the right triangle. Round to two decimal places.

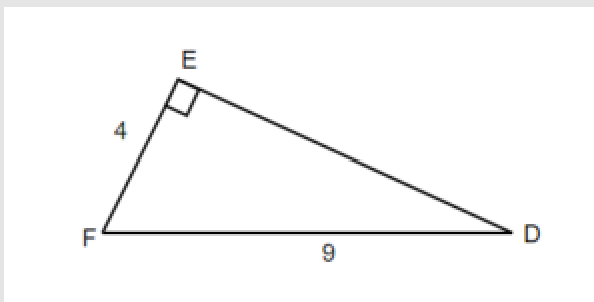


Figure 6.5.19

Solution

***Step 1:* Read problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.**

A drawing is given. Let angle D be our reference angle, $d = 4$ is the opposite side, f is the adjacent side, and $e = 9$ is the hypotenuse.

***Step 2:* Identify what you are looking for.**

a. Angle D.
b. The adjacent.

***Step 3:* Label what we are looking for by choosing a variable to represent it.**

$$\text{a. } \angle D = ?$$

$$\text{b. } f = ?$$

Step 4: **Find the required trigonometric ratio.**

$$\text{a. } sin(D) = \frac{4}{9}$$

$$\text{b. } 4^2 + f^2 = 9^2$$

Step 5: **Solve the ratio using good algebra techniques.**

$$\begin{aligned} \text{a.} \qquad sin(D) &= 0.4444 \\ D &= sin{-1}0.4444 \\ \angle D &= 26.3850^\circ \end{aligned}$$

$$\begin{aligned} \text{b.} \qquad 16 + f^2 &= 81 \\ f^2 &= 81 - 16 \\ f^2 &= 65 \\ f &= \sqrt{65} \\ f &= 8.06 \end{aligned}$$

Step 6: **Check the answer in the problem and by making sure it makes sense.**

$$\begin{aligned} \text{a.} \qquad sin(26.3850^\circ) &\stackrel{?}{=} \frac{4}{9} \\ 0.4444 &= 0.4444\checkmark \end{aligned}$$

$$\begin{aligned} \text{b.} \qquad 4^2 + 8.06^2 &\stackrel{?}{=} 9^2 \\ 81 &= 81\checkmark \end{aligned}$$

Step 7: **Answer the question with a complete sentence.**

a. The missing angle D is 26.39°.
b. The adjacent side is 8.06. *Rounded to two decimal places.*

The missing angle $\begin{aligned} F &= 180^\circ - 90^\circ - 26.39^\circ \\ F &= 63.61^\circ \end{aligned}$.

We solved the right triangle

$$\begin{aligned}\angle D &= 26.39^\circ \\ \angle E &= 90^\circ \\ \angle F &= 63.61^\circ\end{aligned}$$

$$\begin{aligned}d &= 4 \\ e &= 9 \\ f &= 8.06\end{aligned}$$

Try It

11) Solve the right triangle. Round to one decimal place.

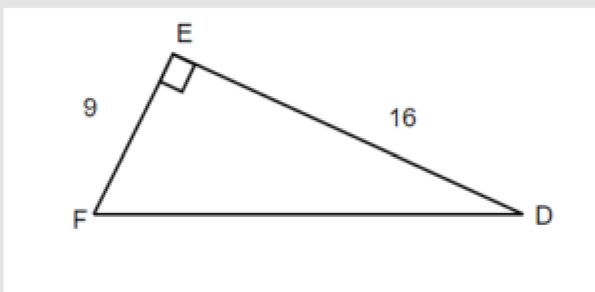


Figure 6.5.20

$$\begin{aligned}\angle D &= 29.3^\circ \\ \angle E &= 90^\circ \\ \angle F &= 60.7^\circ\end{aligned}$$

Solution

$$\begin{aligned}d &= 29.4 \\ e &= 18.4 \\ f &= 60.6\end{aligned}$$

Solve Applications Using Trigonometric Ratios

In the previous examples we were able to find missing sides and missing angles of a right triangle. Now, let's use the trigonometric ratios to solve real-life problems.

Many applications of trigonometric ratios involve understanding of an angle of elevation or angle of depression.

The angle of elevation is an angle between the horizontal line (ground) and the observer's line of sight.

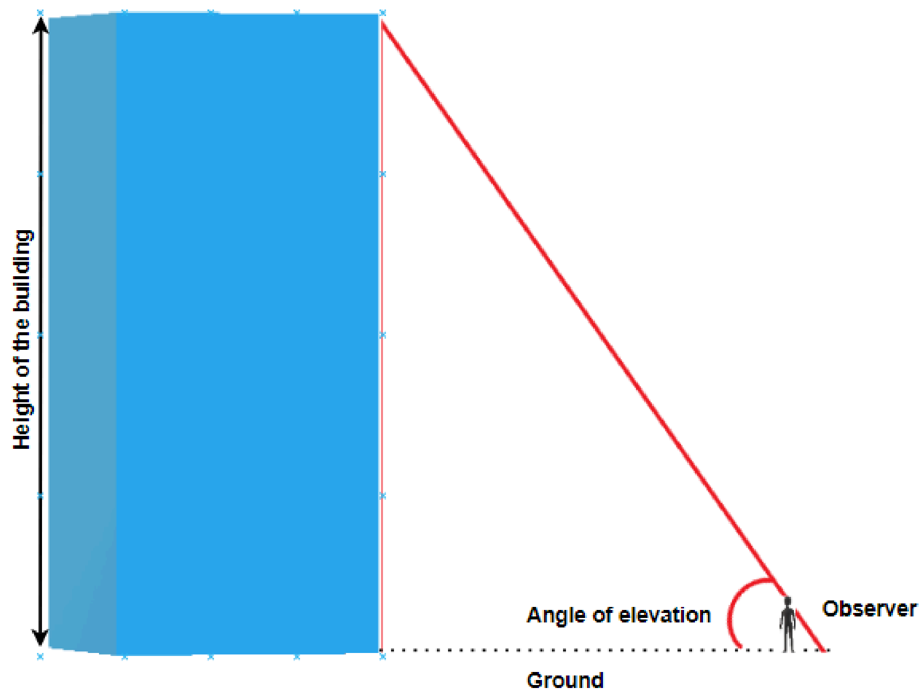


Figure 6.5.21

The angle of depression is the angle between horizontal line (that is parallel to the ground) and the observer's line of sight.

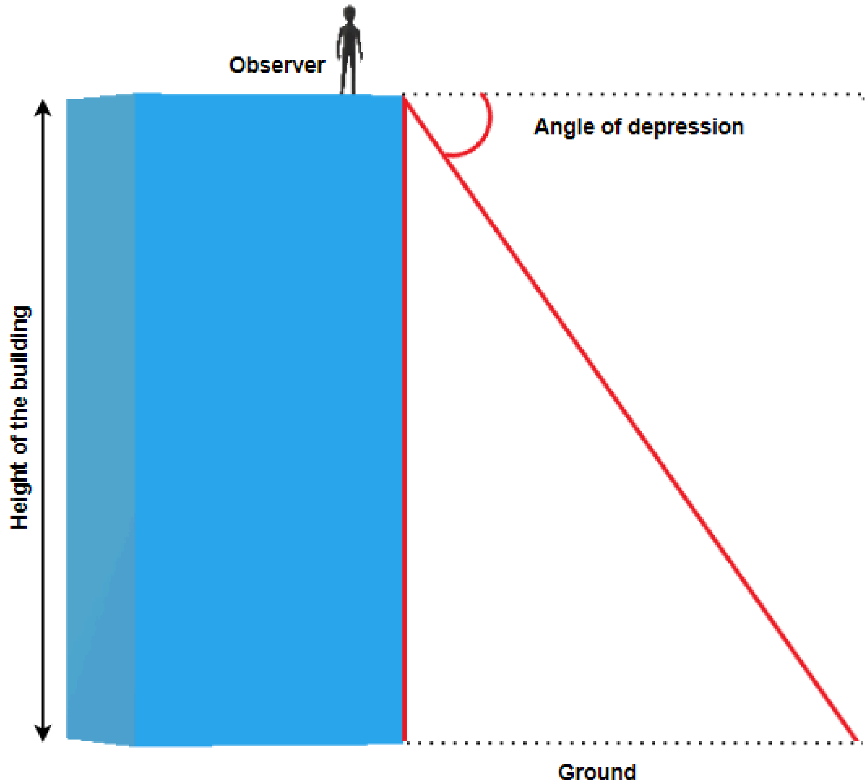


Figure 6.5.22

Example 6.5.12

James is standing 31 metres away from the base of the Harbour Centre in Vancouver. He looks up to the top of the building at a $78°$ angle. How tall is the Harbour Centre?

Solution

Step 1: **Read problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.**

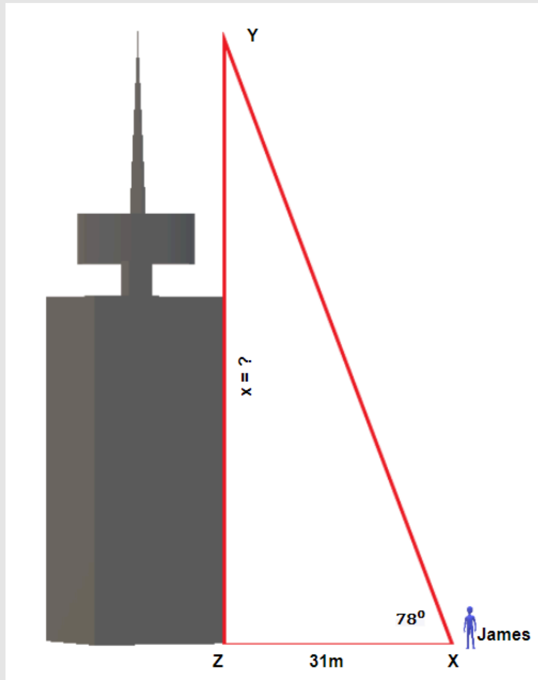


Figure 6.5.23

Angle X is our reference angle, x is opposite side, $y = 31$m is the adjacent side, and z is the hypotenuse.

Step 2: **Identify what you are looking for.**

The opposite side.

Step 3: **Label what we are looking for by choosing a variable to represent it.**

$$x = ?$$

Step 4: **Find the required trigonometric ratio.**

$$tan(78^\circ) = \frac{x}{31}$$

Step 5: **Solve the ratio using good algebra techniques.**

$$4.7046 = \frac{x}{31}$$
$$x = 145.8426$$

Step 6: **Check the answer in the problem and by making sure it makes sense.**

$$4.7046 \stackrel{?}{=} \frac{145.8426}{31}$$
$$4.7046 = 4.7046 \checkmark$$

***Step 7:* Answer the question with a complete sentence.**

The Harbour Centre is 145.8426 metres or rounded to 146 metres.

Try It

12) Marta is standing 23 metres away from the base of the tallest apartment building in Prince George and looks at the top of the building at a 62° angle. How tall is the building?

Solution

43.3 metres

Example 6.5.13

Thomas is standing at the top of the building that is 45 metres high and looks at her friend that is standing on the ground, 22 metres from the base of the building. What is the angle of depression?

Solution

***Step 1:* Read problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.**

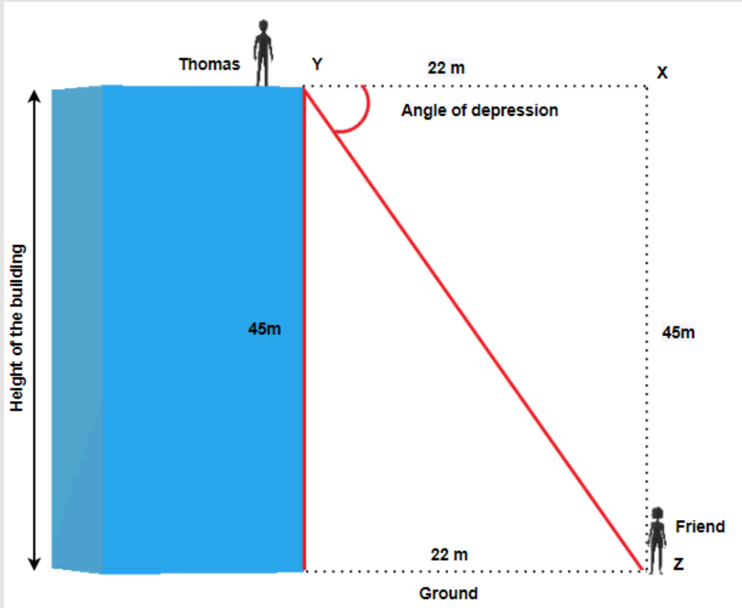


Figure 6.5.24

Angle Y is our reference angle, $y = 45$m is the opposite side, $z = 22$m is the adjacent side, and x is the hypotenuse.

Step 2: **Identify what you are looking for.**

The angle Y.

Step 3: **Label what we are looking for by choosing a variable to represent it.**

$$\angle Y = ?$$

Step 4: **Find the required trigonometric ratio.**

$$tan(Y) = \frac{45}{22}$$

Step 5: **Solve the ratio using good algebra techniques.**

$$\begin{aligned} tan(Y) &= 2.0455 \\ Y &= tan^{-1}(2.0455) \\ \angle\, Y &= 63.9470^{\circ} \end{aligned}$$

Step 6: **Check the answer in the problem and by making sure it makes sense.**

$$\begin{aligned} tan(63.9470^{\circ}) &\stackrel{?}{=} 2.0455 \\ 2.0455 &= 2.0455 \checkmark \end{aligned}$$

***Step 7:* Answer the question with a complete sentence.**

The angle of depression is $63.9470°$ or $64.0°$ rounded to one decimal place.

Try It

13) Hemanth is standing on the top of a cliff 250 feet above the ground and looks at his friend that is standing on the ground, 40 feet from the base of the cliff. What is the angle of depression?

Solution

$80.9°$

Key Concepts

- **Three Basic Trigonometric Ratios:** (Where θ is the measure of a reference angle measured in degrees).
  - sine\;\theta = \frac{\text{the length of the opposite side}}{\text{the length of the hypotenuse side}}
  - cosine\;\theta = \frac{\text{the length of the adjacent side}}{\text{the length of the hypotenuse side}}
  - tangent\;\theta = \frac{\text{the length of the opposite side}}{\text{the length of the adjacent side}}
- **Problem-Solving Strategy for Trigonometry Applications**

1. Read the problem and make sure all the words and ideas are understood. Draw the right triangle and label the given parts.
2. Identify what we are looking for.
3. Label what we are looking for by choosing a variable to represent it.
4. Find the required trigonometric ratio.
5. Solve the ratio using good algebra techniques.
6. Check the answer by substituting it back into the ratio solved in step 5 and by making sure it makes sense in the context of the problem.
7. Answer the question with a complete sentence.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://ecampusontario.pressbooks.pub/prehealthsciencesmath1/?p=6064#h5p-35

b. Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

6.6 UNIT SOURCES

Unit 6 Sources

6.1 "Use Properties of Angles, Triangles, and the Pythagorean Theorem" from Prealgebra 2e by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

6.2 "Use Properties of Rectangles, Triangles, and Trapezoids" from Prealgebra 2e by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

6.3 "Solve Geometry Applications: Circles and Irregular Figures" from Prealgebra 2e by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

6.4 "Solve Geometry Applications: Volume and Surface Area" from Prealgebra 2e by Open Stax – Rice University is licensed under a Creative Commons Attribution 4.0 International License.

6.5 "Solve Applications: Sine, Cosine and Tangent Ratios" from Business/Technical Mathematics by Izabela Mazur and Kim Moshenko is licensed under a Creative Commons Attribution 4.0 International License, except where otherwise noted.

VERSIONING HISTORY

This page provides a record of edits and changes made to this book since its initial publication. Whenever edits or updates are made in the text, we provide a record and description of those changes here. If the change is minor, the version number increases by 0.1. If the edits involve a number of changes, the version number increases to the next full number.

The files posted alongside this book always reflect the most recent version.

Version	Date	Change	Affected Web Page
1.0	December, 2022	First Publication	N/A